1. [40 pt] A bipartite graph is a graph \( G = (V, E) \) whose vertices can be partitioned into two sets \( (V = V_1 \cup V_2 \text{ and } V_1 \cap V_2 = \emptyset) \) such that there are no edges between vertices in the same set (for instance, if \( u, v \in V_1 \), then there is no edge between \( u \) and \( v \)). Give a linear-time algorithm to determine whether an undirected graph is bipartite.

2. [40 pt] We have three containers whose sizes are 10 pints, 7 pints, and 4 pints, respectively. The 7-pint and 4-pint containers start out full of water, but the 10-pint container is initially empty. We are allowed one type of operation: pouring the contents of one container into another, stopping only when the source container is empty or the destination container is full. We want to know if there is a sequence of pourings that leaves exactly 2 pints in the 7- or 4-pint container.
   
   (a) Model this as a graph problem: give a precise definition of the graph involved and state the specific question about this graph that needs to be answered.
   
   (b) What algorithm should be applied to solve the problem?

3. [20 pt] Prove that if \( G \) is a connected undirected graph with \( n \) vertices and \( n - 1 \) edges, then \( G \) is a tree.

4. [25 pt] Develop a nondeterministic algorithm of complexity \( O(n) \) to determine whether there is a subset of \( n \) numbers \( a_i, 1 \leq i \leq n \), that sums to \( m \). Assume that the set \( a_1, \ldots, a_n \) is given.

5. [25 pt] Show that the knapsack optimization problem reduces to the knapsack decision problem when all the \( p \)'s, \( w \)'s, and \( m \) are integer and the complexity is measured as a function of input length.
   
   Hint: If the input length is \( q \), then \( \sum p_i \leq n2^q \), where \( n \) is the number of objects. Use a binary search to determine the optimal solution value.

What to handin

Handin a hardcopy of all the sources, readme, makefile(s), and results. Create your programs in a directory called \textit{username.5} where \textit{username} is your login name on admiral. Once you are done with everything, remove the executables and object files, and issue the following commands:

\begin{verbatim}
% cd
% ~bhatias/bin/handin cs5130 5
\end{verbatim}