Morphological Image Processing

Morphology

- Identification, analysis, and description of the structure of the smallest unit of words
- Theory and technique for the analysis and processing of geometric structures
  - Based on set theory, lattice theory, topology, and random functions
  - Extract image components useful in the representation and description of region shape such as boundaries, skeletons, and convex hulls
  - Input in the form of images, output in the form of attributes extracted from those images
  - Attempt to extract the meaning of the images

Preliminaries

- Set theory in the context of image processing
  - Sets of pixels represent objects in the image
  - Set of all white pixels in a binary image is a complete morphological description of the image
- Sets in binary images
  - Members of the 2D integer space $\mathbb{Z}^2$
  - Each element of the set is a 2-tuple whose coordinates are the $(x, y)$ coordinates of a white pixel in the image
  - Gray scale images can be represented as a set of 3-tuples in $\mathbb{Z}^3$
  - Set reflection $\hat{B}$
    $$\hat{B} = \{ w | w = -b, \text{ for } b \in B \}$$
    * In binary image, $\hat{B}$ is the set of points in $B$ whose $(x, y)$ coordinates have been replaced by $(-x, -y)$
    * Figure 9.1a
  - Set translation
    * Translation of a set $B$ by point $z = (z_1, z_2)$ is denoted by $(B)_z$
    $$(B)_z = \{ c | c = b + z, \text{ for } b \in B \}$$
    * In binary image, $(B)_z$ is the set of points in $B$ whose $(x, y)$ coordinates have been replaced by $(x + z_1, y + z_2)$
    * Figure 9.1c
  - Set reflection and set translation are used to formulate operations based on so-called structuring elements
    * Small sets or subimages used to probe an image for properties of interest
    * Figure 9.2
    * Preference for ses to be rectangular arrays
    * Some locations are such that it does not matter whether they are part of the SE
      - Such locations are flagged by $\times$ in the SE
    * The origin of the SE must also be specified
      - Indicated by $\bullet$ in Figure 9.2
      - If SE is symmetric and no $\bullet$ is shown, the origin is assumed to be at the center of SE
  - Using ses in morphology
* Figure 9.3 – A simple set A and an SE B
* Convert A to a rectangular array by adding background elements
* Make background border large enough to accommodate the entire SE when the origin is on the border of original A
* Fill in the SE with the smallest number of background elements to make it a rectangular array
* Operation of set A using SE B
  * Create a new set by running B over A
  * Origin of B visits every element of A
  * If B is completely contained in A, mark that location as a member of the new set; else it is not a member of the new set
  * Results in eroding the boundary of A

Erosion and dilation

- Erosion
  * With A and B as sets in $Z^2$, erosion of A by B, denoted by $A \ominus B$ is defined as
    $$A \ominus B = \{ z \mid (B)_z \subseteq A \}$$
  * Set of all points z such that B, translated by z, is contained in A
  * B does not share any common elements with the background
    $$A \ominus B = \{ z \mid (B)_z \cap A^c = \emptyset \}$$
  * Figure 9.4
  * Example: Figure 9.5
    * Erosion shrinks or thins objects in a binary image
    * Morphological filter in which image details smaller than the SE are filtered/removed from the image

- Dilation
  * With A and B as sets in $Z^2$, dilation of A by B, denoted by $A \oplus B$ is defined as
    $$A \oplus B = \{ z \mid (\hat{B})_z \cap A \neq \emptyset \}$$
  * Reflect B about the origin, and shift the reflection by z
  * Dilation is the set of all displacements z such that B and A overlap by at least one element
    * An equivalent formulation is
      $$A \oplus B = \{ z \mid [(\hat{B})_z \cap A] \subseteq A \}$$
  * Grows or thickens objects in a binary image
  * Figure 9.6
  * Example: Figure 9.7
    * Bridging gaps in broken characters
    * Lowpass filtering produces a grayscale image; morphological operation produces a binary image

- Erosion and dilation are based on set operations and therefore, are nonlinear

- Duality
– Erosion and dilation are duals of each other with respect to set complementation and reflection

\[(A \ominus B)^c = A^c \oplus \hat{B} \]

\[(A \oplus B)^c = A^c \ominus \hat{B} \]

– Duality property is especially useful when \(SE\) is symmetric with respect to its origin so that \(\hat{B} = B\)

* Allows for erosion of an image by dilating its background \((A^c)\) using the same \(SE\) and complementing the results

– Proving duality

* Definition for erosion can be written as

\[(A \ominus B)^c = \{z \mid (B)_z \subseteq A\}^c \]

* \((B)_z \subseteq A \Rightarrow (B)_z \cap A^c = \emptyset\)

* So, the previous expression yields

\[(A \ominus B)^c = \{z \mid (B)_z \cap A^c = \emptyset\}^c \]

* The complement of the set of \(z\)'s that satisfy \((B)_z \cap A^c = \emptyset\) is the set of \(z\)'s such that \((B)_z \cap A^c \neq \emptyset\)

* This leads to

\[ (A \ominus B)^c = \{z \mid (B)_z \cap A^c \neq \emptyset\} \]

\[ = A^c \oplus \hat{B} \]

Opening and closing

- Opening smoothes the contours of an object, breaks narrow isthmuses, and eliminates thin protrusions
- Closing smoothes sections of contours, fusing narrow breaks and long thin gulfs, eliminates small holes, and fills gaps in the contour
- Opening of a set \(A\) by \(SE\ \hat{B}\), denoted by \(A \circ B\), is defined by

\[ A \circ B = (A \ominus B) \oplus B \]

- Closing of a set \(A\) by \(SE\ \hat{B}\), denoted by \(A \bullet B\), is defined by

\[ A \bullet B = (A \oplus B) \ominus B \]

- Geometric interpretation of opening expressed as a fitting process such that

\[ A \circ B = \bigcup \{(B)_z \mid (B)_z \subseteq A\} \]

– Union of all translates of \(B\) that fit into \(A\)
– Figure 9.8

- Similar interpretation of closing in Figure 9.9
- Example – Figure 9.10
- Duality property

\[(A \bullet B)^c = (A^c \circ \hat{B}) \]

\[(A \circ B)^c = (A^c \bullet \hat{B}) \]
• Opening operation satisfies the following properties
  1. $A \circ B \subseteq A$
  2. $C \subseteq D \Rightarrow C \circ B \subseteq D \circ B$
  3. $(A \circ B) \circ B = A \circ B$

• Similarly, closing operation satisfies
  1. $A \subseteq A \bullet B$
  2. $C \subseteq D \Rightarrow C \bullet B \subseteq D \bullet B$
  3. $(A \bullet B) \bullet B = A \bullet B$

  In both the above cases, multiple application of opening and closing has no effect after the first application

• Example: Removing noise from fingerprints
  – Figure 9.11
  – Noise as random light elements on a dark background

**Hit-or-miss transformation**

• Basic tool for shape detection in a binary image
  – Uses the morphological erosion operator and a pair of disjoint SES
  – First SES fits in the foreground of input image; second SES misses it completely
  – The pair of two SES is called composite structuring element

• Figure 9.12
  – Three disjoint shapes denoted $C$, $D$, and $E$
    * $A = C \cup D \cup E$
  – Objective: To find the location of one of the shapes, say $D$
  – Origin/location of each shape given by its center of gravity
  – Let $D$ be enclosed by a small window $W$
  – *Local background* of $D$ defined by the set difference $(W - D)$
    * Note that $D$ and $W - D$ provide us with the two disjoint SES

  $$D \cap (W - D) = \emptyset$$

  – Compute $A^c$
  – Compute $A \ominus D$
  – Compute $A^c \ominus (W - D)$
  – Set of locations where $D$ exactly fits inside $A$ is $(A \ominus D) \cap (A^c \ominus (W - D))$
    * The exact location of $D$
  – If $B$ is the set composed of $D$ and its background, the match of $B$ in $A$ is given by

  $$A \ominus B = (A \ominus D) \cap [A^c \ominus (W - D)]$$

• The above can be generalized to the composite SES being defined by $B = (B_1, B_2)$ leading to

  $$A \ominus B = (A \ominus B_1) \cap (A^c \ominus B_2)$$
- $B_1$ is the set formed from elements of $B$ associated with the object; $B_1 = D$
- $B_2 = (W - D)$

- A point $z$ in universe $A$ belongs to the output if $(B_1)_z$ fits in $A$ (hit) and $(B_2)_z$ misses $A$

**Some basic morphological algorithms**

- Useful in extracting image components for representation and description of shape
- **Boundary extraction**
  - Boundary of a set $A$
    * Denoted by $\beta(A)$
    * Extracted by eroding $A$ by a suitable se $B$ and computing set difference between $A$ and its erosion
      \[\beta(A) = A - (A \ominus B)\]
  - Figure 9.13
    * Using a larger se will yield a thicker boundary
    - Figure 9.14
- **Hole filling**
  - Hole
    * Background region surrounded by a connected border of foreground pixels
    - Algorithm based on set dilation, complementation, and intersection
    - Let $A$ be a set whose elements are 8-connected boundaries, each boundary enclosing a background (hole)
    - Given a point in each hole, we want to fill all holes
    - Start by forming an array $X_0$ of 0s of the same size as $A$
      * The locations in $X_0$ corresponding to the given point in each hole are set to 1
    - Let $B$ be a symmetric se with 4-connected neighbors to the origin
      \[
      \begin{array}{ccc}
      0 & 1 & 0 \\
      1 & 1 & 1 \\
      0 & 1 & 0 \\
      \end{array}
      \]
        - Compute $X_k = (X_{k-1} \oplus B) \cap A^c$  $k = 1, 2, 3, \ldots$
        - Algorithm terminates at iteration step $k$ if $X_k = X_{k-1}$
        - $X_k$ contains all the filled holes
        - $X_k \cup A$ contains all the filled holes and their boundaries
        - The intersection with $A^c$ at each step limits the result to inside the ROI
          * Also called *conditioned dilation*
    - Figure 9.15
    - Example: Figure 9.16
      * Thresholded image of polished spheres (ball bearings)
      * Eliminate reflection by hole filling
      * Points inside the background selected manually
- **Extraction of connected components**
  - Let $A$ be a set containing one or more connected components
– Form an array $X_0$ of the same size as $A$
  * All elements of $X_0$ are 0 except for one point in each connected component set to 1
– Select a suitable SE $B$, possibly an 8-connected neighborhood as

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

– Start with $X_0$ and find all connected components using the iterative procedure

\[
X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \ldots
\]

– Procedure terminates when $X_k = X_{k-1}$; $X_k$ contains all the connected components in the input image
– The only difference from the hole-filling algorithm is the intersection with $A$ instead of $A^c$
  * This is because here, we are searching for foreground points while in hole filling, we looked for background points (holes)

– Figure 9.17
– Example: Figure 9.18
  * X-ray image of chicken breast with bone fragments
  * Objects of “significant size” can be selected by applying erosion to the thresholded image
  * We may apply labels to the extracted components (region labeling)

• Convex hull

  – Convex set $A$
    * Straight line segment joining any two points in $A$ lies entirely within $A$
  – Convex hull $H$ of an arbitrary set of points $S$ is the smallest convex set containing $S$
  – Set difference $H - S$ is called the convex deficiency of $S$
  – Convex hull and convex deficiency are useful to describe objects
  – Algorithm to compute convex hull $C(A)$ of a set $A$
    * Figure 9.19
    * Let $B^i, i = 1, 2, 3, 4$ represent the four structuring elements in the figure
      * $B^i$ is a clockwise rotation of $B^{i-1}$ by $90^\circ$
    * Implement the equation

\[
X^i_k = (X_{k-1} \oplus B^i) \cup A \quad i = 1, 2, 3, 4 \text{ and } k = 1, 2, 3, \ldots
\]

  with $X^i_0 = A$
  * Apply hit-or-miss with $B^1$ till $X_k = X_{k-1}$, then, with $B^2$ over original $A$, $B^3$, and $B_4$
  * Procedure converges when $X^i_k = X^i_{k-1}$ and we let $D^i = X^i_k$
  * Convex hull of $A$ is given by

\[
C(A) = \bigcup_{i=1}^{4} D^i
\]

– Shortcoming of the above procedure
  * Convex hull can grow beyond the minimum dimensions required to guarantee convexity
  * May be fixed by limiting growth to not extend past the bounding box for the original set of points
  * Figure 9.20

• Thinning
– Transformation of a digital image into a simple topologically equivalent image
  * Remove selected foreground pixels from binary images
  * Used to tidy up the output of edge detectors by reducing all lines to single pixel thickness
– Thinning of a set $A$ by se $B$ is denoted by $A \otimes B$
– Defined in terms of hit-or-miss transform as

$$A \otimes B = A - (A \ast B) = A \cap (A \ast B)^c$$

– Only need to do pattern matching with se; no background operation required in hit-or-miss transform
– A more useful expression for thinning $A$ symmetrically based on a sequence of ses

$$\{B\} = \{B^1, B^2, \ldots, B^n\}$$

where $B^i$ is a rotated version of $B^{i-1}$
– Define thinning by a sequence of ses as

$$A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n)$$

– Figure 9.21
  * Iterate over the procedure till convergence

• Thickening
– Morphological dual of thinning defined by

$$A \circ B = A \cup (A \ast B)$$

– ses complements of those used for thinning
– Thickening can also be defined as a sequential operation

$$A \circ \{B\} = ((\ldots((A \circ B^1) \circ B^2) \ldots) \circ B^n)$$

– Figure 9.22
– Usual practice to thin the background and take the complement
  * May result in disconnected points
  * Post-process to remove the disconnected points

• Skeletons
– Figure 9.23
  * Skeleton $S(A)$ of a set $A$
  * Deductions
    1. If $z$ is a point of $S(A)$ and $(D)_z$ is the largest disk centered at $z$ and contained in $A$, one cannot find a larger disk (not necessarily centered at $z$) containing $(D)_z$ and included in $A$; $(D)_z$ is called a maximum disk
    2. Disk $(D)_z$ touches the boundary of $A$ at two or more different places
– Skeleton can be expressed in terms of erosions and openings

$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

where

$$S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$$
\[ (A \ominus kB) = (((A \ominus B) \ominus B) \ominus B) \ominus B \]

* \( K \) is the last iterative step before \( A \) erodes to an empty set

\[ K = \max\{k \mid (A \ominus kB) \neq \emptyset\} \]

* \( S(A) \) can be obtained as the union of skeleton subsets \( S_k(A) \)

* \( A \) can be reconstructed from the subsets using the equation

\[ \bigcup_{k=0}^{K} (S_k(A) \oplus kB) \]

where \( (S_k(A) \oplus kB) \) denotes \( k \) successive dilations of \( S_k(A) \)

\[ (S_k(A) \oplus kB) = (((S_k(A) \oplus B) \oplus B) \oplus \ldots) \oplus B \]

* Figure 9.24

**Pruning**

- Complement to thinning and skeletonizing algorithms to remove unwanted parasitic components
- Automatic recognition of hand-printed characters
  * Analyze the shape of the skeleton of each character
  * Skeletons characterized by “spurs” or parasitic components
  * Spurs caused during erosion by non-uniformities in the strokes
  * Assume that the length of a spur does not exceed a specific number of pixels
- Figure 9.25 – Skeleton of hand-printed “a”
  * Suppress a parasitic branch by successively eliminating its end point
  * Assumption: Any branch with \( \leq 3 \) pixels will be removed
  * Achieved with thinning of an input set \( A \) with a sequence of SEs designed to detect only end points

\[ X_1 = A \otimes \{B\} \]

* Figure 9.25d – Result of applying the above thinning three times
* Restore the character to its original form with the parasitic branches removed
* Form a set \( X_2 \) containing all end points in \( X_1 \)

\[ X_2 = \bigcup_{k=1}^{8} (X_1 \oplus B^k) \]

* Dilate end points three times using set \( A \) as delimiter

\[ X_3 = (X_2 \oplus H) \cap A \]

where \( H \) is a \( 3 \times 3 \) SE of 1s and intersection with \( A \) is applied after each step
* The final result comes from

\[ X_4 = X_1 \cup X_3 \]

**Morphological Reconstruction**

- Works on two images and an SE
– One image is called the **marker** and contains the starting points for transformation
– Second image is called the **mask** and contains the transformation or constraint
– SE is used to define connectivity

• Geodesic dilation and erosion

– Let $F$ be the marker image and $G$ be the mask image
– $F$ and $G$ are binary images and $F \subseteq G$
– Geodesic dilation
  * Geodesic dilation of size 1 of $F$ with respect to $G$ is defined as
    $$D_G^{(1)}(F) = (F \oplus B) \cap G$$
  * Geodesic dilation of size $n$ of $F$ with respect to $G$ is defined as
    $$D_G^{(n)}(F) = D_G^{(1)} \left[ D_G^{(n-1)}(F) \right]$$
    with $D_G^{(0)}(F) = F$
    - Set intersection is performed at each step of recursion
    - Mask $G$ limits the growth of marker $F$
  * Figure 9.26
– Geodesic erosion
  * Geodesic erosion of size 1 of $F$ with respect to $G$ is defined as
    $$E_G^{(1)}(F) = (F \ominus B) \cup G$$
  * Geodesic erosion of size $n$ of $F$ with respect to $G$ is defined as
    $$E_G^{(n)}(F) = E_G^{(1)} \left[ E_G^{(n-1)}(F) \right]$$
    with $E_G^{(0)}(F) = F$
    - Set union is performed at each step of recursion
    - Guarantees that geodesic erosion of an image remains greater than or equal to its mask
  * Figure 9.27
    - Bottom leftmost pixel of $F$ should be white
– Geodesic dilation and erosion are duals with respect to set complementation
– Both operations converge after a finite number of iterative steps

• Morphological reconstruction by dilation and erosion

– Morphological reconstruction by dilation
  * Given mask image $G$ and marker image $F$
  * Denoted by $R_G^D(F)$
  * Defined as the geodesic dilation of $F$ with respect to $G$ iterated till stability is achieved
    $$R_G^D(F) = D_G^{(k)}(F)$$
    with $k$ such that $D_G^{(k)}(F) = D_G^{(k+1)}(F)$
  * Figure 9.28
– Morphological reconstruction by erosion
  * Given mask image $G$ and marker image $F$
* Denoted by $R_E^G(F)$
* Defined as the geodesic erosion of $F$ with respect to $G$ iterated till stability is achieved

$$R_E^G(F) = E_G^{(k)}(F)$$

with $k$ such that $E_G^{(k)}(F) = E_G^{(k+1)}(F)$

- Reconstruction by dilation and erosion are duals with respect to set complementation

- Sample applications
  - Opening by reconstruction
    * Morphological opening
      - Erosion removes small objects
      - Dilation attempts to restore the shape of objects that remain
      * Accuracy dependent on the shape of objects and SE
    * Opening by reconstruction restores exactly the shape of objects that remain
    * Opening by reconstruction of size $n$ of an image $F$ is defined as the reconstruction by dilation of $F$ from the erosion of size $n$ of $F$

$$O_R^{(n)}(F) = R_D^F[F \ominus nB]$$

$F$ is used as a mask
* Figure 9.29
  - Extract characters containing long vertical strokes

- Filling holes
  * Earlier algorithm based on knowledge of a starting point for each hole
  * Now, we develop a fully automated procedure based on morphological reconstruction
  * Input binary image $I(x,y)$
  * Marker image

$$F(x,y) = \begin{cases} 
1 - I(x,y) & \text{if } (x,y) \text{ is on the border of } I \\
0 & \text{otherwise}
\end{cases}$$

* The output binary image with all holes filled is given by

$$H = \left[R_D^I(F)\right]^c$$

* Figure 9.30
* Figure 9.31

- Border clearing
  * Remove objects that touch a border of image so that only the objects that are completely enclosed in the picture remain
  * Use original image $I(x,y)$ as the mask
  * Marker image

$$F(x,y) = \begin{cases} 
I(x,y) & \text{if } (x,y) \text{ is on the border of } I \\
0 & \text{otherwise}
\end{cases}$$

* Compute the image $X$ as

$$X = I - R_D^I(F)$$

$X$ has no objects touching the border
* Figure 9.32

**Gray-scale morphology**
• Gray scale image $f(x, y)$, under the assumptions followed so far

• SE $b(x, y)$
  – The coefficients of SE may be in $\mathbb{Z}$ or $\mathbb{R}$
  – SE performs the same basic functions as binary counterparts; used as probes to examine a given image for specific properties
  – Figure 9.34 – Nonflat and flat SE
  – Used infrequently in practice
  – Reflection of an SE in gray scale morphology is denoted by

$$\hat{b}(x, y) = b(-x, -y)$$

• Erosion and dilation
  – Erosion
    * Erosion of $f$ by a flat SE $b$ at any location $(x, y)$ is defined as minimum value of the image coincident with $b$ when the origin $b$ is at $(x, y)$

$$[f \ominus b](x, y) = \min_{(s, t) \in b} \{ f(x + s, y + t) \}$$

  – Dilation
    * Dilation of $f$ by a flat SE $b$ at any location $(x, y)$ is defined as maximum value of the image coincident with $b$ when the origin $\hat{b}$ is at $(x, y)$

$$[f \oplus b](x, y) = \max_{(s, t) \in \hat{b}} \{ f(x + s, y + t) \}$$

where $\hat{b} = b(-x, -y)$
  – Figure 9.35