Morphological Image Processing

Morphology

- Identification, analysis, and description of the structure of the smallest unit of words
- Theory and technique for the analysis and processing of geometric structures
  - Based on set theory, lattice theory, topology, and random functions
  - Extract image components useful in the representation and description of region shape such as boundaries, skeletons, and convex hull
  - Input in the form of images, output in the form of attributes extracted from those images
  - Attempt to extract the *meaning* of the images

Preliminaries

- Set theory in the context of image processing
  - Sets of pixels represent objects in the image
  - Set of all white pixels in a binary image is a complete morphological description of the image
- Sets in binary images
  - Members of the 2D integer space Z^2
  - Each element of the set is a 2-tuple whose coordinates are the (x, y) coordinates of a white pixel in the image
    - Gray scale images can be represented as a set of 3-tuples in Z^3
    - Higher dimensions can be used to represent other attributes such as color
  - Morphological operations
    - Defined in terms of sets: objects and structuring elements
    - Objects defined as sets of foreground pixels
    - SEs specified in terms of both foreground and background pixels
      - SEs may contain "don't care" elements
    - Sets are embedded in rectangular arrays due to rectangular nature of images
      - Figure 9.1
  - Set reflection $\hat{B}$

$$\hat{B} = \{w | w = -b, \ \text{for } b \in B\}$$

    - In binary image, $\hat{B}$ is the set of points in B whose (x, y) coordinates have been replaced by $(-x, -y)$
    - Figure 9.2
  - Set translation
    - Translation of a set B by point $z = (z_1, z_2)$ is denoted by $(B)_z$

$$(B)_z = \{c | c = b + z, \ \text{for } b \in B\}$$

    - In binary image, $(B)_z$ is the set of points in B whose (x, y) coordinates have been replaced by $(x + z_1, y + z_2)$
    - Figure 9.1c
  - Set reflection and set translation are used to formulate operations based on so-called *structuring elements*
    - Small sets or subimages used to probe an image for properties of interest

    * Figure 9.2
    * Preference for SEs to be rectangular arrays
    * Some locations are such that it does not matter whether they are part of the SE
      · Such locations are flagged by $\times$ in the SE
    * The origin of the SE must also be specified
      · Indicated by $\bullet$ in Figure 9.2
      · If SE is symmetric and no $\bullet$ is shown, the origin is assumed to be at the center of SE
  - Using SEs in morphology
    * Figure 9.3 – A simple set A and an SE B
    * Convert A to a rectangular array by adding background elements
    * Make background border large enough to accommodate the entire SE when the origin is on the border of original A
    * Fill in the SE with the smallest number of background elements to make it a rectangular array
    * Operation of set A using SE B
      · Create a new set by running B over A
      · Origin of B visits every element of A
      · If B is completely contained in A, mark that location as a member of the new set; else it is not a member of the new set
      · Results in *eroding* the boundary of A

Erosion and dilation

- Erosion
  - With A and B as sets in Z^2, erosion of A by B, denoted by $A \ominus B$ is defined as

$$A \ominus B = \{z \mid (B)_z \subseteq A\}$$

  - Set of all points z such that B, translated by z, is contained in A
  - B does not share any common elements with the background

$$A \ominus B = \{z \mid (B)_z \cap A^c = \emptyset$$

  - Figure 9.4
  - Example: Figure 9.5
    * Erosion shrinks or thins objects in a binary image
    * Morphological filter in which image details smaller than the SE are filtered/removed from the image
- Dilation
  - With A and B as sets in Z^2, dilation of A by B, denoted by $A \oplus B$ is defined as

$$A \oplus B = \{z \mid (\hat{B})_z \cap A \neq \emptyset\}$$

  - Reflect B about the origin, and shift the reflection by z
  - Dilation is the set of all displacements z such that B and A overlap by at least one element
  - An equivalent formulation is

$$A \oplus B = \{z \mid [(\hat{B})_z \cap A] \subseteq A\}$$

  - Grows or thickens objects in a binary image
  - Figure 9.6

- Example: Figure 9.7
  * Bridging gaps in broken characters
  * Lowpass filtering produces a grayscale image; morphological operation produces a binary image
- Erosion and dilation are based on set operations and therefore, are nonlinear
- Duality
  - Erosion and dilation are duals of each other with respect to set complementation and reflection

$$\begin{aligned}(A \ominus B)^c &= A^c \oplus \hat{B} \\ (A \oplus B)^c &= A^c \ominus \hat{B}\end{aligned}$$

  - Duality property is especially useful when SE is symmetric with respect to its origin so that $\hat{B} = B$
    * Allows for erosion of an image by dilating its background (A^c) using the same SE and complementing the results
  - Proving duality
    * Definition for erosion can be written as

$$(A \ominus B)^c = \{z \mid (B)_z \subseteq A\}^c$$

    * $(B)_z \subseteq A \Rightarrow (B)_z \cap A^c = \emptyset$
    * So, the previous expression yields

$$(A \ominus B)^c = \{z \mid (B)_z \cap A^c = \emptyset\}^c$$

    * The complement of the set of z's that satisfy $(B)_z \cap A^c = \emptyset$ is the set of z's such that $(B)_z \cap A^c \neq \emptyset$
    * This leads to

$$\begin{aligned}(A \ominus B)^c &= \{z \mid (B)_z \cap A^c \neq \emptyset\} \\ &= A^c \oplus \hat{B}\end{aligned}$$

Opening and closing

- Opening smoothes the contours of an object, breaks narrow isthmuses, and eliminates thin protrusions
- Closing smoothes sections of contours, fusing narrow breaks and long thin gulfs, eliminates small holes, and fills gaps in the contour
- Opening of a set A by SE B, denoted by $A \circ B$, is defined by

$$A \circ B = (A \ominus B) \oplus B$$

- Closing of a set A by SE B, denoted by $A \bullet B$, is defined by

$$A \bullet B = (A \oplus B) \ominus B$$

- Geometric interpretation of opening expressed as a fitting process such that

$$A \circ B = \bigcup \{(B)_z \mid (B)_z \subseteq A\}$$

  - Union of all translates of B that fit into A
  - Figure 9.8
- Similar interpretation of closing in Figure 9.9

- Example – Figure 9.10
- Duality property

$$\begin{aligned}(A \bullet B)^c &= (A^c \circ \hat{B})\\ (A \circ B)^c &= (A^c \bullet \hat{B})\end{aligned}$$

- Opening operation satisfies the following properties
  1. $A \circ B \subseteq A$
  2. $C \subseteq D \Rightarrow C \circ B \subseteq D \circ B$
  3. $(A \circ B) \circ B = A \circ B$
- Similarly, closing operation satisfies
  1. $A \subseteq A \bullet B$
  2. $C \subseteq D \Rightarrow C \bullet B \subseteq D \bullet B$
  3. $(A \bullet B) \bullet B = A \bullet B$

  – In both the above cases, multiple application of opening and closing has no effect after the first application
- Example: Removing noise from fingerprints
  – Figure 9.11
  – Noise as random light elements on a dark background

Hit-or-miss transformation

- Basic tool for shape detection in a binary image
  – Uses the morphological erosion operator and a pair of disjoint SEs B_1 and B_2; $B_2 = B_1^c$
  – First SE fits in the foreground of input image I; second SE misses it completely
  – The pair of two SEs is called *composite structuring element*
  – The operator is defined as

$$\begin{aligned}I \circledast B_{1,2} &= \{z \mid (B_1)_z \subseteq A \text{ and } (B_2)_z \subseteq A^c\}\\ &= (A \ominus B_1) \cap (A^c \ominus B_2)\end{aligned}$$

- Figure 9.12
  – Three disjoint shapes denoted C, D, and E
    * $A = C \cup D \cup E$
  – Objective: To find the location of one of the shapes, say D
  – Origin/location of each shape given by its center of gravity
  – Let D be enclosed by a small window W
  – *Local background* of D defined by the set difference $(W - D)$
    * Note that D and $W - D$ provide us with the two disjoint SEs

$$D \cap (W - D) = \emptyset$$

  – Compute A^c

  - Compute $A \ominus D$
  - Compute $A^c \ominus (W - D)$
  - Set of locations where D exactly fits inside A is $(A \ominus D) \cap (A^c \ominus (W - D))$
    * The exact location of D
  - If B is the set composed of D and its background, the match of B in A is given by

$$A \circledast B = (A \ominus D) \cap [A^c \ominus (W - D)]$$

- The above can be generalized to the composite SE being defined by $B = (B_1, B_2)$ leading to

$$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$$

  - B_1 is the set formed from elements of B associated with the object; $B_1 = D$
  - $B_2 = (W - D)$
- A point z in universe A belongs to the output if $(B_1)_z$ fits in A (hit) and $(B_2)_z$ misses A
- The object can be directly detected if we can process both foreground and background pixels simultaneously
  - Remake the SE to restate the transform as

$$I \circledast B = \{z | (B)_z \subseteq I\}$$

  - B is made up of both foreground and background
  - Figure 9.13

Some basic morphological algorithms

- Useful in extracting image components for representation and description of shape
- Boundary extraction
  - Boundary of a set A
    * Denoted by $\beta(A)$
    * Extracted by eroding A by a suitable SE B and computing set difference between A and its erosion

$$\beta(A) = A - (A \ominus B)$$

  - Figure 9.13
    * Using a larger SE will yield a thicker boundary
  - Figure 9.14
- Hole filling
  - Hole
    * Background region surrounded by a connected border of foreground pixels
  - Algorithm based on set dilation, complementation, and intersection
  - Let A be a set whose elements are 8-connected boundaries, each boundary enclosing a background (hole)
  - Given a point in each hole, we want to fill all holes
  - Start by forming an array X_0 of 0s of the same size as A
    * The locations in X_0 corresponding to the given point in each hole are set to 1
  - Let B be a symmetric SE with 4-connected neighbors to the origin

0	1	0
1	1	1
0	1	0

  - Compute $X_k = (X_{k-1} \oplus B) \cap A^c \quad k = 1, 2, 3, \ldots$
  - Algorithm terminates at iteration step k if $X_k = X_{k-1}$
  - X_k contains all the filled holes
  - $X_k \cup A$ contains all the filled holes and their boundaries
  - The intersection with A^c at each step limits the result to inside the ROI
    * Also called *conditioned dilation*
  - Figure 9.15
  - Example: Figure 9.16
    * Thresholded image of polished spheres (ball bearings)
    * Eliminate reflection by hole filling
    * Points inside the background selected manually

- Extraction of connected components
  - Let A be a set containing one or more connected components
  - Form an array X_0 of the same size as A
    * All elements of X_0 are 0 except for one point in each connected component set to 1
  - Select a suitable SE B, possibly an 8-connected neighborhood as

1	1	1
1	1	1
1	1	1

  - Start with X_0 and find all connected components using the iterative procedure

$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \ldots$$

  - Procedure terminates when $X_k = X_{k-1}$; X_k contains all the connected components in the input image
  - The only difference from the hole-filling algorithm is the intersection with A instead of A^c
    * This is because here, we are searching for foreground points while in hole filling, we looked for background points (holes)
  - Figure 9.17
  - Example: Figure 9.18
    * X-ray image of chicken breast with bone fragments
    * Objects of "significant size" can be selected by applying erosion to the thresholded image
    * We may apply labels to the extracted components (region labeling)

- Convex hull
  - Convex set A
    * Straight line segment joining any two points in A lies entirely within A
  - Convex hull H of an arbitrary set of points S is the smallest convex set containing S
  - Set difference $H - S$ is called the *convex deficiency* of S
  - Convex hull and convex deficiency are useful to describe objects
  - Algorithm to compute convex hull $C(A)$ of a set A
    * Figure 9.19

    - Let $B^i, i = 1, 2, 3, 4$ represent the four structuring elements in the figure
        - B^i is a clockwise rotation of B^{i-1} by 90°
    - Implement the equation

$$X_k^i = (X_{k-1} \circledast B^i) \cup A \qquad i = 1, 2, 3, 4 \text{ and } k = 1, 2, 3, \ldots$$

      with $X_0^i = A$
    - Apply hit-or-miss with B^1 till $X_k == X_{k-1}$, then, with B^2 over original A, B^3, and B_4
    - Procedure converges when $X_k^i = X_{k-1}^i$ and we let $D^i = X_k^i$
    - Convex hull of A is given by

$$C(A) = \bigcup_{i=1}^{4} D^i$$

  - Shortcoming of the above procedure
    - Convex hull can grow beyond the minimum dimensions required to guarantee convexity
    - May be fixed by limiting growth to not extend past the bounding box for the original set of points
    - Figure 9.20

- Thinning
  - Transformation of a digital image into a simple topologically equivalent image
    - Remove selected foreground pixels from binary images
    - Used to tidy up the output of edge detectors by reducing all lines to single pixel thickness
  - Thinning of a set A by SE B is denoted by $A \otimes B$
  - Defined in terms of hit-or-miss transform as

$$\begin{aligned} A \otimes B &= A - (A \circledast B) \\ &= A \cap (A \circledast B)^c \end{aligned}$$

  - Only need to do pattern matching with SE; no background operation required in hit-or-miss transform
  - A more useful expression for thinning A symmetrically based on a sequence of SEs

$$\{B\} = \{B^1, B^2, \ldots, B^n\}$$

    where B^i is a rotated version of B^{i-1}
  - Define thinning by a sequence of SEs as

$$A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n)$$

  - Figure 9.21
    - Iterate over the procedure till convergence

- Thickening
  - Morphological dual of thinning defined by

$$A \odot B = A \cup (A \circledast B)$$

  - SEs complements of those used for thinning
  - Thickening can also be defined as a sequential operation

$$A \odot \{B\} = ((\ldots((A \odot B^1) \odot B^2) \ldots) \odot B^n)$$

  - Figure 9.22

- Usual practice to thin the background and take the complement
  * May result in disconnected points
  * Post-process to remove the disconnected points

- Skeletons
  - Figure 9.23
    * Skeleton $S(A)$ of a set A
    * Deductions
      1. If z is a point of $S(A)$ and $(D)_z$ is the largest disk centered at z and contained in A, one cannot find a larger disk (not necessarily centered at z) containing $(D)_z$ and included in A; $(D)_z$ is called a *maximum disk*
      2. Disk $(D)_z$ touches the boundary of A at two or more different places
  - Skeleton can be expressed in terms of erosions and openings

$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

where

$$S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$$

* $A \ominus kB$ indicates k successive erosions of A

$$(A \ominus kB) = ((\ldots((A \ominus B) \ominus B) \ominus \ldots) \ominus B)$$

* K is the last iterative step before A erodes to an empty set

$$K = \max\{k \mid (A \ominus kB) \neq \emptyset\}$$

* $S(A)$ can be obtained as the union of skeleton subsets $S_k(A)$
* A can be reconstructed from the subsets using the equation

$$\bigcup_{k=0}^{K} (S_k(A) \oplus kB)$$

where $(S_k(A) \oplus kB)$ denotes k successive dilations of $S_k(A)$

$$(S_k(A) \oplus kB) = ((\ldots((S_k(A) \oplus B) \oplus B) \oplus \ldots) \oplus B)$$

* Figure 9.24

- Pruning
  - Complement to thinning and sketonizing algorithms to remove unwanted parasitic components
  - Automatic recognition of hand-printed characters
    * Analyze the shape of the skeleton of each character
    * Skeletons characterized by "spurs" or parasitic components
    * Spurs caused during erosion by non-uniformities in the strokes
    * Assume that the length of a spur does not exceed a specific number of pixels
  - Figure 9.25 – Skeleton of hand-printed "a"
    * Suppress a parasitic branch by successively eliminating its end point
    * Assumption: Any branch with ≤ 3 pixels will be removed

        * Achieved with thinning of an input set A with a sequence of SEs designed to detect only end points

$$X_1 = A \otimes \{B\}$$

        * Figure 9.25d – Result of applying the above thinning three times
        * Restore the character to its original form with the parasitic branches removed
        * Form a set X_2 containing all end points in X_1

$$X_2 = \bigcup_{k=1}^{8} (X_1 \circledast B^k)$$

        * Dilate end points three times using set A as delimiter

$$X_3 = (X_2 \oplus H) \cap A$$

          where H is a 3×3 SE of 1s and intersection with A is applied after each step
        * The final result comes from

$$X_4 = X_1 \cup X_3$$

Morphological Reconstruction

- Works on two images and an SE
    - One image is called the *marker* and contains the starting points for transformation
    - Second image is called the *mask* and contains the transformation or constraint
    - SE is used to define connectivity
- Geodesic dilation and erosion
    - Let F be the marker image and G be the mask image
    - F and G are binary images and $F \subseteq G$
    - Geodesic dilation
        * Geodesic dilation of size 1 of F with respect to G is defined as

$$D_G^{(1)}(F) = (F \oplus B) \cap G$$

        * Geodesic dilation of size n of F with respect to G is defined as

$$D_G^{(n)}(F) = D_G^{(1)}\left[D_G^{(n-1)}(F)\right]$$

          with $D_G^{(0)}(F) = F$
            · Set intersection is performed at each step of recursion
            · Mask G limits the growth of marker F
        * Figure 9.26
    - Geodesic erosion
        * Geodesic erosion of size 1 of F with respect to G is defined as

$$E_G^{(1)}(F) = (F \ominus B) \cup G$$

        * Geodesic erosion of size n of F with respect to G is defined as

$$E_G^{(n)}(F) = E_G^{(1)}\left[E_G^{(n-1)}(F)\right]$$

          with $E_G^{(0)}(F) = F$

        - Set union is performed at each step of recursion
        - Guarantees that geodesic erosion of an image remains greater than or equal to its mask
    - Figure 9.27
        - Bottom leftmost pixel of F should be white
  - Geodesic dilation and erosion are duals with respect to set complementation
  - Both operations converge after a finite number of iterative steps

- Morphological reconstruction by dilation and erosion
  - Morphological reconstruction by dilation
    - Given mask image G and marker image F
    - Denoted by $R_G^D(F)$
    - Defined as the geodesic dilation of F with respect to G iterated till stability is achieved

$$R_G^D(F) = D_G^{(k)}(F)$$

      with k such that $D_G^{(k)}(F) = D_G^{(k+1)}(F)$
    - Figure 9.28
  - Morphological reconstruction by erosion
    - Given mask image G and marker image F
    - Denoted by $R_G^E(F)$
    - Defined as the geodesic erosion of F with respect to G iterated till stability is achieved

$$R_G^E(F) = E_G^{(k)}(F)$$

      with k such that $E_G^{(k)}(F) = E_G^{(k+1)}(F)$
  - Reconstruction by dilation and erosion are duals with respect to set complementation

- Sample applications
  - Opening by reconstruction
    - Morphological opening
        - Erosion removes small objects
        - Dilation attempts to restore the shape of objects that remain
        - Accuracy dependent on the shape of objects and SE
    - Opening by reconstruction restores exactly the shape of objects that remain
    - Opening by reconstruction of size n of an image F is defined as the reconstruction by dilation of F from the erosion of size n of F

$$O_R^{(n)}(F) = R_F^D[F \ominus nB]$$

      F is used as a mask
    - Figure 9.29
        - Extract characters containing long vertical strokes
  - Filling holes
    - Earlier algorithm based on knowledge of a starting point for each hole
    - Now, we develop a fully automated procedure based on morphological reconstruction
    - Input binary image $I(x, y)$
    - Marker image

$$F(x,y) = \begin{cases} 1 - I(x,y) & \text{if } (x,y) \text{ is on the border of } I \\ 0 & \text{otherwise} \end{cases}$$

- * The output binary image with all holes filled is given by

$$H = [R^D_{I^c}(F)]^c$$

  * Figure 9.30
  * Figure 9.31
- Border clearing
  * Remove objects that touch a border of image so that only the objects that are completely enclosed in the picture remain
  * Use original image $I(x,y)$ as the mask
  * Marker image

$$F(x,y) = \begin{cases} I(x,y) & \text{if } (x,y) \text{ is on the border of } I \\ 0 & \text{otherwise} \end{cases}$$

  * Compute the image X as

$$X = I - R^D_I(F)$$

    X has no objects touching the border
  * Figure 9.32

Gray-scale morphology

- Gray scale image $f(x,y)$, under the assumptions followed so far
- SE $b(x,y)$
  - The coefficients of SE may be in $\mathcal{Z}$ or $\mathcal{R}$
  - SE performs the same basic functions as binary counterparts; used as *probes* to examine a given image for specific properties
  - Figure 9.34 – Nonflat and flat SE
  - Used infrequently in practice
  - Reflection of an SE in gray scale morphology is denoted by

$$\hat{b}(x,y) = b(-x,-y)$$

- Erosion and dilation
  - Erosion
    * Erosion of f by a *flat* SE b at any location (x,y) is defined as *minimum* value of the image coincident with b when the origin b is at (x,y)

$$[f \ominus b](x,y) = \min_{(s,t)\in b}\{f(x+s,y+t)\}$$

  - Dilation
    * Dilation of f by a *flat* SE b at any location (x,y) is defined as *maximum* value of the image coincident with b when the origin $\hat{b}$ is at (x,y)

$$[f \oplus b](x,y) = \max_{(s,t)\in \hat{b}}\{f(x+s,y+t)\}$$

    where $\hat{b} = b(-x,-y)$
  - Figure 9.35