Morphological Image Processing

Binary image processing

- In binary images, we conventionally take background as black (0) and foreground objects as white (1 or 255)
  - Figure 4.1 – objects on a conveyor belt
  - Thresholding may not produce a perfect separation between foreground and background
  - Characterized by noisy results

Morphology

- Provide a powerful technique to clean up the noise in binary images
- Identification, analysis, and description of the structure of the smallest unit of words
- Theory and technique for the analysis and processing of geometric structures
  - Based on set theory, lattice theory, topology, and random functions
  - Extract image components useful in the representation and description of region shape such as boundaries, skeletons, and convex hull
  - Input in the form of images, output in the form of attributes extracted from those images
  - Attempt to extract the meaning of the images

Preliminaries

- Set theory in the context of image processing
  - Sets of pixels represent objects in the image
  - Set of all white pixels in a binary image is a complete morphological description of the image
- Sets in binary images
  - Members of the 2D integer space $Z^2$
  - Each element of the set is a 2-tuple whose coordinates are the $(x, y)$ coordinates of a white pixel in the image
  - Gray scale images can be represented as a set of 3-tuples in $Z^3$
  - Set reflection $\hat{B}$
    \[ \hat{B} = \{ w | w = -b, \text{ for } b \in B \} \]
    - In binary image, $\hat{B}$ is the set of points in $B$ whose $(x, y)$ coordinates have been replaced by $(-x, -y)$
    - Figure 9.1a
  - Set translation
    - Translation of a set $B$ by point $z = (z_1, z_2)$ is denoted by $(B)_z$
    \[ (B)_z = \{ c | c = b + z, \text{ for } b \in B \} \]
    - In binary image, $(B)_z$ is the set of points in $B$ whose $(x, y)$ coordinates have been replaced by $(x + z_1, y + z_2)$
    - Figure 9.1c
  - Set reflection and set translation are used to formulate operations based on so-called structuring elements
    - Small sets or subimages used to probe an image for properties of interest
Preference for SES to be rectangular arrays

- Some locations are such that it does not matter whether they are part of the SE
  - Such locations are flagged by \( \times \) in the SE

- The origin of the SE must also be specified
  - Indicated by \( \bullet \) in Figure 9.2
  - If SE is symmetric and no \( \bullet \) is shown, the origin is assumed to be at the center of SE

Using SES in morphology

- Figure 9.3 – A simple set \( A \) and an SE \( B \)
- Convert \( A \) to a rectangular array by adding background elements
- Make background border large enough to accommodate the entire SE when the origin is on the border of original \( A \)
- Fill in the SE with the smallest number of background elements to make it a rectangular array
- Operation of set \( A \) using SE \( B \)
  - Create a new set by running \( B \) over \( A \)
  - Origin of \( B \) visits every element of \( A \)
  - If \( B \) is completely contained in \( A \), mark that location as a member of the new set; else it is not a member of the new set
  - Results in eroding the boundary of \( A \)

Erosion and dilation

- Erosion
  - With \( A \) and \( B \) as sets in \( \mathbb{Z}^2 \), erosion of \( A \) by \( B \), denoted by \( A \ominus B \) is defined as
    \[
    A \ominus B = \{ z | (B)_z \subseteq A \}
    \]
  - Set of all points \( z \) such that \( B \), translated by \( z \), is contained in \( A \)
  - \( B \) does not share any common elements with the background
    \[
    A \ominus B = \{ z | (B)_z \cap A^c = \emptyset \}
    \]
  - Figure 9.4
  - Example: Figure 9.5
    - Erosion shrinks or thins objects in a binary image
    - Morphological filter in which image details smaller than the SE are filtered/removed from the image

- Dilation
  - With \( A \) and \( B \) as sets in \( \mathbb{Z}^2 \), dilation of \( A \) by \( B \), denoted by \( A \oplus B \) is defined as
    \[
    A \oplus B = \{ z | (\hat{B})_z \cap A \neq \emptyset \}
    \]
  - Reflect \( B \) about the origin, and shift the reflection by \( z \)
  - Dilation is the set of all displacements \( z \) such that \( B \) and \( A \) overlap by at least one element
  - An equivalent formulation is
    \[
    A \oplus B = \{ z | [(\hat{B})_z \cap A] \subseteq A \}
    \]
  - Grows or thickens objects in a binary image
  - Figure 9.6
Example: Figure 9.7

- Bridging gaps in broken characters
- Lowpass filtering produces a grayscale image; morphological operation produces a binary image

- Erosion and dilation are based on set operations and therefore, are nonlinear

- Duality
  - Erosion and dilation are duals of each other with respect to set complementation and reflection
    \[(A \ominus B)^c = A^c \oplus \hat{B}\]
    \[(A \oplus B)^c = A^c \ominus \hat{B}\]
  - Duality property is especially useful when SE is symmetric with respect to its origin so that \(\hat{B} = B\)
    * Allows for erosion of an image by dilating its background \((A^c)\) using the same SE and complementing the results
  - Proving duality
    * Definition for erosion can be written as
      \[(A \ominus B)^c = \{z | (B)_z \subseteq A\}\]
      * \((B)_z \subseteq A \Rightarrow (B)_z \cap A^c = \emptyset\)
      * So, the previous expression yields
        \[(A \oplus B)^c = \{z | (B)_z \cap A^c = \emptyset\}\]
    * The complement of the set of z’s that satisfy \((B)_z \cap A^c = \emptyset\) is the set of z’s such that \((B)_z \cap A^c \neq \emptyset\)
    * This leads to
      \[(A \ominus B)^c = \{z | (B)_z \cap A^c \neq \emptyset\}\]
      \[= A^c \oplus \hat{B}\]

Opening and closing

- Opening smooths the contours of an object, breaks narrow isthmuses, and eliminates thin protrusions
- Closing smooths sections of contours, fusing narrow breaks and long thin gulfs, eliminates small holes, and fills gaps in the contour
- Opening of a set \(A\) by SE \(B\), denoted by \(A \diamond B\), is defined by
  \[A \diamond B = (A \ominus B) \oplus B\]
- Closing of a set \(A\) by SE \(B\), denoted by \(A \bullet B\), is defined by
  \[A \bullet B = (A \oplus B) \ominus B\]
- Geometric interpretation of opening expressed as a fitting process such that
  \[A \diamond B = \bigcup \{(B)_z | (B)_z \subseteq A\}\]
  - Union of all translates of \(B\) that fit into \(A\)
  - Figure 9.8
- Similar interpretation of closing in Figure 9.9
Example – Figure 9.10

Duality property

\[(A \bullet B)^c = (A^c \circ B)\]
\[(A \circ B)^c = (A^c \bullet B)\]

Opening operation satisfies the following properties

1. \(A \circ B \subseteq A\)
2. \(C \subseteq D \Rightarrow C \circ B \subseteq D \circ B\)
3. \((A \circ B) \circ B = A \circ B\)

Similarly, closing operation satisfies

1. \(A \subseteq A \bullet B\)
2. \(C \subseteq D \Rightarrow C \bullet B \subseteq D \bullet B\)
3. \((A \bullet B) \bullet B = A \bullet B\)

In both the above cases, multiple application of opening and closing has no effect after the first application

Example: Removing noise from fingerprints

– Figure 9.11
– Noise as random light elements on a dark background

Hit-or-miss transformation

Basic tool for shape detection in a binary image

– Uses the morphological erosion operator and a pair of disjoint SEs
– First SE fits in the foreground of input image; second SE misses it completely
– The pair of two SEs is called composite structuring element

Figure 9.12

– Three disjoint shapes denoted \(C\), \(D\), and \(E\)
  * \(A = C \cup D \cup E\)
– Objective: To find the location of one of the shapes, say \(D\)
– Origin/location of each shape given by its center of gravity
– Let \(D\) be enclosed by a small window \(W\)
– Local background of \(D\) defined by the set difference \((W - D)\)
  * Note that \(D\) and \(W - D\) provide us with the two disjoint SEs

\[D \cap (W - D) = \emptyset\]

– Compute \(A^c\)
– Compute \(A \ominus D\)
– Compute \(A^c \ominus (W - D)\)
– Set of locations where \(D\) exactly fits inside \(A\) is \((A \ominus D) \cap (A^c \ominus (W - D))\)
The exact location of $D$

If $B$ is the set composed of $D$ and its background, the match of $B$ in $A$ is given by

$$A \odot B = (A \ominus D) \cap [A^c \ominus (W - D)]$$

The above can be generalized to the composite $\text{se}$ being defined by $B = (B_1, B_2)$ leading to

$$A \odot B = (A \ominus B_1) \cap (A^c \ominus B_2)$$

- $B_1$ is the set formed from elements of $B$ associated with the object; $B_1 = D$
- $B_2 = (W - D)$

A point $z$ in universe $A$ belongs to the output if $(B_1)_z$ fits in $A$ (hit) and $(B_2)_z$ misses $A$

### Some basic morphological algorithms

- Useful in extracting image components for representation and description of shape
- Boundary extraction
  - Boundary of a set $A$
    * Denoted by $\beta(A)$
    * Extracted by eroding $A$ by a suitable $\text{se} B$ and computing set difference between $A$ and its erosion
      $$\beta(A) = A - (A \ominus B)$$
  - Figure 9.13
    * Using a larger $\text{se}$ will yield a thicker boundary
    - Figure 9.14
- Hole filling
  - Hole
    * Background region surrounded by a connected border of foreground pixels
  - Algorithm based on set dilation, complementation, and intersection
  - Let $A$ be a set whose elements are 8-connected boundaries, each boundary enclosing a background (hole)
  - Given a point in each hole, we want to fill all holes
  - Start by forming an array $X_0$ of 0s of the same size as $A$
    * The locations in $X_0$ corresponding to the given point in each hole are set to 1
  - Let $B$ be a symmetric $\text{se}$ with 4-connected neighbors to the origin

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- Compute $X_k = (X_{k-1} \oplus B) \cap A^c$  
  $k = 1, 2, 3, \ldots$
- Algorithm terminates at iteration step $k$ if $X_k = X_{k-1}$
- $X_k$ contains all the filled holes
- $X_k \cup A$ contains all the filled holes and their boundaries
- The intersection with $A^c$ at each step limits the result to inside the ROI
  * Also called conditioned dilation
- Figure 9.15
- Example: Figure 9.16
  * Thresholded image of polished spheres (ball bearings)
  * Eliminate reflection by hole filling
  * Points inside the background selected manually

- Extraction of connected components
  - Let $A$ be a set containing one or more connected components
  - Form an array $X_0$ of the same size as $A$
    * All elements of $X_0$ are 0 except for one point in each connected component set to 1
  - Select a suitable se $B$, possibly an 8-connected neighborhood as
    \[
    \begin{array}{ccc}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    \end{array}
    \]
  - Start with $X_0$ and find all connected components using the iterative procedure
    \[
    X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \ldots
    \]
  - Procedure terminates when $X_k = X_{k-1}$; $X_k$ contains all the connected components in the input image
  - The only difference from the hole-filling algorithm is the intersection with $A$ instead of $A^c$
    * This is because here, we are searching for foreground points while in hole filling, we looked for background points (holes)
  - Figure 9.17
  - Example: Figure 9.18
    * X-ray image of chicken breast with bone fragments
    * Objects of “significant size” can be selected by applying erosion to the thresholded image
    * We may apply labels to the extracted components (region labeling)

- Convex hull
  - Convex set $A$
    * Straight line segment joining any two points in $A$ lies entirely within $A$
  - Convex hull $H$ of an arbitrary set of points $S$ is the smallest convex set containing $S$
  - Set difference $H - S$ is called the convex deficiency of $S$
  - Convex hull and convex deficiency are useful to describe objects
  - Algorithm to compute convex hull $C(A)$ of a set $A$
    * Figure 9.19
    * Let $B^i$, $i = 1, 2, 3, 4$ represent the four structuring elements in the figure
      - $B^i$ is a clockwise rotation of $B^{i-1}$ by $90^\circ$
    * Implement the equation
      \[
      X_k^i = (X_{k-1} \oplus B^i) \cup A \quad i = 1, 2, 3, 4 \text{ and } k = 1, 2, 3, \ldots
      \]
      with $X_0^i = A$
    * Apply hit-or-miss with $B^1$ till $X_k = X_{k-1}$, then, with $B^2$ over original $A$, $B^3$, and $B_4$
    * Procedure converges when $X_k^i = X_{k-1}^i$ and we let $D^i = X_k^i$
Convex hull of $A$ is given by
\[ C(A) = \bigcup_{i=1}^{4} D^i \]

- Shortcoming of the above procedure
  * Convex hull can grow beyond the minimum dimensions required to guarantee convexity
  * May be fixed by limiting growth to not extend past the bounding box for the original set of points
  * Figure 9.20

**Thinning**

- Transformation of a digital image into a simple topologically equivalent image
  * Remove selected foreground pixels from binary images
  * Used to tidy up the output of edge detectors by reducing all lines to single pixel thickness
- Thinning of a set $A$ by se $B$ is denoted by $A \otimes B$
- Defined in terms of hit-or-miss transform as
  \[ A \otimes B = A - (A \ast B) \]
  \[ = A \cap (A \ast B)^c \]

- Only need to do pattern matching with se; no background operation required in hit-or-miss transform
- A more useful expression for thinning $A$ symmetrically based on a sequence of ses
  \[ \{B\} = \{B^1, B^2, \ldots, B^n\} \]
  where $B^i$ is a rotated version of $B^{i-1}$
- Define thinning by a sequence of ses as
  \[ A \otimes \{B\} = (\ldots((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n) \]
  - Figure 9.21
    * Iterate over the procedure till convergence

**Thickening**

- Morphological dual of thinning defined by
  \[ A \circ B = A \cup (A \ast B) \]
- ses complements of those used for thinning
- Thickening can also be defined as a sequential operation
  \[ A \circ \{B\} = (\ldots((A \circ B^1) \circ B^2) \ldots) \circ B^n) \]
  - Figure 9.22
    - Usual practice to thin the background and take the complement
      * May result in disconnected points
      * Post-process to remove the disconnected points

**Skeletons**

- Figure 9.23
  * Skeleton $S(A)$ of a set $A$
  * Deductions
1. If \( z \) is a point of \( S(A) \) and \( (D)_z \) is the largest disk centered at \( z \) and contained in \( A \), one cannot find a larger disk (not necessarily centered at \( z \)) containing \( (D)_z \) and included in \( A \); \( (D)_z \) is called a maximum disk.

2. Disk \( (D)_z \) touches the boundary of \( A \) at two or more different places.

- Skeleton can be expressed in terms of erosions and openings

\[
S(A) = \bigcup_{k=0}^{K} S_k(A)
\]

where

\[
S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B
\]

* \( A \ominus kB \) indicates \( k \) successive erosions of \( A \)

\[
(A \ominus kB) = (((A \ominus B) \ominus B) \ominus \ldots) \ominus B
\]

* \( K \) is the last iterative step before \( A \) erodes to an empty set

\[
K = \max\{k \mid (A \ominus kB) \neq \emptyset\}
\]

* \( S(A) \) can be obtained as the union of skeleton subsets \( S_k(A) \)

* \( A \) can be reconstructed from the subsets using the equation

\[
\bigcup_{k=0}^{K} (S_k(A) \oplus kB)
\]

where \( (S_k(A) \oplus kB) \) denotes \( k \) successive dilations of \( S_k(A) \)

\[
(S_k(A) \oplus kB) = (((S_k(A) \oplus B) \oplus B) \oplus \ldots) \oplus B
\]

* Figure 9.24

- Pruning

  - Complement to thinning and skeletonizing algorithms to remove unwanted parasitic components

  - Automatic recognition of hand-printed characters

    * Analyze the shape of the skeleton of each character
    * Skeletons characterized by “spurs” or parasitic components
    * Spurs caused during erosion by non-uniformities in the strokes
    * Assume that the length of a spur does not exceed a specific number of pixels

  - Figure 9.25 – Skeleton of hand-printed “a”

    * Suppress a parasitic branch by successively eliminating its end point
    * Assumption: Any branch with \( \leq 3 \) pixels will be removed
    * Achieved with thinning of an input set \( A \) with a sequence of SEs designed to detect only end points

\[
X_1 = A \ominus \{B\}
\]

* Figure 9.25d – Result of applying the above thinning three times

  * Restore the character to its original form with the parasitic branches removed

  * Form a set \( X_2 \) containing all end points in \( X_1 \)

\[
X_2 = \bigcup_{k=1}^{8} (X_1 \oplus B^k)
\]
* Dilate end points three times using set $A$ as delimiter

$$X_3 = (X_2 \oplus H) \cap A$$

where $H$ is a $3 \times 3$ SE of 1s and intersection with $A$ is applied after each step

* The final result comes from

$$X_4 = X_1 \cup X_3$$

### Morphological Reconstruction

- Works on two images and an SE
  - One image is called the *marker* and contains the starting points for transformation
  - Second image is called the *mask* and contains the transformation or constraint
  - SE is used to define connectivity

- Geodesic dilation and erosion
  - Let $F$ be the marker image and $G$ be the mask image
  - $F$ and $G$ are binary images and $F \subseteq G$
  - Geodesic dilation
    * Geodesic dilation of size 1 of $F$ with respect to $G$ is defined as
      $$D_G^{(1)}(F) = (F \oplus B) \cap G$$
    * Geodesic dilation of size $n$ of $F$ with respect to $G$ is defined as
      $$D_G^{(n)}(F) = D_G^{(1)} \left[D_G^{(n-1)}(F)\right]$$
      with $D_G^{(0)}(F) = F$
      - Set intersection is performed at each step of recursion
      - Mask $G$ limits the growth of marker $F$
    - Figure 9.26
  - Geodesic erosion
    * Geodesic erosion of size 1 of $F$ with respect to $G$ is defined as
      $$E_G^{(1)}(F) = (F \ominus B) \cup G$$
    * Geodesic erosion of size $n$ of $F$ with respect to $G$ is defined as
      $$E_G^{(n)}(F) = E_G^{(1)} \left[E_G^{(n-1)}(F)\right]$$
      with $E_G^{(0)}(F) = F$
      - Set union is performed at each step of recursion
      - Guarantees that geodesic erosion of an image remains greater than or equal to its mask
    - Figure 9.27
      - Bottom leftmost pixel of $F$ should be white
  - Geodesic dilation and erosion are duals with respect to set complementation
  - Both operations converge after a finite number of iterative steps

- Morphological reconstruction by dilation and erosion
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– Morphological reconstruction by dilation
  * Given mask image \( G \) and marker image \( F \)
  * Denoted by \( R_D^G(F) \)
  * Defined as the geodesic dilation of \( F \) with respect to \( G \) iterated till stability is achieved
    \[
    R_D^G(F) = D_G^{(k)}(F)
    \]
    with \( k \) such that \( D_G^{(k)}(F) = D_G^{(k+1)}(F) \)
  * Figure 9.28

– Morphological reconstruction by erosion
  * Given mask image \( G \) and marker image \( F \)
  * Denoted by \( R_E^G(F) \)
  * Defined as the geodesic erosion of \( F \) with respect to \( G \) iterated till stability is achieved
    \[
    R_E^G(F) = E_G^{(k)}(F)
    \]
    with \( k \) such that \( E_G^{(k)}(F) = E_G^{(k+1)}(F) \)
  * Reconstruction by dilation and erosion are duals with respect to set complementation

- Sample applications
  - Opening by reconstruction
    * Morphological opening
      - Erosion removes small objects
      - Dilation attempts to restore the shape of objects that remain
      - Accuracy dependent on the shape of objects and \( SE \)
    * Opening by reconstruction restores exactly the shape of objects that remain
    * Opening by reconstruction of size \( n \) of an image \( F \) is defined as the reconstruction by dilation of \( F \) from the erosion of size \( n \) of \( F \)
      \[
      O_R^{(n)}(F) = R_D^F[F \ominus nB]
      \]
    * Figure 9.29
    - Extract characters containing long vertical strokes
  - Filling holes
    * Earlier algorithm based on knowledge of a starting point for each hole
    * Now, we develop a fully automated procedure based on morphological reconstruction
    * Input binary image \( I(x, y) \)
    * Marker image
      \[
      F(x, y) = \begin{cases} 
      1 - I(x, y) & \text{if } (x, y) \text{ is on the border of } I \\
      0 & \text{otherwise}
      \end{cases}
      \]
    * The output binary image with all holes filled is given by
      \[
      H = [R_I^F(F)]^c
      \]
    * Figure 9.30
    * Figure 9.31
  - Border clearing
    * Remove objects that touch a border of image so that only the objects that are completely enclosed in the picture remain
* Use original image $I(x, y)$ as the mask
* Marker image

$$F(x, y) = \begin{cases} 
I(x, y) & \text{if } (x, y) \text{ is on the border of } I \\
0 & \text{otherwise} 
\end{cases}$$

* Compute the image $X$ as

$$X = I - R_I^D(F)$$

$X$ has no objects touching the border
* Figure 9.32

Gray-scale morphology

- Gray scale image $f(x, y)$, under the assumptions followed so far
- se $b(x, y)$
  - The coefficients of se may be in $\mathbb{Z}$ or $\mathbb{R}$
  - se performs the same basic functions as binary counterparts; used as probes to examine a given image for specific properties
  - Figure 9.34 – Nonflat and flat se
  - Used infrequently in practice
  - Reflection of an se in gray scale morphology is denoted by

$$\hat{b}(x, y) = b(-x, -y)$$

- Erosion and dilation
  - Erosion
    * Erosion of $f$ by a flat se $b$ at any location $(x, y)$ is defined as minimum value of the image coincident with $b$ when the origin $\hat{b}$ is at $(x, y)$

$$[f \ominus b](x, y) = \min_{(s, t) \in \hat{b}} \{ f(x + s, y + t) \}$$

  - Dilation
    * Dilation of $f$ by a flat se $b$ at any location $(x, y)$ is defined as maximum value of the image coincident with $b$ when the origin $\hat{b}$ is at $(x, y)$

$$[f \oplus b](x, y) = \max_{(s, t) \in \hat{b}} \{ f(x + s, y + t) \}$$

  where $\hat{b} = b(-x, -y)$
  - Figure 9.35