Image Enhancement in Frequency Domain

Background

- Any periodic function can be represented as the sum of sines and cosines of different frequencies, multiplied by a different coefficient (Fig 4-01)
  - The sum is called Fourier series
- Functions that are not periodic but whose area under the curve is finite can be represented as the integral of sines and cosines multiplied by a weight function
  - Known as Fourier transform
- Function represented by Fourier transform can be completely reconstructed by an inverse transform with no loss of information
  - Allows working in Fourier domain and return to the original domain without any loss of information

Fourier transform and frequency domain

- One-dimensional Fourier transform and its inverse
  - Given single variable continuous function $f(x)$
  - Fourier transform $F(u)$ is given by
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$
    where $j = \sqrt{-1}$
  - The inverse of the transform is given by
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$
    * The above two functions form the Fourier transform pair
  - Extension of the transforms to two variables
$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy \\ f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv \end{aligned}$$
  - Fourier transform of a discrete function of one variable $f(x), x = 0, 1, 2, \ldots, M-1$ (discrete Fourier transform, or DFT)
$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad \text{for } u = 0, 1, 2, \ldots, M-1$$
  - Inverse DFT
$$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad \text{for } x = 0, 1, 2, \ldots, M-1$$
  - By summing for all values of x for each value of u, you get the complete Fourier transform
    * Approximately M^2 summations and multiplications to compute the DFT
    * Like $f(x)$, $F(u)$ is discrete with same number of components as $f(x)$

- Finite values for digital domain ensure the existence of DFT and its inverse
- Concept of a frequency domain follows from Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- Substituting into the expression for $F(u)$ and noting that $\cos(-\theta) = \cos(\theta)$, we get

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos(2\pi ux/M) - j\sin(2\pi ux/M)]$$

  for $u = 0, 1, 2, \ldots, M-1$
  * Each term in Fourier transform is composed of the sum of all values of the function $f(x)$
  * Values of $f(x)$ are multiplied by sines and cosines of various frequencies
  * The domain of values of $F(u)$ is called the frequency domain
    · u determines the frequency of the components of the transform
    · x's are summed out and make the same contribution for each value of u
  * Each of the M terms of $F(u)$ is called a frequency component of the transform
  * Frequency component and frequency domain are also referred to as time component and time domain
- Fourier transform as a "mathematical prism" to separate function into various components
- Expressing $F(u)$ in polar coordinates

$$F(u) = |F(u)|e^{-j\phi(u)}$$

  * Magnitude $|F(u)| = [R^2(u) + I^2(u)]^{1/2}$
  * Phase angle $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
  * $R(u)$ and $I(u)$ are real and imaginary parts of $F(u)$
- Power spectrum
  * Square of the Fourier spectrum

$$\begin{aligned} P(u) &= |F(u)|^2 \\ &= R^2(u) + I^2(u) \end{aligned}$$

  * Also referred to as spectral density
- Simple 1D example of DFT: Fig 4-2
  * Discrete $f(x)$ and $F(u)$
  * $M = 1024$, $K = 8$, $A = 1$
  * Spectrum centered at $u = 0$, accomplished by multiplying $f(x)$ by $(-1)^x$ before computing transform
  * Bottom figures with $K = 16$
  * Height of spectrum doubles as area under the curve in spatial domain doubles
  * Number of zeroes in the spectrum in the same interval doubles as the length of the function doubles

• Two-dimensional DFT and its inverse
- In 2D, DFT of an image $f(x, y)$ of dimensions $M \times N$ is given by

$$F(u, v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)e^{-j2\pi(ux/M+vy/N)}$$

- The inverse Fourier transform is given by

$$f(x, y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u, v)e^{j2\pi(ux/M+vy/N)}$$

- Fourier spectrum, phase angle, and power spectrum are defined by

$$\begin{aligned} |F(u,v)| &= [R^2(u,v) + I^2(u,v)]^{1/2} \\ \phi(u,v) &= \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right] \\ P(u,v) &= |F(u,v)|^2 \\ &= R^2(u,v) + I^2(u,v) \end{aligned}$$

- Common practice to multiply the input image by $(-1)^{x+y}$ before computing the Fourier transform
- Due to properties of exponentials, we have the Fourier transform as

$$\Im[f(x,y)(-1)^{x+y}] = F(u - M/2, v - N/2)$$

  * The origin of the Fourier transform $F(0,0)$ is located at $u = M/2$ and $v = N/2$, or center of the area occupied by 2D DFT
  * This area of frequency domain is called frequency rectangle
  * Guaranteed by requiring that M and N are even integers
- At point $(0,0)$ we have

$$F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

  which is the average of the entire image
  * DC component of the spectrum
- Figure 4-03
  * White rectangle of 20×40 pixels superimposed on a black background of 512×512 pixels
  * Image multiplied by $(-1)^{x+y}$ before computing Fourier transform to center the spectrum
  * Separation of spectrum zeroes in u-direction is exactly twice the separation of zeroes in v-direction due to 1:2 size ratio of rectangle in the original image

- Filtering in the frequency domain
  - Basic properties of frequency domain
    * *Each* term of $F(u,v)$ contains *all* values of $f(x,y)$ modified by the values of exponential terms
    * Not easy to make direct association between specific components of an image and its transform
    * Frequency is related to rate of change
      · Frequencies in Fourier transform can be associated with patterns of intensity variations in image
      · Slowest varying frequency component $(u = v = 0)$ corresponds to average gray level of image
      · As you move away from origin of transform, low frequencies correspond to slowly varying components of image
      · Farther away from origin, you have higher frequencies corresponding to faster gray level changes
    * Fig 4-04
      · Fourier spectrum shows prominent components along $\pm 45°$ direction corresponding to sharp edges
  - Basics of filtering in frequency domain
    * Based on following steps
      1. Multiply input image by $(-1)^{x+y}$ to center the transform
      2. Compute DFT of the image $F(u,v)$
      3. Multiply $F(u,v)$ by a filter function $H(u,v)$
      4. Compute inverse DFT of the result
      5. Obtain the real part of the result

6. Multiply the result by $(-1)^{x+y}$

* Filter or filter transfer function
  - Suppresses certain frequencies in transform while leaving others unchanged
* Fourier transform of the output image

$$G(u,v) = H(u,v) \cdot F(u,v)$$

  - Multiplication of H and F is only between corresponding elements
  - $G(0,0) = H(0,0) \cdot F(0,0)$
* Zero-phase-shift filters
  - Each component of H multiplies both the real and imaginary part of F
  - Filters do not change the phase of the transform
* Filtered image $= \Im^{-1}[G(u,v)]$
  - Only retain the real part and multiply that by $(-1)^{x+y}$
  - For real input image and real filter function, the imaginary components of inverse transform are all zero
  - In practice, imaginary components may not be zero due to round-off errors and should be ignored
* Fig 4-05
  - Preprocessing: crop image to its closest even dimensions;
  - Multiply input image by $(-1)^{x+y}$ to center the transform

– Some basic filters and their properties
* Changing the average value of an image to zero
  - Make $F(0,0)$ as zero

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$$

  - Filter is called *notch filter*; constant function with a hole/notch at the origin
  - Fig 4-06
  - Drop in overall gray level
  - Average cannot be zero because negative values cannot be handled; most negative is changed to zero with other values scaled up from that
  - Useful when it is possible to identify spatial image effects caused by specific, localized frequency domain components
* Low frequencies depict smooth areas while high frequencies show edges and noise
  - Lowpass and highpass filters
  - Fig 4-07
  - Adding a constant to image resulting from high-pass filter; Fig 4-08

• Correspondence between filtering in spatial and frequency domains

– Convolution theorem
* Convolution based on the process of moving a mask over pixels to compute a predefined quantity
* Discrete convolution of two functions $f(x,y)$ and $h(x,y)$ of size $M \times N$ is defined by

$$f(x,y) * h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

* Function h is mirrored about the origin
* Let $F(u,v)$ and $H(u,v)$ denote the Fourier transforms of $f(x,y)$ and $h(x,y)$, respectively
* $f(x,y) * h(x,y)$ and $F(u,v)H(u,v)$ constitute a Fourier transform pair, or formally

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

    * Analogous result states that convolution in frequency domain reduces to multiplication in the spatial domain, and vice versa
$$f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$
    * The two equivalences constitute the convolution theorem
  - Impulse function of strength A located at (x_0, y_0) is denoted by $A\delta(x - x_0, y - y_0)$ and is defined by
$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y) A\delta(x - x_0, y - y_0) = As(x_0, y_0)$$
    * Summation of a function $s(x, y)$ multiplied by an impulse is simply the value of the function at the location of the impulse, multiplied by the strength of the impulse
    * Limits of the summation are the same as the limits spanned by the function
    * $A\delta(x - x_0, y - y_0)$ is also an image of size $M \times N$
      · Image composed of all zeroes except at (x_0, y_0) where the value of image is A
    * Sifting property of the impulse function
      · Let either f or h in the definition of discrete convolution be an impulse function
      · Convolution of a function with an impulse copies the value of that function to the location of the impulse
      · Unit impulse located at the origin is denoted $\delta(x, y)$
$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y)\delta(x,y) = s(0,0)$$
  - Tie between filtering in spatial and frequency domains
    * Fourier transform of unit impulse at the origin
$$\begin{aligned} F(u,v) &= \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \delta(x,y) e^{-j2\pi(ux/M + vy/N)} \\ &= \frac{1}{MN} \end{aligned}$$
    * Since there is no exponent component, the Fourier transform of an impulse at the origin is a real constant, with phase angle 0
    * Impulse located at a point other than origin has complex component
    * Magnitude of non-origin impulse is the same, with the translation reflected in nonzero phase angle
  - Let $f(x, y) = \delta(x, y)$ and carry out the discrete convolution
    * Again, $\delta(x, y)$ is the unit impulse at the origin
    * We have
$$\begin{aligned} f(x,y) * h(x,y) &= \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \delta(m,n) h(x - m, y - n) \\ &= \frac{1}{MN} h(x,y) \end{aligned}$$
      since the summation variables are m and n
    * Furthermore
$$\begin{aligned} f(x,y) * h(x,y) &\Leftrightarrow F(u,v)H(u,v) \\ \delta(x,y) * h(x,y) &\Leftrightarrow \Im[\delta(x,y)]H(u,v) \\ h(x,y) &\Leftrightarrow H(u,v) \end{aligned}$$

- Implications
  - Filters in the spatial and frequency domains constitute a Fourier transform pair
- All functions discussed above are the same size $M \times N$
  - Filtering in the frequency domain does not help from computational standpoint
  - Smaller filters in spatial domain
  - Filtering more intuitive in frequency domain
  - Fourier transform and its inverse are linear processes
- Filters based on Gaussian functions
  - Shape of the function is easily specified
  - Both forward and inverse Fourier transforms of a Gaussian function are real Gaussian functions
  - Discussion limited to 1D functions
  - Gaussian filter function in frequency domain $H(u)$

    $$H(u) = Ae^{-u^2/2\sigma^2}$$

    where σ is the standard deviation of the Gaussian curve
  - Corresponding filter in spatial domain is

    $$h(x) = \sqrt{2\pi}\sigma Ae^{-2\pi^2\sigma^2x^2}$$

  - Important observations
    1. Both components of Fourier transform pair are Gaussian and real
    2. Functions behave reciprocally with respect to one another
       - When $H(u)$ has a broad profile (large value of σ), $h(x)$ has a narrow profile, and vice versa
       - As σ approaches infinity, $H(u)$ tends towards a constant function and $h(x)$ tends towards an impulse
  - Fig 4-09
    - Positive values of H and h in both frequency and spatial domains for low pass filter (4.09a and 4.09c)
    - Lowpass filtering can be implemented using a mask with all positive coefficients
    - Narrow frequency domain filter attenuates lower frequencies implying a wider filter in the spatial domain
    - Complex filters, or a highpass filter as a difference of Gaussians
    - In frequency domain (4.09b)

      $$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

      where $A \geq B$ and $\sigma_1 > \sigma_2$
    - Corresponding filter in spatial domain (4.09d)

      $$h(x) = \sqrt{2\pi}\sigma_1 A^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 B^{-2\pi^2\sigma_2^2x^2}$$

    - Now the spatial filter has both negative and positive values and once the values turn negative, they never turn positive again (moving away from center in 4.09d)
- Computational complexity

Smoothing frequency-domain filters

- Noise and sharp transitions are the high-frequency component
- Remove noise by attenuating the specified range of high frequency components
- Basic model of filtering in frequency domain

$$G(u,v) = H(u,v)F(u,v)$$

- Basic problem to determine the filter $H(u, v)$
- Ideal lowpass filters
  - Cut off all high frequency components of the Fourier transform that are at a distance greater than a specified distance D_0 from the origin
  - Called 2D ideal lowpass filter (ILPF) with transfer function

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{otherwise} \end{cases}$$

    where D_0 is a specified nonnegative quantity, and $D(u, v)$ is the distance from point (u, v) to the center of the frequency rectangle
  - Distance is taken to be Euclidean distance, and for an $M \times N$ image

$$D(u, v) = \sqrt{(u - M/2)^2 + (v - n/2)^2}$$

  - Figure 4-10
    - All frequencies inside the circle of radius D_0 are passed with no attenuation
    - Radially symmetric filter
  - Cutoff frequency
    - Point of transition between $H(u, v) = 0$ abd $H(u, v) = 1$
    - Not realizable through electronic components but ok in computers
    - Nonphysical filter
  - Computing cutoff frequency
    - Compute circles that enclose specified amounts of total image power P_T
    - Obtained by summation of components of power spectrum at each point as

$$P_T = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} P(u, v)$$

      - Power spectrum at a point is given by

$$P(u, v) = R^2(u, v) + I^2(u, v)$$

    - If transform is centered, a circle of radius r with origin at the center of the frequency rectangle encloses α percent of the power

$$\alpha = 100 \left[\sum_u \sum_v P(u, v)/P_T \right]$$

      with summation taken over the points that lie inside the circle
    - Figure 4-11: Circles of radius on power spectrum
    - Figure 4-12: Applying ideal lowpass filters with cutoff frequencies from Figure 4-11
      - Ringing phenomenon in filtered images with smaller radius
    - Ringing and blurring properties of ILPF
      - Using convolution theorem, Fourier transform of original image $(f(x, y)$ and blurred image $g(x, y)$ are related in frequency domain by

$$G(u, v) = H(u, v)F(u, v)$$

      - Corresponding process in spatial domain is given by

$$g(x, y) = h(x, y) * f(x, y)$$

- Figure 4-13a: ILPF of radius 5; the spatial filter function is shown in 4-13b
- Filter $h(x, y)$ has a dominant component at the origin and concentric, circular components about the center component
- Center component is responsible for blurring
- Concentric components are responsible for ringing characteristics of ideal filters
- Radius of center component and number of circles per unit distance from the origin are inversely proportional to the value of cutoff frequency of ideal filter
- Filtered image can have negative values, and may require scaling
- Figures 4-13c and d show the application of filter on an image with five bright pixels

- Butterworth lowpass filters
  - Transfer function of a Butterworth lowpass filter (BLPF) of order n with cutoff frequency D_0 from the origin is defined as

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$

  - Figure 4-14
  - BLPF does not have a sharp discontinuity like ILPF
  - In the above equation, $H(u, v) = 0.5$ when $D(u, v) = D_0$
  - Figure 4-15: Application of BLPF
  - Ringing observed in filters of higher order but not when $n = 1$ or 2
    - Figure 4-16
- Gaussian lowpass filters
  - Gaussian lowpass filter (GLPF) in two dimensions is given by

$$H(u, v) = e^{-D^2(u,v)/2\sigma^2}$$

    - $D(u, v)$ is the distance from the origin (center of image)
    - σ is a measure of spread of Gaussian curve
  - If $\sigma = D_0$, the filter can be expressed as

$$H(u, v) = e^{-D^2(u,v)/2D_0^2}$$

  - When $D(u, v) = D_0$, the filter is down to 0.607 of its maximum value
  - Observations
    - Inverse Fourier transform of a Gaussian lowpass filter is also Gaussian
    - A spatial Gaussian filter, obtained by computing the inverse Fourier transform, will have no ringing
    - Figure 4-17
  - Figure 4-18
    - Not as much smoothing as BLPF of order 2 but no ringing either
- Additional examples of lowpass filtering
  - Figure 4-19
    - Optical character recognition
    - Repair broken characters
  - Figure 4-20 – Cosmetic processing
  - Figure 4-21 – Remote sensing
    - Scan lines due to sensor artifacts along the direction of scan

Sharpening frequency domain filters

- Image sharpening achieved by highpass filtering process
  - Attenuates the low frequency components without disturbing the high frequency information in Fourier transform
  - Reverse of lowpass filter
$$H_{\text{hp}}(u,v) = 1 - H_{\text{lp}}(u,v)$$
  - Figure 4-22 – ideal, Butterworth, and Gaussian highpass filters
- Ideal highpass filters
  - A 2D ideal highpass filter (IHPF) is defined as
$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0 \\ 1 & \text{otherwise} \end{cases}$$
    where D_0 is a specified nonnegative quantity, and $D(u,v)$ is the distance from point (u,v) to the center of the frequency rectangle
  - Ringing problem persists due to the same reason as in ILPF
  - Figure 4-23, Figure 4-24
- Butterworth highpass filters
  - Transfer function of a Butterworth highpass filter (BHPF) of order n with cutoff frequency D_0 from the origin is defined as
$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$
  - Figure 4-22 (middle row)
  - Figure 4-25
- Gaussian highpass filters
  - Gaussian highpass filter (GHPF) in two dimensions is given by
$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$
  - Figure 4-26
- Laplacian in frequency domain
  - It can be shown that
$$\Im\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$$
  - It follows that
$$\begin{aligned} \Im\left[\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right] &= (ju)^2 F(u,v) + (jv)^2 F(u,v) \\ &= -(u^2+v^2)F(u,v) \end{aligned}$$
    * The expression on the left side is the Laplacian of $f(x,y)$ and we have
$$\Im[\nabla^2 f(x,y)] = -(u^2+v^2)F(u,v)$$
    * Laplacian can be implemented in frequency domain by the filter
$$H(u,v) = -(u^2+v^2)$$

- Assume that the origin of $F(u,v)$ is centered by performing the operation $f(x,y)(-1)^{x+y}$ before taking the transform
- Center of filter function can be shifted by

$$H(u,v) = -((u - M/2)^2 + (v - N/2)^2)$$

- Laplacian filtered image in spatial domain is obtained by computing the inverse Fourier transform of $H(u,v)F(u,v)$ as

$$\nabla^2 f(x,y) = \Im^{-1}(-((u - M/2)^2 + (v - N/2)^2)F(u,v))$$

- Conversely, we can apply convolution theorem to get the Fourier transform pair notation

$$\nabla^2 f(x,y) \Leftrightarrow -((u - M/2)^2 + (v - N/2)^2)F(u,v)$$

- Interesting properties of spatial domain Laplacian filter by taking the inverse Fourier transform of $H(u,v)$ defined above
  - Figure 4-27
- Subtracting the Laplacian from the original image gives you the enhanced image

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

- The filter can be combined into one operation as

$$H(u,v) = 1 + ((u - M/2)^2 + (v - N/2)^2)$$

- The enhanced image is obtained with a single transform by

$$g(x,y) = \Im^{-1}((1 + ((u - M/2)^2 + (v - N/2)^2))F(u,v))$$

- Figure 4-28

- Unsharp masking, high-boost filtering and high-frequency emphasis filtering
  - High-boost filtering
    - Increase the contribution of the original image to overall filtered result
    - Generalization of unsharp masking
      - Generate a sharp image by subtracting its blurred version from itself
      - Obtain a highpass-filtered image by subtracting its lowpass filtered version from itself

$$f_{\text{hp}}(x,y) = f(x,y) - f_{\text{lp}}(x,y)$$

    - For high-boost filtering

$$f_{\text{hb}}(x,y) = Af(x,y) - f_{\text{lp}}(x,y)$$

      where $A \geq 1$
    - You can also write it as

$$\begin{aligned} f_{\text{hb}}(x,y) &= (A-1)f(x,y) + f(x,y) - f_{\text{lp}}(x,y) \\ &= (A-1)f(x,y) + f_{\text{hp}}(x,y) \end{aligned}$$

    - For $A = 1$, high-boost filtering reduces to highpass filtering
    - For $A >> 1$, image contribution becomes more dominant
    - Also make sure that you normalize the result back by dividing the coefficients by A
  - Fourier domain
    - $F_{\text{hp}}(u,v) = F(u,v) - F_{\text{lp}}(u,v)$

- $F_{\text{lp}}(u,v) = H_{\text{lp}}(u,v)F(u,v)$
- The composite filter in frequency domain is

$$H_{\text{hp}}(u,v) = 1 - H_{\text{lp}}(u,v)$$

- High-boost filtering in frequency domain is

$$H_{\text{hb}}(u,v) = (A-1) + H_{\text{hp}}(u,v)$$

- Figure 4-29

– High-frequency emphasis
  - Multiply highpass filter by a constant and add an offset so that the DC term is not eliminated by the filter
  - Transfer function given by

$$H_{\text{hfe}}(u,v) = a + bH_{\text{hp}}(u,v)$$

  where $a \geq 0$ and $b > a$
  - Reduces to high-boost filtering when $a = (A-1)$ and $b = 1$
  - When $b > 1$, high frequencies are emphasized
  - Figure 4-30