ARITHMETICALLY COHEN-MACAULAY BUNDLES ON THREE DIMENSIONAL HYPERSURFACES

N. MOHAN KUMAR, A. P. RAO, AND G. V. RAVINDRA

Abstract. We prove that any rank two arithmetically Cohen-Macaulay vector bundle on a general hypersurface of degree at least six in $\mathbb{P}^4$ must be split.

1. Introduction

An arithmetically Cohen-Macaulay (ACM for short) vector bundle on a hypersurface $X \subset \mathbb{P}^n$ is a bundle $E$ for which $H^i(X, E(k)) = 0$ for $0 < i < n - 1$ and for all integers $k$. ACM bundles of large rank, which are not split as a sum of line bundles, exist on any hypersurface $X$ of degree $> 1$ (see [2]), and it is also conjectured by Buchweitz-Greuel-Schreyer (op. cit.) that low rank ACM bundles on smooth hypersurfaces should be split. For example, it is well known [7] that there are no non-split ACM bundles of rank two on a smooth hypersurface in $\mathbb{P}^6$. In [6], it was proved that on a general hypersurface of degree $\geq 3$ in $\mathbb{P}^5$, there are no non-split ACM bundles of rank two.

In the current paper, we extend this result to general hypersurfaces of degree $d \geq 6$ in $\mathbb{P}^4$:

Main Theorem. Fix $d \geq 6$. There is a non-empty Zariski open set of hypersurfaces of degree $d$ in $\mathbb{P}^4$, none of which support an indecomposable ACM rank two bundle.

The special case when $d = 6$ was proved by Chiantini and Madonna [3]. The result we prove is optimal and we refer the reader to [6] for more details.

Our result can also be translated into a statement about curves on $X$: on a general hypersurface $X$ in $\mathbb{P}^4$ of degree $d \geq 6$, any arithmetically Gorenstein curve on $X$ is a complete intersection of $X$ with two other hypersurfaces in $\mathbb{P}^4$. Yet another translation of the result is that the defining equation of such a hypersurface cannot be expressed as the Pfaffian of a skew-symmetric matrix in a non-trivial way.
In the current paper, we will need some of the results from [6]. We will also use the relation between rank two ACM bundles on hypersurfaces and Pfaffians that was observed by Beauville in [1] and which was not needed in [6].

The results from [6] that are important for our proof here are paraphrased below (combining theorem 1.1 (3) and corollary 2.3 of that article). As usual, we will let $H^i(X, E(k))$ denote the graded module $\oplus_{k \in \mathbb{Z}} H^i(X, E(k))$.

**Theorem 1.** Let $E$ be an indecomposable rank two ACM bundle on a smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^4$. Then $H^2(X, End(E))$ is a non-zero cyclic module of finite length, with the generator living in degree $-d$. If $d \geq 5$ and $X$ is general, then $H^2(X, End(E)) = 0$.

## 2. ACM bundles and Pfaffians

We work over an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d$. Let $E$ be an ACM vector bundle of rank two on $X$. By Horrocks’ criterion [5], this is equivalent to saying that $E$ has a resolution,

$$0 \rightarrow F_1 \xrightarrow{\Phi} F_0 \xrightarrow{\sigma} E \rightarrow 0,$$

where the $F_i$’s are direct sums of line bundles on $\mathbb{P}^n$. We will assume that this resolution is minimal, with $F_0 = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^n}(-a_i)$ where $a_1 \leq a_2 \leq \cdots \leq a_n$. Using [1], we may write $F_1$ as $F_1^\vee (e - d)$, where $e$ is the first Chern class of $E$, and we may assume that $\Phi$ is a skew-symmetric $n \times n$ matrix with $n$ even. The $(i,j)$-th entry $\phi_{ij}$ of $\Phi$ has degree $d - e - a_i - a_j$. The condition of minimality implies that there are no non-zero scalar entries in $\Phi$ and thus every degree zero entry must be zero.

We quote some facts about Pfaffians and refer the reader to [9] for more details. Let $\Phi = (\phi_{ij})$ be an $n \times n$ even-sized skew symmetric matrix and let $Pf(\Phi)$ denote its Pfaffian. Then $Pf(\Phi)^2 = \det \Phi$. Let $\Phi(i,j)$ be the matrix obtained from $\Phi$ by removing the $i$-th and $j$-th rows and columns. Let $\Psi$ be the skew-symmetric matrix of the same size with entries $\psi_{ij} = (-1)^{i+j} Pf(\Phi(i,j))$ for $0 \leq i < j \leq n$. We shall refer to $Pf(\Phi(i,j))$ as the $(i,j)$-Pfaffian of $\Phi$. The product $\Phi \Psi = Pf(\Phi) I_n$ where $I_n$ is the identity matrix.

**Example 1.** Let $n = 4$ above. Then

$$Pf(\Phi) = \phi_{12} \phi_{34} - \phi_{13} \phi_{24} + \phi_{14} \phi_{23}.$$
The following lemma shows the relation between skew-symmetric matrices, ACM rank 2 bundles and the equation defining the hypersurface.

**Lemma 1.** Let $E$ be a rank 2 ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d$ and let $\Phi : F_1 \to F_0$ be the minimal skew-symmetric matrix associated to $E$. Then $X = X_\Phi$, the zero locus of Pf$(\Phi)$. Conversely, let $\Phi : F_1 \to F_0$ be a minimal skew-symmetric matrix such that the hypersurface $X_\Phi$ defined by Pf$(\Phi)$ is smooth of degree $d$. Then $E_\Phi$, the cokernel of $\Phi$, is a rank 2 ACM bundle on $X_\Phi$.

**Proof.** Let $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be the polynomial defining $X$. Since $E$ is supported along $X$, $\det \Phi = f^n$ for some $n$ up to a non-zero constant where $\Phi$ is as in resolution (1). Locally $E$ is a sum of two line bundles and so the matrix $\Phi$ is locally the diagonal matrix $(f, f, 1, \cdots, 1)$. Since the determinant of this diagonal matrix is $f^2$, we get $f = \text{Pf}(\Phi)$ (up to a non-zero constant).

To see the converse: let $\Phi$ be any skew-symmetric matrix and $\Psi$ be defined as above. Let $f = \text{Pf}(\Phi)$ be the Pfaffian. Since $\Phi \Psi = f I_n$, this implies that the composite $F_0(-d) \to F_0 \to E_\Phi$ is zero. Thus $E_\Phi$ is annihilated by $f$ and so is supported on the hypersurface $X_\Phi$ defined by $f$. Since $X_\Phi$ is smooth, by the Auslander-Buchsbaum formula, $E_\Phi$ is a vector bundle on $X_\Phi$. Therefore locally $\Phi$ is a diagonal matrix of the form $(f, \cdots, f, 1, \cdots, 1)$ where the number of $f$’s in the diagonal is equal to the rank of $E$. Since $\det(\Phi) = f^2$, we conclude that rank of $E_\Phi$ is 2. \qed

Let $V \subset \text{Hom}(F_1, F_0)$ be the subspace consisting of all minimal skew-symmetric homomorphisms, where $F_i$’s are as above. The following is an easy consequence of the above lemma.

**Lemma 2.** Let $\Phi_0 \in V$ be an element such that $E_{\Phi_0}$ is a rank 2 ACM bundle on a smooth hypersurface $X_{\Phi_0}$. Then there exists a Zariski open neighbourhood $U$ of $\Phi_0$ such that for any $\Phi \in U$, $X_\Phi$ is a smooth hypersurface and $E_\Phi$ is a rank two ACM bundle supported on $X_\Phi$.

### 3. Special cases

The proof of the main theorem will require the study of some special cases, which are listed below.

**Lemma 3.** Consider the following three types of curves in $\mathbb{P}^4$:

- a curve $C$ which is the complete intersection of three general hypersurfaces, two of which are of degree $\leq 2$. 


• a curve \( D \) which is the locus of vanishing of the principal \( 4 \times 4 \) sub-Pfaffians of a general \( 5 \times 5 \) skew-symmetric matrix \( \chi \) of linear forms.

• a curve \( C_r, r \geq 0 \), which is the locus of vanishing of the \( 2 \times 2 \) minors of a general \( 4 \times 2 \) matrix \( \Delta \) with one row consisting of forms of degree \( 1 + r \), and the remaining three rows consisting of linear forms.

The general hypersurface \( X \) in \( \mathbb{P}^4 \) of degree \( \geq 6 \) cannot contain any curve of the first two types. The general hypersurface \( X \) of degree \( d \geq \max\{6, r + 4\} \) cannot contain any curve of the third type.

Proof. The curve \( C \) is smooth if the hypersurfaces are general. If \( \chi \) is general, the curve \( D \) is smooth (see [10], page 432 for example). If \( \Delta \) is general, the curve \( C_r \) is smooth (see op. cit. page 425).

The proof of the lemma is a straightforward dimension count. By counting the dimension of the set of all pairs \((Y, X)\) where \( Y \) is a smooth curve of the described type and \( X \) is a hypersurface of degree \( d \) containing \( Y \), it suffices to show that this dimension is less than the dimension of the set of all hypersurfaces \( X \) of degree \( d \) in \( \mathbb{P}^4 \). This can be done by showing that if \( S \) denotes the (irreducible) subset of the Hilbert scheme of curves in \( \mathbb{P}^4 \) parameterizing all such smooth curves \( Y \), then the dimension of \( S \) is at most \( h^0(\mathcal{O}_Y(d)) - 1 \).

This argument was carried out in [8] where \( Y \) is any complete intersection curve in \( \mathbb{P}^4 \). The case where \( Y \) equals the first type of curve \( D \) in the list above is Case 2 of [8]. Hence we will only consider the types of curves \( D \) and \( C_r \) here.

If \( Y \) is of type \( D \) in the list, the sheaf \( \mathcal{I}_D \) has the following free resolution ([10], page 427):

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 5} \to \mathcal{I}_D \to 0.
\]

Computing Hilbert polynomials, we see that \( D \) is a smooth elliptic quintic in \( \mathbb{P}^4 \), and it easily computed that that \( h^0(N_D) = 25 \). Since \( h^0(\mathcal{O}_D(d)) = 5d \), for \( d \geq 6 \), we get \( \dim S \leq h^0(N_D) \leq h^0(\mathcal{O}_D(d)) - 1 \).

If \( Y \) is of type \( C_r \) in the list, we may analyze the dimension of the parameter space of all such \( C_r \)’s as follows. Let \( S \) be the cubic scroll in \( \mathbb{P}^4 \) given by the vanishing of the two by two minors of the linear \( 3 \times 2 \) submatrix \( \theta \) of the \( 4 \times 2 \) matrix \( \Delta \). The ideal sheaf of the determinantal surface \( S \) has resolution:

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^n}(-2)^3 \to \mathcal{I}_S \to 0.
\]

From this one computes the dimension of the set of such cubic scrolls to be 18, since the 30 dimensional space of all \( 3 \times 2 \) linear matrices is acted
on by automorphisms of \( \mathcal{O}_{\mathbb{P}^n}(-3)^2 \) and \( \mathcal{O}_{\mathbb{P}^n}(-2)^3 \), with scalars giving the stabilizer of the action. Furthermore, by dualizing the resolution, we get a resolution for \( \omega_S \):

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 2} \to \omega_S \to 0.
\]

A section of \( \omega_S(r + 3) \) gives a lift \( \mathcal{O}_{\mathbb{P}^n}(-r - 3) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 2} \), and we obtain a \( 4 \times 2 \) matrix \( \left( \begin{array}{c} \theta \\ \alpha_\vee \end{array} \right) \) of the required type. Hence \( C_r \) is a curve on \( S \) in the linear series \( |K_S + (r + 3)H| \), where \( H \) is the hyperplane section on \( S \). Intersection theory on \( S \) gives

\[
K_S.K_S = 8, \quad K_S.H = -5, \quad H.H = 3.
\]

Using this, we may compute the dimension of the linear system of \( C_r \) on \( S \), and we get the dimension of the set \( S \) of all such \( C_r \) in \( \mathbb{P}^4 \) to be 21 if \( r = 0 \) and \( \left( \frac{3}{2} \right)r^2 + \left( \frac{13}{2} \right)r + 24 \) otherwise.

The ideal sheaf of \( C_r \) has a free resolution given by the Eagon-Northcott complex \([4]\)

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-r - 4)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(-r - 3)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^n}(-r - 2)^{\oplus 3} \to \mathcal{I}_{C_r} \to 0.
\]

Let \( d \geq \max\{6, r + 4\} \) be chosen as in the statement of the lemma. Then \( d = r + s + 4 \) where \( s \geq 0 \) (\( s \geq 2 \) when \( r = 0 \); \( s \geq 1 \) when \( r = 1 \)). Using the above resolution, a calculation gives

\[
h^0(\mathcal{O}_{C_r}(d)) = \frac{3}{2}r^2 + \frac{29}{2}r + 3rs + 4s + 17.
\]

The required inequality \( \dim S < h^0(\mathcal{O}_{C_r}(d)) \) is now evident. \( \square \)

4. PROOF OF MAIN THEOREM

In this section, \( E \) will be an indecomposable ACM bundle of rank two and first Chern class \( e \) on a smooth hypersurface \( X \) of degree \( d \) in \( \mathbb{P}^4 \). The minimal resolution (1) gives \( \sigma : F_0 \to E \to 0 \), and we may describe \( \sigma \) as \([s_1, s_2, \ldots, s_n]\) where \( s_1, s_2, \ldots, s_n \) is a set of minimal generators of the graded module \( H^0(E) \) of global sections of \( E \), with degrees \( a_1 \leq a_2 \leq \cdots \leq a_n \).

**Lemma 4.** If \( E \) is an indecomposable rank 2 ACM bundle with first Chern class \( e \) on a general hypersurface \( X \subset \mathbb{P}^4 \) of degree \( d \geq 6 \), then there is a relation in degree \( 3 - e \) among the minimal generators of \( S^2E \).

**Proof.** Consider the short exact sequence

\[
0 \to \mathcal{O}_X \to \mathcal{E}nd(E) \to (S^2E)(-e) \to 0.
\]
\( S^2E(-e) \) has the same intermediate cohomology as \( \mathcal{E}nd(E) \) since the sequence splits in characteristic zero.

Choose a minimal resolution of \( S^2E \):
\[
0 \to B \to C \to S^2E \to 0,
\]
where \( C \) is a direct sum of line bundles on \( X \) and \( B \) is a bundle on \( X \) with \( H^1(X, B) = 0 \).

We first show that \( B^\vee(e + d - 5) \) is not regular. For this, consider the dual sequence
\[
0 \to (S^2E)^\vee \to C^\vee \to B^\vee \to 0.
\]
By Serre duality and Theorem 1
\[
H^1(X, (S^2E)^\vee(d + e - 5)) = 0.
\]
Therefore
\[
H^0(X, C^\vee(d + e - 5)) \to H^0(X, B^\vee(d + e - 5))
\]
is onto. If \( B^\vee(d + e - 5) \) were regular, the same would be true for
\[
H^0(X, C^\vee(d + e - 5 + k)) \to H^0(X, B^\vee(d + e - 5 + k)) \quad \forall k \geq 0.
\]
However, this is false for \( k = d \) since by Serre duality and Theorem 1, \( H^1(X, (S^2E)^\vee(2d + e - 5)) \neq 0 \). Thus \( B^\vee(e + d - 5) \) is not regular. Now
\[
H^1(X, B^\vee(e + d - 6)) \cong H^2(X, B(1 - e)) \cong H^1(X, S^2E(1 - e)) \cong H^1(X, \mathcal{E}nd(E)(1)).
\]
By Serre duality,
\[
H^1(X, \mathcal{E}nd(E)(1)) \cong H^2(X, \mathcal{E}nd(E)(d - 6))
\]
which by Theorem 1 equals zero for \( d \geq 6 \) (this is the main place where we use the hypothesis that \( d \geq 6 \)). Furthermore, \( H^2(X, B^\vee(e + d - 7)) = 0 \) since \( H^1(X, B) = 0 \). Since \( B^\vee(e + d - 5) \) is not regular, we must have \( H^3(X, B^\vee(e + d - 8)) \neq 0 \).

In conclusion, \( H^0(X, B(3 - e)) \neq 0 \). In other words, there is a relation in degree \( 3 - e \) among the minimal generators of \( S^2E \).

\[\square\]

**Lemma 5.** Let \( E \) be as above. Then \( 1 \leq a_1 + a_2 + e \leq a_1 + a_3 + e \leq 2 \).

**Proof.** The resolution (1) for \( E \) gives an exact sequence of vector bundles on \( X \):
\[
0 \to G \to \mathcal{F}_0 \to E \to 0,
\]
where \( \mathcal{F}_0 = F_0 \otimes \mathcal{O}_X \) and \( G \) is the kernel. This yields a long exact sequence,
\[
0 \to \wedge^2 G \to \mathcal{F}_0 \otimes G \to S^2\mathcal{F}_0 \to S^2E \to 0.
\]
From the arguments after Lemma 2.1 of [6] (using formula (5)), it follows that \( H^2(\wedge^2 G) = 0 \). Hence the map \( S^2\mathcal{F}_0 \to S^2E \) is surjective on global sections. The image of this map picks out the sections \( s_is_j \) of degree \( a_i + a_j \) in \( S^2E \). Observe that the lowest degree minimal sections
$s_1, s_2$ of $E$ induce an inclusion of sheaves $O_X(-a_1) \oplus O_X(-a_2) [s_1, s_2] \hookrightarrow E$ whose cokernel is supported on a surface in the linear system $|O_X(a_1 + a_2 + e)|$ on $X$ (a nonempty surface when $E$ is indecomposable). Hence $1 \leq a_1 + a_2 + e$. There is an induced inclusion

$$S^2[O_X(-a_1) \oplus O_X(-a_2)] \hookrightarrow S^2E.$$ 

Therefore the three sections of $S^2E$ given by $s_1^2, s_1 s_2, s_2^2$ cannot have any relations amongst them. Since these are also three sections of $S^2E$ of the lowest degrees, they can be taken as part of a minimal system of generators for $S^2E$. It follows that the relation in degree $3 - e$ among the minimal generators of $S^2E$ obtained in the previous lemma must include minimal generators other than $s_1^2, s_1 s_2, s_2^2$. Since the other minimal generators have degree at least $a_1 + a_3$, and since we are considering a relation amongst minimal generators, we get the inequality $a_1 + a_3 \leq 2 - e$. □

**Lemma 6.** For any choice of $1 \leq i < j \leq n$, the $(i, j)$-Pfaffian of $\Phi$, is non-zero. Consequently, its degree (which is $a_i + a_j + e$) is at least $(n - 2)/2$.

**Proof.** On $X$, $E$ has an infinite resolution

$$\cdots \to F_0(e - 2d) \xrightarrow{\sigma} F_0(-d) \xrightarrow{\Phi} F_0(e - d) \xrightarrow{\Phi} F_0 \to E \to 0.$$ 

We also have

$$\begin{array}{cccc}
F_0(e - 2d) & \xrightarrow{\sigma} & F_0(-d) & \sigma \\
\downarrow & & \downarrow \cong & \\
E'(e - d) & \xrightarrow{\sigma^\vee} & F_0'(e - d) & \xrightarrow{\Phi^\vee} F_0.
\end{array}$$

Let $\Theta = \sigma^\vee \alpha \sigma$. Since $\sigma = (s_1, \cdots, s_n)$, we may express the $(i, j)$-th entry of $\Theta$ as $\theta_{ij} = s_i^\vee s_j$ (suppressing the canonical isomorphism $\alpha$). $\Phi^\vee = -\Phi$ and $\alpha \sigma : F_0(-d) \to E'(e - d)$ is surjective on global sections. Hence we have a commuting diagram

$$\begin{array}{cccc}
F_0'(e - 2d) & \xrightarrow{\Phi^\vee} & F_0'(e - d) & \xrightarrow{\Phi^\vee} F_0 \to E \to 0 \\
\downarrow B & & \downarrow \cong & \\
F_0'(e - 2d) & \xrightarrow{\Phi^\vee} & F_0'(e - d) & \xrightarrow{\Phi^\vee} F_0 \to E \to 0
\end{array}$$

It is easy to see that $B$ is an isomorphism. As a result, every column of $B$ has a non-zero scalar entry.

Now suppose that $\psi_{ij} = 0$ for some $i, j$ so that $\sum_k s_i^\vee s_k b_{kj} = 0$. Let $Y_i$ be the curve given by the vanishing of the minimal section $s_i$ with
the exact sequence
\[ 0 \to \mathcal{O}_X(-a_i) \xrightarrow{s_i} E \xrightarrow{s_i'} I_{Y_i/X}(a_i + e) \to 0. \]
Hence \( s_i's_i = 0 \) and \( s_i's_k \) for \( k \neq i \) give minimal generators for \( I_{Y_i/X} \).
It follows that no \( b_{kj} \) can be a non-zero scalar for \( k \neq i \). Hence \( b_{ij} \) has to be a non-zero scalar and the only one in the \( j \)-th column. However, \( \psi_{jj} = 0 \). So by the same argument, \( b_{jj} \) is the only non-zero scalar. To avoid contradiction, \( \psi_{ij} \) and hence \( \psi_{ij} \neq 0 \) for \( i \neq j \).

We now complete the proof of the Main Theorem. As in the previous lemmas, assume that \( X \) is general of degree \( d \geq 6 \), with \( E \) an indecomposable rank two ACM bundle on \( X \). We will show that the inequalities of Lemma 5 lead us to the special cases of Lemma 3, giving a contradiction.

Let \( \mu = a_1 + a_2 + e \). By Lemma 5, \( 1 \leq \mu \leq 2 \).

\textbf{Case} \( \mu = 1 \). In this case, in order for the \( (1,2) \)-Pfaffian of \( \Phi \) to be linear, by Lemma 6, \( n \) must equal 4. In the \( 4 \times 4 \) matrix \( \Phi \), the \( (1,2) \)-Pfaffian is the entry \( \phi_{34} \) which we are claiming is linear. Likewise the \( (1,3) \)-Pfaffian is the entry \( \phi_{24} \) which by Lemma 5 has degree \( a_1 + a_3 + e \leq 2 \). By Lemma 2, we may assume that \( \phi_{14}, \phi_{24}, \phi_{34} \) define a smooth complete intersection curve and \( X \) contains this curve by example 1. By Lemma 3, \( X \) cannot be general.

\textbf{Case} \( \mu = 2 \). In this case \( a_2 = a_3 \). By Lemma 6, \( n \) must be 4 or 6. The case \( n = 4 \) is ruled out again by the arguments of the above paragraph since \( \Phi \) has two entries of degree 2 in its last column. We will therefore assume that \( n = 6 \). The matrix
\[
\Phi = \begin{pmatrix}
0 & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} \\
* & 0 & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} \\
* & * & 0 & \phi_{34} & \phi_{35} & \phi_{36} \\
* & * & * & 0 & \phi_{45} & \phi_{46} \\
* & * & * & * & 0 & \phi_{56} \\
* & * & * & * & * & 0
\end{pmatrix}
\]
is skew-symmetric and by our choice of ordering of the \( a_i \)'s, the degrees of the upper triangular entries are non-increasing as we move to the right or down.

As remarked before, the degree of \( \phi_{ij} \) is \( d - e - a_i - a_j \). The \( (1,2) \)-Pfaffian (which is a non-zero quadric when \( \mu = 2 \)) is given by the expression (see example 1)
\[
Pf(\Phi(1,2)) = \phi_{34}\phi_{56} - \phi_{35}\phi_{46} + \phi_{36}\phi_{45}.
\]
We shall consider the following two sub-cases, one where \( \phi_{56} \) has positive degree (and hence can be chosen non-zero by Lemma 2) and the other where it has non-positive degree (and hence is forced to be zero):

\[
d - e - a_5 - a_6 > 0.
\]

Since \( \phi_{34} \cdot \phi_{56} \) is one term in the \((1,2)\)-Pfaffian of \( \Phi \), and since degree \( \phi_{34} \) is at least degree \( \phi_{56} \), they are both forced to be linear. Therefore \( a_3 = a_4 = a_5 \). Likewise, \( a_3 = a_4 = a_5 = a_6 \). Hence \( \Phi \) has a principal \( 5 \times 5 \) submatrix \( \chi \) (obtained by deleting the first row and column in \( \Phi \)) which is a skew symmetric matrix of linear terms, while its first row and first column have entries of degree \( 1 + r, r \geq 0 \).

By Lemma 2 we may assume that the ideal of the \( 4 \times 4 \) Pfaffians of \( \chi \) defines a smooth curve \( C \). \( X \) is then a degree \( d = 3 + r \) hypersurface containing \( C \). By Lemma 3, \( X \) cannot be general when \( d \geq 6 \).

\[
d - e - a_5 - a_6 \leq 0.
\]

In this case, the entry \( \phi_{56} = 0 \). Suppose \( \phi_{46} \) is also zero. Then both \( \phi_{36} \) and \( \phi_{45} \) must be linear and non-zero since the \((1,2)\)-Pfaffian of \( \Phi \) (see equation 2) is a non-zero quadric. Since \( a_2 = a_3 \), \( \phi_{26} \) is also linear. Thus using Lemma 2, \( X \) contains the complete intersection curve given by the vanishing of \( \phi_{16} \) and the two linear forms \( \phi_{36}, \phi_{26} \). By Lemma 3, \( X \) cannot be general.

So we may assume that \( \phi_{46} \neq 0 \). Since \( \phi_{35} \) is also non-zero, both must be linear. Hence \( a_3 + a_5 = a_4 + a_6 \), and so \( a_3 = a_4 \) and \( a_5 = a_6 \).

After twisting \( E \) by a line bundle, we may assume that \( a_2 = a_3 = a_4 = 0 \leq a_5 = a_6 = b \). The linearity of the entry \( \phi_{46} \) gives \( d - e - b = 1 \). The condition \( d - e - a_5 - a_6 \leq 0 \) yields \( 1 \leq b \). Taking first Chern classes in resolution (1) gives \( e = 2 - a_1 \).

Let \( r = -a_1, s = b - 1 \). Then \( r, s \geq 0 \), and \( d = r + s + 4 \). If we inspect the matrix \( \Phi \), the non-zero rows in columns 5 and 6 give a \( 4 \times 2 \) matrix \( \Delta \) with top row of degree \( 1 + r \) and the other entries all linear. By Lemma 2, we may assume that the \( 2 \times 2 \) minors of this \( 4 \times 2 \) matrix define a smooth curve \( C_r \) as described in Lemma 3. Since \( X \) contains this curve, \( X \) cannot be general when \( d \geq 6 \).

**References**


Department of Mathematics, Washington University in St. Louis, St. Louis, Missouri, 63130
E-mail address: kumar@wustl.edu
URL: http://www.math.wustl.edu/~kumar

Department of Mathematics, University of Missouri-St. Louis, St. Louis, Missouri 63121
E-mail address: raoa@umsl.edu

Department of Mathematics, Indian Institute of Science, Bangalore–560012, India
E-mail address: ravindra@math.iisc.ernet.in