Existence and perfect simulation of one-dimensional loss networks

Nancy L. Garcia, *UNICAMP* *

Nevena Marić, *USP* †

May 10, 2006

Abstract

Perfect simulation of an one-dimensional loss network on $\mathbb{R}$ with length distribution π and cable capacity C can performed using the clan of ancestors method. Domination of the clan of ancestors by a branching process with dependency in two generations improves the known sufficient conditions for the perfect scheme to be applicable.

Key words: clan of ancestors, multitype branching process, perfect simulation

AMS Classification: Primary: 93E30, 60G55; Secondary: 15A18

1 Introduction

Kelly (1991) introduced a continuous unbounded loss network described as follows. Imagine that users are arranged along an infinitely long cable and that a call between two points on the cable

*Corresponding author: IMECC/UNICAMP, Caixa Postal 6065, 13.081-970 - Campinas SP BRAZIL, nancy@ime.unicamp.br

†IME/USP, Caixa Postal 66281, 05311-970 - São Paulo SP BRAZIL, nevena@ime.usp.br

s_1, $s_2 \in \mathbb{R}$ involves just that section of the cable between s_1 and s_2. Past any point along its length the cable has the capacity to carry simultaneously up to C calls: a call attempt between s_1, $s_2 \in \mathbb{R}$, $s_1 < s_2$, is lost if past any point of the interval $[s_1, s_2]$ the cable is already carrying C calls. Suppose that calls are attempted at points in $\mathbb{R}$ following a homogeneous Poisson process with rate λ. Assume that the section of the cable demanded by a call has distribution π with finite mean ρ_1 and the duration of a call has exponential distribution with mean one. Assume that the location of a call, the cable section needed and its duration are independent. Let $m(s,t)$ be the number of calls in progress past point s on the cable at time t. Kelly (1991) conjectured that $((m(s,t), s \in \mathbb{R}), t \geq 0)$ has a unique invariant measure, given by a stationary $M/G/\infty$ queue (Markov arrivals, general service time and infinite servers) conditioned to have at most C clients at all times. Ferrari and Garcia (1998) used a continuous (non-oriented) percolation argument to prove the above conjecture whenever π has finite third moment and the arrival rate λ is sufficiently small. Fernández, Ferrari and Garcia (2002) using an oriented percolation argument improved this bound to

$$\lambda_c^{FFG} = \frac{1}{(\rho_2 + \rho_1 + 1)} \tag{1.1}$$

where ρ_1 and ρ_2 are the first and second moment of distribution π respectively. This argument is based on a graphical representation of the birth and death process and it is the basis for the perfect simulation scheme "*Backward-Forward Algorithm*", described in Fernández, Ferrari and Garcia (2002). This algorithm involves the "thinning" of a marked λ-homogeneous Poisson process —the *free process*— which dominates the birth-and-death process, and it involves a time-backward and a time-forward sweep. The initial stage of the construction is done *toward the past*, starting with a finite window and retrospectively looking to *ancestors*, namely to those births in the past that could have (had) an influence on the current birth. The construction of the *clan of ancestors* constitutes the time-backward sweep of the algorithm. Once this clan is completely constructed, the algorithm proceeds in a time-forward fashion "cleaning up" successive generations according to appropriate penalization schemes. The relation "being ancestor of" induces a backward in time *contact/oriented percolation* process. The algorithm is applicable as long as this oriented percolation process is sub-critical. Garcia and Marić (2003) using the Perron-Frobenius theory for sub-criticality of branching process obtained a new bound given by

$$\lambda_c^* = \frac{1}{(\sqrt{\rho_2} + \rho_1)}. \tag{1.2}$$

The implementation of Backward-Forward Algorithm is presented in Garcia and Marić (2002) for continuous one-dimensional loss networks. In that work, the characteristics of the clan of ancestors is studied through simulations and it becames clear that the domination by the branching process is far from sharp. They show that for fixed length calls and Beta distributed calls the estimated critical value is approximately $2.8\lambda_c^*$.

In fact, the number of ancestors is much smaller than the total number of the population in the branching process and the clan of ancestors can be finite even though the branching is supercritical. This comes from the fact that in the dominating branching process the offspring distribution is independent for each individual. Therefore, in general, there is generation of several Poisson processes in the same region and the overall rate of the Poisson process is multiplied inhomogeneously. On the other hand, for the clan of ancestors, in order to keep the rate constant, the Poisson process with rate λ is generated in disjoint regions.

To overcome this unnecessary increase in the rate, we can prevent some births by looking further into the past. In the simulation scheme, if a region was used to construct the free Poisson process in one generation, it is eliminated for further births. In this work, we propose to dominate the clan of ancestors by a 2-generation branching process, that is a Markov process with order 2 so that we eliminate some points of the branching process that are not necessary to dominate the clan of ancestors. In Section 4, we improve bound (1.2) to

$$\lambda_c^{**} = \frac{4}{3\rho_1 + \sqrt{\rho_1^2 + 8\rho_2}} \tag{1.3}$$

and it stands that $\lambda_c^{**} \geq (4/3)\lambda_c^*$.

We believe that this approach can be extended using higher order branching processes leading to a sequence of approximations leading to better bounds.

2 Loss networks

Let $\mathcal{G}$ be a family of intervals of the line γ ($\gamma = (x, x+u), x, u \in \mathbb{R}$), which will be named *calls*, and consider a state space $\mathcal{S} = \{\xi \in \mathbb{N}^{\mathcal{G}} : \xi(\gamma) \neq 0$ only for a countable set of $\gamma \in \mathcal{G}\}$.

Loss networks are a particular class of spatial birth-and-death processes. The evolution of these processes in time are given either by the *birth* of a new call to be added to the actual configuration or by the *death* of an existing call that will be eliminated from the actual configuration. Moreover, they have the Markovian property in time that, the probability of a change depends only on the actual configuration of the system. A loss network η_t is defined by a marked Poisson process where births are controlled by a *birth rate*, a non-negative measurable function $\mathbf{b}(\gamma, \eta)$ such that

$$\int_B \mathbf{b}(\gamma, \eta) d\gamma < \infty$$

for each B, bounded Borel set, and for all $\eta \in \mathcal{S}$. The births are regulated by the exclusion principle, depending on the capacity (C) of the network. The marks include a life-time exponentially distributed with mean one and the length of the call which is distributed according to a distribution π with finite first and second moments ρ_1 and ρ_2 respectively. The *death rate* also a non-negative measurable function $\mathbf{d} : \mathbb{R}^d \times \mathcal{S} \to [0, \infty)$. In this work, we are going to assume that the death rate is always equal to one, that is, $\mathbf{d}(\gamma, \eta) = 1$ if $\eta(\gamma) = 1$. The generator of the process is given by

$$Af(\eta) = \int (f(\eta + \delta_\gamma) - f(\eta)) b(\gamma, \eta) d\gamma + \int (f(\eta - \delta_\gamma) - f(\eta)) \eta(d\gamma) \tag{2.1}$$

where $\eta \in \{0,1\}^{\mathcal{B}(\mathbb{R})}$. In the second term in the RHS we included the death rate to be 1 and we use the following notation $\int g(\gamma)\eta(d\gamma) = \sum_{\gamma : \eta(\gamma) \geq 1} g(\gamma)$.

For a process α_t with rate densities which are independent of the actual configuration there exists $\omega : \mathcal{G} \to [0, \infty)$ such that

$$\mathbf{b}(\gamma, \alpha) = \omega(\gamma) \tag{2.2}$$

we call this process a *free process*. Such a process is just a space-time marked Poisson process. It exists and is ergodic whichever the choice of w. In the particular case where $\omega(\gamma) = \lambda$ the invariant measure is the λ-homogeneous Poisson process. For the one-dimensional loss networks (described in Section 1) the birth rate is uniformly bounded and it can be decomposed as

$$\mathbf{b}((x, x+u), \eta) = \lambda \ \pi(u) \ M((x, x+u), \eta) \tag{2.3}$$

where, $0 \le M(\gamma, \eta) \le 1$. The first factor represents a basic birth-rate density due to an "internal" Poissonian clock and the last factor acts as an unnormalized probability for the individual to be actually born once the internal clock has rang. The birth is hindered or reinforced according to the configuration η.

In this case, for capacity $C = 1$,

$$M(\gamma, \eta) = \prod_{\theta : \eta(\theta) \neq 0} (1 - I(\gamma, \theta)) \tag{2.4}$$

$$I(\gamma, \theta) = \begin{cases} 1 & \gamma \cap \theta \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

where γ, θ are of the form $(x, x+u)$. For $C > 1$, the expression is less simple

$$M((x, x+u), \eta) = \begin{cases} 1 & \text{otherwise} \\ 0 & \text{there exists } y \in (x, x+u) \ \textit{and} \\ & \theta_1, \ldots, \theta_C \text{ such that } \eta(\theta_i) = 1 \\ & \text{and } y \in \theta_i \ \text{ for all } i = 1, \ldots, C. \end{cases}$$

3 Graphical construction for the loss networks

Let $\mathcal{N} = \{ \ (\xi_1, T_1), \ (\xi_2, T_2), \ldots \}$ be a homogeneous Poisson Process with rate λ in $\mathbb{R} \times [0, \infty)$, $S_1, S_2, \ldots$ be i.i.d. random variables exponentially distributed with mean one and $U_1, U_2, \ldots$ be i.i.d. random variables with common distribution π. Assume the family of variables $\{S_1, S_2, \ldots\}$, $\{U_1, U_2, \ldots\}$ and the Poisson process are all independent. Consider the random rectangles

$$R_i = \{(x, y); \xi_i \le x \le \xi_i + U_i, T_i \le y \le T_i + S_i\}.$$

Then $\{R_i, i \ge 1\} = \{(\xi_i, T_i) + D_i, i \ge 1\}$ is a Boolean model in $\mathbb{R}^2$ where $D_i = [0, U_i] \times [0, S_i]$ and represents the free process of calls. Boolean models have the property that the number of sets $C \in \mathcal{C}$ that cover a fixed point $x \in \mathbb{R}^d$ is a Poisson random variable with mean $\lambda \mathbb{E}(\text{vol}(S))$. For more details about coverage processes see Hall (1988).

Now, for each rectangle R_i we associate an independent mark $Z_i \sim U(0,1)$, and each marked rectangle we identify with the marked point $(\xi_i, T_i, S_i, U_i, Z_i)$. We recognize in the marked point process $\mathbf{R} = \{(\xi_i, T_i, S_i, U_i, Z_i),\ i = 1, 2, \ldots\}$ a graphical representation of the birth and death process with constant birth rate λ, and constant death rate, equal to 1. We call this free process α and Z_i will serve as a flag of allowed births. Calling $R = (\xi, \tau, s, u, z)$, we use the notation

$$\text{Basis}\,(R) = (\xi, \xi + u), \quad \text{Birth}\,(R) = \tau, \quad \text{Life}\,(R) = [\tau, \tau + s], \quad \text{Flag}\,(R) = z. \tag{3.1}$$

We also define, for two rectangles R and R',

$$R' \nsim R, \text{ if } R' \cap R \ne \emptyset$$
$$R' \sim R, \text{ otherwise.}$$

We need a series of definitions:

- For an arbitrary point $(x, t) \in \mathbb{R}^2$ define the collection of all rectangles in $\mathbf{R}$ that contain this point
$$\mathbf{A}_1^{(x,t)} = \{R \in \mathbf{R} |\ x \in \text{Basis}\,(R), t \in \text{Life}\,(R)\} \tag{3.2}$$

- For each rectangle R define its ancestor set

$$\mathbf{A}_1^R = \{R' \in \mathbf{R} |\ \text{Birth}\,(R') \le \text{Birth}\,(R),\ R' \nsim R\} \tag{3.3}$$

- Define recursively the generations ($n > 1$) of the above sets that is, the nth generation of ancestors:

$$\mathbf{A}_n^{(x,t)} = \{R'' | R'' \in \mathbf{A}_1^{R'} \text{ for some } R' \in \mathbf{A}_{n-1}^{(x,t)}\} \tag{3.4}$$

$$\mathbf{A}_n^{R} = \{R'' | R'' \in \mathbf{A}_1^{R'} \text{ for some } R' \in \mathbf{A}_{n-1}^{R}\} \tag{3.5}$$

We say that there is *backward oriented percolation* if there exists one point (x, t) such that $\mathbf{A}_n^{(x,t)} \neq \emptyset$ for all n, that is, if there exists one point with an infinite number of ancestors. Call *clan of ancestors* of (x, t) the union of all its ancestors:

$$\mathbf{A}^{(x,t)} = \bigcup_{n \geq 1} \mathbf{A}_n^{(x,t)} \tag{3.6}$$

and $\mathbf{R}[0, \mathrm{t}] = \{\mathrm{R} \in \mathbf{R} | \text{ Birth}\,(\mathrm{R}) \in [0, \mathrm{t}]\}$.

The existence of the process in infinite volume for any time interval is guaranteed as long as the process do not explode, that is, no rectangle has an infinite number of ancestors in a finite time. The following theorem is proved in Fernández, Ferrari and Garcia (2001).

Theorem 3.7 *If $\mathbf{A}^{(x,t)} \cap \mathbf{R}[0, \mathrm{t}]$ is finite with probability one, for any $x \in \mathbb{R}$ and $t \geq 0$, then for all $\Lambda \subseteq \mathbb{R}$ the loss network process defined in Λ is well-defined and has at least one invariant measure μ^Λ.*

For the existence of the process in infinite time, it is needed that the clan of ancestors of all rectangles are finite with probability one, that is, there is no backward oriented percolation. In order to construct the invariant measure for stationary Markov processes it is usual to construct the process beginning at $-\infty$ with an arbitrary configuration and look at the process at time 0. If the configuration at time 0 does not depend on the initial configuration then we have a sample of invariant measure. The graphical construction described above allow us to construct the process η_t by a thinning of the free process α_t for all $t \in \mathbb{R}$. Moreover, the same argument shows that the distribution of η_0 does not depend on the initial configuration. The next theorem summarizes the results about the process, see Fernández et al. (2001, 2002).

Theorem 3.8 *If with probability one there is no backward oriented percolation in* $\mathbf{R}$, *then the loss network process can be constructed in* $(-\infty, \infty)$ *in such a way that the marginal distribution of* η_t *is invariant. Moreover, this distribution is unique and the velocity of convergence is exponential.*

One way of determining the lack of percolation is the domination through a branching process. Establishing sub-criticality conditions for the branching process we obtain sufficient conditions for lack of percolation. Looking backward, the ancestors will be the branches. The time of the death will be the birth time for the branching process. The clan of ancestors in itself is not a branching process because the lack of independence.

Let R be a rectangle with basis $\gamma = (x, x+u)$ with length u, born at time 0. Define $\tilde{b}_n^u(v)$ as the number of rectangles in the nth generation of ancestors of R having basis with length v:

$$\tilde{b}_n^u(v) = |\{R^{'} \in \mathbf{A}_n^R| \ |\text{Basis}\,(R^{'})| = v\}|. \tag{3.9}$$

The process $\tilde{b}_n$ is not a Galton-Watson process but it can be dominated by one (call it b_n) as described by Fernández et al. (2001), where each call length represents a type and it has as offspring distribution the same one as $\tilde{b}_1$. The number of types can be finite, countable or uncountable depending upon the distribution π.

For the one-dimensional loss network the offspring distribution of b_n is Poisson distributed with mean

$$m(u, v) = \lambda \ \pi(v) \ (u + v) \tag{3.10}$$

where $m(u, v)$ is the mean number of children type v for parents type u. In this case, Garcia and Marić (2003) used the Perron–Frobenius theory for sub-criticality of branching process obtained a new bound given by

$$\lambda(\sqrt{\rho_2} + \rho_1) < 1. \tag{3.11}$$

In Section 4 we are going to improve this bound dominating the clan of ancestors by a branching process with order 2, that is, the reproducing mechanism is governed not only by the parents but also by the grandparents.

4 Dominating the clan of ancestors by a 2-generation branching process. Critical value.

To construct the Galton-Watson process b_n mentioned by the end of Section 3, Fernández et al. (2001) used a multitype branching process $\mathbf{B}_n$, in the set of cylinders, which dominates $\mathbf{A}_n$. To do this they looked "backward in time" and let "ancestors" play the role of "branches". In particular, births in the original marked Poisson process correspond to disappearance of branches. Like them, we reserve the words "birth" and "death" for the original forward-time Poisson process. This construction can be done by enlarging the probability space and defining, for any given set $\{R_1, \ldots, R_k\}$, *independent* random sets $\mathbf{B}_1^{R_i}$ with the same marginal distribution as $\mathbf{A}_1^{R_i}$. The important point here is that

$$\bigcup_{i=1}^{k} \mathbf{A}_1^{R_i} \subset \bigcup_{i=1}^{k} \mathbf{B}_1^{R_i}. \tag{4.1}$$

The procedure defined by $\mathbf{B}_1$ naturally induces a multitype branching process in the space of rectangles. The n-th generation of the branching process is defined by

$$\mathbf{B}_n^R = \{\mathbf{B}_1^{R'} : R' \in \mathbf{B}_{n-1}^R\} \tag{4.2}$$

where for all R', $\mathbf{B}_1^{R'}$ has the same distribution as $\mathbf{A}_1^{R'}$ and are independent random sets depending only on R'. Then, for all $n \geq 1$ it follows

$$\mathbf{A}_n^R \subset \mathbf{B}_n^R. \tag{4.3}$$

It is introduced then a multitype branching process in the set of calls $\mathcal{G}$. For an initial rectangle R with the basis of size u, b_n^u is defined as the number of rectangles in the n-th generation of $\mathbf{B}_n$ that have basis with length v:

$$\mathbf{b}_n^u(v) = |\{R^{'} \in \mathbf{B}_n^R| \ |\text{Basis}\,(R^{'})| = v\}|. \tag{4.4}$$

The following relation is then established

$$|\mathbf{A}_n^R| \leq |\mathbf{B}_n^R| = \sum_v b_n^u(v). \tag{4.5}$$

We will have to further enlarge the set of types of $\mathbf{B}_n$ by creating a branching process in the set of cylinders $\mathbf{B}_n^*$ with a new type called Color $\in \{\textsc{b}, \textsc{g}\}$ which is inherited by a cylinder not only from the types of the first generation (parent) but also from the 2nd generation (grand-parent). The types $\textsc{b}, \textsc{g}$ are chosen to associate to *black* and *green*.
The idea is to identify some rectangles in the branching process $\mathbf{B}_n$ that could never be present in the $\mathbf{A}_n$. These rectangles we have in mind as "black". Then, if the total number of "green" (not-black) rectangles is finite implies that the clan of ancestors is also finite. We expect then that counting only "green" rectangles one can obtain better critical value for λ. That is what we are going to prove in the following.

For a rectangle $V = (\xi_V, \tau_V, s_V, u_V)$ in the nth generation of the branching process $\mathbf{B}_n$, we define an extra type to be its color as follows. For $n = 1$ or 2, Color$(V) = \textsc{g}$. For $n > 2$, let $R \in \mathbf{B}_{n-1}^*$ and $R' \in \mathbf{B}_{n-2}^*$ such that $V \in \mathbf{B}_1^R$ and $R \in \mathbf{B}_1^{R'}$. Denote $R = (\xi, \tau, s, u, c)$ and $R' = (\xi', \tau', s', u', c')$, where $c =$ Color(R) and $c' =$ Color(R'). Therefore,

- if $c = \textsc{b}$ then Color$(V) = \textsc{b}$,
- if $c = \textsc{g}$ then Color$(V) = \textsc{b}$ if and only if $(\xi_V, \tau_V + s_V) \in D(u_V, R, R')$ where $D(u_V, R, R') = L(\xi, u, \xi', u', u_V) \times [0, \tau']$ and

$$L(\xi, u, \xi', u', u_V) = [\max(\xi - u_V, \xi' - u_V), \min(\xi + u, \xi' + u')]. \tag{4.6}$$

This way, for every rectangle is defined its bi-dimensional type Type$(\cdot) \in \mathcal{G} \times \{\textsc{b}, \textsc{g}\}$. Figure 4.1 shows an example for birth of green and black rectangles from a green parent. Notice that the black rectangle is an ancestor of both R and R' but it will count as green for R' and black for R. We stress the fact that a black parent produces only black descendents.

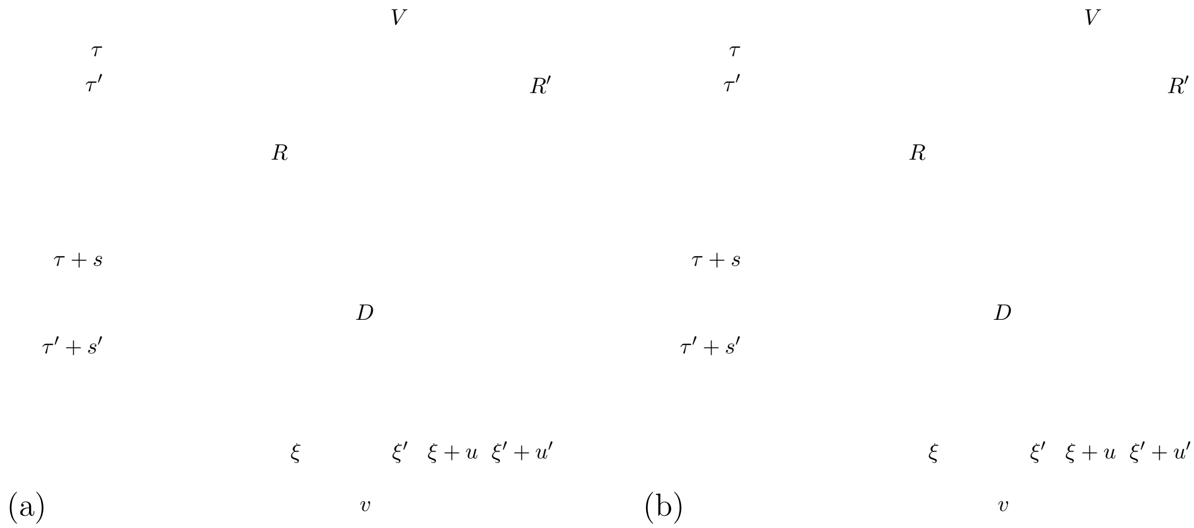

Figure 4.1: Green parent R and grandparent R' giving birth to (a) green and (b) black rectangles V

Consider $d_n^{(w,\ell)}((u,c),(v,k))$ to be a 2-generation multitype branching process defined by

$$\begin{aligned} & d_n^{(w,\ell)}((u,c),(v,k)) \\ & \quad = |\{V : V \in B_1^R \text{ and } R \in B_{n-1}^C; \text{Type}\,(C) = (w,\ell), \text{Type}\,(R) = (u,c), \text{Type}\,(V) = (v,k)\}|, \end{aligned} \tag{4.7}$$

that is, it is the number of cylinders of type (v,k) in the nth generation of a cylinder of type (w,ℓ) which have an ancestor in the $n-1$ generation of type (u,c). We can think of this branching process as a first order branching process with enlarged state space given by $\mathcal{S} = \{((i,u,c),(j,v,k))\}$ and offspring distribution to be Poisson distributed.

From the definition of the process d_n and the relation (4.5) it is clear that

$$|\mathbf{A}_n^R| \le \sum_{u,v} d_n^{(w,\mathrm{G})}((u,\mathrm{G}),(v,\mathrm{G})) \le \sum_v b_n^w(v). \tag{4.8}$$

We suppose from now on that π has discrete support.

Proposition 4.9 *In the process d_n the mean number of offspring of type (v,k) of an individual* *of type* (u,c) *with the parent of type* (u',c') *is given by*

$$m(((u',c'),(u,c));((u,c),(v,k))) = \begin{cases} 0, & \textit{if } c=\text{B}, k=\text{G} \textit{ or if } c'=\text{B}, c=\text{G} \\ \lambda\pi(v)(u+v), & \textit{if } c=\text{B}, k=\text{B} \end{cases} \tag{4.10}$$

and

$$m(((u',\text{G}),(u,\text{G}));((u,\text{G}),(v,\text{B})) \geq \frac{1}{2}\lambda\pi(v)v, \tag{4.11}$$

$$m(((u',\text{G}),(u,\text{G}));((u,\text{G}),(v,\text{G})) \leq \lambda\pi(v)(u+\frac{v}{2}) \tag{4.12}$$

Proof: For (4.10) it is enough to observe that from the definition of Color follows that is impossible to have green children from black parents. Since a black individual has all its children black, *independently of its own parent type*, the number of its children has the same low as the first generation in branching process b_n defined by (4.4). The total number of children of a green individual has also the same low as the above one, but in this case it is possible to have children of both colors. Recall from (3.10) that $\mathbb{E}(b^u(v)) = \lambda\pi(v)(u+v)$ so that

$$m(((u',\text{G}),(u,\text{G}));((u,\text{G}),(v,\text{B}))) + m(((u',\text{G}),(u,\text{G}));((u,\text{G}),(v,\text{G}))) = \lambda\pi(v)(u+v). \tag{4.13}$$

Therefore, to prove (4.11) and (4.12) it is sufficient to find the mean number of black children from green parent (and necessarily green grandparent): $m(((u',\text{G}),(u,\text{G}));((u,\text{G}),(v,\text{B})))$.

Consider two incompatible rectangles $R = (\xi,\tau,s,u,\text{G}) \in \mathbf{A}_2$ and $R' = (\xi',\tau',s',u',\text{G}) \in \mathbf{A}_1$ such that $R \in \mathbf{A}_1^{R'}$, that is R is an ancestor of R'. Since R' belongs to the first generation, we have $\tau' < 0$ and $\tau' + s' > 0$. By construction of the clan of ancestors $\mathbf{A}_n$ and the branching process $\mathbf{B}_n$, rectangles R' in the first generation have Birth $(R') < 0$ and Death $(R') > 0$. We want to compute the $X(v,R,R')$– number of **black** ancestors of v-type, namely the number of those $V = (\xi_V,\tau_V,s_V,v,\text{B}) \in \mathbf{A}_3$ with length v such that $V \in \mathbf{A}_1^R$ and $R \in \mathbf{A}_1^{R'}$.

The same argument as used by Garcia and Marić (2003) to conclude that $b_1^u(v)$ is Poisson distributed can be used here. In this case, given R, R', $X(v,R,R')$ is a Poisson random variable with rate that has to be calculated. To accomplish this, for fixed R, R', let $\Lambda = L \times (\tau', \infty)$ where $L = L(\xi, u, \xi', u', v)$ is given by (4.6). For the set Λ, call the ancestors of Λ all rectangles $W = (\xi_W, \tau_W, s_W, u_W, c_W)$ in the free process α such that $(\xi_W, \tau_W + s_W) \in \Lambda$ and $\tau_W < \tau'$.

Then, $X(v, R, R')$ is the number of rectangles with length v born before time τ which and died after τ' in area Λ. Therefore, $X(v, R, R')$ is the difference between the total number of ancestors of Λ and the ancestors of Λ born after time τ. Fix $t < \tau'$, let $W = (\xi_W, \tau_W, s_W, v, c_W)$ belong to the free process α with $\tau_W > t$, and recall that in the free process $\xi_W \sim U(L)$ and $\tau_W \sim U(t, \tau')$. Then, we have

$$\mathbb{P}\left(W \text{ is an ancestor of } \Lambda \mid R, R'\right) \;=\; \pi(v)\mathbb{P}(\tau_W + s_W > \tau') \;=\; \pi(v)\frac{1 - e^{-(\tau' - t)}}{\tau' - t}\,. \tag{4.14}$$

Since the free process is constructed from a homogeneous Poisson process, it is immediate that the number of ancestors of Λ born before and after time τ are independent Poisson random variables.

Applying equation (4.14) for the case $t = \tau$ we obtain that the mean of number of ancestors of Λ born after time τ equals to

$$\lambda|L|(\tau' - \tau)\pi(v)\frac{1 - e^{-(\tau' - \tau)}}{\tau' - \tau} \;=\; \lambda\pi(v)|L|(1 - e^{-(\tau' - \tau)})\,.$$

For the same reason as above, letting $t \to -\infty$ in equation (4.14), we obtain that that the total number of ancestors of Λ is Poisson distributed with mean $\lambda\pi(v)|L|$.

Now, it is straight forward to conclude that, given R and R', $X(v, R, R')$ is Poisson distributed with mean

$$\lambda\pi(v)|L| \;-\; \lambda\pi(v)|L|(1 - e^{-(\tau' - \tau)}) \;=\; \lambda\pi(v)|L|e^{-(\tau' - \tau)}\,. \tag{4.15}$$

Notice that $\tau < \tau'$ and we are given the parent relation, so $\tau' - \tau$ has exponential distribution with mean one. Hence, $\mathbb{E}(e^{-(\tau' - \tau)}) = 1/2$.

Furthermore, since $\xi \in [\xi' - u, \xi' + u']$

$$|L| = \min(\xi + u, \xi' + u') - \max(\xi - v, \xi' - v) \quad \geq \quad v\,. \tag{4.16}$$

From (4.15) and (4.16), (4.11) follows immediately as

$$m(((u',\mathrm{G}),(u,\mathrm{G}));((u,\mathrm{G}),(v,\mathrm{B})))=\mathbb{E}(X(v,R,R'))\ \geq\ \frac{1}{2}\lambda\pi(v)v\,. \tag{4.17}$$

□

Sub-criticality: On account of the relation (4.8) we are interested in sub-criticality conditions for the green-type population of the branching process d_n described above. Remember that green individuals may appear in the n-th generation only as descendants of a branch made of green individuals only. Therefore, our aim is to establish conditions for the convergence of the series

$$\sum_{n\geq 1}\sum_{u_{n-1},v} m^{(n)}(((w,\mathrm{G}),(u,\mathrm{G}));((u_{n-1},\mathrm{G}),(v,\mathrm{G}))) \tag{4.18}$$

where $m^{(1)}(\cdot\,;\cdot)=m(\cdot\,;\cdot)$ is given by (4.11) and for $n>1$

$$\begin{aligned} &m^{(n)}(((w,c'),(u,c));((u_1,c_1),(v,k))) \qquad (4.19)\\ &\quad=\sum_{u_2,c_2} m^{(n-1)}(((w,c'),(u,c));((u_2,c_2),(u_1,c_1)))m(((u_2,c_2),(u_1,c_1));((u_1,c_1),(v,k))). \end{aligned}$$

To simplify notation let

$$m_*(u,v)\quad=\lambda\pi(v)(u+\tfrac{v}{2}) \tag{4.20}$$

and it follows from Proposition (4.9) that for all u,v and independently of w

$$m(((w,\mathrm{G}),(u,\mathrm{G}));((u,\mathrm{G}),(v,\mathrm{G})))\leq m_*(u,v). \tag{4.21}$$

Therefore, using the notation above and (3.10) we can easily estimate the series (4.18) to

$$\begin{aligned} \sum_{u_{n-1},u} m^{(n)}(((w,\mathrm{G}),(u,\mathrm{G}));((u_{n-1},\mathrm{G}),(v,\mathrm{G})))&\leq\sum_{v,u_1\ldots u_{n-1}} m(u,u_1)m_*(u_1,u_2)\cdots m_*(u_{n-1},v)\\ &=\lambda^n\textstyle\sum_v\sum_{u_1}\cdots\sum_{u_{n-1}}\pi(u_1)(u+u_1)\pi(u_2)\left(u_1+\frac{u_2}{2}\right)\cdots\pi(v)\left(u_{n-1}+\frac{v}{2}\right) \qquad (4.22) \end{aligned}$$

Observe that

$$\sum_v\pi(v)\left(u_{n-1}+\frac{v}{2}\right)=u_{n-1}+\frac{\rho_1}{2}=f_1^*+g_1^*u_{n-1} \tag{4.23}$$

where $f_1^* = \frac{\rho_1}{2}$ and $g_1^* = 1$, and define inductively

$$\sum_{u_{n-i+1}} \pi(u_{n-i+1}) \left(u_{n-i} + \frac{u_{n-i+1}}{2}\right) (f_{i-1}^* + g_{i-1}^* u_{n-i+1}) = f_i^* + g_i^* u_{n-i}. \tag{4.24}$$

Then we have

$$\begin{bmatrix} f_{j+1}^* \\ g_{j+1}^* \end{bmatrix} = \begin{bmatrix} \rho_1/2 & \rho_2/2 \\ 1 & \rho_1 \end{bmatrix} \cdot \begin{bmatrix} f_j^* \\ g_j^* \end{bmatrix} = \begin{bmatrix} \rho_1/2 & \rho_2/2 \\ 1 & \rho_1 \end{bmatrix}^j \cdot \begin{bmatrix} \rho_1/2 \\ 1 \end{bmatrix} \tag{4.25}$$

and consequently, the series (4.22) is equal to

$$\lambda^n \sum_{u_1} \pi(u_1)(u + u_1)(g_{n-1}^* v_1 + f_{n-1}^*) = \lambda^n (u(g_{n-1}^* \rho_1 + f_{n-1}^*) + \rho_2 g_{n-1}^* + \rho_1 f_{n-1}^*). \tag{4.26}$$

In order to find f_n^*, g_n^* we exponentiate $T^* = \begin{bmatrix} \rho_1/2 & \rho_2/2 \\ 1 & \rho_1 \end{bmatrix}$. For this operation suffices the eigenvalues of T^*, ε_1 and ε_2 given by

$$\varepsilon_{1,2} = \frac{3\rho_1 \pm \sqrt{\rho_1^2 + 8\rho_2}}{4}. \tag{4.27}$$

and two corresponding normalized eigenvectors

$$\frac{1}{\sqrt{1 + (\varepsilon_1 - \rho_1)^2}} \begin{bmatrix} \varepsilon_1 - \rho_1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{1 + (\varepsilon_2 - \rho_1)^2}} \begin{bmatrix} \varepsilon_2 - \rho_1 \\ 1 \end{bmatrix}. \tag{4.28}$$

From (4.25) it follows

$$f_n^* = \frac{1}{\varepsilon_1 - \varepsilon_2} (\varepsilon_1^n (\varepsilon_1 - \rho_1) + \varepsilon_2^n (\rho_1 - \varepsilon_2)) \tag{4.29}$$

$$g_n^* = \frac{1}{\varepsilon_1 - \varepsilon_2} (\varepsilon_1^n - \varepsilon_2^n). \tag{4.30}$$

Using the fact that $|\varepsilon_2/\varepsilon_1| \leq 1$ in (4.26) and the Cauchy-Hadamard formula, we obtain that the radius of convergence of the series (4.22) is

$$\lambda_c^{**} = \frac{1}{\varepsilon_1} = \frac{4}{3\rho_1 + \sqrt{\rho_1^2 + 8\rho_2}} \tag{4.31}$$

Therefore, as long as $\lambda < \lambda_c^{**}$ the series (4.18) is absolutely convergent and consequently the green-type population of the process d_n is almost surely finite.

Calculation being done for countable number of types is not a real limitation. Tetzlaff (2003) studies conditions for almost sure extintion for multitype branching process with unbounded and/or continuous types. It is immediate to adapt his results to our set-up.

Suppose that the distribution of the length of the calls is continuous with distribution π with support Π (the set of all possible types). The mean number of green-type offspring in all generations is less then

$$\sum_{n\geq 1} \int_{\Pi} m_*^{(n)}(u, dv) \tag{4.32}$$

where the expected number of green-type offspring in the n-th generation belonging to the set A is given by the recursive equation (see Harris (1963), Chapter 3 or Mode (1971), Chapter 6)

$$\int_A m_*^{(n)}(u, dv) = \int_{\Pi} \left[\int_A m_*(u', dv) \right] m_*^{(n-1)}(u, du')\,. \tag{4.33}$$

Adapting the expression provided by Tetzlaff (2003), the expected number of green-children of a green-parent type u with types in the set A is given by

$$\int_A \lambda(\frac{v}{2} + u)\pi(dv) \tag{4.34}$$

and the computation is completely analogous to the discrete case.

We summarize the results proved in this section in the following theorem:

Theorem 4.35 *If $\lambda < \lambda_c^{**}$, where λ_c^{**} is given by (4.31), then with probability one there is no backward oriented percolation in* $\mathbf{R}$.

Remark: If π is the $U(0,1)$ density the condition becomes $\lambda < 1.247$

Observe the following relations:

$$\frac{2}{3\rho_1} \geq \lambda_c^{**} \geq \frac{4}{3(\rho_1 + \sqrt{\rho_2})} = \frac{4}{3}\lambda_c^*. \tag{4.36}$$

The upper estimate in (4.36)is achieved in the case of just one type (fixed call length). The last inequality proves that a better bound for the critical value is obtained.

Acknowledgments We would like to thank Pablo Ferrari for many fruitful discussions and Marcelo Freire for nice drawings. We also thank the anonymous referee for comments that greatly improved the presentation. This work was partially funded by FAPESP Grant 00/01375-8 and CNPq Grant 301054/93-2.

References

[1] Fernández, R., Ferrari, P. A., Garcia, N. L. (2001). Loss network representation of Peierls contours, *Annals of Probability* **29**(2): 902-937.

[2] Fernández, R., Ferrari, P. A., Garcia, N. L. (2002). Perfect simulation for interacting point processes, loss networks and Ising models. *Stoch. Process. Appl.* **102**(1): 63-88.

[3] Ferrari P. A., Garcia, N. L. (1998). One-dimensional loss networks and conditioned $M/G/\infty$ queues. *J. Appl. Probab.* **35**(4): 963-975.

[4] Garcia, N. L., Marić, N. (2002) Perfect simulation of a continuous one-dimensional loss network. Relatório Técnico, IMECC UNICAMP, available at http://www.ime.unicamp.br/rel-pesq/2002/rp18-02.html.

[5] Garcia, N. L., Marić, N. (2003). Ergodicity conditions for a continuous one-dimensional loss network. *Bull. Braz. Math. Soc.* **34**(3): 349-360.

[6] Hall, P. (1988). *Introduction to the theory of coverage processes* . John Wiley & Sons Inc., New York

[7] Harris, T. E. (1963) *The theory of branching processes*, Springer Verlag, Berlin.

[8] Kelly, F. P. (1991). Loss networks, *Ann. Appl. Probab.* **1**(3): 319-378.

[9] Mode, C.J. (1971) *Multitype branching processes: theory and applications*, American Elsevier Publishing Company, New York.

[10] Tetzlaff, G.T. (2003) Criticality in discrete time branching processes with not uniformly bounded types *Rev. Mat. Apl* **24**: 25-36.