Compactly Supported Orthogonal and Biorthogonal $\sqrt{5}$-refinement Wavelets with 4-fold Symmetry

Qingtang Jiang

***Abstract*—Recently $\sqrt{5}$-refinement hierarchical sampling has been studied and $\sqrt{5}$-refinement has been used for surface subdivision. Compared with other refinements such as the dyadic or quincunx refinement, $\sqrt{5}$-refinement has a special property that the nodes in a refined lattice form groups of five nodes with these five nodes having different *x* and *y* coordinates. This special property has been shown to be very useful to represent adaptively and render complex and procedural geometry. When $\sqrt{5}$-refinement is used for multiresolution data processing, $\sqrt{5}$-refinement filter banks and wavelets are required. While the construction of 2-dimensional nonseparable (bi)orthogonal wavelets with the dyadic or quincunx refinement has been studied by many researchers, the construction of (bi)orthogonal wavelets with $\sqrt{5}$-refinement has not been investigated. The main goal of this paper is to construct compactly supported orthogonal and biorthogonal wavelets with $\sqrt{5}$-refinement. In this paper we obtain block structures of orthogonal and biorthogonal $\sqrt{5}$-refinement FIR filter banks with 4-fold rotational symmetry. We construct compactly supported orthogonal and biorthogonal wavelets based on these block structures.**

***Index Terms*—$\sqrt{5}$-refinement, $\sqrt{5}$-refinement multiresolution decomposition/reconstruction, $\sqrt{5}$-refinement orthogonal and biorthogonal wavelets, orthogonal and biorthogonal filter banks with 4-fold symmetry.**

EDICS Category: TEC-MRS Multiresolution Processing of Images & Video

I. INTRODUCTION

The construction of compactly supported 2-dimensional nonseparable orthogonal and biorthogonal wavelets has been studied by many researchers, see e.g. [1]-[18]. The refinement considered in those papers is either the dyadic or quincunx refinement. See Fig.1 for a square lattice (top-left), its refined lattices by the dyadic refinement (top-right) and by the quincunx refinement (bottom). Recently $\sqrt{5}$-refinement hierarchical sampling has been studied in [19], [20], and $\sqrt{5}$-refinement has been used for surface subdivision in [21], [22].

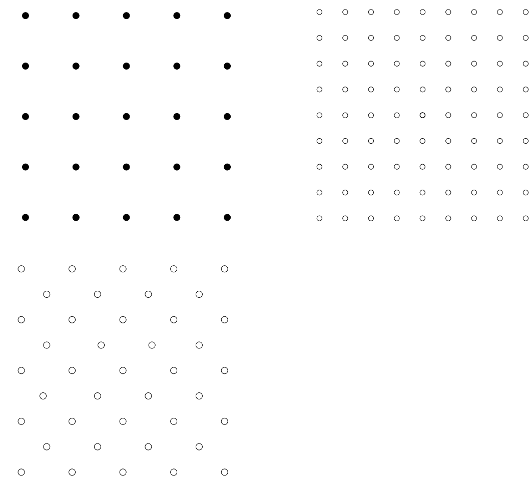

Fig. 1. *Square lattice (top-left), refined lattice by dyadic refinement (top-right) and refined lattice by quincunx refinement (bottom)*

To describe $\sqrt{5}$-refinement or $\sqrt{5}$-subdivision, let us consider the square lattice of $\mathbf{Z}^2$, as the coarse lattice. To obtain the refined lattice, for each $\mathbf{k} \in \mathbf{Z}^2$, select four points $\mathbf{k}+(2/5,1/5), \mathbf{k}+(4/5,2/5), \mathbf{k}+(3/5,4/5), \mathbf{k}+(1/5,3/5)$ (to be called new nodes) within the square with nodes $\mathbf{k}, \mathbf{k}+(1,0), \mathbf{k}+(1,1)$ and $\mathbf{k}+(0,1)$. See the picture in the top-left of Fig.2 for the case $\mathbf{k}=(0,0)$. Then the new nodes and the old nodes $\mathbf{k}$ of the coarse lattice $\mathbf{Z}^2$ form a new (refined) square lattice shown in the top-right of Fig.2. Observe that in the refined lattice, an old node and its four neighbors form

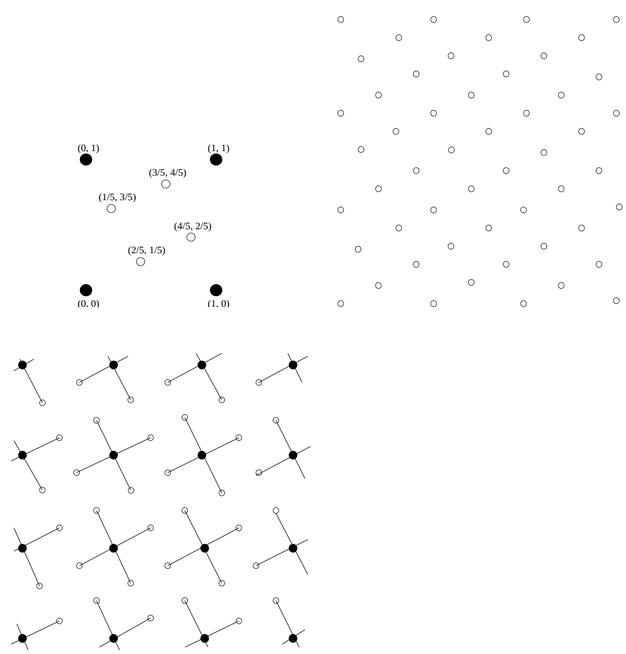

Fig. 2. *4 new points added in each square (top-left), refined lattice after one $\sqrt{5}$ refinement (top-right), five points aggregated as a group (bottom)*

Manuscript received March 28, 2008; revised July 28, 2008. This work was supported by UM Research Board 10/05 and UMSL Research Award 10/06. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Mario Figueiredo.

The author is with the Department of Mathematics and Computer Science, University of Missouri-St. Louis, St. Louis, MO 63121 USA (e-mail: jiangq@umsl.edu).

a group of five nodes with these five nodes having different x and y coordinates. See the bottom picture of Fig.2. This special property has been shown to be very useful in [19] when $\sqrt{5}$-refinement is used to represent adaptively and render complex and procedural geometry. When $\sqrt{5}$-refinement is used for multiresolution data processing, $\sqrt{5}$-refinement filter banks and wavelets are required. To the best of the author's knowledge, there are no $\sqrt{5}$-refinement filter banks or wavelets available in the literature. The main goal of this paper is to investigate the construction of compactly supported orthogonal and biorthogonal wavelets with $\sqrt{5}$-refinement.

This paper is organized as follows. In Section II, we discuss the $\sqrt{5}$-refinement and its associated dilation matrices, and give a brief review on (bi)orthogonality of $\sqrt{5}$-refinement filter banks and wavelets. In that section, we also provide $\sqrt{5}$-refinement multiresolution decomposition/reconstruction algorithm. In Section III, we present a block structure of FIR filter banks with 4-fold rotational symmetry. The construction of orthogonal and biorthogonal FIR filter banks of 4-fold rotational symmetry is studied in Sections IV and V, resp.

In this paper, for a positive integer n, we use I_n to denote the $n \times n$ identity matrix. For a matrix M, M^* denotes its conjugate transpose $\overline{M^T}$, and for a nonsingular matrix M, M^{-T} denotes $(M^{-1})^T$. In the following, a point $\mathbf{x}$ in $\mathbb{R}^2$ is written as a vector: $\mathbf{x} = [x_1, x_2]^T$. For $\mathbf{x} = [x_1, x_2]^T, \mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$, $\mathbf{x} \cdot \mathbf{y}$ denotes their dot (inner) product $\mathbf{x}^T\mathbf{y}$.

II. Compactly supported $\sqrt{5}$-refinement wavelets

In this section, first we discuss $\sqrt{5}$-refinement and its associated dilation matrix. After that we give a brief review on (bi)orthogonality of $\sqrt{5}$-refinement filter banks and wavelets, and present $\sqrt{5}$-refinement multiresolution decomposition/reconstruction algorithm.

A. $\sqrt{5}$-*refinement*

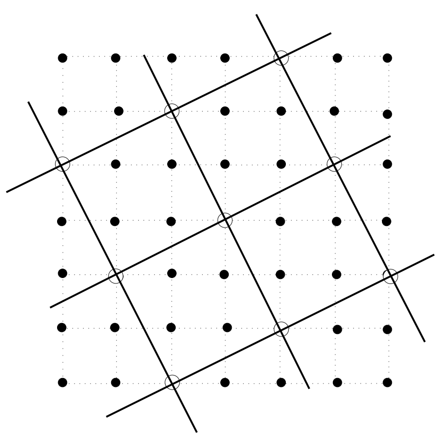

Fig. 3. *Lattice $\mathbf{Z}^2$ (with nodes $\bullet$ and $\circ$) and coarse lattice (with nodes $\circ$), and their associated grids*

Let $\mathbf{Z}^2_{1/5}$ denote the refined (subdivided) lattice of $\mathbf{Z}^2$ after one $\sqrt{5}$-refinement, namely, $\mathbf{Z}^2_{1/5}$ is the lattice shown in the top-right of Fig.2. In general, let $\mathbf{Z}^2_{5^{-n}}$ denote the refined lattice after nth step of $\sqrt{5}$-refinement (subdivision) iterations. We use $\mathbf{Z}^2_{5^n}$, $n > 0$, to denote the coarse lattice of $\mathbf{Z}^2$ after nth $\sqrt{5}$-refinement ($\sqrt{5}$-subsampling) iterations. $\mathbf{Z}^2_5$ (with nodes denoted by circles $\circ$) is shown in Fig.3, where two square grids formed by connecting nodes $\mathbf{k}$ of $\mathbf{Z}^2$ and nodes of $\mathbf{Z}^2_5$ to their four neighbors are also provided.

For an (input) image sampled on $\mathbf{Z}^2$, the nodes of $\mathbf{Z}^2_{5^n}$ can be considered as the sampling points of the subsampling image when the multiresolution decomposition algorithm is applied n times to the input image. To provide the multiresolution image decomposition/reconstruction algorithm, we need to choose a 2×2 matrix M, called the *dilation matrix*, such that it maps the lattice $\mathbf{Z}^2_{5^{j-1}}$ onto its coarse lattice $\mathbf{Z}^2_{5^j}, j \in \mathbf{Z}$. There are several choices for such a matrix M. For example, we may choose M to be one of the matrices:

$$M_1 = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}. \qquad (1)$$

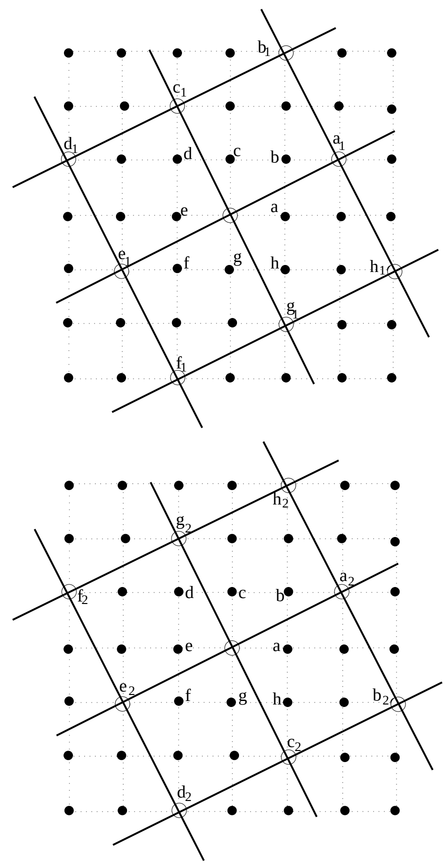

Fig. 4. $\sqrt{5}$ *spiraling (top) and togging (bottom) subsampling*

When M_1 is applied once to $\mathbf{Z}_{5^{j-1}}$, the axes of $M_1\mathbf{Z}_{5^{j-1}}$, the image of $\mathbf{Z}_{5^{j-1}}$ with M_1, keeps the orientation but is rotated counterclockwise about 26.6° ($\arctan(\frac{1}{2})$) with respect to the axes of $\mathbf{Z}_{5^{j-1}}$, see the top part of Fig.4, where $\mathbf{a}_1, \cdots, \mathbf{h}_1$ are the images of $\mathbf{a}, \cdots, \mathbf{h}$ with M_1. When M_2 is applied once to $\mathbf{Z}_{5^{j-1}}$, the axes of $M_2\mathbf{Z}_{5^{j-1}}$ are rotated and reflected from those of $\mathbf{Z}_{5^{j-1}}$, see the bottom part of Fig.4, where $\mathbf{a}_2, \cdots, \mathbf{h}_2$ are the images of $\mathbf{a}, \cdots, \mathbf{h}$ with M_2. When M_2 is applied twice, the axes of $(M_2)^2\mathbf{Z}_{5^{j-1}}$ are the same as those of $\mathbf{Z}_{5^{j-1}}$ since $(M_2)^2$ is $5I_2$. In [20], the subsampling with M_1 and M_2 is called the spiraling subsampling and toggling subsampling, resp.

B. $\sqrt{5}$-refinement (bi)orthogonal wavelets and multiresolution decomposition/reconstruction algorithm

In this subsection we recall (bi)orthogonality for $\sqrt{5}$-refinement wavelets, and provide $\sqrt{5}$-refinement multiresolution decomposition/reconstruction algorithm.

Let M be a $\sqrt{5}$-refinement dilation matrix, a 2×2 integer matrix which maps $\mathbf{Z}^2$ onto $\mathbf{Z}_5^2$. For example, we may choose M to be M_1 or M_2 in (1). Functions $\psi^{(1)},\cdots,\psi^{(4)}\in L^2(\mathbb{R}^2)$ are called $\sqrt{5}$-refinement *orthogonal wavelets* (with dilation matrix M) if $\{\psi^{(1)}_{j,\mathbf{k}}(\mathbf{x}),\cdots,\psi^{(4)}_{j,\mathbf{k}}(\mathbf{x}) : j\in\mathbf{Z}, k\in\mathbf{Z}^2\}$ forms an orthonormal basis of $L^2(\mathbb{R}^2)$, where for a function f on $\mathbb{R}^2$, $f_{j,\mathbf{k}}(\mathbf{x}) = 5^{j/2}f(M^j\mathbf{x}-\mathbf{k})$. Two sets of functions $\{\psi^{(1)},\cdots,\psi^{(4)}\}$ and $\{\widetilde{\psi}^{(1)},\cdots,\widetilde{\psi}^{(4)}\}$ on $\mathbb{R}^2$ are called biorthogonal wavelets (with dilation matrix M) if they generate biorthogonal bases of $L^2(\mathbb{R}^2)$: $\{\psi^{(\ell)}_{j,\mathbf{k}}(\mathbf{x}) : j\in\mathbf{Z},\mathbf{k}\in\mathbf{Z}^2, 1\le\ell\le 4\}$ and $\{\widetilde{\psi}^{(\ell)}_{j,\mathbf{k}}(\mathbf{x}) : j\in\mathbf{Z},\mathbf{k}\in\mathbf{Z}^2, 1\le\ell\le 4\}$ are Riesz bases of $L^2(\mathbb{R}^2)$ and they are biorthogonal to each other:

$$\int_{\mathbb{R}^2}\psi^{(\ell)}_{j,\mathbf{k}}(\mathbf{x})\overline{\widetilde{\psi}^{(\ell)}_{j',\mathbf{k}'}(\mathbf{x})}d\mathbf{x} = \delta_{\ell-\ell'}\delta_{j-j'}\delta_{\mathbf{k}-\mathbf{k}'},$$

for $1\le\ell,\ell'\le 4, j,j'\in\mathbf{Z},\mathbf{k},\mathbf{k}'\in\mathbf{Z}^2$, where δ is the kronecker-delta sequence.

(Bi)orthogonal wavelet construction is associated with the multiresolution analysis [23], [24]. With this approach, biorthogonal wavelets $\psi^{(\ell)},\widetilde{\psi}^{(\ell)}$ are given by

$$\psi^{(\ell)}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} q^{(\ell)}_{\mathbf{k}}\phi(M\mathbf{x}-\mathbf{k}), \widetilde{\psi}^{(\ell)}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2}\widetilde{q}^{(\ell)}_{\mathbf{k}}\widetilde{\phi}(M\mathbf{x}-\mathbf{k}) \tag{2}$$

where $q^{(\ell)}_{\mathbf{k}},\widetilde{q}^{(\ell)}_{\mathbf{k}}\in\mathbb{R}$ with finitely many $q^{(\ell)}_{\mathbf{k}}\neq 0,\widetilde{q}^{(\ell)}_{\mathbf{k}}\neq 0$, ϕ and $\widetilde{\phi}$ are scaling functions satisfying

$$\phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} p_{\mathbf{k}}\phi(M\mathbf{x}-\mathbf{k}), \widetilde{\phi}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^2}\widetilde{p}_{\mathbf{k}}\widetilde{\phi}(M\mathbf{x}-\mathbf{k}) \tag{3}$$

with $p_{\mathbf{k}},\widetilde{p}_{\mathbf{k}}\in\mathbb{R}$ and $p_{\mathbf{k}}\neq 0,\widetilde{p}_{\mathbf{k}}\neq 0$ for finitely many $\mathbf{k}$. To construct biorthogonal wavelets, we first construct ϕ and $\widetilde{\phi}$ such that they are biorthogonal duals:

$$\int_{\mathbb{R}^2}\phi(\mathbf{x})\overline{\widetilde{\phi}(\mathbf{x}-\mathbf{k})}\,d\mathbf{x} = \delta_{\mathbf{k}}, \quad \mathbf{k}\in\mathbf{Z}^2.$$

Let $p(\omega)$ and $\widetilde{p}(\omega)$ be the finite impulse response (*FIR*) filters with their impulse response coefficients $p_{\mathbf{k}}$ and $\widetilde{p}_{\mathbf{k}}$ (here a factor $1/5$ is added for convenience):

$$p(\omega) = 1/5\sum_{\mathbf{k}\in\mathbf{Z}^2}p_{\mathbf{k}}e^{-i\mathbf{k}\cdot\omega}, \quad \widetilde{p}(\omega) = 1/5\sum_{\mathbf{k}\in\mathbf{Z}^2}\widetilde{p}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\omega}.$$

One can show that if ϕ and $\widetilde{\phi}$ are biorthogonal duals, then $p(\omega)$ and $\widetilde{p}(\omega)$ satisfy

$$\sum_{0\le j\le 4} p(\omega+2\pi M^{-T}\eta_j)\overline{\widetilde{p}(\omega+2\pi M^{-T}\eta_j)} = 1 \tag{4}$$

for $\omega\in\mathbb{R}^2$, where $\eta_j, 0\le j\le 4$, are the representatives of the group $\mathbf{Z}^2/(M^T\mathbf{Z}^2)$. For example, when M is the dilation matrix M_1 or M_2 in (1), $\eta_j, 0\le j\le 4$ are

$$\begin{cases} \eta_0=[0,0]^T,\ \eta_1=[1,0]^T,\ \eta_2=[0,1]^T \\ \eta_3=[-1,0]^T,\ \eta_4=[0,-1]^T. \end{cases} \tag{5}$$

For a (lowpass) filter $p(\omega) = 1/5\sum_{\mathbf{k}\in\mathbf{Z}^2}p_{\mathbf{k}}e^{-i\mathbf{k}\cdot\omega}$, we say that $p(\omega)$ has sum rule order K if

$$p(0)=1,\ D_1^{\alpha_1}D_2^{\alpha_2}p(2\pi M^{-T}\eta_j)=0,\ 1\le j\le 4 \tag{6}$$

for all nonnegative integers α_1,α_2 with $0\le\alpha_1+\alpha_2<K$, where D_1 and D_2 denote the differential operators $\frac{\partial}{\partial\omega_1}$ and $\frac{\partial}{\partial\omega_2}$ resp. Under certain conditions, sum rule order of $p(\omega)$ is equivalent to the approximation order and accuracy of the scaling function ϕ associated with $p(\omega)$. The reader sees [25], [26] and the references therein for the details. When M is M_1 or M_2, the condition for sum rule order K of $p(\omega)$ is equivalent to that $\sum_{\mathbf{k}}p_{\mathbf{k}}=5$ and for all $0\le\alpha_1+\alpha_2<K$,

$$\begin{aligned}
&\sum_{\mathbf{k}}(Mk_1)^{\alpha_1}(Mk_2)^{\alpha_2}p_{(Mk_1,Mk_2)}\\
&=\sum_{\mathbf{k}}(Mk_1+1)^{\alpha_1}(Mk_2)^{\alpha_2}p_{(Mk_1+1,Mk_2)}\\
&=\sum_{\mathbf{k}}(Mk_1)^{\alpha_1}(Mk_2+1)^{\alpha_2}p_{(Mk_1,Mk_2+1)}\\
&=\sum_{\mathbf{k}}(Mk_1-1)^{\alpha_1}(Mk_2)^{\alpha_2}p_{(Mk_1-1,Mk_2)}\\
&=\sum_{\mathbf{k}}(Mk_1)^{\alpha_1}(Mk_2-1)^{\alpha_2}p_{(Mk_1,Mk_2-1)}.
\end{aligned}$$

For an FIR lowpass filter $p(\omega) = 1/5\sum_{\mathbf{k}\in\mathbf{Z}^2}p_{\mathbf{k}}e^{-i\mathbf{k}\cdot\omega}$, let T_p denote its transition operator matrix $T_p = [P_{M\mathbf{k}-\mathbf{j}}]_{\mathbf{k},\mathbf{j}\in[-N,N]^2}$, where $P_{\mathbf{j}} = 1/5\sum_{\mathbf{n}\in\mathbf{Z}^2}p_{\mathbf{n}-\mathbf{j}}p_{\mathbf{n}}$ and N is a suitable positive integer depending on the filter length of p and the dilation matrix M. We say T_p to satisfy Condition E if 1 is its simple eigenvalue and all other eigenvalues λ of T_p satisfy $|\lambda|<1$. It was shown that (refer to [27]-[29]), if $p(\omega),\widetilde{p}(\omega)$ satisfy (4), both $p(\omega)$ and $\widetilde{p}(\omega)$ have sum rule order (at least) 1, and the transition operator matrices T_p and $T_{\widetilde{p}}$ associated with p and $\widetilde{p}$ satisfy Condition E, then ϕ and $\widetilde{\phi}$ are biorthogonal duals. Furthermore, if $q^{(\ell)}(\omega),\widetilde{q}^{(\ell)}(\omega), 1\le\ell\le 4$ are FIR filters satisfying

$$\sum_{0\le j\le 4} p(\omega+2\pi M^{-T}\eta_j)\overline{\widetilde{q}^{(\ell)}(\omega+2\pi M^{-T}\eta_j)} = 0 \tag{7}$$

$$\sum_{0\le j\le 4} q^{(\ell')}(\omega+2\pi M^{-T}\eta_j)\overline{\widetilde{q}^{(\ell)}(\omega+2\pi M^{-T}\eta_j)} = \delta_{\ell'-\ell}, \tag{8}$$

for $1\le\ell,\ell'\le 4$ and $\omega\in\mathbb{R}^2$, then $\psi^{(\ell)}$ and $\widetilde{\psi}^{(\ell)}, 1\le\ell\le 4$ define by (2) with $q^{(\ell)}_{\mathbf{k}}$ and $\widetilde{q}^{(\ell)}_{\mathbf{k}}$ resp. are biorthogonal wavelets (see e.g. [30], [31]). Filter banks $\{p,q^{(1)},\cdots,q^{(4)}\}$ and $\{\tilde{p},\tilde{q}^{(1)},\cdots,\tilde{q}^{(4)}\}$ are commonly said to be *biorthogonal* (with dilation matrix M) if they satisfy (4), (7) and (8); and a filter bank $\{p,q^{(1)},\cdots,q^{(4)}\}$ is commonly referred to be *orthogonal* (with dilation matrix M) if it satisfies (4), (7) and (8) with $\tilde{p}=p,\tilde{q}^{(\ell)}=q^{(\ell)}, 1\le\ell\le 4$.

For filter banks $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$, the *multiresolution decomposition algorithm* with a dilation matrix M for an input image $c_{\mathbf{k},0}$ is

$$\begin{cases} c_{\mathbf{n},j+1} = (1/\sqrt{5}) \sum_{\mathbf{k}\in\mathbf{Z}^2} p_{\mathbf{k}-M\mathbf{n}} c_{\mathbf{k},j} \\ d^{(\ell)}_{\mathbf{n},j+1} = (1/\sqrt{5}) \sum_{\mathbf{k}\in\mathbf{Z}^2} q^{(\ell)}_{\mathbf{k}-M\mathbf{n}} c_{\mathbf{k},j} \end{cases}$$

with $\mathbf{n} \in \mathbf{Z}^2, \ell = 1, \cdots, 4$, where $j = 0, 1, \cdots, J-1$ for some positive integer J, and the *reconstruction algorithm* is

$$\hat{c}_{\mathbf{k},j} = \frac{1}{\sqrt{5}} \Big(\sum_{\mathbf{n}\in\mathbf{Z}^2} \widetilde{p}_{\mathbf{k}-M\mathbf{n}} \hat{c}_{\mathbf{n},j+1} + \sum_{1\le\ell\le4} \sum_{\mathbf{n}\in\mathbf{Z}^2} \widetilde{q}^{(\ell)}_{\mathbf{k}-M\mathbf{n}} d^{(\ell)}_{\mathbf{n},j+1} \Big)$$

for $j = J-1, J-2, \cdots, 0$, where $\hat{c}_{\mathbf{n},J} = c_{\mathbf{n},J}$. If $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ are biorthogonal, then $\hat{c}_{\mathbf{k},j} = c_{\mathbf{k},j},\ 0 \le j \le J-1$ for any input image $c_{\mathbf{k},0}$. Therefore, a biorthogonal filter banks is also referred to be *perfect reconstruction filter banks*.

Thus, to construct $\sqrt{5}$-refinement (bi)orthogonal wavelets, first, we choose a dilation matrix M which maps $\mathbf{Z}^2$ onto $\mathbf{Z}^2_5$. After that we construct (bi)orthogonal filter banks $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \tilde{q}^{(4)}\}$. If these filter banks are given by some parameters, then we select parameters such that the resulting p and $\widetilde{p}$ have sum rule order (at least) 1 and the corresponding T_p and $T_{\widetilde{p}}$ satisfy Condition E.

In the following sections we obtain block structures of orthogonal/biorthogonal FIR filter banks with 4-fold rotational symmetry. These orthogonal/biorthogonal filter banks are given by some free parameters. We are going to select the parameters such that the resulting $p(\omega)$ and $\widetilde{p}(\omega)$ have sum rule of certain order, and that the associated scaling functions $\phi, \widetilde{\phi}$ have (locally) optimal (Sobolev) smoothness. For $s \ge 0$, $W^s(\mathbb{R}^2)$ denotes the Sobolev space consisting of functions $f(\mathbf{x})$ on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2}(1+|\omega|^2)^s|\hat{f}(\omega)|^2 d\omega < \infty$, where $\hat{f}$ denotes the Fourier transform of f: $\hat{f}(\omega) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\omega} d\mathbf{x}$. [32], [33] provide the Sobolev smoothness formulas for scaling functions/vectors. The reader refers to [34] for algorithms and Matlab routines to find the Sobolev smoothness order.

Remark 1: Observe that $\{M_1^{-T}\eta_j : 0 \le j \le 4\}$=$\{M_2^{-T}\eta_j : 0 \le j \le 4\}$. Thus we can conclude that if a pair of filter banks are biorthogonal with one dilation matrix of M_1 and M_2, then they are also biorthogonal with the other dilation matrix. Furthermore, if a lowpass filter has sum rule order K with one of M_1 and M_2, then it also has sum rule order K with the other dilation matrix.

In the rest of this paper, without loss of the generality, M denotes the matrix M_1 or M_2 in (1).

III. Filter banks with 4-fold rotational symmetry

It is desirable that the filter banks designed have certain symmetry so that we have simpler algorithms and efficient computations. In this paper we consider 4-fold symmetry.

Definition 1: A filter bank $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is said to have 4*-fold rotational symmetry* if the coefficients $p_\mathbf{k}$ of the lowpass filter $p(\omega)$ is invariant under rotations of $\pi/2$, π, $3\pi/2$, and the coefficients $q^{(2)}_\mathbf{k}, q^{(3)}_\mathbf{k}, q^{(4)}_\mathbf{k}$ of three highpass filters $q^{(2)}$, $q^{(3)}, q^{(4)}$ are resp. $\pi/2$, π and $3\pi/2$ (counterclockwise) rotations of the coefficients $q^{(1)}_\mathbf{k}$ of the highpass filter $q^{(1)}$.

Therefore, a filter bank with 4-fold rotational symmetry is actually given by two filters.

Let $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ denote the rotation matrix. Denote $R_1 = R(\frac{\pi}{2}),\ R_j = (R_1)^j,\ j = 2, 3$. That is R_1, R_2, R_3 are the (clockwise) rotation matrices of $\pi/2$, π, $3\pi/2$, resp. More precisely,

$$R_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ R_2 = -I_2,\ R_3 = -R_1.$$

Then 4-fold rotational symmetry of a filter bank $\{p, q^{(1)}, \cdots, q^{(4)}\}$ means that

$$p_{R_j\mathbf{k}} = p_\mathbf{k},\ q^{(j+1)}_\mathbf{k} = q^{(1)}_{R_j\mathbf{k}},\ 1 \le j \le 3,\ \mathbf{k} \in \mathbf{Z}^2. \qquad (9)$$

Clearly, (9) is equivalent to that for $\omega \in \mathbb{R}^2$, $1 \le j \le 3$,

$$p(R_j^{-T}\omega) = p(\omega), q^{(j+1)}(\omega) = q^{(1)}(R_j^{-T}\omega). \qquad (10)$$

First, we show that the scaling function and wavelets associated with a filter bank of 4-fold rotational symmetry also have such a symmetry. Recall that the dilation matrix M we consider is M_1 or M_2 defined in (1).

Proposition 1: Suppose $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is a filter bank with 4-fold symmetry. Let ϕ be the associated scaling function with dilation matrix $M = M_1$ or $M = M_2$ and $\psi^{(\ell)}, 1 \le \ell \le 4$ be the functions define by (2) with $q^{(\ell)}$. Then

$$\phi(R_j\mathbf{x}) = \phi(\mathbf{x}),\ 1 \le j \le 3, \qquad (11)$$

and

$$\psi^{(j+1)}(\mathbf{x}) = \psi^{(1)}(R_j\mathbf{x}),\ 1 \le j \le 3, (\text{if } M = M_1) \qquad (12)$$

or

$$\begin{cases} \psi^{(2)}(\mathbf{x}) = \psi^{(1)}(R_3\mathbf{x}),\ \psi^{(3)}(\mathbf{x}) = \psi^{(1)}(R_2\mathbf{x}) \\ \psi^{(4)}(\mathbf{x}) = \psi^{(1)}(R_1\mathbf{x}), (\text{if } M = M_2). \end{cases} \qquad (13)$$

Proof: The proof is based on the facts: for $k > 0$,

$$\begin{aligned} &(M_1^{-T})^k R_j^{-T} (M_1^T)^k = R_j^{-T},\ 1 \le j \le 3; \\ &(M_2^{-T})^k R_1^{-T} (M_2^T)^k = \begin{cases} R_3^{-T}, \text{ if } k \text{ is odd} \\ R_1^{-T}, \text{ if } k \text{ is even}, \end{cases} \\ &(M_2^{-T})^k R_2^{-T} (M_2^T)^k = R_2^{-T} \\ &(M_2^{-T})^k R_3^{-T} (M_2^T)^k = \begin{cases} R_1^{-T}, \text{ if } k \text{ is odd} \\ R_3^{-T}, \text{ if } k \text{ is even}. \end{cases} \end{aligned}$$

In the following we give the proof of (11) with $M = M_2$ and (13) for $\psi^{(2)}$. The proof of others is similar and it is omitted.

From (3) (with $M = M_2$), we have $\widehat{\phi}(\omega) = p(M_2^{-T}\omega)\widehat{\phi}(M_2^{-T}\omega)$. Thus $\widehat{\phi}(\omega) = \Pi_{k=1}^\infty p((M_2^{-T})^k\omega)\widehat{\phi}(0)$. Since for each j with $1 \le j \le 3$ and $k > 0$, $(M_2^{-T})^k R_j^{-T} (M_2^T)^k = R_{j'}^{-T}$ for some j', $1 \le j' \le 3$, and $p(R_{j'}^{-T}\omega) = p(\omega)$, we have

$$\begin{aligned} \widehat{\phi}(R_j^{-T}\omega) &= \Pi_{k=1}^\infty p((M_2^{-T})^k R_j^{-T}\omega)\widehat{\phi}(0) \\ &= \Pi_{k=1}^\infty p(R_{j'}^{-T}(M_2^{-T})^k\omega)\widehat{\phi}(0) \\ &= \Pi_{k=1}^\infty p((M_2^{-T})^k\omega)\widehat{\phi}(0) = \widehat{\phi}(\omega). \end{aligned}$$

Therefore, $\phi(R_j\mathbf{x}) = \phi(\mathbf{x})$.

Next, we show that $\psi^{(2)}(\mathbf{x}) = \psi^{(1)}(R_3\mathbf{x})$. From (2) with $M = M_2$, $\widehat{\psi}^{(\ell)}(\omega) = q^{(\ell)}(M_2^{-T}\omega)\widehat{\phi}(M_2^{-T}\omega)$. Then using the fact $M_2^{-T}R_3^{-T}M_2^T = R_1^{-T}$, we have

$$\begin{aligned}\widehat{\psi}^{(1)}(R_3^{-T}\omega) &= q^{(1)}(M_2^{-T}R_3^{-T}\omega)\widehat{\phi}(M_2^{-T}R_3^{-T}\omega)\\ &= q^{(1)}(R_1^{-T}M_2^{-T}\omega)\widehat{\phi}(R_1^{-T}M_2^{-T}\omega)\\ &= q^{(2)}(M_2^{-T}\omega)\widehat{\phi}(M_2^{-T}\omega) = \widehat{\psi}^{(2)}(\omega).\end{aligned}$$

Thus $\psi^{(2)}(\mathbf{x}) = \psi^{(1)}(R_3\mathbf{x})$, as desired. ■

The reader refers to [35] for the relationship between the symmetry of a lowpass filter and that of the associated scaling function with a general dilation matrix.

Next, we have the following proposition which gives a simpler condition for the 4-fold symmetry of a filter bank.

Proposition 2: A filter bank $\{p, q^{(1)}, \cdots, q^{(4)}\}$ has 4-fold rotational symmetry if and only if (iff) it satisfies

$$\Big[p,\ q^{(1)},\ \cdots,\ q^{(4)}\Big]^T(R_1\omega) = M_0\Big[p(\omega),\ q^{(1)}(\omega),\ \cdots,\ q^{(4)}(\omega)\Big]^T, \tag{14}$$

where

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

Proof: The facts that $R_j = R_1^j, 1 \le j \le 3$, $R_1^4 = I_2$ and $R_1^{-T} = R_1$ imply the equivalence of (10) and (14). ■

After giving these two propositions about 4-fold symmetry, we consider the filter bank $\{p, q^{(1)}, \cdots, q^{(4)}\}$ to be given by the product of block matrices. Assume that we can write $[p(\omega), q^{(1)}(\omega), \cdots, q^{(4)}(\omega)]^T$ as $B(\omega)[p_s(\omega), q_s^{(1)}(\omega), \cdots, q_s^{(4)}(\omega)]^T$, where $B(\omega)$ is a 5×5 matrix with entries of trigonometric polynomials, and $\{p_s, q_s^{(1)}, \cdots, q_s^{(4)}\}$ is another FIR filter bank with a shorter filter length. If both $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{p_s, q_s^{(1)}, \cdots, q_s^{(4)}\}$ have 4-fold rotational symmetry, then Proposition 2 leads to that $B(\omega)$ satisfies

$$B(R_1\omega) = M_0 B(\omega) M_0^{-1}, \tag{16}$$

where M_0 is the matrix defined by (15). Observe that 1-tap filter bank

$$\{1, e^{-i\omega_1}, e^{-i\omega_2}, e^{i\omega_1}, e^{i\omega_2}\}$$

has 4-fold rotational symmetry and hence, it could be used as the initial filter bank. For the choice of basic block matrix $B(\omega)$, we should choose such a $B(\omega)$ that it can be written as $C(M^T\omega)$ for some 5×5 matrix $C(\omega)$ with each entry being a trigonometric polynomial. The reason is that the filter banks generated by such basic block matrices will easily yield (bi)orthogonal wavelets. Denote

$$E(\omega) = \mathrm{diag}(1,\ e^{-i(2\omega_1+\omega_2)},\ e^{i(\omega_1-2\omega_2)},\ e^{i(2\omega_1+\omega_2)},\ e^{i(-\omega_1+2\omega_2)}). \tag{17}$$

Then, with $R_1[\omega_1, \omega_2]^T = [\omega_2, -\omega_1]^T$, we have that

$$E(R_1\omega) = \mathrm{diag}(1,\ e^{i(\omega_1-2\omega_2)},\ e^{i(2\omega_1+\omega_2)},\ e^{i(-\omega_1+2\omega_2)},\ e^{-i(2\omega_1+\omega_2)}).$$

Thus $E(\omega)$ satisfies (16). Furthermore, $E(\omega)$ can be written as

$$E(\omega) = D_1(M_1^T\omega), E(\omega) = D_2(M_2^T\omega),$$

where

$$D_1(\omega) = \mathrm{diag}(1, e^{-i\omega_1}, e^{-i\omega_2}, e^{i\omega_1}, e^{i\omega_2}), \tag{18}$$

and

$$D_2(\omega) = \mathrm{diag}(1, e^{-i\omega_1}, e^{i\omega_2}, e^{i\omega_1}, e^{-i\omega_2}). \tag{19}$$

Therefore, $E(\omega)$ could be used to build block matrices for 4-fold rotational symmetric filter banks which yield scaling functions and wavelets with both dilation matrix M_1 and dilation matrix M_2. Next we use $B(\omega) = BE(\omega)$ as the block matrix, where B is a 5×5 (real) constant matrix. Clearly, $B(\omega)$ satisfies (16) iff B satisfies $M_0 B M_0^{-1} = B$, which is equivalent to that B has the form:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{12} & b_{12} & b_{12}\\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25}\\ b_{21} & b_{25} & b_{22} & b_{23} & b_{24}\\ b_{21} & b_{24} & b_{25} & b_{22} & b_{23}\\ b_{21} & b_{25} & b_{24} & b_{25} & b_{22} \end{bmatrix}. \tag{20}$$

Based on the above discussion, we reach the following result on the filter banks with 4-fold rotational symmetry.

Theorem 1: If $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is given by

$$\begin{aligned}&[p(\omega), q^{(1)}(\omega), \cdots, q^{(4)}(\omega)]^T =\\ &(1/\sqrt{5})B_n E(\omega) B_{n-1} E(\omega)\cdots B_1 E(\omega) B_0\cdot\\ &[1,\ e^{-i\omega_1},\ e^{-i\omega_2},\ e^{i\omega_1},\ e^{i\omega_2}]^T,\end{aligned} \tag{21}$$

where $n \in \mathbf{Z}_+, E(\omega)$ is defined by (17) and $B_0, B_1, \cdots, B_n$ are constant matrices of the form (20), then $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is an FIR filter bank with 4-fold rotational symmetry.

Next two sections show that block structure (21) yields (bi)orthogonal FIR filter banks with 4-fold symmetry.

IV. Compactly supported orthogonal wavelets with 4-fold rotational symmetry

In this section, we study the construction of orthogonal filter banks with 4-fold rotational symmetry. For an FIR filter bank $\{p, q^{(1)}, \cdots, q^{(4)}\}$, denote $q^{(0)}(\omega) = p(\omega)$. Let $U(\omega)$ be the 5×5 matrix defined by

$$U(\omega) = \Big[q^{(\ell)}(\omega + \eta_j)\Big]_{0\le \ell, j\le 4}, \tag{22}$$

where $\eta_0, \eta_1, \cdots, \eta_4$ are given in (5). Then $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is orthogonal iff $U(\omega)$ is unitary for all $\omega \in \mathbb{R}^2$, namely,

$$U(\omega)U(\omega)^* = I_5, \quad \omega \in \mathbb{R}^2. \tag{23}$$

Next, we write $q^{(\ell)}(\omega), 0 \le \ell \le 4$ as

$$q^{(\ell)}(\omega) = \frac{1}{\sqrt{5}}\Big(q_0^{(\ell)}(M^T\omega) + q_1^{(\ell)}(M^T\omega)e^{-i\omega_1} + q_2^{(\ell)}(M^T\omega)e^{-i\omega_2} + q_3^{(\ell)}(M^T\omega)e^{i\omega_1} + q_4^{(\ell)}(M^T\omega)e^{i\omega_2}\Big),$$

where $q_k^{(\ell)}(\omega), 0 \le k \le 4$ are trigonometric polynomials. Let $V(\omega)$ denote the polyphase matrix of $\{p(\omega), q^{(1)}(\omega), \cdots, q^{(4)}(\omega)\}$:

$$V(\omega) = \left[q_k^{(\ell)}(\omega)\right]_{0\le \ell, k\le 4}. \qquad (24)$$

Clearly,

$$[p(\omega), q^{(1)}(\omega), \cdots, q^{(4)}(\omega)]^T = (1/\sqrt{5})V(M^T\omega)[1, e^{-i\omega_1}, e^{-i\omega_2}, e^{i\omega_1}, e^{i\omega_2}]^T.$$

Let $l(\omega) = [1, e^{-i\omega_1}, e^{-i\omega_2}, e^{i\omega_1}, e^{i\omega_2}]^T$, and denote

$$L(\omega) = 1/\sqrt{5}[l(\omega + 2\pi M^{-T}\eta_0), \cdots, l(\omega + 2\pi M^{-T}\eta_4)].$$

One can verify that for M to be M_1 or M_2 defined in (1), the 5×5 matrix $L(\omega)$ is unitary for all $\omega \in \mathbb{R}^2$. This fact and that $U(\omega) = V(M^T\omega)L(\omega)$ lead to that (23) holds iff $V(M^T\omega)$ is unitary for all $\omega \in \mathbb{R}^2$, namely, $V(\omega)$ satisfies

$$V(\omega)V(\omega)^* = I_5, \quad \omega \in \mathbb{R}^2. \qquad (25)$$

Therefore, to construct an orthogonal filter bank $\{p, q^{(1)}, \cdots, q^{(4)}\}$, we need only to construct such a trigonometric polynomial matrix $V(\omega)$ that satisfies (25).

If $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is given by (21), then its polyphase matrix $V(\omega)$ for $M = M_1$ in (1) is

$$V(\omega) = B_n D_1(\omega)B_{n-1}D_1(\omega)\cdots B_1 D_1(\omega)B_0, \qquad (26)$$

and the polyphase matrix $V(\omega)$ of $\{p, q^{(1)}, \cdots, q^{(4)}\}$ for $M = M_2$ in (1) is

$$V(\omega) = B_n D_2(\omega)B_{n-1}D_2(\omega)\cdots B_1 D_2(\omega)B_0. \qquad (27)$$

Since both $D_1(\omega)$ and $D_2(\omega)$ are unitary, we know that if constant matrices $B_k, 0 \le k \le n$, are orthogonal, then $V(\omega)$ is unitary, namely, it satisfies (25).

Next, we consider the orthogonality of a matrix B of the from (20). To this regard, denote

$$R = \frac{\sqrt{2}}{2}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Then R is an orthogonal matrix and $RBR^T = \begin{bmatrix} O_1 & 0_{3\times 2} \\ 0_{2\times 3} & O_2 \end{bmatrix}$, where

$$O_1 = \begin{bmatrix} b_{11} & \sqrt{2}b_{12} & \sqrt{2}b_{12} \\ \sqrt{2}b_{21} & u_1 & u_2 \\ \sqrt{2}b_{21} & u_2 & u_1 \end{bmatrix}, O_2 = \begin{bmatrix} v_1 & v_2 \\ -v_2 & v_1 \end{bmatrix} \qquad (28)$$

with

$$\begin{array}{l} u_1 = b_{22} + b_{24},\ u_2 = b_{23} + b_{25} \\ v_1 = b_{22} - b_{24},\ v_2 = b_{23} - b_{25}. \end{array} \qquad (29)$$

Since R is orthogonal, we have that B is orthogonal iff both O_1 and O_2 are orthogonal. O_2 is orthogonal iff v_1, v_2 can be given as

$$v_1 = \pm(1-s^2)/(1+s^2),\ v_2 = 2s/(1+s^2)$$

for some $s \in \mathbb{R}$, while a matrix O_1 with the form in (28), is orthogonal if it can be expressed as

$$O_1 = \frac{1}{1+4t^2}\begin{bmatrix} 1-4t^2 & 2\sqrt{2}t & 2\sqrt{2}t \\ \pm 2\sqrt{2}t & \mp 1 & \pm 4t^2 \\ \pm 2\sqrt{2}t & \pm 4t^2 & \mp 1 \end{bmatrix};$$

or

$$O_1 = \frac{1}{1+4t^2}\begin{bmatrix} 1-4t^2 & 2\sqrt{2}t & 2\sqrt{2}t \\ \pm 2\sqrt{2}t & \pm 4t^2 & \mp 1 \\ \pm 2\sqrt{2}t & \mp 1 & \pm 4t^2 \end{bmatrix}.$$

Therefore, an orthogonal B of the form (20) has two parameters with its entries b_{ij} given by

$$\begin{cases} b_{11} = \frac{1-4t^2}{1+4t^2},\ b_{12} = b_{21} = \frac{2t}{1+t^2} \\ b_{22} = \frac{1}{2}(-\frac{1}{1+4t^2} \pm \frac{1-s^2}{1+s^2}),\ b_{23} = \frac{1}{2}(\frac{4t^2}{1+4t^2} + \frac{2s}{1+s^2}) \\ b_{24} = \frac{1}{2}(-\frac{1}{1+4t^2} \mp \frac{1-s^2}{1+s^2}),\ b_{25} = \frac{1}{2}(\frac{4t^2}{1+4t^2} - \frac{2s}{1+s^2}); \end{cases} \qquad (30)$$

$$\begin{cases} b_{11} = \frac{1-4t^2}{1+4t^2},\ b_{12} = b_{21} = \frac{2t}{1+t^2} \\ b_{22} = \frac{1}{2}(\frac{4t^2}{1+4t^2} \pm \frac{1-s^2}{1+s^2}),\ b_{23} = \frac{1}{2}(-\frac{1}{1+4t^2} + \frac{2s}{1+s^2}) \\ b_{24} = \frac{1}{2}(\frac{4t^2}{1+4t^2} \mp \frac{1-s^2}{1+s^2}),\ b_{25} = \frac{1}{2}(-\frac{1}{1+4t^2} - \frac{2s}{1+s^2}); \end{cases} \qquad (31)$$

$$\begin{cases} b_{11} = \frac{1-4t^2}{1+4t^2},\ b_{12} = \frac{2t}{1+t^2},\ b_{21} = -b_{12} \\ b_{22} = \frac{1}{2}(\frac{1}{1+4t^2} \pm \frac{1-s^2}{1+s^2}),\ b_{23} = \frac{1}{2}(-\frac{4t^2}{1+4t^2} + \frac{2s}{1+s^2}) \\ b_{24} = \frac{1}{2}(\frac{1}{1+4t^2} \mp \frac{1-s^2}{1+s^2}),\ b_{25} = \frac{1}{2}(-\frac{4t^2}{1+4t^2} - \frac{2s}{1+s^2}); \text{or} \end{cases} \qquad (32)$$

$$\begin{cases} b_{11} = \frac{1-4t^2}{1+4t^2},\ b_{12} = \frac{2t}{1+t^2},\ b_{21} = -b_{12} \\ b_{22} = \frac{1}{2}(-\frac{4t^2}{1+4t^2} \pm \frac{1-s^2}{1+s^2}),\ b_{23} = \frac{1}{2}(\frac{1}{1+4t^2} + \frac{2s}{1+s^2}) \\ b_{24} = \frac{1}{2}(-\frac{4t^2}{1+4t^2} \mp \frac{1-s^2}{1+s^2}),\ b_{25} = \frac{1}{2}(\frac{1}{1+4t^2} - \frac{2s}{1+s^2}). \end{cases} \qquad (33)$$

Theorem 2: Suppose $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is given by (21). If each B_k, $0 \le k \le n$ is orthogonal and of the form (20), namely its entries b_{ij} are given by (30), (31), (32) or (33), then $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is an orthogonal FIR filter bank with 4-fold rotational symmetry.

Next we construct two sets of orthogonal wavelets based on this structure.

Example 1: Let $\{p, q^{(1)}, \cdots, q^{(4)}\}$ be the orthogonal filter bank with 4-fold rotational symmetry given by (21) with $n = 0$: $(1/\sqrt{5})B_0[1, e^{-i\omega_1}, e^{-i\omega_2}, e^{i\omega_1}, e^{i\omega_2}]^T$, where the parameters b_{ij} of B_0 are given by (30) for some t, s (with choice of $\pm$ to be $+$). This is a 5-tap filter bank. The lowpass filter $p(\omega)$ is given by one free parameter t. We can choose this parameter $t = (\sqrt{5}-1)/4$ such that the resulting $p(\omega)$ has sum rule order 1 (with both M_1 and M_2). In this case for both dilation matrices M_1 and M_2, the corresponding scaling function ϕ is in $W^{0.31739}(\mathbb{R}^2)$. The nonzero $p_{\mathbf{k}}$ are

$$p_{00} = p_{10} = p_{01} = p_{-10} = p_{0-1} = 1.$$

If we set $s = 0$, then the nonzero $q_{\mathbf{k}}^{(1)}$ of $q^{(1)}(\omega)$ are

$$\begin{array}{l} q_{00}^{(1)} = 1,\ q_{-10}^{(1)} = -(1+3\sqrt{5})/4 \\ q_{10}^{(1)} = q_{01}^{(1)} = q_{0-1}^{(1)} = (\sqrt{5}-1)/4. \end{array}$$

The impulse responses $p_{\mathbf{k}}, q_{\mathbf{k}}^{(1)}$ of $p(\omega), q^{(1)}(\omega)$ are displayed in Fig.5, while $q_{\mathbf{k}}^{(j+1)}$ are $\pi j/2$ rotations of $q_{\mathbf{k}}^{(1)}$, $1 \le j \le 3$.

Example 2: Let $\{p, q^{(1)}, \cdots, q^{(4)}\}$ be the orthogonal filter bank with 4-fold rotational symmetry given by (21) with $n = 1$: $(1/\sqrt{5})B_1E(\omega)B_0[1, e^{-i\omega_1}, e^{-i\omega_2}, e^{i\omega_1}, e^{i\omega_2}]^T$, where

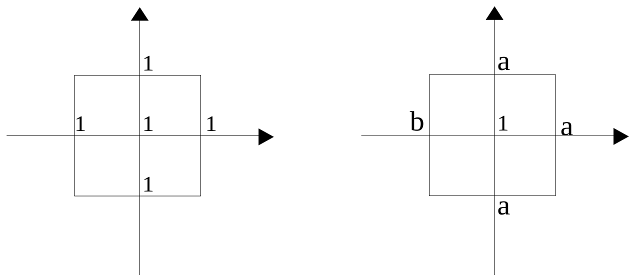

Fig. 5. *Impulse responses $p_{\mathbf{k}}$ (left) and $q^{(1)}_{\mathbf{k}}$ with $a = (\sqrt{5}-1)/4, b = -(1+3\sqrt{5})/4$ (right); $q^{(j+1)}_{\mathbf{k}}$ are $\pi j/2$ rotations of $q^{(1)}_{\mathbf{k}}$, $1 \le j \le 3$.*

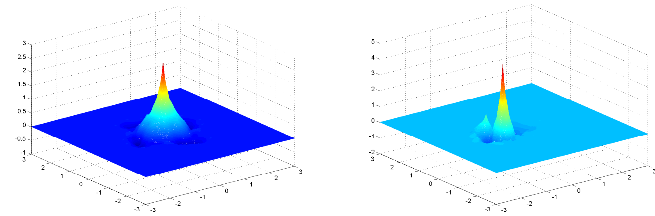

Fig. 6. *ϕ (left) and $\psi^{(1)}$ (right) with $M = M_1$*

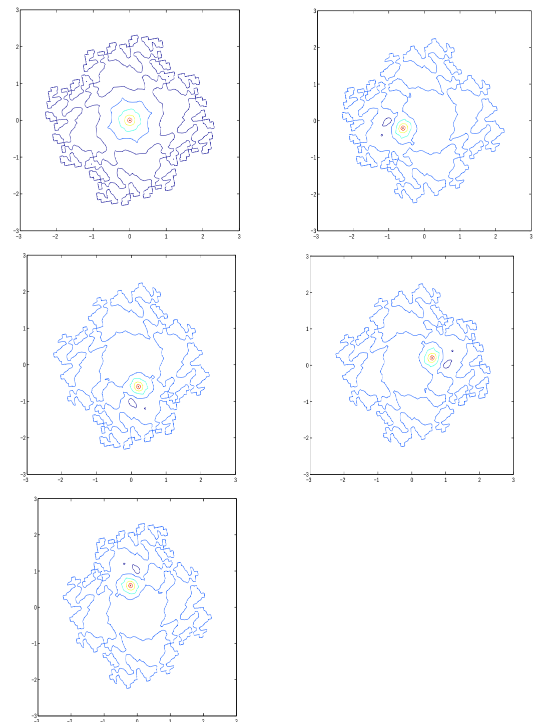

Fig. 7. *Contours of $\phi, \psi^{(2)}, \psi^{(4)}$ (left column from top) and $\psi^{(1)}, \psi^{(3)}$ (right column from top)*

B_1 and B_0 are given by (30) (both with choice of $\pm$ to be $+$) for some t_1, s_1 and t_0, s_0 resp. The lowpass filter $p(\omega)$ depends on t_0, s_0 and t_1. If $t_0 = (\sqrt{21}-\sqrt{5})(\sqrt{5}-1)/16, s_0 = \sqrt{5}-2, t_1 = (\sqrt{21}-5)/4$, then resulting $p(\omega)$ has sum rule order 2 (with both M_1 and M_2), and the corresponding scaling function with dilation matrix M_1 is in $W^{0.95435}(\mathbb{R}^2)$, while the associated scaling function with dilation matrix M_2 is in $W^{0.97640}(\mathbb{R}^2)$. There is one free parameter s_1 left for the highpass filters of this orthogonal filter bank. In the following we let $s_1 = 0$. In Fig.6 we show the pictures of ϕ and $\psi^{(1)}$ with dilation matrix $M = M_1$. The contours of $\phi, \psi^{(\ell)}, 1 \le \ell \le 4$ are provided in Fig.7. For the convenience to the reader, the nonzero $p_{\mathbf{k}}, q^{(\ell)}_{\mathbf{k}}$ of the resulting filter bank are provided below:

$$
\begin{aligned}
&p_{00} = (21 + 4\sqrt{21})/25\\
&p_{10} = p_{01} = p_{-10} = p_{0-1} = (21 - \sqrt{21})/25\\
&p_{11} = p_{-11} = p_{-1-1} = p_{1-1} = (24 + 5\sqrt{5} + \sqrt{21})/100\\
&p_{20} = p_{02} = p_{-20} = p_{0-2} = (14 - 5\sqrt{5} + \sqrt{21})/100\\
&p_{21} = p_{-12} = p_{-2-1} = p_{1-2} = (1 - \sqrt{21})/25\\
&p_{22} = p_{-22} = p_{-2-2} = p_{2-2} = (\sqrt{21} - 6 - 5\sqrt{5})/100\\
&p_{31} = p_{-13} = p_{-3-1} = p_{1-3} = (\sqrt{21} - 16 + 5\sqrt{5})/100;\\
&q^{(1)}_{00} = (-4 - \sqrt{21})/25\\
&q^{(1)}_{10} = q^{(1)}_{01} = q^{(1)}_{-10} = q^{(1)}_{0-1} = (1 - \sqrt{21})/25\\
&q^{(1)}_{11} = q^{(1)}_{1-1} = q^{(1)}_{-11} = (19\sqrt{21} + 5\sqrt{105} - 99 - 25\sqrt{5})/400\\
&q^{(1)}_{-1-1} = (381 + 75\sqrt{5} + 39\sqrt{21} + 5\sqrt{105})/400\\
&q^{(1)}_{20} = q^{(1)}_{02} = q^{(1)}_{0-2} = (25\sqrt{5} - 49 + 9\sqrt{21} - 5\sqrt{105})/400\\
&q^{(1)}_{-20} = (231 - 75\sqrt{5} + 29\sqrt{21} - 5\sqrt{105})/400\\
&q^{(1)}_{21} = q^{(1)}_{-12} = q^{(1)}_{1-2} = (-13 + 3\sqrt{21})/50\\
&q^{(1)}_{-2-1} = (-3 - 7\sqrt{21})/50\\
&q^{(1)}_{22} = q^{(1)}_{-22} = q^{(1)}_{2-2} = (51 + 25\sqrt{5} - 11\sqrt{21} - 5\sqrt{105})/400\\
&q^{(1)}_{-2-2} = (9\sqrt{21} - 69 - 75\sqrt{5} - 5\sqrt{105})/400\\
&q^{(1)}_{31} = q^{(1)}_{-13} = q^{(1)}_{1-3} = (101 - 25\sqrt{5} - 21\sqrt{21} + 5\sqrt{105})/400\\
&q^{(1)}_{-3-1} = (-219 + 75\sqrt{5} - \sqrt{21} + 5\sqrt{105})/400;
\end{aligned}
$$

and $q^{(2)}_{\mathbf{k}}, q^{(3)}_{\mathbf{k}}, q^{(4)}_{\mathbf{k}}$ are $\pi/2, \pi, 3\pi/2$ rotations of $q^{(1)}_{\mathbf{k}}$.

We apply this filter bank to the 512×512 Lena image in Fig.8. The decomposed images with the lowpass filter and highpass filters (with $M = M_1$) are shown on the left of Fig.9 and in Fig.10 resp. These images are indeed rotated about $26.6°$ with respect to the original image. The decomposed image with the lowpass filter applied twice is shown on the right of Fig.9. $\diamondsuit$

To construct scaling functions and wavelets with higher smoothness order, we need to use more blocks $B_k E(\omega)$ in (21). In the next section, we consider compactly supported biorthogonal wavelets of 4-fold rotational symmetry.

Fig. 8. *Original image*

Fig. 9. *Decomposed images with lowpass filter P with one (left) and two (right) steps of decomposition algorithm*

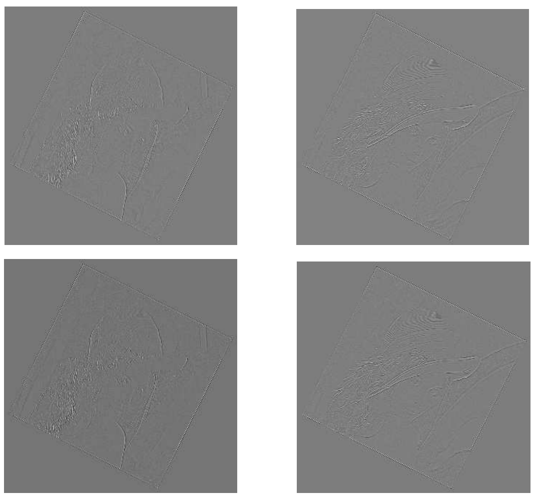

Fig. 10. *Decomposed images with highpass filters $Q^{(1)}, Q^{(2)}$ (1st row from left) and with $Q^{(3)}, Q^{(4)}$ (2nd row from left)*

V. Compactly supported biorthogonal wavelets with 4-fold rotational symmetry

Biorthogonal wavelets can be constructed by the method of the lifting scheme, see [36] (see also [37] for the similar concept to the lifting scheme). Here we use the block structure (21) to construct $\sqrt{5}$-refinement biorthogonal filter banks and wavelets with 4-fold rotational symmetry.

Suppose $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ are a pair of filter banks. Let $U(\omega)$ be the matrix defined by (22). With $\widetilde{q}^{(0)}(\omega) = \widetilde{p}(\omega)$, denote $\widetilde{U}(\omega) = \left[\widetilde{q}^{(\ell)}(\omega + \eta_j)\right]_{0 \le \ell, j \le 4}$. Then $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ are biorthogonal iff $U(\omega)\widetilde{U}(\omega)^* = I_5, \ \omega \in \mathbb{R}^2$. Let $V(\omega)$ and $\widetilde{V}(\omega)$ be polyphase matrices of $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ defined as in (24). Then by the facts that $U(\omega) = V(M^T\omega)L(\omega)$, $\widetilde{U}(\omega) = \widetilde{V}(M^T\omega)L(\omega)$ and that $L(\omega)L(\omega)^* = I_5$, we know $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ are biorthogonal iff $V(\omega)$ and $\widetilde{V}(\omega)$ satisfy

$$V(\omega)\widetilde{V}(\omega)^* = I_5, \ \omega \in \mathbb{R}^2. \tag{34}$$

Next theorem shows that if $\{p, q^{(1)}, \cdots, q^{(4)}\}$ is the FIR filter bank given by (21) for some 5×5 real nonsingular matrices B_k of the form (20), then it has an FIR biorthogonal dual which is also given by (21) with B_k replaced by B_k^{-T}.

Theorem 3: Let $\{p, q^{(1)}, \cdots, q^{(4)}\}$ be the FIR filter bank given by (21), where $B_k, 0 \le k \le n$ are nonsingular constant matrices of the form (20). Let $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ be the FIR filter bank given by

$$\begin{aligned}&[\widetilde{p}(\omega), \widetilde{q}^{(1)}(\omega), \cdots, \widetilde{q}^{(4)}(\omega)]^T = \\ &(1/\sqrt{5})B_n^{-T}E(\omega)B_{n-1}^{-T}E(\omega)\cdots B_1^{-T}E(\omega)B_0^{-T} \cdot \\ &[1, e^{-i\omega_1}, e^{-i\omega_2}, e^{i\omega_1}, e^{i\omega_2}]^T.\end{aligned} \tag{35}$$

Then $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \tilde{q}^{(4)}\}$ is an FIR filter bank biorthogonal to $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and it has 4-fold rotational symmetry.

Proof: Let $V(\omega)$ and $\widetilde{V}(\omega)$ be the polyphase matrices of $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \tilde{q}^{(4)}\}$ resp. defined by (24). Then $V(\omega)$ is given by (26) if $M = M_1$ and by (27) if $M = M_2$. On the other hand, for $M = M_1$

$$\widetilde{V}(\omega) = B_n^{-T}D_1(\omega)B_{n-1}^{-T}D_1(\omega)\cdots B_1^{-T}D_1(\omega)B_0^{-T},$$

and for $M = M_2$,

$$\widetilde{V}(\omega) = B_n^{-T}D_2(\omega)B_{n-1}^{-T}D_2(\omega)\cdots B_1^{-T}D_2(\omega)B_0^{-T}.$$

This and the fact $D_j(\omega)D_j(\omega)^* = I_5, j = 1, 2$, imply $V(\omega)\widetilde{V}(\omega)^* = I_5$. Hence, $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \tilde{q}^{(4)}\}$ is biorthogonal to $\{p, q^{(1)}, \cdots, q^{(4)}\}$.

Since $M_0B_kM_0^{-1} = B_k$ and $M_0^T = M_0^{-1}$, we know that $M_0B_k^{-T}M_0^{-1} = B_k^{-T}$, i.e., B_k^{-T} also has the form of (20). Therefore, Proposition 1 implies that $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \tilde{q}^{(4)}\}$ has 4-fold rotational symmetry. ∎

From the family of biorthogonal FIR filter banks given in Theorem 3, one can choose parameters for nonsingular matrices B_k of the form (20) to design filter banks for one's specific applications. In the following we construct biorthogonal wavelets by selecting the parameters of the filter banks such that the resulting primal and dual wavelets have certain smoothness.

Example 3: Let $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ be the biorthogonal filter banks with 4-fold rotational symmetry given by Theorem 3 with $n = 1$ for nonsingular matrices B_0, B_1 of the form (20). We can choose the free parameters for B_0 and B_1 such that $p(\omega)$ and $\widetilde{p}(\omega)$ have sum rule order 2 and 1, and the resulting scaling functions (with dilation matrix M_1) $\phi \in W^{1.35885}(\mathbb{R}^2), \tilde{\phi} \in W^{0.56932}(\mathbb{R}^2)$, and the the resulting scaling functions (with dilation matrix M_2) $\phi \in W^{1.38793}(\mathbb{R}^2), \tilde{\phi} \in W^{0.58255}(\mathbb{R}^2)$. The selected parameters, denoted as a_{ij}, for B_0 are

$$\begin{aligned}&a_{11} = -.8142362882, \ a_{12} = -.5123117764 \\ &a_{21} = -.1491660034, \ a_{22} = -.2015353408 \\ &a_{23} = -.2306845383, \ a_{24} = .6519338759 \\ &a_{25} = .1960500700,\end{aligned}$$

and selected parameters, denoted as b_{ij}, for B_1 are

$$\begin{aligned}&b_{11} = -.7028342827, b_{12} = .2095979969 \\ &b_{21} = -.1637602755, \ b_{22} = .4616178091 \\ &b_{23} = -.6306789060, \ b_{24} = -1.1580817015 \\ &b_{25} = -.4317778159. \diamondsuit\end{aligned}$$

Example 4: Let $\{p, q^{(1)}, \cdots, q^{(4)}\}$ and $\{\tilde{p}, \tilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\}$ be the biorthogonal filter banks with 4-fold rotational symmetry given by Theorem 3 with $n = 2$ for nonsingular matrices B_0, B_1, B_2 of the form (20). In this case we can choose the free parameters such that $p(\omega)$ and $\widetilde{p}(\omega)$ have sum rule orders 2 and 1, and the resulting scaling functions (with dilation matrix M_1) $\phi \in W^{1.74086}(\mathbb{R}^2), \tilde{\phi} \in W^{0.57518}(\mathbb{R}^2)$, and the the resulting scaling functions (with dilation matrix M_2) $\phi \in W^{1.74645}(\mathbb{R}^2), \tilde{\phi} \in W^{0.58213}(\mathbb{R}^2)$. The selected parameters, denoted as a_{ij}, for B_0 are

$$\begin{aligned} &a_{11} = -.7990918368,\ a_{12} = -.4746214511\\ &a_{21} = -.2386636281,\ a_{22} = -.4506816068\\ &a_{23} = -.3049002942,\ a_{24} = 1.3307611157\\ &a_{25} = .0865617975, \end{aligned}$$

the selected parameters, denoted as b_{ij}, for B_1 are

$$\begin{aligned} &b_{11} = -.8078649634,\ b_{12} = .1608905843\\ &b_{21} = -.0105323863,\ b_{22} = 1.3196936112\\ &b_{23} = -.9365346463,\ b_{24} = -1.0156985962\\ &b_{25} = .5753507070; \end{aligned}$$

and the selected parameters, denoted as c_{ij}, for B_2 are

$$\begin{aligned} &c_{11} = .9122240147,\ c_{12} = -.0177565295\\ &c_{21} = -.0029166441,\ c_{22} = .7638905933\\ &c_{23} = -.5955888499,\ c_{24} = .7634910809\\ &c_{25} = .7639648549. \end{aligned}$$

We show $\phi, \psi^{(1)}$ (with $M = M_1$) and their contours in Fig.11, and show the pictures and contours of $\widetilde{\phi}, \widetilde{\psi}^{(1)}$ in Fig.12. $\diamondsuit$

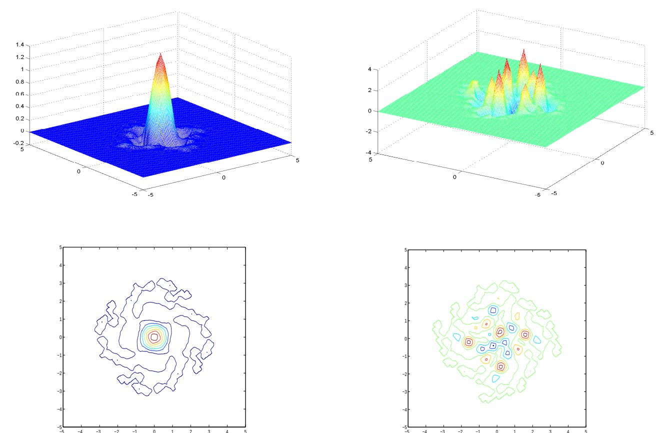

Fig. 11. *ϕ (top-left) and $\psi^{(1)}$ (top-right) and their contours*

Using more blocks $B_k E(\omega)$ and $B_k^{-T} E(\omega)$ in (21) and (35), we can construct biorthogonal wavelets with higher smoothness orders. For example, when we use the block structure with $n = 3$ for some nonsingular matrices $B_0, \cdots, B_3$ of the form (20), we can choose the free parameters such that the resulting scaling functions ϕ and $\widetilde{\phi}$ (with dilation matrix M_1) are in $W^{1.99820}(\mathbb{R}^2)$ and $W^{0.73147}(\mathbb{R}^2)$ resp. The selected parameters are not provided here.

VI. Conclusion

In this paper we study the construction of compactly supported $\sqrt{5}$-refinement orthogonal and biorthogonal wavelets.

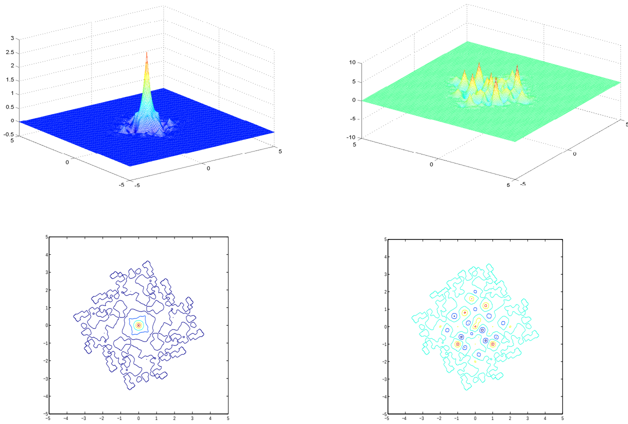

Fig. 12. *$\widetilde{\phi}$ (top-left) and $\widetilde{\psi}^{(1)}$ (top-right) and their contours*

We obtain block structures of orthogonal and biorthogonal $\sqrt{5}$-refinement FIR filter banks with 4-fold rotational symmetry. Based on these block structures, we construct compactly supported orthogonal and biorthogonal wavelets with $\sqrt{5}$-refinement. The $\sqrt{5}$-refinement FIR filter banks and compactly supported orthogonal and biorthogonal wavelets provided in this paper will have potential applications in representing and rendering complex and procedural geometry, and in multiresolution image/data processing.

Acknowledgments. The author would like to thank the anonymous reviewers for their valuable suggestions and comments.

References

[1] J. Kovačević and M. Vetterli, "Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbb{R}^n$", *IEEE Trans. Info. Theory*, vol. 38, no. 2, pp. 533–555, Mar. 1992.

[2] S.D. Riemenschneider and Z.W. Shen, "Wavelets and pre-wavelets in low dimensions", *J. Approx. Theory*, vol. 71, no. 1, pp. 18–38, Oct. 1992.

[3] A. Cohen and I. Daubechies, "Nonseparable dimensional wavelet bases", *Rev. Math. Iberoamericana*, vol. 9, no. 1, pp. 51–137, 1993.

[4] J. Kovačević and M. Vetterli, "Nonseparable two- and three-dimensional wavelets", *IEEE Trans. Signal Proc.*, vol. 43, no. 5, pp. 1260–1273, May 1995.

[5] P. Maass, "Families of orthogonal two-dimensional wavelets", *SIAM J. Math. Anal.*, vol. 27, no. 5, pp. 1454–1481, Sep. 1996.

[6] M. Lounsbery, T. DeRose, and J. Warren, "Multiresolution analysis for surfaces of arbitrary topological type", *ACM Trans. Graphics*, vol. 16, no. 1, pp. 34–73, Jan. 1997.

[7] D. Stanhill and Y. Y. Zeevi, "Two dimensional orthogonal filter banks and wavelets with linear phase", *IEEE Trans. Signal Proc.*, vol. 46, no. 1, pp. 183–190, Jan. 1998.

[8] A. Ayache, "Construction of non-separable dyadic compactly supported orthonormal wavelet bases for $L^2(\mathbb{R}^2)$ of arbitrarily high regularity", *Rev. Mat. Iberoamericana*, vol. 15, no. 1, pp. 37–58, 1999.

[9] E. Belogay and Y. Wang, "Arbitrarily smooth orthogonal non-separable wavelets in R^2", *SIAM J. Math. Anal.*, vol. 30, no. 3, pp. 678–697, May 1999.

[10] B. Han, "Analysis and construction of optimal multivariate biorthogonal wavelets with compact support", *SIAM J. Math. Anal.*, vol. 31, no. 2, pp. 274–304, 1999/2000.

[11] W.J. He and M.J. Lai, "Examples of bivariate nonseparable compactly supported orthonormal continuous wavelets", *IEEE Trans. Image Proc.*, vol. 9, no. 5, pp. 949–953, May 2000.

[12] B. Han and R.Q. Jia, "Quincunx fundamental refinable functions and quincunx biorthogonal wavelets", *Math. Comput.*, vol. 71, no. 237, pp. 165–196, 2002.

[13] B. Han, “Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix”, *J. Comput. Appl. Math.*, vol. 155, no. 1, pp. 43–67, Jun. 2003.

[14] M. Bertram, “Biorthogonal Loop-subdivision Wavelets”, *Computing*, vol. 72, no. 12, pp. 29-39, Apr. 2004.

[15] M. Bertram, M.A. Duchaineau, B. Hamann, and K.I. Joy, “Generalized B-Spline subdivision-surface wavelets for geometry compression”, *IEEE Trans. Visualization and Computer Graphics*, vol. 10, no. 3, pp. 326–338, May/Jun. 2004.

[16] M.J. Lai, “Construction of multivariate compactly supported orthonormal wavelets”, *Adv. Comput. Math.*, vol. 25, no. 1-3, pp. 41–56, Jul. 2006.

[17] H.W. Wang, K.H. Qin, and K. Tang, “Efficient wavelet construction with Catmull-Clark subdivision”, *The Visual Computer*, vol. 22, nos. 9–11, pp. 874-884, Sep. 2006.

[18] H.W. Wang, K.H. Qin, and H.Q. Sun, “$\sqrt{3}$-subdivision-based biorthogonal wavelets”, *IEEE Trans. Visualization and Computer Graphics*, vol. 13, no. 5, pp. 914–925, Sep./Oct. 2007.

[19] M. Stamminger and G. Drettakis, “Interactive sampling and rendering for complex and procedural geometry”, In *Proceedings of the 12th Eurographics Workshop on Rendering Techniques* (June 25–27, 2001), (S.J. Gortler and K. Myszkowski, Eds.), Springer-Verlag, London, 2001, pp. 151–162.

[20] A. Lundmark, N. Wadströmer, and H. Li, “Hierarchical subsample giving fractal region”, *IEEE Trans. Image Proc.*, vol. 10, no. 1, pp. 167–173, Jan. 2001.

[21] I.P. Ivrissimtzis, M.A. Sabin, and N.A. Dodgson, “$\sqrt{5}$-subdivision”, in *Advances in Multiresolution for Geometric Modelling*, Springer Berlin Heidelberg, 2005, pp. 285–299.

[22] C.K. Chui, Q.T. Jiang, and R.N. Ndao, “Triangular $\sqrt{7}$ and quadrilateral $\sqrt{5}$ subdivision schemes: regular case”, *J. Math. Anal. Appl.*, vol. 338, no. 2, pp. 1204–1223, Feb. 2008.

[23] S. Mallat, “Multiresolution approximations and wavelet orthonormal basis of $L^2(\mathbb{R})$”, *Trans. Amer. Math. Soc.*, vol. 315, no. 1, pp. 69-87, Sep. 1989.

[24] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, UK, 1992.

[25] R.Q. Jia, “Approximation properties of multivariate wavelets”, *Math. Comp.*, vol. 67, no. 222, pp. 647–665, Apr. 1998.

[26] R.Q. Jia, and Q.T. Jiang, “Approximation power of refinable vectors of functions”, in *Wavelet Analysis and Applications*, AMS/IP Stud. Adv. Math., Vol. 25, Amer. Math. Soc., Providence, RI, 2002, pp. 155–178.

[27] A. Cohen and I. Daubechies, “A stability criterion for biorthogonal wavelet bases and their related subband coding scheme”, *Duke Math. J.*, vol. 68, no. 2, pp. 313–335, 1992.

[28] W. Lawton, S.L. Lee, and Z.W. Shen, “Stability and orthonormality of multivariate refinable functions”, *SIAM J. Math. Anal.*, vol. 28, no. 4, pp. 999–1014, Jul. 1997.

[29] R.Q. Jia, “Convergence of vector subdivision schemes and construction of biorthogonal multiple wavelets”, In *Advances in Wavelets*, Springer-Verlag, Singapore, 1999, pp. 199–227.

[30] C. de Boor, R. DeVore and A. Ron, “The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$”, *J. Funct. Anal.*, vol. 119, no. 1, pp. 37–78, 1994.

[31] R.Q. Jia and Z.W. Shen, “Multiresolution and wavelets”, *Proc. Edinburgh Math. Soc.*, vol. 37, no. 2, pp. 271–300, 1994.

[32] R.Q. Jia and S.R. Zhang, “Spectral properties of the transition operator associated to a multivariate refinement equation”, *Linear Algebra Appl.*, vol. 292, no. 1, pp. 155–178, May 1999.

[33] R.Q. Jia and Q.T. Jiang, “Spectral analysis of transition operators and its applications to smoothness analysis of wavelets”, *SIAM J. Matrix Anal. Appl.*, vol. 24, no. 4, pp. 1071–1109, 2003.

[34] Q.T. Jiang and P. Oswald, “Triangular $\sqrt{3}$-subdivision schemes: the regular case”, *J. Comput. Appl. Math.*, vol. 156, no. 1, pp. 47–75, Jul. 2003.

[35] B. Han, “Symmetry property and construction of wavelets with a general dilation matrix”, *Linear Algebra and its Applications*, vol. 353, no. 1, pp. 207–225, Sep. 2002.

[36] W. Sweldens, “The lifting scheme: a custom-design construction of biorthogonal wavelets”, *Appl. Comput. Harmonic Anal.*, vol. 3, no. 2, pp. 186–200, Apr. 1996.

[37] W. Dahmen, “Decomposition of refinable spaces and applications to operator equations”, *Numer. Algor.*, vol. 5, no. 5, pp. 229–245, May 1993.

PLACE PHOTO HERE

Qingtang Jiang received the B.S. and M.S. degrees from Hangzhou University, Hangzhou, China, in 1986 and 1989, respectively, and the Ph.D. degree from Peking University, Beijing, China, in 1992, all in mathematics.

He was with Peking University from 1992 to 1995. He was an NSTB postdoctoral fellow and then a research fellow at the National University of Singapore from 1995 to 1999. Before he joined the University of Missouri-St. Louis, in 2002, he held visiting positions at University of Alberta, Canada, and West Virginia University, Morgantown. He is now a Professor in the Department of Math and Computer Science, University of Missouri-St. Louis. His current research interests include time-frequency analysis, wavelet theory and its applications, filter bank design, signal classification, image processing, and surface subdivision.