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SPECTTRAL ANALYSIS OF THE TRANSITION OPERATOR AND ITS APPLICATIONS TO SMOOTHNESS ANALYSIS OF WAVELETS *

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Abstract. The purpose of this paper is to investigate spectral properties of the transition operator associated to a multivariate vector refinement equation and their applications to the study of smoothness of the corresponding refinable vector of functions.

Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R}^s)$ satisfying the refinement equation $\Phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\Phi(M \cdot - \alpha)$, where M is an expansive integer matrix. We assume that M is isotropic, *i.e.*, M is similar to a diagonal matrix $\operatorname{diag}(\sigma_1, \ldots, \sigma_s)$ with $|\sigma_1| = \cdots = |\sigma_s|$. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, define $\sigma^{-\mu} := \sigma_1^{-\mu_1} \cdots \sigma_s^{-\mu_s}$. The smoothness of Φ is measured by the critical exponent

$$\lambda(\Phi) := \sup\Big\{ \lambda : \phi_j \in W_2^\lambda(\mathbb{R}^s) \ \text{ for all } \ j = 1, \ldots, r \Big\},$$

where $W_2^\lambda(\mathbb{R}^s)$ denotes the Sobolev space $\{f \in L_2(\mathbb{R}^s) : \int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 (1 + |\xi|^\lambda)^2 \, d\xi < \infty\}$. We assume that the mask a is finitely supported, *i.e.*, the set $\operatorname{supp} a := \{\alpha \in \mathbb{Z}^s : a(\alpha) \neq 0\}$ is finite. Note that each $a(\alpha)$ is an $r \times r$ complex matrix. Let $A := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)/d$, where $d := |\det M|$. We assume that $\operatorname{spec}(A)$ (the spectrum of A) is $\{\eta_1, \eta_2 \ldots, \eta_r\}$, where $\eta_1 = 1$ and $\eta_j \neq 1$ for $j = 2, \ldots, r$. For $\alpha \in \mathbb{Z}^s$, let $b(\alpha) := \sum_{\beta \in \mathbb{Z}^s} \overline{a(\beta)} \otimes a(\alpha + \beta)/d$, where $\otimes$ denotes the (right) Kronecker product. Suppose the highest degree of polynomials reproduced by Φ is $k - 1$. Let

$$E_k := \{\eta_j \overline{\sigma^{-\mu}}, \overline{\eta_j} \sigma^{-\mu} : |\mu| < k, j = 2, \ldots, r\} \cup \{\sigma^{-\mu} : |\mu| < 2k\}.$$

The main result of this paper asserts that if Φ is stable, then $\lambda(\Phi) = -\Big(\log_d \rho_k\Big) s/2$, where

$$\rho_k := \max\bigg\{ |\nu| : \nu \in \operatorname{spec}\Big(b(M\alpha - \beta)\Big)_{\alpha, \beta \in K} \setminus E_k \bigg\},$$

and K is the set $\mathbb{Z}^s \cap \sum_{n=1}^\infty M^{-n}(\operatorname{supp} b)$. This result is obtained through an extensive use of linear algebra and matrix theory. Three examples are provided to illustrate the general theory. All these examples have background of practical applications.

Key words. refinement equations, wavelets, subdivision operators, transition operators, polynomial reproducibility, spectral analysis, smoothness analysis

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1. Introduction.

The purpose of this paper is to investigate spectral properties of the transition operator associated to a multivariate vector refinement equation and their

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applications to the study of smoothness of the corresponding refinable vector of functions. This study is important to applications of wavelets to image processing, computer aided geometric design, and numerical solutions to partial differential equations.

Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}^s$ the s-dimensional Euclidean space. An element of $\mathbb{R}^s$ is also viewed as an $r \times 1$ vector of real numbers. The inner product of two vectors x and y in $\mathbb{R}^s$ is denoted by $x \cdot y$. The norm of x is $|x| := \sqrt{x \cdot x}$.

Let f be a (Lebesgue) measurable function from $\mathbb{R}^s$ to $\mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers. For $1 \le p < \infty$, let

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p \, dx \right)^{1/p}.$$

For $p = \infty$, let $\|f\|_\infty$ be the essential supremum of $|f|$ on $\mathbb{R}^s$. By $L_p(\mathbb{R}^s)$ we denote the Banach space of all measurable functions f such that $\|f\|_p < \infty$. A function f is said to be integrable if f lies in $L_1(\mathbb{R}^s)$.

The Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi} \, dx, \qquad \xi \in \mathbb{R}^s,$$

where i denotes the imaginary unit. The domain of the Fourier transform can be naturally extended to $L_2(\mathbb{R}^s)$.

Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0$ the set of nonnegative integers. An s-tuple $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is called a **multi-index**. The length of μ is $|\mu| := \mu_1 + \cdots + \mu_s$, and the factorial of μ is $\mu! := \mu_1! \cdots \mu_s!$. For $\mu, \nu \in \mathbb{N}_0^s$, $\nu \le \mu$ means $\nu_j \le \mu_j$, $j = 1, \ldots, s$. If $\nu \le \mu$ and $\nu \ne \mu$, we write $\nu < \mu$. For $\nu \le \mu$, we define

$$\binom{\mu}{\nu} := \frac{\mu!}{\nu!(\mu - \nu)!}.$$

For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$, define

$$x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}.$$

The function $x \mapsto x^\mu$ ($x \in \mathbb{R}^s$) is called a monomial and its (total) degree is $|\mu|$. A polynomial is a linear combination of monomials. The degree of a polynomial $q = \sum_\mu c_\mu x^\mu$ is defined to be $\deg q := \max\{|\mu| : c_\mu \ne 0\}$. For $k \in \mathbb{N}_0$, we use Π_k to denote the linear space of all polynomials of degree at most k.

Let $\mathbb{Z}$ denote the set of integers. By $\ell(\mathbb{Z}^s)$ we denote the linear space of all sequences on $\mathbb{Z}^s$. A sequence a on $\mathbb{Z}^s$ is said to be finitely supported if $a(\alpha) \ne 0$ only for finitely many α. Let $\ell_0(\mathbb{Z}^s)$ denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$. Let $u \in \ell(\mathbb{Z}^s)$. For $1 \le p < \infty$, we define

$$\|u\|_p := \left(\sum_{\alpha \in \mathbb{Z}^s} |u(\alpha)|^p \right)^{1/p}.$$

For $p = \infty$, define $\|u\|_\infty$ to be the supremum of $|u|$ on $\mathbb{Z}$. For $1 \le p \le \infty$, let $\ell_p(\mathbb{Z}^s)$ denote the Banach space of all sequences u for which $\|u\|_p < \infty$.

For positive integers m and n, by $\mathbb{C}^{m\times n}$ we denote the collection of all $m \times n$ complex matrices. The transpose of a matrix A is denoted by A^T. When $n = 1$, $\mathbb{C}^{m\times 1}$ is abbreviated as $\mathbb{C}^m$. The linear span of a set E of vectors is denoted by $\operatorname{span}(E)$.

We use $\ell(\mathbb{Z}^s \to \mathbb{C}^{m\times n})$ to denote the linear space of all sequences of $m \times n$ matrices. Similarly, we use $\ell_0(\mathbb{Z}^s \to \mathbb{C}^{m\times n})$ to denote the linear space of all finitely supported sequences of $m \times n$ matrices. For simplicity, $\ell(\mathbb{Z}^s \to \mathbb{C}^{m\times n})$ and $\ell_0(\mathbb{Z}^s \to \mathbb{C}^{m\times n})$ will be abbreviated as $\ell^{m\times n}(\mathbb{Z}^s)$ and $\ell_0^{m\times n}(\mathbb{Z}^s)$, respectively. When $n = 1$, $\ell^{m\times 1}(\mathbb{Z}^s)$ and $\ell_0^{m\times 1}(\mathbb{Z}^s)$ will be further abbreviated as $\ell^m(\mathbb{Z}^s)$ and $\ell_0^m(\mathbb{Z}^s)$, respectively. For a subset $K \subseteq \mathbb{Z}^s$, $\ell^{m\times n}(K)$ denotes the linear space of those elements $u \in \ell^{m\times n}(\mathbb{Z}^s)$ for which $u(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \setminus K$.

The **symbol** of an element $v \in \ell_0(\mathbb{Z}^s)$, denoted $\hat{v}$, is the trigonometric polynomial given by

$$\hat{v}(\xi) := \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s.$$

The symbol of an element in $\ell_0^{m\times n}(\mathbb{Z}^s)$ is defined accordingly.

By $\mathbb{T}(\mathbb{R}^s)$ we denote the set of all trigonometric polynomials on $\mathbb{R}^s$. Accordingly, by $\mathbb{T}^{m\times n}(\mathbb{R}^s)$ we denote the set of all $m \times n$ matrices of trigonometric polynomials on $\mathbb{R}^s$.

The **spectrum** of a square matrix A is denoted by $\operatorname{spec}(A)$ and it is understood to be the multiset of its eigenvalues. In other words, multiplicities of eigenvalues are counted in the spectrum. The multiset of nonzero eigenvalues of a square matrix A is denoted by $\operatorname{spec}'(A)$. By $\rho(A)$ we denote the spectral radius of A. Clearly, if $\operatorname{spec}'(A)$ is not empty,

$$\rho(A) = \max\{|\nu| : \nu \in \operatorname{spec}(A)\} = \max\{|\nu| : \nu \in \operatorname{spec}'(A)\}.$$

Let M be an $s \times s$ integer matrix. We assume that M is **expansive**, *i.e.*, all the eigenvalues of M are greater than 1 in modulus.

An $r \times 1$ vector $\Phi = (\phi_1, \ldots, \phi_r)^T$ of compactly supported functions in $L_p(\mathbb{R}^s)$ is said to be M-refinable if Φ satisfies the following vector refinement equation

$$\Phi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\Phi(M\cdot - \alpha), \tag{1.1}$$

where $a \in \ell_0^{r\times r}(\mathbb{Z}^s)$. We call a the (refinement) **mask**. Taking Fourier transform of both sides of (1.1), we obtain

$$\hat{\Phi}(\xi) = A\big((M^T)^{-1}\xi\big)\hat{\Phi}\big((M^T)^{-1}\xi\big), \quad \xi \in \mathbb{R}^s, \tag{1.2}$$

where

$$A(\xi) := \frac{1}{|\det M|} \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)e^{-i\alpha\cdot\xi}. \tag{1.3}$$

It follows from (1.2) that $\hat{\Phi}(0) = A(0)\hat{\Phi}(0)$, where

$$A(0) = \frac{1}{d} \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) \quad \text{and} \quad d := |\det M|. \tag{1.4}$$

Our goal is to determine the smoothness of Φ in the L_2 norm strictly in terms of the mask a. For $\lambda \ge 0$, we denote by $W_2^\lambda(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 (1 + |\xi|^\lambda)^2 \, d\xi < \infty.$$

The smoothness of $\Phi = (\phi_1, \ldots, \phi_r)^T$ is measured by the critical exponent $\lambda(\Phi)$, which is defined by

$$\lambda(\Phi) := \sup\bigl\{\lambda : \phi_j \in W_2^\lambda(\mathbb{R}^s) \ \text{ for all } \ j = 1, \ldots, r\bigr\}.$$

The smoothness of refinable functions is an important issue in all multi-resolution analyses and has a strong impact on applications of wavelets to image processing and geometric modelling, *e.g.*, subdivision schemes.

The smoothness order of refinable functions has been studied extensively. For the scalar case ($r = 1$), a characterization of the critical exponent of a refinable function in terms of the corresponding mask was given in [12], [45], and [5]. In particular, it was shown that the critical exponent of a refinable function could be calculated in terms of the spectral radius of a transition matrix associated to the mask.

The aforementioned results rely on factorization of the symbol of the mask. In the multivariate case $s > 1$, however, the symbol of the refinement mask is often irreducible. This difficulty was overcome in [21] by considering certain invariant subspaces of the transition operator associated to the mask. Based on the characterization of smoothness of multivariate refinable functions given in [21], a useful algorithm for calculation of the critical exponent was given in [29]. These results are valid when the matrix M is isotropic. In the case when M is anisotropic, smoothness of multivariate refinable functions was investigated in [7].

For the vector case ($r > 1$), smoothness of univariate refinable vectors of functions was studied in [6] and [36] on the basis of a factorization technique. A different approach was employed in [28] to give the optimal smoothness of refinable vectors of functions. Smoothness of multivariate refinable vectors of functions were analyzed in [30] and [31]. Also, see [41] and [26] for a recent study of the Sobolev regularity of refinable functions without the requirement of stability.

The study of smoothness of Φ is related to properties of shift-invariant spaces. Suppose $\Phi = (\phi_1, \ldots, \phi_r)^T$ is an $r \times 1$ vector of compactly supported functions in $L_p(\mathbb{R}^s)$. We use $\mathbb{S}(\Phi)$ to denote the shift-invariant space generated from Φ, which is the linear space of functions of the form

$$\sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} u_j(\alpha) \phi_j(\cdot - \alpha),$$

where $u_1, \ldots, u_r \in \ell(\mathbb{Z}^s)$. The (multi-integer) shifts of $\phi_1, \ldots, \phi_r$ are said to be stable, if there exist two positive constants C_1 and C_2 such that the inequalities

$$C_1 \Bigl(\sum_{j=1}^r \|u_j\|_p \Bigr) \le \Bigl\| \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} u_j(\alpha) \phi_j(\cdot - \alpha) \Bigr\|_p \le C_2 \Bigl(\sum_{j=1}^r \|u_j\|_p \Bigr)$$

are valid for all $u_1, \ldots, u_r \in \ell_p(\mathbb{R}^s)$. If this is the case, we simply say that Φ is stable. It was proved in [27] and [19] that the shifts of $\phi_1, \ldots, \phi_r$ are stable if and only if, for every $\xi \in \mathbb{R}^s$,

$$\operatorname{span}\{\hat{\Phi}(\xi + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r.$$

The Kronecker product of two matrices is a useful tool in our study of vector refinement equations. Let us recall some basic properties of the Kronecker product. Suppose $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ and $B = (b_{ij})_{1\le i\le r, 1\le j\le s}$ are two matrices. The (right) Kronecker product of A and B, written $A \otimes B$, is defined to be the block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

For three matrices A, B, and C of the same type, we have

$$(A + B) \otimes C = (A \otimes C) + (B \otimes C) \quad \text{and} \quad A \otimes (B + C) = (A \otimes B) + (A \otimes C).$$

If A, B, C, D are four matrices such that the products AC and BD are well defined, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Moreover, if $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of an $r \times r$ matrix A and $\mu_1, \ldots, \mu_s$ are the eigenvalues of an $s \times s$ matrix B, then the eigenvalues of the Kronecker product $A \otimes B$ are $\lambda_m \mu_n$, $m = 1, \ldots, r, n = 1, \ldots, s$. See [34, Chap. 12] for a proof of these results.

For a matrix $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$, the vector

$$(a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn})^T$$

is said to be the vec-function of A and written as $\operatorname{vec} A$. If A, X, and B are three matrices such that the product AXB is well defined, then

$$\operatorname{vec}(AXB) = (B^T \otimes A)\operatorname{vec} X. \tag{1.5}$$

For two functions f, g in $L_2(\mathbb{R}^s)$, $f \odot g$ is defined as follows:

$$f \odot g(x) := \int_{\mathbb{R}^s} f(x + y)\overline{g(y)}\, dy, \qquad x \in \mathbb{R}^s,$$

where $\overline{g(y)}$ stands for the complex conjugate of $g(y)$. In other words, $f \odot g$ is the convolution of f with the function $y \mapsto \overline{g(-y)}$, $y \in \mathbb{R}^s$. It is easily seen that $f \odot g$ lies in $C_0(\mathbb{R}^s)$, the space of continuous functions on $\mathbb{R}$ which vanish at ∞. In particular, $f \odot g$ is uniformly continuous.

Suppose $\Phi = (\phi_1, \ldots, \phi_r)^T$ is an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R}^s)$ satisfying the refinement equation (1.1). Let

$$\Phi \odot \Phi^T := \begin{bmatrix} \phi_1 \odot \phi_1 & \phi_1 \odot \phi_2 & \cdots & \phi_1 \odot \phi_r \\ \phi_2 \odot \phi_1 & \phi_2 \odot \phi_2 & \cdots & \phi_2 \odot \phi_r \\ \vdots & \vdots & \ddots & \vdots \\ \phi_r \odot \phi_1 & \phi_r \odot \phi_2 & \cdots & \phi_r \odot \phi_r \end{bmatrix}.$$

It follows from (1.1) that

$$\Phi \odot \Phi^T = \sum_{\alpha \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} a(\alpha)\Phi(M \cdot - \alpha) \odot \Phi^T(M \cdot - \beta)\overline{a(\beta)}^T.$$

Let $F := \mathrm{vec}(\Phi \odot \Phi^T)$. With the help of (1.5) we obtain

$$F = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) F(M \cdot - \alpha),$$

where $b \in \ell_0^{r^2 \times r^2}(\mathbb{Z}^s)$ is given by

$$b(\alpha) := \frac{1}{d} \sum_{\beta \in \mathbb{Z}^s} \overline{a(\beta)} \otimes a(\alpha + \beta), \quad \alpha \in \mathbb{Z}^s. \tag{1.6}$$

For a bounded subset H of $\mathbb{R}^s$, the set $\sum_{n=1}^{\infty} M^{-n} H$ is defined as

$$\Big\{ \sum_{n=1}^{\infty} M^{-n} h_n \ : \ h_n \in H \ \text{ for } \ n = 1, 2, \ldots \Big\}.$$

If H is a compact set, then $\sum_{n=1}^{\infty} M^{-n} H$ is also compact. By $\mathrm{supp}\, b$ we denote the set $\{\alpha \in \mathbb{Z}^s \ : \ b(\alpha) \neq 0\}$. Let

$$K := \Big(\sum_{n=1}^{\infty} M^{-n} (\mathrm{supp}\, b) \Big) \cap \mathbb{Z}^s.$$

We assume that M is isotropic, *i.e.*, M is similar to a diagonal matrix $\mathrm{diag}(\sigma_1, \ldots, \sigma_s)$ with $|\sigma_1| = \cdots = |\sigma_s|$. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s$, define

$$\sigma^\mu := \sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s}.$$

Suppose $r = 1$ and Φ is stable. Let k be the largest integer such that $\mathbb{S}(\Phi) \supset \Pi_{k-1}$. It was proved in [29] that $\lambda(\Phi) = -(\log_d \rho_k)\, s/2$, where

$$\rho_k := \max \Big\{ \ |\nu| : \nu \in \mathrm{spec}\big(b(M\alpha - \beta)\big)_{\alpha, \beta \in K} \setminus \{\sigma^{-\mu} : |\mu| < 2k\} \ \Big\}.$$

A straightforward generalization of this result to the vector case ($r > 1$) does not work. See §8 for a counterexample. In fact, in the vector case, a correct formula for $\lambda(\Phi)$ must involve the spectrum of the $r\times r$ matrix $A(0)$ given in (1.4). Suppose $\operatorname{spec}(A(0)) = \{\eta_1, \eta_2, \ldots, \eta_r\}$. We assume that $\eta_1 = 1$ and $\eta_j \neq 1$ for $j = 2, \ldots, r$. The following theorem is the main result of this paper.

THEOREM 1.1. *Let Φ be an $r\times 1$ vector of compactly supported functions in $L_2(\mathbb{R}^s)$. Suppose Φ satisfies the refinement equation (1.1) with mask a. Let k be the largest integer such that $\mathbb{S}(\Phi) \supset \Pi_{k-1}$. Let*

$$E_k := \{\eta_j\overline{\sigma^{-\mu}}, \overline{\eta_j}\sigma^{-\mu} : |\mu| < k, j = 2, \ldots, r\} \cup \{\sigma^{-\mu} : |\mu| < 2k\}.$$

If, in addition, Φ is stable, then

$$\lambda(\Phi) = -\big(\log_d \rho_k\big)\, s/2,$$

where

$$\rho_k := \max\Big\{\, |\nu| : \nu \in \operatorname{spec}\big(b(M\alpha-\beta)\big)_{\alpha,\beta\in K} \setminus E_k \,\Big\}.$$

Here is an outline of the paper. Section 2 is devoted to a study of subdivision and transition operators. The fact that the subdivision operator is the algebraic adjoint of the transition operator will be employed to derive useful spectral properties of these linear operators. In Section 3 we will review polynomial reproducibility of refinable vectors of functions and introduce certain invariant subspaces of the subdivision and transition operators, which will be needed in the smoothness analysis of refinable functions. In Section 4 we will give a characterization of the smoothness order of a refinable vector Φ of functions in terms of the corresponding mask a. This characterization is difficult to implement. Thus, in Section 5, we will give a formula for the critical exponent of Φ in terms of the spectral radius of the transition operator T_b restricted to a certain invariant subspace, where b is obtained from a by (1.6). In order to calculate this spectral radius, we will carefully analyze the relevant invariant subspaces and spectra of the subdivision operator and the transition operator in Sections 6 and 7. This analysis enables us to prove Theorem 1.1 and other related results. Finally, in Section 8, we will provide three examples to illustrate the general theory. These examples demonstrate usefullness of Theorem 1.1 to various applications such as multi-wavelets, numerical solutions of partial differential equations, and computer aided geometric design.

In relation with their study of $\sqrt{3}$-subdivision schemes (see [38]), Jiang and Oswald [32] developed Matlab software to calculate $\lambda(\Phi)$ in Theorem 1.1. It can be downloaded at `http://cm.bell-labs.com/who/poswald` and at `http://www.math.umsl.edu/~jiang`. The reader is referred to [32] for explanations of the Matlab routines.

2. Subdivision and transition operators.

To each $a \in \ell_0^{r\times r}(\mathbb{Z}^s)$ we associate two linear operators: the subdivision operator S_a and the transition operator T_a. The subdivision operator S_a is the linear operator on $\ell^{1\times r}(\mathbb{Z}^s)$ defined by

$$S_a u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} u(\beta)a(\alpha - M\beta), \quad \alpha\in\mathbb{Z}^s,\ u\in\ell^{1\times r}(\mathbb{Z}^s).$$

The transition operator T_a is the linear operator on $\ell_0^r(\mathbb{Z}^s)$ defined by

$$T_a v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(M\alpha-\beta)v(\beta), \quad \alpha\in\mathbb{Z}^s,\ v\in\ell_0^r(\mathbb{Z}^s).$$

This section is devoted to a study of the subdivision and transition operators. See [3] and [10] for some earlier work on these operators.

We introduce a bilinear form on a pair of the linear spaces $\ell_0^r(\mathbb{Z}^s)$ and $\ell^{1\times r}(\mathbb{Z}^s)$ as follows:

$$\langle u,v\rangle := \sum_{\alpha\in\mathbb{Z}^s} u(-\alpha)v(\alpha), \quad u\in\ell^{1\times r}(\mathbb{Z}^s),\ v\in\ell_0^r(\mathbb{Z}^s).$$

Then $\ell^{1\times r}(\mathbb{Z}^s)$ is the algebraic dual of $\ell_0^r(\mathbb{Z}^s)$ with respect to this bilinear form. For $u\in\ell^{1\times r}(\mathbb{Z}^s)$ and $v\in\ell_0^r(\mathbb{Z}^s)$, we have

$$\begin{aligned}\langle S_a u,v\rangle &= \sum_{\alpha\in\mathbb{Z}^s}(S_a u)(\alpha)v(-\alpha) = \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} u(\beta)a(\alpha-M\beta)v(-\alpha)\\ &= \sum_{\beta\in\mathbb{Z}^s}\sum_{\alpha\in\mathbb{Z}^s} u(-\beta)a(M\beta-\alpha)v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} u(-\beta)(T_a v)(\beta) = \langle u, T_a v\rangle.\end{aligned}$$

Consequently, S_a is the algebraic adjoint of T_a.

The annihilator of a linear subspace U of $\ell^{1\times r}(\mathbb{Z}^s)$ is defined by

$$U^\perp := \{v\in\ell_0^r(\mathbb{Z}^s) : \langle u,v\rangle = 0\ \forall\, u\in U\}.$$

Similarly, the annihilator of a linear subspace V of $\ell_0^r(\mathbb{Z}^s)$ is defined by

$$V^\perp := \{u\in\ell^{1\times r}(\mathbb{Z}^s) : \langle u,v\rangle = 0\ \forall\, v\in V\}.$$

Clearly, $U\subseteq(U^\perp)^\perp$. If U is a *finite* dimensional subspace of $\ell^{1\times r}(\mathbb{Z}^s)$, then $(U^\perp)^\perp = U$. This comes from the Theorem on Linear Dependence (see [33, p. 7]), which states that a linear functional f is a linear combination of a finite set $\{f_1,\dots,f_n\}$ of linear functionals if and only if the null space of f contains the intersection of the null spaces of $f_1,\dots,f_n$. Indeed, an element $u\in\ell^{1\times r}(\mathbb{Z}^s)$ can be viewed as a linear functional on $\ell_0^r(\mathbb{Z}^s)$. Suppose $\{u_1,\dots,u_n\}$ is a basis for U. Then $u\in(U^\perp)^\perp$ means the null space of u contains the intersection of the null spaces of $u_1,\dots,u_n$. Hence, by the Theorem on Linear Dependence, u lies in U.

Moreover, if V is a linear subspace of $\ell_0^r(\mathbb{Z}^s)$, then $(V^\perp)^\perp = V$. In this case, V is not required to be finite dimensional. Clearly, $V\subseteq(V^\perp)^\perp$. The inclusion relation $(V^\perp)^\perp\subseteq V$ can be proved by a version of Hahn-Banach theorem. Suppose $w\in\ell_0^r(\mathbb{Z}^s)\setminus V$. Let W be the linear span of V and w. Then we can find a linear functional f on W such that f vanishes on V and $f(w)=1$. This linear functional can be extended to a linear functional on $\ell_0^r(\mathbb{Z}^s)$. Since $\ell^{1\times r}(\mathbb{Z}^s)$ is the algebraic dual of $\ell_0^r(\mathbb{Z}^s)$, this means that there exists some element u in $\ell^{1\times r}(\mathbb{Z}^s)$ such that $u\in V^\perp$ and $\langle u,w\rangle = 1$. Hence, $w\notin(V^\perp)^\perp$. This shows $(V^\perp)^\perp\subseteq V$.

LEMMA 2.1. *Let U be a finite dimensional linear subspace of $\ell^{1\times r}(\mathbb{Z}^s)$, and let $V := U^\perp$. Then U is invariant under the subdivision operator S_a if and only if V is invariant under the transition operator T_a.*

Proof. Suppose U is invariant under S_a. For $v \in V$ we have

$$\langle u, T_a v\rangle = \langle S_a u, v\rangle = 0 \quad \forall\, u \in U.$$

Hence, $T_a v \in U^\perp = V$. This shows that V is invariant under T_a.

Suppose V is invariant under T_a. For $u \in U$ we have

$$\langle S_a u, v\rangle = \langle u, T_a v\rangle = 0 \quad \forall\, v \in V.$$

Hence, $S_a u \in V^\perp = U$. This shows that U is invariant under S_a. ∎

It was proved in [15] that T_a has only finitely many nonzero eigenvalues. The following is an outline of the proof. By $\operatorname{supp} a$ we denote the set $\{\alpha \in \mathbb{Z}^s : a(\alpha) \neq 0\}$. Similarly, for $v \in \ell_0^r(\mathbb{Z}^s)$, $\operatorname{supp} v$ stands for the set $\{\alpha \in \mathbb{Z}^s : v(\alpha) \neq 0\}$. By the definition of T_a we see that $T_a v(\alpha) \neq 0$ if and only if

$$M\alpha - \beta \in \operatorname{supp} a \ \text{ for some } \ \beta \in \operatorname{supp} v.$$

Hence,

$$\operatorname{supp}(T_a v) \subseteq M^{-1}\operatorname{supp} a + M^{-1}\operatorname{supp} v.$$

Applying the above argument repeatedly, we obtain

$$\operatorname{supp}(T_a^n v) \subseteq \sum_{j=1}^{n} M^{-j}\operatorname{supp} a + M^{-n}\operatorname{supp} v. \tag{1.7}$$

Let

$$K := \mathbb{Z}^s \bigcap \left(\sum_{n=1}^{\infty} M^{-n}(\operatorname{supp} a)\right). \tag{1.8}$$

The preceding discussion tells us that $\operatorname{supp} v \subseteq K$ implies $\operatorname{supp}(T_a v) \subseteq K$. Therefore, $\ell^r(K)$ is invariant under T_a. Suppose v is an arbitrary element in $\ell_0^r(\mathbb{Z}^s)$. Comparing (1.7) with (1.8), we see that there exists a positive integer N such that, for $n \geq N$ and each $\alpha \in \operatorname{supp}(T_a^n v)$, the distance from the point α to the set K is less than $1/2$. But $\alpha \in \mathbb{Z}^s$ and $K \subset \mathbb{Z}^s$, so α lies in K. This shows that $T_a^n v \in \ell^r(K)$ for sufficiently large n.

Now suppose θ is a nonzero eigenvalue of T_a and $T_a v = \theta v$ for some $v \in \ell_0^r(\mathbb{Z}^s)$. For sufficiently large n we have $\theta^n v = T_a^n v \in \ell^r(K)$. It follows that $v \in \ell^r(K)$. Since $\ell^r(K)$ is finite dimensional, T_a has only finitely many nonzero eigenvalues.

The following lemma extends the above results.

LEMMA 2.2. *Let V and W be two invariant subspaces of the transition operator T_a. Suppose W is finite dimensional and $V \cap \ell^r(K) \subseteq W \subseteq V$, where K is the set given in (1.8). Then*

$$\operatorname{spec}'\big(T_a|_W\big) = \operatorname{spec}'\big(T_a|_{V\cap \ell^r(K)}\big).$$

Proof. Let $\tilde{T}_a$ denote the quotient linear operator induced by T_a on the quotient space $W/(V \cap \ell^r(K))$. Clearly,

$$\operatorname{spec}\bigl(T_a|_W\bigr) = \operatorname{spec}(\tilde{T}_a) \cup \operatorname{spec}\bigl(T_a|_{V\cap\ell^r(K)}\bigr).$$

Thus, it suffices to show that all the eigenvalues of $\tilde{T}_a$ are zero. Let θ be an eigenvalue of $\tilde{T}_a$. Then there exists some $v \in W \setminus (V \cap \ell^r(K))$ such that

$$\tilde{T}_a(v + V \cap \ell^r(K)) = \theta(v + V \cap \ell^r(K)).$$

It follows that

$$T_a v - \theta v \in V \cap \ell^r(K).$$

Since $V \cap \ell^r(K)$ is invariant under T_a, for $n \in \mathbb{N}$ we have

$$T_a^n v - \theta^n v = (T_a^{n-1} + \cdots + \theta^{n-1})(T_a v - \theta v) \in V \cap \ell^r(K).$$

For sufficiently large n, $T_a^n v \in \ell^r(K)$. Hence, $\theta^n v \in V \cap \ell^r(K)$ for sufficiently large n. But $v \notin V \cap \ell^r(K)$. Therefore, $\theta = 0$. The proof is complete. ∎

Lemma 2.2 tells us

$$\rho\bigl(T_a|_W\bigr) = \rho\bigl(T_a|_{V\cap\ell^r(K)}\bigr).$$

This motivates us to define the spectral radius of $T_a|_V$ as $\rho\bigl(T_a|_{V\cap\ell^r(K)}\bigr)$.

LEMMA 2.3. *Let U be a finite dimensional invariant subspace of the subdivision operator S_a, and let $V := U^\perp$. Then*

$$\rho(T_a|_V) = \max\bigl\{|\nu| : \nu \in \operatorname{spec}\bigl((a(M\alpha - \beta))_{\alpha,\beta\in K}\bigr) \setminus \operatorname{spec}(S_a|_U)\bigr\}, \tag{1.9}$$

where $K = \mathbb{Z}^s \cap \sum_{n=1}^\infty M^{-n}(\operatorname{supp} a)$.

Proof. Suppose $\{u_1, \ldots, u_N\}$ is a basis for U. Then there exist $v_1, \ldots, v_N \in \ell_0^r(\mathbb{Z}^s)$ such that

$$\langle u_j, v_m\rangle = \delta_{jm} \quad \text{for} \ \ j, m = 1, \ldots, N, \tag{1.10}$$

where δ stands for the Kronecker sign. Let G be a bounded subset of $\mathbb{R}^s$ such that

$$G \supseteq \{0\} \cup \operatorname{supp} a \cup \bigl(\cup_{m=1}^N \operatorname{supp}(Mv_m)\bigr),$$

and let $J := \mathbb{Z}^s \cap \bigl(\sum_{n=1}^\infty M^{-n} G\bigr)$. Then $K \subseteq J$ and $v_1, \ldots, v_N \in \ell^r(J)$. Moreover, $\ell^r(J) \cap V$ is an invariant subspace of T_a.

Consider the quotient space $\ell^r(J)/(\ell^r(J) \cap V)$. For $v \in \ell^r(J)$, let $\tilde{v}$ denote the coset $v + \ell^r(J) \cap V$. We claim that $\{\tilde{v}_1, \ldots, \tilde{v}_N\}$ forms a basis for $\ell^r(J)/(\ell^r(J) \cap V)$. Indeed, for $v \in \ell^r(J)$ we have

$$v - \sum_{j=1}^N \langle u_j, v\rangle\, v_j \in \ell^r(J) \cap V.$$

Consequently, $\tilde{v}$ lies in the span of $\{\tilde{v}_1, \ldots, \tilde{v}_N\}$. Furthermore, suppose $\sum_{m=1}^N c_m \tilde{v}_m = 0$. Then $\sum_{m=1}^N c_m v_m \in \ell^r(J) \cap V$. It follows that, for $j = 1, \ldots, N$,

$$c_j = \left\langle u_j,\ \sum_{m=1}^N c_m v_m \right\rangle = 0.$$

Hence, $\tilde{v}_1, \ldots, \tilde{v}_N$ are linearly independent. This justifies our claim.

Let $\tilde{T}_a$ denote the linear quotient operator induced by T_a on the quotient space $\ell^r(J)/(\ell^r(J) \cap V)$, that is, $\tilde{T}_a$ is defined by $\tilde{T}_a \tilde{v} := \widetilde{T_a v}$. Suppose

$$S_a u_j = \sum_{m=1}^N b_{jm} u_m \quad \text{and} \quad \tilde{T}_a \tilde{v}_j = \sum_{m=1}^N c_{jm} \tilde{v}_m, \quad j = 1, \ldots, N.$$

By (1.10) we have

$$b_{jm} = \langle S_a u_j, v_m \rangle = \langle u_j, T_a v_m \rangle = c_{mj}, \quad j, m = 1, \ldots, N.$$

Therefore,

$$\operatorname{spec}(\tilde{T}_a) = \operatorname{spec}\big(S_a|_U\big).$$

Consequently, we have

$$\operatorname{spec}(T_a|_{\ell^r(J)}) = \operatorname{spec}(\tilde{T}_a) \cup \operatorname{spec}(T_a|_{\ell^r(J)\cap V}) = \operatorname{spec}(S_a|_U) \cup \operatorname{spec}(T_a|_{\ell^r(J)\cap V}).$$

It follows that

$$\operatorname{spec}'(T_a|_{\ell^r(J)}) = \operatorname{spec}'(S_a|_U) \cup \operatorname{spec}'(T_a|_{\ell^r(J)\cap V}).$$

By Lemma 2.2,

$$\operatorname{spec}'(T_a|_{\ell^r(J)}) = \operatorname{spec}'(T_a|_{\ell^r(K)}) \quad \text{and} \quad \operatorname{spec}'(T_a|_{\ell^r(J)\cap V}) = \operatorname{spec}'(T_a|_{\ell^r(K)\cap V}).$$

Hence,

$$\operatorname{spec}'(T_a|_{\ell^r(K)}) = \operatorname{spec}'(S_a|_U) \cup \operatorname{spec}'(T_a|_{\ell^r(K)\cap V}). \tag{1.11}$$

Note that

$$\rho(T_a|_V) = \rho(T_a|_{\ell^r(K)\cap V}) = \max\{|\nu| : \nu \in \operatorname{spec}'(T_a|_{\ell^r(K)\cap V})\}.$$

In light of (1.11) we have

$$\rho(T_a|_V) = \max\{|\nu| : \nu \in \operatorname{spec}'(T_a|_{\ell^r(K)}) \setminus \operatorname{spec}'(S_a|_U)\}.$$

Finally,

$$\rho(T_a|_V) = \max\{|\nu| : \nu \in \operatorname{spec}(T_a|_{\ell^r(K)}) \setminus \operatorname{spec}(S_a|_U)\}.$$

But $\operatorname{spec}(T_a|_{\ell^r(K)}) = \operatorname{spec}\big((a(M\alpha - \beta))_{\alpha,\beta\in K}\big)$. Taking this into account, we obtain the desired formula (1.9). $\square$

3. Polynomial reproducibility. Let Φ be an $r \times 1$ vector $(\phi_1, \ldots, \phi_r)^T$, where $\phi_1, \ldots, \phi_r$ are compactly supported integrable functions on $\mathbb{R}^s$. If there exists a (finite) linear combination ψ of shifts of $\phi_1, \ldots, \phi_r$ such that

$$\sum_{\alpha \in \mathbb{Z}^s} q(\alpha)\psi(\cdot - \alpha) = q \quad \forall\, q \in \Pi_{k-1}, \tag{1.12}$$

then we say that Φ reproduces all polynomials of degree at most $k-1$. In this section we review results on polynomial reproducibility relevant to our study of smoothness of refinable vectors of functions.

For $j = 1, \ldots, s$, let e_j denote the jth column of the $s \times s$ identity matrix. We may view $e_1, \ldots, e_s$ as the coordinate unit vectors in $\mathbb{R}^s$. By D_j we denote the partial derivative with respect to the jth coordinate. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, D^μ stands for the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$.

The conditions in (1.12) are equivalent to the following conditions:

$$D^\mu \hat{\psi}(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \qquad \forall\, |\mu| < k \text{ and } \beta \in \mathbb{Z}^s.$$

If this is the case, then we say that Φ satisfies the Strang-Fix conditions of order k (see [44]). In [8] Dahmen and Micchelli investigated approximation order on the basis of the Strang-Fix conditions.

It is easily seen that Φ satisfies the Strang-Fix conditions of order k if and only if there exists a $1 \times r$ vector y of trigonometric polynomials such that

$$D^\mu (y\hat{\Phi})(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \qquad \forall\, |\mu| < k \text{ and } \beta \in \mathbb{Z}^s. \tag{1.13}$$

If y satisfies the conditions in (1.13), then we have

$$\frac{x^\mu}{\mu!} = \sum_{\alpha \in \mathbb{Z}^s} u_\mu(\alpha)\Phi(x - \alpha), \quad x \in \mathbb{R}^s,\ |\mu| < k, \tag{1.14}$$

where

$$u_\mu(\alpha) := \sum_{\nu \le \mu} \frac{(-iD)^{\mu-\nu} y(0)}{(\mu - \nu)!}\, \frac{\alpha^\nu}{\nu!}, \quad \alpha \in \mathbb{Z}^s. \tag{1.15}$$

See the recent survey paper [24] for a proof of this result.

Now suppose Φ satisfies the refinement equation (1.1). Naturally, we wish to find the optimal order of the Strang-Fix conditions satisfied by Φ in terms of the mask. There has been a lot of research on this problem. See [17] and [39] for the univariate case ($s = 1$), and [1], [2] and [31] for the multivariate case ($s > 1$). The results in these papers can be summarized as follows (see [24]). Suppose $\Phi = (\phi_1, \ldots, \phi_r)^T$ satisfies the refinement equation (1.1) with a being its mask. Let $A(\xi)$ ($\xi \in \mathbb{R}^s$) be the $r \times r$ matrix given in (1.3). Let y be a $1 \times r$ vector of trigonometric polynomials, and let $g(\xi) := y(M^T\xi)A(\xi)$, $\xi \in \mathbb{R}^s$. Then (1.13) is valid, provided the following three conditions are satisfied:

(P1) $y(0)\hat{\Phi}(0) = 1$;

(P2) $D^\mu g(2\pi(M^T)^{-1}\omega) = 0$ for all $|\mu| < k$ and $\omega \in \mathbb{Z}^s \setminus (M^T\mathbb{Z}^s)$;

(P3) $D^\mu g(0) = D^\mu y(0)$ for all $|\mu| < k$.

Conversely, if Φ is stable and (1.13) is valid, then the above conditions (P1), (P2), and (P3) are satisfied.

For the special case $k = 1$, it is known (see, *e.g.*, [24]) that conditions (P2) and (P3) together are equivalent to

$$y_0 \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) = y_0 \quad \forall\, \alpha \in \mathbb{Z}^s, \tag{1.16}$$

where $y_0 := y(0) \in \mathbb{C}^{1\times r}$. If this is the case, we say that a satisfies the basis sum rule with respect to y_0. For the general case $k \geq 1$, conditions (P2) and (P3) can also be expressed as sum rules involving a. Thus, we say that a satisfies the sum rules of order k with respect to y if y and $g : \xi \mapsto y(M^T\xi)A(\xi)$ $(\xi \in \mathbb{R}^s)$ satisfy conditions (P2) and (P3). If the meaning of y is clear from the context, then the reference to y may be omitted. We always assume that $y(0) \neq 0$.

Let Ω be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M^T\mathbb{Z}^s$. We assume $0 \in \Omega$. Clearly, $\#\Omega$ (the number of elements in Ω) is equal to $d := |\det M|$. Note that condition (P2) can be restated as $D^\mu g(2\pi(M^T)^{-1}\omega) = 0$ for all $|\mu| < k$ and $\omega \in \Omega \setminus \{0\}$. For $v \in \ell_0^r(\mathbb{Z}^s)$ and $\alpha \in \mathbb{Z}^s$ we have

$$\begin{aligned}\sum_{\omega\in\Omega} \hat{v}\big((M^T)^{-1}(\xi + 2\pi\omega)\big) &= \sum_{\omega\in\Omega}\sum_{\alpha\in\mathbb{Z}^s} v(\alpha) e^{-i\alpha\cdot((M^T)^{-1}(\xi+2\pi\omega))}\\ &= \sum_{\alpha\in\mathbb{Z}^s} v(\alpha) e^{-iM^{-1}\alpha\cdot\xi} \sum_{\omega\in\Omega} e^{-2\pi i M^{-1}\alpha\cdot\omega}.\end{aligned}$$

With the help of the following identity (see, *e.g.*, [20, Lemma 3.2])

$$\sum_{\omega\in\Omega} e^{-2\pi i M^{-1}\alpha\cdot\omega} = \begin{cases} d & \text{if } \alpha \in M\mathbb{Z}^s,\\ 0 & \text{if } \alpha \notin M\mathbb{Z}^s,\end{cases}$$

we obtain

$$\sum_{\omega\in\Omega} \hat{v}\big((M^T)^{-1}(\xi + 2\pi\omega)\big) = d \sum_{\alpha\in\mathbb{Z}^s} v(M\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s.$$

The convolution of $u \in \ell^{m\times n}(\mathbb{Z}^s)$ and $v \in \ell_0^n(\mathbb{Z}^s)$ is the element in $\ell^m(\mathbb{Z}^s)$ given by

$$u{*}v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} u(\alpha - \beta) v(\beta), \quad \alpha \in \mathbb{Z}^s.$$

Suppose $v \in \ell_0^r(\mathbb{Z}^s)$. By the definition of the transition operator T_a, we have

$$(T_a v)(\alpha) = (a{*}v)(M\alpha), \quad \alpha \in \mathbb{Z}^s.$$

Hence

$$(T_a v)\hat{\ }(\xi) = \sum_{\alpha\in\mathbb{Z}^s} (a{*}v)(M\alpha) e^{-i\alpha\cdot\xi} = \frac{1}{d}\sum_{\omega\in\Omega} (a{*}v)\hat{\ }\big((M^T)^{-1}(\xi + 2\pi\omega)\big), \quad \xi \in \mathbb{R}^s.$$

It follows that

$$(T_a v)\hat{\ }(\xi) = \sum_{\omega\in\Omega} A\big((M^T)^{-1}(\xi+2\pi\omega)\big)\,\hat{v}\big((M^T)^{-1}(\xi+2\pi\omega)\big), \quad \xi\in\mathbb{R}^s. \tag{1.17}$$

LEMMA 3.1. *Let $a\in\ell_0^{r\times r}(\mathbb{Z}^s)$. Suppose a satisfies the sum rules of order k with respect to $y\in\mathbb{T}^{1\times r}(\mathbb{R}^s)$. Then the linear space H_j $(0\le j<k)$ given by*

$$H_j := \big\{\, v\in\ell_0^r(\mathbb{Z}^s) : D^\mu(y\hat{v})(0)=0\ \forall\,|\mu|=j\,\big\}$$

is invariant under the transition operator T_a.

Proof. By (1.17) we have

$$(T_a v)\hat{\ }(M^T\xi) = \sum_{\omega\in\Omega} A\big(\xi+2\pi(M^T)^{-1}\omega\big)\hat{v}\big(\xi+2\pi(M^T)^{-1}\omega\big), \quad \xi\in\mathbb{R}^s.$$

It follows that

$$y(M^T\xi)(T_a v)\hat{\ }(M^T\xi) = \sum_{\omega\in\Omega} y(M^T\xi)A\big(\xi+2\pi(M^T)^{-1}\omega\big)\hat{v}\big(\xi+2\pi(M^T)^{-1}\omega\big), \quad \xi\in\mathbb{R}^s.$$

For $\omega\in\Omega\setminus\{0\}$, we have by (P2)

$$D^\mu\big(y(M^T\xi)A(\xi+2\pi(M^T)^{-1}\omega)\big)|_{\xi=0} = D^\mu g\big(2\pi(M^T)^{-1}\omega\big)=0 \quad \forall\,|\mu|<k.$$

For $\omega=0$, we have $D^\mu g(0)=D^\mu y(0)$ for all $|\mu|<k$. Hence,

$$D^\mu\big(y(M^T\xi)A(\xi)\hat{v}(\xi)\big)|_{\xi=0} = D^\mu\big(g(\xi)\hat{v}(\xi)\big)|_{\xi=0} = D^\mu(y\hat{v})(0).$$

But $v\in H_j$ implies $D^\mu(y\hat{v})(0)=0$ for all $|\mu|=j$. Therefore,

$$D^\mu\big(y(M^T\xi)(T_a v)\hat{\ }(M^T\xi)\big)|_{\xi=0}=0 \quad \forall\,|\mu|=j.$$

Let $f(\xi):=y(\xi)(T_a v)\hat{\ }(\xi)$, $\xi\in\mathbb{R}^s$. We use $f\circ M^T$ to denote the composition of f and M^T. The above equation tells us that, for all $|\mu|=j$, $D^\mu(f\circ M^T)(0)=0$. Clearly, $f=(f\circ M^T)\circ(M^T)^{-1}$. By the chain rule, $D^\mu f$ is a linear combination of $D^\nu(f\circ M^T)$, $|\nu|=j$. Therefore, $D^\mu(y(\widehat{T_a v}))(0)=D^\mu f(0)=0$ for all $|\mu|=j$, *i.e.*, $T_a v\in H_j$. This shows that H_j is invariant under T_a. ∎

By the Leibniz rule for differentiation we have

$$(-iD)^\mu(y\hat{v})(0) = \sum_{\nu\le\mu}\binom{\mu}{\nu}(-iD)^{\mu-\nu}y(0)(-iD)^\nu\hat{v}(0).$$

But

$$(-iD)^\nu\hat{v}(\xi) = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)(-\alpha)^\nu e^{-i\alpha\cdot\xi}, \quad \xi\in\mathbb{R}^s.$$

It follows that

$$(-iD)^{\nu}\hat{v}(0) = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)(-\alpha)^{\nu}.$$

Hence,

$$\frac{(-iD)^{\mu}(y\hat{v})(0)}{\mu!} = \sum_{\alpha\in\mathbb{Z}^s}\sum_{\nu\le\mu} \frac{(-iD)^{\mu-\nu}y(0)}{(\mu-\nu)!}\,\frac{(-\alpha)^{\nu}}{\nu!}\,v(\alpha) = \sum_{\alpha\in\mathbb{Z}^s} u_\mu(-\alpha)v(\alpha) = \langle u_\mu, v\rangle,$$

where u_μ is the element in $\ell^{1\times r}(\mathbb{Z}^s)$ as defined in (1.15). Consequently, v lies in H_j if and only if $\langle u_\mu, v\rangle = 0$ for all $|\mu| = j$. In other words, $H_j = G_j^{\perp}$, where

$$G_j := \operatorname{span}\{u_\mu : |\mu| = j\}.$$

Let $U_k := \operatorname{span}\{u_\mu : |\mu| < k\}$ and

$$V_k := \{v \in \ell_0^r(\mathbb{Z}^s) : D^{\mu}(y\hat{v})(0) = 0 \;\forall\, |\mu| < k\}. \tag{1.18}$$

Then $V_k = U_k^{\perp}$. Moreover,

$$U_k = G_0 + G_1 + \cdots + G_{k-1} \quad\text{and}\quad V_k = H_0 \cap H_1 \cap \cdots \cap H_{k-1}.$$

We may write u_μ as $\sum_{\nu\le\mu} y_{\mu-\nu}q_\nu$, where $y_{\mu-\nu} := (-iD)^{\mu-\nu}y(0)/(\mu-\nu)!$ and q_ν is the sequence given by $q_\nu(\alpha) = \alpha^{\nu}/\nu!$, $\alpha\in\mathbb{Z}^s$. If $y_0 \ne 0$, then the set $\{u_\mu : |\mu| < k\}$ is linearly independent. To justify our claim, let c_μ $(|\mu| < k)$ be complex numbers such that $\sum_{|\mu|<k} c_\mu u_\mu = 0$. It follows that

$$\sum_{|\mu|=k-1} c_\mu y_0 q_\mu + \sum_{|\nu|<k-1} h_\nu q_\nu = 0,$$

where h_ν $(|\nu| < k-1)$ are some elements in $\mathbb{C}^{1\times r}$. Since q_μ $(|\mu| < k)$ are linearly independent, we have $c_\mu y_0 = 0$ for all $|\mu| = k-1$. But $y_0 \ne 0$. Hence, $c_\mu = 0$ for all $|\mu| = k-1$. By using this argument repeatedly, we see that $c_\mu = 0$ for all $|\mu| = j$, $j = k-1, k-2, \ldots, 0$. This shows that $\{u_\mu : |\mu| < k\}$ is linearly independent. Consequently, $\{u_\mu : |\mu| = j\}$ is a basis for G_j $(j < k)$.

For $\gamma\in\mathbb{Z}^s$, the difference operator ∇_γ on the space $\ell^{m\times n}(\mathbb{Z}^s)$ is defined by

$$\nabla_\gamma u = u - u(\cdot - \gamma), \quad u \in \ell^{m\times n}(\mathbb{Z}^s).$$

Let us consider $\nabla_\gamma u_\mu$. For $\alpha\in\mathbb{Z}^s$ we have

$$u_\mu(\alpha) - u_\mu(\alpha-\gamma) = \sum_{\nu\le\mu} \frac{1}{\nu!}\big[\alpha^{\nu} - (\alpha-\gamma)^{\nu}\big] y_{\mu-\nu} = \sum_{\nu\le\mu}\sum_{0<\tau\le\nu} -\frac{1}{\nu!}\binom{\nu}{\tau}(-\gamma)^{\tau}\alpha^{\nu-\tau}y_{\mu-\nu}.$$

It follows that

$$\nabla_\gamma u_\mu(\alpha) = \sum_{0<\tau\le\mu} -\frac{(-\gamma)^{\tau}}{\tau!}\sum_{\tau\le\nu\le\mu}\frac{\alpha^{\nu-\tau}}{(\nu-\tau)!}y_{(\mu-\tau)-(\nu-\tau)} = \sum_{0<\tau\le\mu} -\frac{(-\gamma)^{\tau}}{\tau!}u_{\mu-\tau}(\alpha). \tag{1.19}$$

Consequently, $\nabla_\gamma u_\mu \in \operatorname{span}\{u_\nu : \nu < \mu\}$.

For $\mu \in \mathbb{N}_0^s$, recall that q_μ is the sequence given by $q_\mu(\alpha) = \alpha^\mu/\mu!$, $\alpha \in \mathbb{Z}^s$. When $\mu \in \mathbb{Z}^s \setminus \mathbb{N}_0^s$, we agree that $q_\mu = 0$. With this convention, we may interpret $D_j q_\mu$ as $q_{\mu - e_j}$. For $\gamma = (\gamma_1, \ldots, \gamma_s) \in \mathbb{Z}^s$, let $D_\gamma := \gamma_1 D_1 + \cdots + \gamma_s D_s$. Then it follows from (1.19) that

$$\nabla_\gamma u_\mu - D_\gamma u_\mu \in \operatorname{span}\{u_\nu : |\nu| \le |\mu| - 2\}.$$

Let Γ be a finite multiset of elements in $\mathbb{Z}^s$. If $\#\Gamma \ge |\mu|$, then the above relation yields

$$\Bigl(\prod_{\gamma\in\Gamma} \nabla_\gamma\Bigr) u_\mu = \Bigl(\prod_{\gamma\in\Gamma} D_\gamma\Bigr) u_\mu. \tag{1.20}$$

Moreover, both sides of the above equation vanish when $\#\Gamma > |\mu|$.

For $j = 1, \ldots, s$, the difference operator ∇_{e_j} is abbreviated as ∇_j. For a multi-index $\tau = (\tau_1, \ldots, \tau_s) \in \mathbb{N}_0^s$, the difference operator ∇^τ is defined as $\nabla_1^{\tau_1} \cdots \nabla_s^{\tau_s}$. As a consequence of (1.20) we have

$$\nabla^\tau u_\mu = D^\tau u_\mu = \delta_{\tau\mu} u_0 \quad \text{for } |\tau| \ge |\mu|. \tag{1.21}$$

Furthermore, it follows from (1.20) that

$$\nabla_{Me_1}^{\tau_1} \cdots \nabla_{Me_s}^{\tau_s} u_\mu = D_{Me_1}^{\tau_1} \cdots D_{Me_s}^{\tau_s} u_\mu \quad \text{for } |\tau| \ge |\mu|. \tag{1.22}$$

Suppose $Me_n = m_{n1} e_1 + \cdots + m_{ns} e_s$ with suitable coefficients m_{nj}, $n, j = 1, \ldots, s$. Then for $|\tau| = j$ we have

$$D_{Me_1}^{\tau_1} \cdots D_{Me_s}^{\tau_s} = \prod_{n=1}^{s} \bigl(m_{n1} D_1 + \cdots + m_{ns} D_s\bigr)^{\tau_n} =: \sum_{|\nu|=j} b_{\tau\nu} D^\nu.$$

Since $\operatorname{spec}(M) = \{\sigma_1, \ldots, \sigma_s\}$, the spectrum of the matrix $(b_{\tau\nu})_{|\tau|=j, |\nu|=j}$ is $\{\sigma^\mu : |\mu| = j\}$ (see [2, Lemma 4.2]). In light of (1.21), (1.22) yields

$$\nabla_{Me_1}^{\tau_1} \cdots \nabla_{Me_s}^{\tau_s} u_\mu = \sum_{|\nu|=j} b_{\tau\nu} D^\nu u_\mu = b_{\tau\mu} u_0 \quad \text{for } |\tau| \ge |\mu|, \tag{1.23}$$

where $b_{\tau\mu}$ is understood to be 0 if $|\tau| > |\mu|$.

Lemma 3.2. *Under the conditions in Lemma 3.1, the linear space G_j $(j < k)$ is invariant under the subdivision operator S_a. If, in addition, $y(0) \ne 0$, then*

$$\operatorname{spec}(S_a|_{G_j}) = \{\sigma^{-\mu} : |\mu| = j\}.$$

Proof. Note that $y_0 = y(0)$ and $u_0 = y_0 q_0$, where $q_0(\alpha) = 1$ for all $\alpha \in \mathbb{Z}^s$. Since a satisfies the basic sum rule with respect to y_0, (1.16) is valid. Hence, for $\alpha \in \mathbb{Z}^s$ we have

$$S_a u_0(\alpha) = \sum_{\beta\in\mathbb{Z}^s} u_0(\beta) a(\alpha - M\beta) = y_0 \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta) = y_0 = u_0(\alpha).$$

This shows $S_a u_0 = u_0$.

Since $G_j^\perp = H_j$ and H_j is invariant under T_a, the linear space G_j is invariant under S_a, by Lemma 2.1. Thus, there exist complex numbers $c_{\mu\nu}$ such that

$$S_a u_\mu = \sum_{|\nu|=j} c_{\mu\nu} u_\nu, \quad |\mu| = j.$$

Let C denote the matrix $(c_{\mu\nu})_{|\mu|=j,|\nu|=j}$. Then $\operatorname{spec}(S_a|_{G_j}) = \operatorname{spec}(C)$.

For $\gamma \in \mathbb{Z}^s$, it can be easily verified that

$$S_a(\nabla_\gamma u_\mu) = \nabla_{M\gamma}(S_a u_\mu).$$

Consequently, for $\tau = (\tau_1, \ldots, \tau_s) \in \mathbb{N}_0^s$ we have

$$S_a(\nabla^\tau u_\mu) = \nabla_{Me_1}^{\tau_1} \cdots \nabla_{Me_s}^{\tau_s}(S_a u_\mu) = \sum_{|\nu|=j} c_{\mu\nu} \big(\nabla_{Me_1}^{\tau_1} \cdots \nabla_{Me_s}^{\tau_s}\big) u_\nu.$$

In light of (1.20) and (1.23), it follows that

$$\delta_{\tau\mu} u_0 = \sum_{|\nu|=j} c_{\mu\nu} b_{\tau\nu} u_0.$$

Hence, $C = (B^T)^{-1}$, where B denotes the matrix $(b_{\tau\nu})_{|\tau|=j,|\nu|=j}$. But the spectrum of B is $\{\sigma^\mu : |\mu| = j\}$. Therefore, $\operatorname{spec}(C) = \{\sigma^{-\mu} : |\mu| = j\}$. This completes the proof. ∎

Recall that U_k is the direct sum of $G_0, \ldots, G_{k-1}$ and $V_k = U_k^\perp$. Hence, we have the following result.

LEMMA 3.3. *Under the conditions in Lemma 3.1, V_k is invariant under the transition operator T_a and U_k is invariant under the subdivision operator S_a. If, in addition, $y(0) \neq 0$, then*

$$\operatorname{spec}(S_a|_{U_k}) = \{\sigma^{-\mu} : |\mu| < k\}.$$

4. Characterization of smoothness. In this section we give a characterization for the smoothness of a refinable vector of functions in terms of the corresponding mask.

Sobolev spaces are related to Lipschitz spaces, which are defined on the basis of the modulus of smoothness. The **modulus of continuity** of a function f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega(f, h)_p := \sup_{|t|\le h} \left\|\nabla_t f\right\|_p, \qquad h \ge 0,$$

where $\nabla_t f := f - f(\cdot - t)$. Let k be a positive integer. The kth **modulus of smoothness** of $f \in L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f, h)_p := \sup_{|t|\le h} \left\|\nabla_t^k f\right\|_p, \qquad h \ge 0.$$

For $1 \le p \le \infty$ and $0 < \lambda \le 1$, the Lipschitz space $\mathrm{Lip}(\lambda, L_p(\mathbb{R}^s))$ consists of all functions $f \in L_p(\mathbb{R}^s)$ for which

$$\omega(f,h)_p \le C\, h^\lambda \qquad \forall\, h > 0,$$

where C is a positive constant independent of h. For $\lambda > 0$ we write $\lambda = m + \eta$, where m is an integer and $0 < \eta \le 1$. The Lipschitz space $\mathrm{Lip}(\lambda, L_p(\mathbb{R}^s))$ consists of those functions $f \in L_p(\mathbb{R}^s)$ for which $D^\mu f \in \mathrm{Lip}(\eta, L_p(\mathbb{R}^s))$ for all multi-indices μ with $|\mu| = m$. For $\lambda > 0$, let k be an integer greater than λ. The generalized Lipschitz space $\mathrm{Lip}^*(\lambda, L_p(\mathbb{R}^s))$ consists of those functions $f \in L_p(\mathbb{R}^s)$ for which

$$\omega_k(f,h)_p \le C\, h^\lambda \qquad \forall\, h > 0,$$

where C is a positive constant independent of h. If $\lambda > 0$ is not an integer, then

$$\mathrm{Lip}(\lambda, L_p(\mathbb{R}^s)) = \mathrm{Lip}^*(\lambda, L_p(\mathbb{R}^s)), \qquad 1 \le p \le \infty.$$

See [11, Chap. 2] for a discussion about Lipschitz spaces.

It is well known that, for $\lambda > \varepsilon > 0$, the inclusion relations

$$\mathrm{Lip}(\lambda, L_2(\mathbb{R}^s)) \subseteq \mathrm{Lip}^*(\lambda, L_2(\mathbb{R}^s)) \subseteq \mathrm{Lip}(\lambda - \varepsilon, L_2(\mathbb{R}^s))$$

and

$$W_2^\lambda(\mathbb{R}^s) \subseteq \mathrm{Lip}(\lambda, L_2(\mathbb{R}^s)) \subseteq W_2^{\lambda-\varepsilon}(\mathbb{R}^s)$$

hold true. See [43, Chap. V] for these facts. Therefore, we have

$$\lambda(f) = \sup\{\lambda : f \in \mathrm{Lip}(\lambda, L_2(\mathbb{R}^s))\} = \sup\{\lambda : f \in \mathrm{Lip}^*(\lambda, L_2(\mathbb{R}^s))\}.$$

The inner product of two functions $f, g \in L_2(\mathbb{R}^s)$ is defined as

$$\langle f, g\rangle := \int_{\mathbb{R}^s} f(x)\, \overline{g(x)}\, dx.$$

This definition still makes sense if f is a compactly supported function in $L_2(\mathbb{R}^s)$ and g is a polynomial on $\mathbb{R}^s$.

By $(L_p(\mathbb{R}^s))^r$ we denote the linear space of all $r \times 1$ vectors $F = (f_1, \ldots, f_r)^T$ such that $f_1, \ldots, f_r \in L_p(\mathbb{R}^s)$. This space is equipped with the norm given by

$$\|F\|_p := \left(\sum_{j=1}^r \|f_j\|_p^p \right)^{1/p}, \quad F = (f_1, \ldots, f_r)^T \in (L_p(\mathbb{R}^s))^r.$$

Suppose $u \in \ell^{m\times n}(\mathbb{Z}^s)$ and $u(\alpha) = (u_{jk}(\alpha))_{1\le j\le m, 1\le k\le n}$ for $\alpha \in \mathbb{Z}^s$. We define

$$\|u\|_p := \left(\sum_{\alpha\in\mathbb{Z}^s} \sum_{1\le j\le m} \sum_{1\le k\le n} |u_{jk}(\alpha)|^p \right)^{1/p}, \quad 1 \le p \le \infty.$$

Let Φ be an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R}^s)$. Suppose Φ satisfies the refinement equation (1.1). We claim that

$$\Phi = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\Phi(M^n \cdot - \alpha), \tag{1.24}$$

where the sequences a_n are given by $a_1 = a$ and, for $n = 2, 3, \ldots$,

$$a_n(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{n-1}(\beta)a(\alpha - M\beta), \quad \alpha \in \mathbb{Z}^s. \tag{1.25}$$

This can be proved by induction on n. Indeed, (1.24) is valid for $n = 1$. Suppose (1.24) holds true for $n - 1$. Then we have

$$\Phi = \sum_{\beta \in \mathbb{Z}^s} a_{n-1}(\beta)\Phi(M^{n-1} \cdot - \beta) = \sum_{\beta \in \mathbb{Z}^s} a_{n-1}(\beta) \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\Phi(M^n \cdot - M\beta - \alpha).$$

It follows that

$$\Phi = \sum_{\alpha \in \mathbb{Z}^s} \left(\sum_{\beta \in \mathbb{Z}^s} a_{n-1}(\beta)a(\alpha - M\beta) \right) \Phi(M^n \cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\Phi(M^n \cdot - \alpha).$$

This completes the induction procedure.

Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R}^s)$ satisfying the refinement equation (1.1) with a being the mask. Recall that $d = |\det M|$. Suppose a satisfies the sum rules of order k with respect to $y \in \mathbb{T}^{1\times r}(\mathbb{R}^s)$ satisfying (1.13). Let

$$V_k := \left\{ v \in \ell_0^r(\mathbb{Z}^s) : D^\mu(y\hat{v})(0) = 0 \;\forall\, |\mu| < k \right\}.$$

THEOREM 4.1. *If for every $v \in V_k$ there exists a positive constant C_v independent of n such that*

$$\|a_n * v\|_2 \le C_v d^{(1/2-\lambda/s)n} \qquad \forall\, n \in \mathbb{N}, \tag{1.26}$$

then $\Phi \in (Lip^*(\lambda, L_2(\mathbb{R}^s)))^r$. Conversely, if $\Phi \in (Lip(\lambda, L_2(\mathbb{R}^s)))^r$, and if Φ is stable, then (1.26) is valid for $v \in V_k$ and $k > \lambda$.

Proof. Recall that $e_1, \ldots, e_s$ are the coordinate unit vectors in $\mathbb{R}^s$. If there exists a constant C such that

$$\left\| \nabla^k_{M^{-n}e_j} \Phi \right\|_2 \le C d^{(-\lambda/s)n} \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad j = 1, \ldots, s, \tag{1.27}$$

then [21, Theorem 2.1] tells us that Φ lies in $(\mathrm{Lip}^*(\lambda, L_2(\mathbb{R}^s)))^r$.

It follows from (1.24) that

$$\nabla_{M^{-n}e_j}\Phi = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\left[\Phi(M^n \cdot - \alpha) - \Phi(M^n \cdot - \alpha - e_j)\right] = \sum_{\alpha \in \mathbb{Z}^s} \nabla_j a_n(\alpha)\Phi(M^n \cdot - \alpha).$$

Applying the difference operator $\nabla_{M^{-n}e_j}$ to (1.24) repeatedly, we obtain

$$\nabla^k_{M^{-n}e_j}\Phi = \sum_{\alpha\in\mathbb{Z}^s} \nabla^k_j a_n(\alpha)\Phi(M^n\cdot-\alpha).$$

Since Φ is compactly supported, it follows that

$$\big\|\nabla^k_{M^{-n}e_j}\Phi\big\|_2 \le Cd^{-n/2}\big\|\nabla^k_j a_n\big\|_2,$$

where C is a constant independent of n. For $m=1,\ldots,r$, let v_m be the element in $\ell_0^r(\mathbb{Z}^s)$ such that $v_m(\alpha)=0$ for all $\alpha\in\mathbb{Z}^s\setminus\{0\}$ and $v_m(0)$ is the mth column of the $r\times r$ identity matrix. We have

$$\big\|\nabla^k_j a_n\big\|_2 \le \sum_{m=1}^r \big\|(\nabla^k_j a_n)*v_m\big\|_2 = \sum_{m=1}^r \big\|a_n*(\nabla^k_j v_m)\big\|_2.$$

We observe that $(\nabla^k_j v_m)\hat{}(\xi) = (1-e^{-i\xi_j})^k\hat{v}_m(\xi)$ for $\xi=(\xi_1,\ldots,\xi_s)\in\mathbb{R}^s$. Hence, for $|\mu|<k$, $D^\mu\big(y(\nabla^k_j v_m)\hat{}\big)(0)=0$ with y as in (1.13). In other words, $\nabla^k_j v_m\in V_k$, $m=1,\ldots,r$.

If (1.26) is valid, then

$$\big\|a_n*(\nabla^k_j v_m)\big\|_2 \le C_m d^{(1/2-\lambda/s)n} \quad \forall\, n\in\mathbb{N},$$

where C_m is a constant independent of n. Combining the above estimates together, we obtain the desired estimate (1.27). Therefore, $\Phi\in(\mathrm{Lip}^*(\lambda,L_2(\mathbb{R}^s)))^r$.

Now suppose $\Phi=(\phi_1,\ldots,\phi_r)^T\in(\mathrm{Lip}(\lambda,L_2(\mathbb{R}^s)))^r$ and Φ is stable. We wish to show that (1.26) is true. For this purpose, we shall use approximation schemes induced by quasi-projection operators (see [35] and [23]).

For $\nu\in\mathbb{N}_0^s$, let q_ν be the monomial given by $q_\nu(x):=x^\nu/\nu!$, $x\in\mathbb{R}^s$. Recall that $y_\nu=(-iD)^\nu y(0)/\nu!$. Each y_ν is a $1\times r$ vector $(y_{\nu 1},\ldots,y_{\nu r})$. There exist real-valued compactly supported functions $g_1,\ldots,g_r$ in $L_2(\mathbb{R}^s)$ such that

$$\langle q_\nu, g_j\rangle = y_{\nu j} \qquad \forall\, |\nu|<k \text{ and } j=1,\ldots,r.$$

For $|\mu|<k$ and $\alpha\in\mathbb{Z}^s$ we have

$$\begin{aligned}\langle q_\mu, g_j(\cdot-\alpha)\rangle = \langle q_\mu(\cdot+\alpha), g_j\rangle &= \sum_{\nu\le\mu}\int_{\mathbb{R}^s}\frac{1}{\mu!}\binom{\mu}{\nu}x^{\mu-\nu}\alpha^\nu g_j(x)\,dx\\ &= \sum_{\nu\le\mu}\frac{\alpha^\nu}{\nu!}\int_{\mathbb{R}^s}\frac{x^{\mu-\nu}}{(\mu-\nu)!}\,g_j(x)\,dx = \sum_{\nu\le\mu}\frac{\alpha^\nu}{\nu!}y_{\mu-\nu,j}.\end{aligned}$$

Let P_Φ be the quasi-projection operator given by

$$P_\Phi f := \sum_{\alpha\in\mathbb{Z}^s}\sum_{j=1}^r\langle f, g_j(\cdot-\alpha)\rangle\phi_j(\cdot-\alpha), \quad f\in L_2(\mathbb{R}^s).$$

For $|\mu| < k$ we have

$$P_\Phi q_\mu = \sum_{\alpha\in\mathbb{Z}^s}\sum_{j=1}^{r}\langle q_\mu, g_j(\cdot-\alpha)\rangle\phi_j(\cdot-\alpha) = \sum_{\alpha\in\mathbb{Z}^s}\sum_{\nu\le\mu}\frac{\alpha^\nu}{\nu!}y_{\mu-\nu}\Phi(\cdot-\alpha) = q_\mu,$$

where (1.14) has been used to derive the last equality. Thus, P_Φ reproduces all polynomials of degree at most $k-1$, *i.e.*, $P_\Phi q = q$ for all $q \in \Pi_{k-1}$. Consequently, for $f \in \mathrm{Lip}(\lambda, L_2(\mathbb{R}^s))$ $(0 < \lambda < k)$ we have

$$\left\| f - \sum_{\alpha\in\mathbb{Z}^s}\sum_{j=1}^{r}\langle f, d^n g_j(M^n\cdot-\alpha)\rangle\phi_j(M^n\cdot-\alpha)\right\|_2 \le C(d^{-1/s})^{\lambda n} \quad \forall\, n \in \mathbb{N}, \tag{1.28}$$

where C is a constant independent of n (see [23]).

Let v be an element in V_k, and let

$$H(x) := \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)h(x-\alpha), \quad x \in \mathbb{R}^s,$$

where h is a compactly supported continuous function on $\mathbb{R}^s$ such that the shifts of h are stable, and $D^\mu\hat{h}(2\pi\beta) = 0$ for all $|\mu| < k$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$. By our choice of H, we have

$$D^\mu(y\hat{H})(2\pi\beta) = D^\mu(y\hat{v}\hat{h})(2\pi\beta) = 0 \quad \forall\, |\mu| < k \text{ and } \beta \in \mathbb{Z}^s.$$

Let $\Psi := \Phi + H$. Taking (1.13) into account, we obtain

$$D^\mu(y\hat{\Psi})(2\pi\beta) = D^\mu(y\hat{\Phi})(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \quad \forall\, |\mu| < k \text{ and } \beta \in \mathbb{Z}^s.$$

Suppose $\Psi = (\psi_1, \ldots, \psi_r)^T$. Let P_Ψ be the quasi-projection operator given by

$$P_\Psi f := \sum_{\alpha\in\mathbb{Z}^s}\sum_{j=1}^{r}\langle f, g_j(\cdot-\alpha)\rangle\psi_j(\cdot-\alpha), \quad f \in L_2(\mathbb{R}^s).$$

Then P_Ψ also reproduces all polynomials of degree at most $k-1$.

For $n = 1, 2, \ldots$, let c_n be the sequence of $r \times r$ matrices given by

$$c_n(\alpha) := \big(\langle \phi_j, d^n g_m(M^n\cdot-\alpha)\rangle\big)_{1\le j,m\le r}, \quad \alpha \in \mathbb{Z}^s.$$

Suppose $\Phi \in (\mathrm{Lip}(\lambda, L_2(\mathbb{R}^s)))^r$ and $0 < \lambda < k$. Since $P_\Phi q = P_\Psi q = q$ for all $q \in \Pi_{k-1}$, the estimate in (1.28) tells us that there exists a positive constant C_1 such that

$$\left\| \Phi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\Phi(M^n\cdot-\alpha)\right\|_2 \le C_1(d^{-1/s})^{\lambda n} \quad \forall\, n \in \mathbb{N} \tag{1.29}$$

and

$$\Big\|\Phi - \sum_{\alpha\in\mathbb{Z}^s} c_n(\alpha)\Psi(M^n\cdot - \alpha)\Big\|_2 \le C_1(d^{-1/s})^{\lambda n} \quad \forall\, n \in \mathbb{N}. \tag{1.30}$$

It follows from (1.24) and (1.29) that

$$\Big\|\sum_{\alpha\in\mathbb{Z}^s} (a_n - c_n)(\alpha)\Phi(M^n\cdot - \alpha)\Big\|_2 \le C_1(d^{-1/s})^{\lambda n} \quad \forall\, n \in \mathbb{N}.$$

Since Φ is stable, we deduce from the above estimate that

$$\|a_n - c_n\|_2 \le C_2 d^{(1/2-\lambda/s)n} \quad \forall\, n \in \mathbb{N},$$

where C_2 is a constant independent of n. This in connection with (1.30) gives

$$\Big\|\Phi - \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\Psi(M^n\cdot - \alpha)\Big\|_2 \le C_3(d^{-1/s})^{\lambda n} \quad \forall\, n \in \mathbb{N},$$

where C_3 is a constant independent of n. But $\Psi = \Phi + H$. So the above inequality together with (1.24) yields

$$\Big\|\sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)H(M^n\cdot - \alpha)\Big\|_2 \le C_3(d^{-1/s})^{\lambda n} \quad \forall\, n \in \mathbb{N}.$$

But

$$\sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)H(M^n\cdot-\alpha) = \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} a_n(\alpha)v(\beta)h(M^n\cdot-\alpha-\beta) = \sum_{\gamma\in\mathbb{Z}^s} (a_n*v)(\gamma)h(M^n\cdot-\gamma).$$

Consequently,

$$\Big\|\sum_{\gamma\in\mathbb{Z}^s} (a_n*v)(\gamma)h(M^n\cdot - \gamma)\Big\|_2 \le C_3(d^{-1/s})^{\lambda n} \quad \forall\, n \in \mathbb{N}.$$

Since the shifts of h are stable, there exists a constant C_v such that (1.26) holds true. ∎

5. Spectral radius. In order to apply the results in the previous section to smoothness analysis of refinable vectors of functions we need to evaluate the limit

$$\lim_{n\to\infty} \|a_n*v\|_2^{1/n}.$$

In this section we shall show that this limit can be evaluated as the spectral radius of a certain (finite) matrix. Some ideas in [13], [22], and [25] will be employed in our discussion.

For $u, v \in \ell_0^r(\mathbb{Z}^s)$, we define $u \odot v^T$ as follows:

$$u \odot v^T(\alpha) := \sum_{\beta \in \mathbb{Z}^s} u(\alpha + \beta)\overline{v(\beta)}^T, \quad \alpha \in \mathbb{Z}^s.$$

Let $u_n := a_n * u$ and $v_n := a_n * v$, where a_n $(n = 1, 2, \ldots)$ are the sequences given in (1.25). Moreover, let $w := \mathrm{vec}(u \odot v^T)$ and $w_n := \mathrm{vec}(u_n \odot v_n^T)$. For $\alpha \in \mathbb{Z}^s$, we have

$$u_n \odot v_n^T(\alpha) = \sum_{\beta \in \mathbb{Z}^s} u_n(\alpha + \beta)\overline{v_n(\beta)}^T = \sum_{\beta \in \mathbb{Z}^s} \sum_{\gamma \in \mathbb{Z}^s} \sum_{\eta \in \mathbb{Z}^s} a_n(\alpha + \beta - \gamma)u(\gamma)\overline{v(\eta)}^T \overline{a_n(\beta - \eta)}^T.$$

It follows by (1.5) that

$$w_n(\alpha) = \sum_{\gamma \in \mathbb{Z}^s} \Bigl(\sum_{\beta \in \mathbb{Z}^s} \overline{a_n(\beta)} \otimes a_n(\alpha + \beta - \gamma) \Bigr) \Bigl(\sum_{\eta \in \mathbb{Z}^s} \mathrm{vec}\bigl(u(\gamma + \eta)\overline{v(\eta)}^T \bigr) \Bigr).$$

Let b_n $(n = 1, 2, \ldots)$ be the sequences given by

$$b_n(\alpha) := \frac{1}{d^n} \sum_{\beta \in \mathbb{Z}^s} \overline{a_n(\beta)} \otimes a_n(\alpha + \beta), \quad \alpha \in \mathbb{Z}^s. \tag{1.31}$$

Consequently,

$$\mathrm{vec}\bigl((a_n * u) \odot (a_n * v)^T \bigr) = d^n b_n * \bigl(\mathrm{vec}\,(u \odot v^T) \bigr). \tag{1.32}$$

Clearly, b_1 is the same as the sequence b given in (1.6). Furthermore, for $n > 1$, it follows from (1.31) and (1.25) that

$$\begin{aligned} d^n b_n(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} \sum_{\eta \in \mathbb{Z}^s} \sum_{\gamma \in \mathbb{Z}^s} \bigl(\overline{a_{n-1}(\eta) a(\beta - M\eta)} \bigr) \otimes \bigl(a_{n-1}(\gamma) a(\alpha + \beta - M\gamma) \bigr) \\ &= \sum_{\gamma \in \mathbb{Z}^s} \Bigl(\sum_{\eta \in \mathbb{Z}^s} \overline{a_{n-1}(\eta)} \otimes a_{n-1}(\eta + \gamma) \Bigr) \Bigl(\sum_{\beta \in \mathbb{Z}^s} \overline{a(\beta)} \otimes a(\alpha + \beta - M\gamma) \Bigr). \end{aligned}$$

It follows that

$$b_n(\alpha) = \sum_{\gamma \in \mathbb{Z}^s} b_{n-1}(\gamma) b(\alpha - M\gamma), \quad \alpha \in \mathbb{Z}^s. \tag{1.33}$$

THEOREM 5.1. *Let $a \in \ell_0^{r \times r}(\mathbb{Z}^s)$, and let a_n $(n = 1, 2, \ldots)$ be given as in (1.25). Then for $v \in \ell_0^r(\mathbb{Z}^s)$,*

$$\lim_{n \to \infty} \|a_n * v\|_2^{1/n} = \sqrt{d\rho(T_b|_W)},$$

where b is the sequence given in (1.6) and W is the minimal invariant subspace of the transition operator T_b generated by $w := \mathrm{vec}(v \odot v^T)$.

Proof. We first establish the following identity for $w \in \ell_0^{r^2}(\mathbb{Z}^s)$:

$$T_b^n w(\alpha) = \sum_{\beta \in \mathbb{Z}^s} b_n(M^n \alpha - \beta) w(\beta), \quad \alpha \in \mathbb{Z}^s. \tag{1.34}$$

This will be proved by induction on n. By the definition of the transition operator T_b, (1.34) is true for $n = 1$. Suppose $n > 1$ and (1.34) is valid for $n-1$. For $\alpha \in \mathbb{Z}^s$ we have

$$\begin{aligned} T_b^n w(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} b_{n-1}(M^{n-1}\alpha - \beta)(T_b w)(\beta) \\ &= \sum_{\beta \in \mathbb{Z}^s} \sum_{\gamma \in \mathbb{Z}^s} b_{n-1}(M^{n-1}\alpha - \beta) b(M\beta - \gamma) w(\gamma) \\ &= \sum_{\gamma \in \mathbb{Z}^s} \left[\sum_{\beta \in \mathbb{Z}^s} b_{n-1}(\beta) b(M^n\alpha - \gamma - M\beta) \right] w(\gamma) \\ &= \sum_{\gamma \in \mathbb{Z}^s} b_n(M^n\alpha - \gamma) w(\gamma), \end{aligned}$$

where (1.33) has been used to derive the last equality. This completes the induction procedure.

Let v be an element in $\ell_0^r(\mathbb{Z}^s)$ and let $w := \mathrm{vec}(v \odot v^T)$. For $n \in \mathbb{N}$, let $v_n := a_n * v$ and $w_n := \mathrm{vec}(v_n \odot v_n^T)$. Then $w_n = d^n b_n * w$, by (1.32). This together with (1.34) yields

$$d^n T_b^n w(\alpha) = d^n b_n * w(M^n\alpha) = w_n(M^n\alpha), \quad \alpha \in \mathbb{Z}^s.$$

Since $w_n = \mathrm{vec}(v_n \odot v_n^T)$, we have

$$d^n \|T_b^n w\|_\infty \le \|w_n\|_\infty \le \|v_n\|_2^2.$$

On the other hand,

$$d^n T_b^n w(0) = w_n(0) = \mathrm{vec}\left(\sum_{\beta \in \mathbb{Z}^s} v_n(\beta) \overline{v_n(\beta)}^T \right).$$

Consequently,

$$\|v_n\|_2^2 \le r d^n \|T_b^n w\|_\infty \le r \|v_n\|_2^2.$$

Therefore,

$$\lim_{n\to\infty} \|a_n * v\|_2^{2/n} = \lim_{n\to\infty} \|v_n\|_2^{2/n} = d \lim_{n\to\infty} \|T_b^n w\|_\infty^{1/n} = d\rho(T_b|_W),$$

where W is the minimal invariant subspace of T_b generated by w. $\square$

Now suppose a satisfies the sum rules of order k with respect to $y \in \mathbb{T}^{1\times r}(\mathbb{R}^s)$. Let

$$W_k := \mathrm{span}\{\mathrm{vec}(u \odot v^T) : u, v \in V_k\},$$

where V_k is the linear space given in (1.18). By Lemma 3.3, V_k is invariant under the transition operator T_a. We claim that W_k is invariant under the transition operator T_b. Suppose $w = \mathrm{vec}(u \odot v^T)$, where $u, v \in V_k$. By (1.32) we have

$$T_b w(\alpha) = b * w(M\alpha) = \frac{1}{d} \mathrm{vec}\big((a*u) \odot (a*v)^T\big)(M\alpha), \quad \alpha \in \mathbb{Z}^s.$$

Let E be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$. Then we have

$$\begin{aligned}\big((a*u)\odot(a*v)^T\big)(M\alpha) &= \sum_{\beta\in\mathbb{Z}^s}(a*u)(M\alpha+\beta)\overline{(a*v)(\beta)}^T \\ &= \sum_{\eta\in E}\sum_{\gamma\in\mathbb{Z}^s}(a*u)(M\alpha+M\gamma+\eta)\overline{(a*v)(M\gamma+\eta)}^T.\end{aligned}$$

But

$$(a*u)(M\alpha+M\gamma+\eta)=T_a\big(u(\cdot+\eta)\big)(\alpha+\gamma),\quad \alpha\in\mathbb{Z}^s.$$

We observe that V_k is shift-invariant, *i.e.*, $u\in V_k$ implies $u(\cdot+\eta)\in V_k$ for $\eta\in\mathbb{Z}^s$. Since V_k is invariant under T_a, we see that $u_\eta := T_a(u(\cdot+\eta))$ lies in V_k. Similarly, $v_\eta := T_a(v(\cdot+\eta))$ lies in V_k. Consequently,

$$\big((a*u)\odot(a*v)^T\big)(M\alpha)=\sum_{\eta\in E}\sum_{\gamma\in\mathbb{Z}^s}u_\eta(\alpha+\gamma)\,\overline{v_\eta(\gamma)}^T,\quad \alpha\in\mathbb{Z}^s.$$

Therefore,

$$T_b w=\frac{1}{d}\sum_{\eta\in E}\mathrm{vec}(u_\eta\odot v_\eta^T)\in W_k.$$

This shows that W_k is invariant under T_b.

Let us consider the special case $k=1$. Suppose a satisfies the basic sum rule with respect to $y_0\neq 0$. In this case, it is easily seen that

$$V_1=\Big\{v\in \ell_0^r(\mathbb{Z}^s): y_0\sum\nolimits_{\alpha\in\mathbb{Z}^s}v(\alpha)=0\Big\}$$

and

$$W_1=\Big\{w\in \ell_0^{r^2}(\mathbb{Z}^s): (\overline{y_0}\otimes y_0)\sum\nolimits_{\alpha\in\mathbb{Z}^s}w(\alpha)=0\Big\}.$$

It was shown in [25] and [4] that the cascade algorithm associated with mask a converges in the L_2 norm if $\lim_{n\to\infty}\|a_n*v\|_2=0$ for each $v\in V_1$. Conversely, suppose $\Phi\in(L_2(\mathbb{R}^s))^r$ is a compactly supported solution to the refinement equation (1.1) and Φ is stable. Then the proof of Theorem 4.1 tells us that $\lim_{n\to\infty}\|a_n*v\|_2=0$ for each $v\in V_1$. Thus, we have the following result.

THEOREM 5.2. *Let $b\in\ell_0^{r^2}(\mathbb{Z}^s)$ be defined as in (1.6). If a satisfies the basic sum rule, and if $\rho(T_b|_{W_1})<1$, then there exists a compactly supported solution $\Phi\in(L_2(\mathbb{R}^s))^r$ to the refinement equation (1.1) with a being the mask. Conversely, if $\Phi\in(L_2(\mathbb{R}^s))^r$ is a compactly supported solution to the refinement equation (1.1) with a being the mask, and if Φ is stable, then a satisfies the basic sum rule and $\rho(T_b|_{W_1})<1$.*

We conclude this section with the following characterization of the critical exponent of Φ in terms of the mask.

THEOREM 5.3. *Let Φ be a $1 \times r$ vector of compactly supported functions in $L_2(\mathbb{R}^s)$ satisfying the refinement equation (1.1). Suppose the mask a satisfies the sum rules of order k and the matrix M is isotropic. Then*

$$\lambda(\Phi) \geq -\big(\log_d \rho(T_b|_{W_k})\big)\, s/2.$$

The equality holds true in the above relation if, in addition, Φ is stable and k is the largest integer such that $\mathbb{S}(\Phi) \supset \Pi_{k-1}$.

Proof. Let $v \in V_k$. Then $w := \mathrm{vec}(v \odot v^T)$ lies in W_k. By Theorem 5.1 we have

$$\lim_{n\to\infty} \|a_n * v\|_2^{2/n} \leq d\rho(T_b|_{W_k}).$$

Write ρ_k for $\rho(T_b|_{W_k})$. For $\varepsilon > 0$, there exists a positive constant C such that

$$\|a_n * v\|_2 \leq C d^{n/2} (\rho_k + \varepsilon)^{n/2} \quad \forall\, n \in \mathbb{N}.$$

Let

$$\lambda_\varepsilon := -(\log_d(\rho_k + \varepsilon))s/2.$$

Then the above inequality can be rewritten as

$$\|a_n * v\|_2 \leq C d^{(1/2 - \lambda_\varepsilon/s)n} \quad \forall\, n \in \mathbb{N}.$$

By Theorem 4.1, Φ lies in $(\mathrm{Lip}(\lambda_\varepsilon, L_2(\mathbb{R}^s)))^r$. Hence,

$$\lambda(\Phi) \geq \lambda_\varepsilon = -(\log_d(\rho_k + \varepsilon))s/2.$$

But $\varepsilon > 0$ could be arbitrarily small. Therefore, we obtain

$$\lambda(\Phi) \geq -(\log_d \rho_k)s/2.$$

Now suppose Φ is stable and k is the largest integer such that $\mathbb{S}(\Phi) \supset \Pi_{k-1}$. We must have $\lambda(\Phi) \leq k$, for otherwise $\lambda(\Phi) > k$ would imply $\mathbb{S}(\Phi) \supset \Pi_k$ (see [40] and [4]). Since Φ is stable and $\mathbb{S}(\Phi) \supset \Pi_{k-1}$, the corresponding mask a satisfies the sum rules of order k with respect to some $y \in \mathbb{T}^{1\times r}(\mathbb{R}^s)$. Let $\lambda_\varepsilon := \lambda(\Phi) - \varepsilon$, where $0 < \varepsilon < \lambda(\Phi)$. Then Φ lies in $(\mathrm{Lip}(\lambda_\varepsilon, L_2(\mathbb{R}^s)))^r$ and $k > \lambda_\varepsilon$. Note that $\rho(T_b|_{W_k}) = \rho(T_b|_{W_k \cap \ell^{r^2}(K)})$, where K is the set $\mathbb{Z}^s \cap \sum_{n=1}^\infty M^{-n}(\mathrm{supp}\, b)$. Since $W_k \cap \ell^{r^2}(K)$ is finite dimensional, we can find $u_j, v_j \in V_k$, $j = 1, \ldots, N$, such that

$$W_k \cap \ell^{r^2}(K) \subseteq \mathrm{span}\big\{\mathrm{vec}(u_j \odot v_j^T) : j = 1, \ldots, N\big\}.$$

Let $w_j := \mathrm{vec}(u_j \odot v_j^T) : j = 1, \ldots, N$. We have

$$\rho_k = \rho(T_b|_{W_k \cap \ell^{r^2}(K)}) \leq \max_{1\leq j\leq N} \Big\{\lim_{n\to\infty} \|T_b^n w_j\|_\infty^{1/n}\Big\}.$$

By (1.32) we have

$$d^n \|b_n * w_j\|_\infty \le \|a_n * u_j\|_2 \, \|a_n * v_j\|_2.$$

Thus, from the proof of Theorem 5.1 we obtain

$$\lim_{n\to\infty} \|T_b^n w_j\|_\infty^{1/n} \le d^{-1} \Big(\lim_{n\to\infty} \|a_n * u_j\|_2^{1/n}\Big) \Big(\lim_{n\to\infty} \|a_n * v_j\|_2^{1/n}\Big).$$

Since $\Phi \in (\mathrm{Lip}(\lambda_\varepsilon, L_2(\mathbb{R}^s)))^r$ with $\lambda_\varepsilon < k$, and since Φ is stable, by Theorem 4.1 we have

$$\lim_{n\to\infty} \|a_n * u_j\|_2^{1/n} \le d^{1/2-\lambda_\varepsilon/s} \quad \text{and} \quad \lim_{n\to\infty} \|a_n * v_j\|_2^{1/n} \le d^{1/2-\lambda_\varepsilon/s}.$$

Therefore,

$$\rho_k \le d^{-1} d^{1/2-\lambda_\varepsilon/s} d^{1/2-\lambda_\varepsilon/s} = d^{-2\lambda_\varepsilon/s}.$$

It follows that

$$\lambda(\Phi) - \varepsilon = \lambda_\varepsilon \le -(\log_d \rho_k) s/2.$$

But $\varepsilon > 0$ could be arbitrarily small. We conclude that $\lambda(\Phi) \le -(\log_d \rho_k)s/2$. This completes the proof. $\square$

6. Invariant subspaces. In the previous section, we reduced calculation of the critical exponent of a refinable vector of functions to the spectral radius of the transition operator T_b restricted to W_k. The purpose of this section is to find a basis for $W_k^\perp$. In this way, we will be able to apply Lemma 2.3 to calculate $\rho(T_b|_{W_k})$.

Let $y \in \mathbb{T}^{1\times r}(\mathbb{R}^s)$. Recall that

$$V_k = \{v \in \ell_0^r(\mathbb{Z}^s) : D^\mu(y\hat{v})(0) = 0 \;\forall\, |\mu| < k\}$$

and

$$W_k = \mathrm{span}\{\mathrm{vec}(u \odot v^T) : u \in V_k, v \in V_k\}. \tag{1.35}$$

For $\xi \in \mathbb{R}^s$, we have

$$\begin{aligned}(u \odot v^T)\hat{}(\xi) &= \sum_{\alpha\in\mathbb{Z}^s} (u\odot v^T)(\alpha) e^{-i\alpha\cdot\xi} = \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} u(\alpha+\beta)\overline{v(\beta)}^T e^{-i(\alpha+\beta)\cdot\xi} e^{i\beta\cdot\xi} \\ &= \sum_{\beta\in\mathbb{Z}^s}\Big(\sum_{\alpha\in\mathbb{Z}^s} u(\alpha+\beta) e^{-i(\alpha+\beta)\cdot\xi}\Big)\overline{v(\beta)e^{-i\beta\cdot\xi}}^T = \hat{u}(\xi)\,\overline{\hat{v}(\xi)}^T.\end{aligned}$$

Let us first consider the special case $r = 1$ and $y = 1$. In this case, we claim that

$$W_k = \{w \in \ell_0(\mathbb{Z}^s) : D^\mu \hat{w}(0) = 0 \;\forall\, |\mu| < 2k\}.$$

Indeed, if $u, v \in V_k$, then $D^\nu \hat{u}(0) = D^\nu \hat{v}(0) = 0$ for all $|\nu| < k$. Hence, $D^\mu(\hat{u}\overline{\hat{v}})(0) = 0$ for all $|\mu| < 2k$. Conversely, suppose $w \in \ell_0(\mathbb{Z}^s)$ and $h := \hat{w}$ satisfies $D^\mu h(0) = 0$ for all $|\mu| < 2k$. The following lemma tells us $w \in W_k$.

LEMMA 6.1. *Let h be a trigonometric polynomial on $\mathbb{R}^s$ such that $D^\mu h(0) = 0$ for all $|\mu| < 2k$. Then*

$$h \in \operatorname{span}\{g_1\overline{g_2} : g_1, g_2 \in \mathbb{T}(\mathbb{R}^s),\ D^\nu g_1(0) = D^\nu g_2(0) = 0\ \forall\, |\nu| < k\}.$$

Proof. For $\beta \in \mathbb{Z}^s$ we use δ_β to denote the sequence on $\mathbb{Z}^s$ given by $\delta_\beta(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s \setminus \{\beta\}$ and $\delta_\beta(\beta) = 1$. Let

$$V := \operatorname{span}\{\nabla^\mu \delta_\beta : |\mu| = 2k, \beta \in \mathbb{Z}^s\},$$

and let $U := V^\perp$. Suppose u is a polynomial sequence of degree at most $2k-1$. For $|\mu| = 2k$ we have

$$\langle u, \nabla^\mu \delta_\beta\rangle = \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\nabla^\mu\delta_\beta(-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} \nabla^\mu u(\alpha)\delta_\beta(-\alpha) = 0.$$

Hence, u lies in U. Conversely, if $u \in U$, then $\langle u, \nabla^\mu\delta_\beta\rangle = 0$ for all $|\mu| = 2k$ and $\beta \in \mathbb{Z}^s$. It follows that $\langle \nabla^\mu u, \delta_\beta\rangle = 0$ for all $\beta \in \mathbb{Z}^s$. Therefore, $\nabla^\mu u = 0$ for all $|\mu| = 2k$. This shows that u is a polynomial sequence of degree at most $2k-1$.

Suppose $h(\xi) = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)e^{-i\alpha\cdot\xi}$, where $v \in \ell_0(\mathbb{Z}^s)$. If $D^\mu h(0) = 0$ for all $|\mu| < 2k$, then

$$\sum_{\alpha\in\mathbb{Z}^s} (-i\alpha)^\mu v(\alpha) = 0 \quad \forall\, |\mu| < 2k.$$

Consequently, $\langle u, v\rangle = 0$ for every polynomial sequence u of degree at most $2k-1$. This shows $v \in U^\perp = (V^\perp)^\perp = V$. Thus, $h = \hat{v}$ lies in $\operatorname{span}\{(\nabla^\mu\delta_\beta)^\wedge : |\mu| = 2k, \beta \in \mathbb{Z}^s\}$. The symbol of $\nabla^\mu\delta_\beta$ is

$$(1 - e^{-i\xi_1})^{\mu_1}\cdots(1 - e^{-i\xi_s})^{\mu_s} e^{-i\beta\cdot\xi}, \quad \xi = (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s.$$

For $|\mu| = 2k$, this expression can be written as $g_1(\xi)\overline{g_2(\xi)}$, where g_1 and g_2 are trigonometric polynomials satisfying $D^\nu g_1(0) = D^\nu g_2(0) = 0$ for all $|\nu| < k$. ∎

The following lemma extends Lemma 6.1 to the general case.

LEMMA 6.2. *Suppose $y = (y_1, \ldots, y_r) \in \mathbb{T}^{1\times r}(\mathbb{R}^s)$ and $y(0) \neq 0$. Let*

$$G := \bigl\{g \in \mathbb{T}^r(\mathbb{R}^s) : D^\nu(yg)(0) = 0 \quad \forall\, |\nu| < k\bigr\},$$

and let H be the set of those $r\times r$ matrices h of trigonometrical polynomials for which $D^\nu(y\,h)(0) = D^\nu(h\,\overline{y}^T)(0) = 0$ for all $|\nu| < k$ and $D^\mu(y\,h\,\overline{y}^T)(0) = 0$ for all $k \le |\mu| < 2k$. Then

$$H = \operatorname{span}\{g_1\overline{g_2}^T : g_1, g_2 \in G\}.$$

Proof. We observe that both G and H are linear spaces. If $g_1, g_2 \in G$, then $h := g_1\overline{g_2}^T$ satisfies

$$D^\nu(yh)(0) = D^\nu(yg_1\overline{g_2}^T)(0) = 0 \quad \forall\, |\nu| < k$$

and

$$D^\nu(h\overline{y}^T)(0) = D^\nu(g_1\overline{g_2}^T\overline{y}^T)(0) = D^\nu(yg_2\overline{g_1}^T)(0) = 0 \quad \forall\, |\nu| < k.$$

Moreover,

$$D^\mu(yh\overline{y}^T)(0) = D^\mu(yg_1\overline{g_2}^T\overline{y}^T)(0) = D^\mu\big((yg_1)(\overline{yg_2})^T\big)(0) = 0 \quad \forall|\mu| < 2k.$$

Hence, $h = g_1\overline{g_2}^T \in H$ for all $g_1, g_2 \in G$.

Conversely, suppose $h = (h_{mn})_{1\le m,n\le r} \in H$. Then $D^\nu(yh)(0) = D^\nu(h\overline{y}^T)(0) = 0$ for all $|\nu| < k$. Consequently, $D^\nu(y\,(h_{1m}, \dots, h_{rm})^T)(0) = D^\nu((h_{m1}, \dots, h_{mr})\,\overline{y}^T)(0) = 0$ for each $m = 1, \dots, r$ and all $|\nu| < k$. Hence, $(h_{1m}, \dots, h_{rm})^T \in G$ and $\overline{(h_{m1}, \dots, h_{mr})}^T \in G$. Without loss of any generality, we may assume that $y_1(0) \ne 0$. Thus, for $m = 2, \dots, r$ we can find $u_m \in \mathbb{T}(\mathbb{R}^s)$ such that

$$D^\nu(y_1u_m + y_m)(0) = 0 \quad \forall\,|\nu| < k.$$

For $m = 2, \dots, r$, consider the vector $(u_m, 0, \dots, 0, 1, 0, \dots, 0)^T$, where 1 is in the mth position. In light of our choice of u_m, we have $(u_m, 0, \dots, 0, 1, 0, \dots, 0)^T \in G$. Let

$$h' := h - \sum_{m=2}^{r}(u_m, 0, \dots, 0, 1, 0, \dots, 0)^T(h_{m1}, \dots, h_{mr}).$$

Recall that $\overline{(h_{m1}, \dots, h_{mr})}^T \in G$. Therefore, h' lies in H. Moreover, for $m = 2, \dots, r$, the mth row of h' vanishes. Suppose the first row of h' is $(h'_{11}, h'_{12}, \dots, h'_{1r})$. Since $h' \in H$, we have $(h'_{1m}, 0, \dots, 0)^T \in G$ for $m = 1, \dots, r$. Let

$$h'' := h' - \sum_{m=2}^{r}(h'_{1m}, 0, \dots, 0)^T\overline{(u_m, 0, \dots, 0, 1, 0, \dots, 0)}.$$

Then $h'' \in H$. All the entries except the $(1,1)$-entry of the matrix h'' are zero. Let h''_{11} be the $(1,1)$-entry of h''. Since $h'' \in H$, we have $D^\nu(y_1h''_{11})(0) = 0$ for all $|\nu| < k$. Moreover, $D^\mu(|y_1|^2h''_{11})(0) = D^\mu(y_1h''_{11}\overline{y_1})(0) = 0$ for $k \le |\mu| < 2k$. But $y_1(0) \ne 0$. Hence, it follows that $D^\mu(h''_{11})(0) = 0$ for all $|\mu| < 2k$. By Lemma 6.1,

$$h''_{11} \in \operatorname{span}\{f_1\overline{f_2} : f_1, f_2 \in \mathbb{T}(\mathbb{R}^s), D^\nu f_1(0) = D^\nu f_2(0) = 0\;\forall\,|\nu| < k\}.$$

If $f_1, f_2 \in \mathbb{T}(\mathbb{R}^s)$ satisfy $D^\nu f_1(0) = D^\nu f_2(0) = 0$ for all $|\nu| < k$, then $g_1 := (f_1, 0, \dots, 0)^T$ and $g_2 := (f_2, 0, \dots, 0)^T$ belong to G. This shows that

$$h'' \in \operatorname{span}\{g_1\overline{g_2}^T : g_1, g_2 \in G\}.$$

Therefore, h itself lies in $\operatorname{span}\{g_1\overline{g_2}^T : g_1, g_2 \in G\}$. ☐

Since $(u \odot v^T)\hat{} = \hat{u}\overline{\hat{v}}^T$, we have

$$\operatorname{span}\{(u \odot v^T)\hat{} : u \in V_k, v \in V_k\} = \operatorname{span}\{\hat{u}\overline{\hat{v}}^T : u \in V_k, v \in V_k\}.$$

By Lemma 6.2, $w \in W_k$ if and only if $\hat{w} = \mathrm{vec}(h)$ for some h satisfying the following conditions:

$$D^\mu(yh)(0) = D^\mu(h\overline{y}^T)(0) = 0 \;\forall\, |\mu| < k \quad \text{and} \quad D^\mu(yh\overline{y}^T)(0) = 0 \;\forall\, k \le |\mu| < 2k.$$

Let $\{t_1, \ldots, t_r\}$ be a basis for $\mathbb{C}^{1\times r}$. It is easily seen that

$$D^\mu(yh)(0) = 0 \Longleftrightarrow D^\mu(yht_m^T) = 0 \quad \forall\, m = 1, \ldots, r.$$

Similarly,

$$D^\mu(h\overline{y}^T)(0) = 0 \Longleftrightarrow D^\mu(t_m h\overline{y}^T) = 0 \quad \forall\, m = 1, \ldots, r.$$

We observe that

$$\mathrm{vec}(yht_m^T) = (t_m \otimes y)\mathrm{vec}(h), \;\; \mathrm{vec}(t_m h\overline{y}^T) = (\overline{y} \otimes t_m)\mathrm{vec}(h), \;\text{and}\; \mathrm{vec}(yh\overline{y}^T) = (\overline{y} \otimes y)\mathrm{vec}(h).$$

Therefore, $u \in W_k$ if and only if

$$D^\mu\big((t_m \otimes y)\hat{w}\big)(0) = D^\mu\big((\overline{y} \otimes t_m)\hat{w}\big)(0) = 0 \quad \forall\, |\mu| < k$$

and

$$D^\mu\big((\overline{y} \otimes y)\hat{w}\big)(0) = 0 \quad \forall\, k \le |\mu| < 2k.$$

By Leibniz rule for differentiation we have

$$\frac{(-iD)^\mu\big((t_m \otimes y)\hat{w}\big)(0)}{\mu!} = \sum_{\nu \le \mu} \frac{(-iD)^{\mu-\nu}(t_m \otimes y)(0)}{(\mu-\nu)!} \, \frac{(-iD)^\nu \hat{w}(0)}{\nu!}.$$

But $(-iD)^\nu \hat{w}(0) = \sum_{\alpha \in \mathbb{Z}^s} (-\alpha)^\nu w(\alpha)$. Hence,

$$\frac{(-iD)^\mu\big((t_m \otimes y)\hat{w}\big)(0)}{\mu!} = \sum_{\alpha \in \mathbb{Z}^s} (t_m \otimes u_\mu)(-\alpha) w(\alpha) = \langle t_m \otimes u_\mu, w\rangle,$$

where u_μ $(|\mu| < k)$ is given by

$$u_\mu := \sum_{\nu \le \mu} \frac{(-iD)^{\mu-\nu} y(0)}{(\mu-\nu)!} \, q_\nu,$$

and $q_\nu(\alpha) = \alpha^\nu/\nu!$, $\alpha \in \mathbb{Z}^s$. Thus,

$$D^\mu\big((t_m \otimes y)\hat{w}\big)(0) = 0 \Longleftrightarrow \langle t_m \otimes u_\mu, w\rangle = 0.$$

Similarly,

$$D^\mu\big((\overline{y} \otimes t_m)\hat{w}\big)(0) = 0 \Longleftrightarrow \langle u'_\mu \otimes t_m, w\rangle = 0,$$

where u'_μ is given by $u'_\mu(\alpha) = \overline{u_\mu(-\alpha)}$, $\alpha \in \mathbb{Z}^s$. Finally, for $|\mu| \le 2k$, let

$$\tilde{u}_\mu := \sum_{\nu \le \mu} \frac{(-iD)^{\mu-\nu}(\overline{y} \otimes y)(0)}{(\mu-\nu)!}\, q_\nu. \tag{1.36}$$

Then

$$D^\mu\big((\overline{y} \otimes y)\hat{w}\big)(0) = 0 \Longleftrightarrow \langle \tilde{u}_\mu, w \rangle = 0.$$

The above discussions are summarized in the following lemma.

LEMMA 6.3. *Suppose y is a $1 \times r$ vector of trigonometric polynomials on $\mathbb{R}^s$ such that $y(0) \ne 0$. Let $\{t_1, \ldots, t_r\}$ be a basis for $\mathbb{C}^{1\times r}$. If W_k is the linear space defined in (1.35), then $W_k = U_k^\perp$, where*

$$U_k := \operatorname{span}\{t_m \otimes u_\mu, u'_\mu \otimes t_m : |\mu| < k \text{ and } m = 1, \ldots, r\} + \operatorname{span}\{\tilde{u}_\mu : k \le |\mu| < 2k\}.$$

Since W_k is invariant under the transition operator T_b, U_k is invariant under the subdivision operator S_b, by Lemma 2.1.

In the above lemma, $\{t_1, \ldots, t_r\}$ could be any basis for $\mathbb{C}^{1\times r}$. But a particular choice of bases will facilitate our study. Recall that $A(0) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)/d$. Suppose

$$\operatorname{spec}(A(0)) = \{\eta_1, \eta_2, \ldots, \eta_r\},$$

where $\eta_1 = 1$ and $\eta_j \ne 1$ for $j = 2, \ldots, r$. We choose a basis $\{t_1, t_2, \ldots, t_r\}$ for $\mathbb{C}^{1\times r}$ such that $t_1 A(0) = t_1$ and

$$t_m A(0) \in \operatorname{span}\{t_2, \ldots, t_r\}, \quad m = 2, \ldots, r.$$

Suppose

$$t_m A(0) = \sum_{n=1}^{r} \eta_{mn} t_n, \quad m = 1, \ldots, r.$$

Then $\eta_{11} = 1$ and $\eta_{m1} = \eta_{1m} = 0$ for $m = 2, \ldots, r$.

7. Spectral analysis. In this section we will establish Theorem 1.1 and other related results. For this purpose we shall first find the spectrum of the subdivision operator S_b restricted to U_k.

Let y be a $1 \times r$ vector of trigonometric polynomials on $\mathbb{R}^s$ such that $y(0) \ne 0$ and $y(0)A(0) = y(0)$. We choose a basis $\{t_1, t_2, \ldots, t_r\}$ for $\mathbb{C}^{1\times r}$ such that $t_1 = y(0)$ and $t_m A(0) \in \operatorname{span}\{t_2, \ldots, t_r\}$, $m = 2, \ldots, r$. Recall that $q_\nu(\alpha) = \alpha^\nu/\nu!$, $\alpha \in \mathbb{Z}^s$, and

$$u_\mu = \sum_{\nu \le \mu} y_{\mu-\nu} q_\nu, \quad |\mu| < k,$$

where $y_{\mu-\nu} = (-iD)^{\mu-\nu} y(0)/(\mu-\nu)!$. In particular, $y_0 = y(0) = t_1$. Moreover,

$$u'_\mu = \sum_{\nu \le \mu} \overline{y_{\mu-\nu}}(-1)^{|\nu|} q_\nu.$$

For $j = 1, \ldots, k$, let

$$U'_j := \operatorname{span}\{t_m \otimes u_\mu, u'_\mu \otimes t_m : |\mu| < j \text{ and } m = 1, \ldots, r\}.$$

LEMMA 7.1. *The set*

$$\{\overline{t_m} \otimes u_\mu : |\mu| < k, m = 1, \ldots, r\} \cup \{u'_\mu \otimes t_m : |\mu| < k, m = 2, \ldots, r\} \tag{1.37}$$

forms a basis for U'_k.

Proof. For $|\mu| = 0$, we have

$$u'_0 \otimes y_0 = \overline{y_0} q_0 \otimes y_0 = \overline{y_0} \otimes y_0 q_0 = \overline{y_0} \otimes u_0.$$

For $|\mu| > 0$ we have

$$\begin{aligned} &u'_\mu \otimes y_0 - (-1)^{|\mu|} \overline{y_0} \otimes u_\mu \\ &= \sum_{\nu \le \mu} (-1)^{|\nu|} \overline{y_{\mu-\nu}} \otimes y_0 q_\nu - (-1)^{|\mu|} \sum_{\nu \le \mu} \overline{y_0} q_\nu \otimes y_{\mu-\nu} \\ &= \sum_{\nu < \mu} (-1)^{|\nu|} \overline{y_{\mu-\nu}} \otimes y_0 q_\nu - (-1)^{|\mu|} \sum_{\nu < \mu} \overline{y_0} q_\nu \otimes y_{\mu-\nu}. \end{aligned}$$

Note that

$$y_0 q_\nu = u_\nu - \sum_{\tau < \nu} y_{\nu-\tau} q_\tau \quad \text{and} \quad (-1)^{|\nu|} \overline{y_0} q_\nu = u'_\nu - \sum_{\tau < \nu} \overline{y_{\nu-\tau}} (-1)^{|\tau|} q_\tau.$$

Hence,

$$u'_\mu \otimes y_0 - (-1)^{|\mu|} \overline{y_0} \otimes u_\mu = \sum_{\nu < \mu} (-1)^{|\nu|} \overline{y_{\mu-\nu}} \otimes u_\nu - \sum_{\nu < \mu} (-1)^{|\mu-\nu|} u'_\nu \otimes y_{\mu-\nu} + J,$$

where

$$J := \sum_{\nu < \mu} \sum_{\tau < \nu} \Big[-(-1)^{|\nu|} \overline{y_{\mu-\nu}} \otimes y_{\nu-\tau} + (-1)^{|\mu-\nu+\tau|} \overline{y_{\nu-\tau}} \otimes y_{\mu-\nu} \Big] q_\tau.$$

It follows that

$$J = \sum_{\tau < \mu} \sum_{\tau < \nu < \mu} \Big[-(-1)^{|\nu|} \overline{y_{\mu-\nu}} \otimes y_{\nu-\tau} + (-1)^{|\mu-\nu+\tau|} \overline{y_{\nu-\tau}} \otimes y_{\mu-\nu} \Big] q_\tau.$$

Replacing ν by $\mu - \nu + \tau$ in the first part of the above inner sum, we obtain

$$\sum_{\tau < \nu < \mu} -(-1)^{|\nu|} \overline{y_{\mu-\nu}} \otimes y_{\nu-\tau} = \sum_{\tau < \nu < \mu} -(-1)^{|\mu-\nu+\tau|} \overline{y_{\nu-\tau}} \otimes y_{\mu-\nu}.$$

This shows $J = 0$. Therefore,

$$u'_\mu \otimes y_0 - (-1)^{|\mu|} \overline{y_0} \otimes u_\mu = \sum_{\nu < \mu} \Big[(-1)^{|\nu|} \overline{y_{\mu-\nu}} \otimes u_\nu - (-1)^{|\mu-\nu|} u'_\nu \otimes y_{\mu-\nu} \Big]. \tag{1.38}$$

In light of (1.38) we see that the set in (1.37) spans U'_k. Actually this set is linearly independent. To justify our claim, we first make the following observation. Suppose $t_1, \dots, t_r$ are linearly independent $1 \times r$ vectors and $w_1, \dots, w_r$ are $1 \times r$ vectors. Then

$$w_1 \otimes t_1 + \cdots + w_r \otimes t_r = 0 \Longrightarrow w_1 = 0, \dots, w_r = 0. \tag{1.39}$$

Indeed, there exist $r \times 1$ vectors v_n $(n = 1, \dots, r)$ such that

$$t_m v_n = \delta_{mn}, \quad m, n = 1, \dots, r,$$

since $t_1, \dots, t_r$ are linearly independent. Let I be the $r \times r$ identity matrix. Then

$$(w_1 \otimes t_1 + \cdots + w_r \otimes t_r)(I \otimes v_n) = 0.$$

But $(w_m \otimes t_m)(I \otimes v_n) = (w_m I) \otimes (t_m v_n) = w_m \delta_{mn}$. Hence, $w_n = 0$ for $n = 1, \dots, r$. This verifies (1.39).

Suppose $c_{j\mu}$ $(|\mu| < k,\ j = 1, \dots, r)$ and $c'_{j\mu}$ $(|\mu| < k,\ j = 2, \dots, r)$ are complex numbers such that

$$\sum_{|\mu|<k} \Big[\sum_{j=1}^{r} c_{j\mu} \overline{t_j} \otimes u_\mu + \sum_{j=2}^{r} c'_{j\mu} u'_\mu \otimes t_j \Big] = 0.$$

We wish to show that all $c_{j\mu} = 0$ and $c'_{j\mu} = 0$. In terms of the expressions of u_μ and u'_μ we have

$$\sum_{|\mu|<k} \sum_{j=1}^{r} \sum_{\nu \le \mu} c_{j\mu} \overline{t_j} \otimes y_{\mu-\nu} q_\nu + \sum_{|\mu|<k} \sum_{j=2}^{r} \sum_{\nu \le \mu} c'_{j\mu} (-1)^\nu \overline{y_{\mu-\nu}} \otimes t_j q_\nu = 0.$$

As sequences on $\mathbb{Z}^s$, q_ν $(|\nu| < k)$ are linearly independent. In the above sums, consider those terms involving q_ν with $|\nu| = k - 1$. Then we have

$$\sum_{|\mu|=k-1} \Big[\sum_{j=1}^{r} c_{j\mu} \overline{t_j} \otimes y_0 + \sum_{j=2}^{r} c'_{j\mu} (-1)^\mu \overline{y_0} \otimes t_j \Big] q_\mu = 0.$$

It follows that

$$\Big(\sum_{j=1}^{r} c_{j\mu} \overline{t_j} \Big) \otimes t_1 + \sum_{j=2}^{r} c'_{j\mu} (-1)^\mu \overline{y_0} \otimes t_j = 0.$$

Since $t_1, t_2, \dots, t_r$ are linearly independent, by (1.39) we have

$$\sum_{j=1}^{r} c_{j\mu} \overline{t_j} = 0 \quad \text{and} \quad c'_{j\mu} (-1)^\mu \overline{y_0} = 0,\ j = 2, \dots, r.$$

Consequently, $c_{j\mu} = 0$ for all $|\mu| = k - 1$ and $j = 1, \dots, r$, and $c'_{j\mu} = 0$ for all $|\mu| = k - 1$ and $j = 2, \dots, r$. By using this argument repeatedly, we see that all $c_{j\mu} = 0$ and $c'_{j\mu} = 0$. Therefore, the set in (1.37) is linearly independent, so it forms a basis for U'_k. $\square$

Recall that $\operatorname{spec}(M) = \{\sigma_1, \ldots, \sigma_s\}$ and $\sigma^\mu = \sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s}$ for $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s$.

LEMMA 7.2. *The spectrum of the subdivision operator S_b restricted to U_k' is*

$$\{\overline{\eta_m}\sigma^{-\mu} : m = 1, \ldots, r, |\mu| < k\} \cup \{\eta_m \overline{\sigma^{-\mu}} : m = 2, \ldots, r, |\mu| < k\}.$$

Proof. Suppose $|\mu| = j < k$. For $m = 1, \ldots, r$ and $\alpha \in \mathbb{Z}^s$, we have

$$\begin{aligned} S_b(\overline{t_m} \otimes u_\mu)(\alpha) &= \sum_{\gamma \in \mathbb{Z}^s} (\overline{t_m} \otimes u_\mu)(\gamma) b(\alpha - M\gamma) \\ &= \frac{1}{d} \sum_{\gamma \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} \big(\overline{t_m} \otimes u_\mu(\gamma)\big)\big(\overline{a(\beta)} \otimes a(\alpha - M\gamma + \beta)\big) \\ &= \frac{1}{d} \sum_{\beta \in \mathbb{Z}^s} (\overline{t_m a(\beta)}) \otimes \big((S_a u_\mu)(\alpha + \beta)\big). \end{aligned}$$

By Lemma 3.2, there are complex numbers $c_{\mu\nu}$ such that

$$S_a u_\mu = \sum_{|\nu| = j} c_{\mu\nu} u_\nu, \quad |\mu| = j.$$

Moreover, the spectrum of the matrix $(c_{\mu\nu})_{|\mu|=j, |\nu|=j}$ is $\{\sigma^{-\mu} : |\mu| = j\}$. For $|\nu| = j$, (1.19) tells us that

$$u_\nu(\alpha + \beta) - u_\nu(\alpha) = \sum_{|\tau| < j} h_{\nu\tau}(\beta) u_\tau(\alpha), \quad \alpha, \beta \in \mathbb{Z}^s,$$

where $h_{\nu\tau} \in \ell(\mathbb{Z}^s)$. Thus, for $\alpha, \beta \in \mathbb{Z}^s$ we have

$$(S_a u_\mu)(\alpha + \beta) = \sum_{|\nu| = j} c_{\mu\nu} u_\nu(\alpha + \beta) = \sum_{|\nu| = j} c_{\mu\nu} u_\nu(\alpha) + \sum_{|\nu| = j} \sum_{|\tau| < j} c_{\mu\nu} h_{\nu\tau}(\beta) u_\tau(\alpha).$$

Hence, there exists some element $w_{m\mu} \in U_j'$ such that

$$S_b(\overline{t_m} \otimes u_\mu) = \frac{1}{d} \sum_{\beta \in \mathbb{Z}^s} (\overline{t_m a(\beta)}) \otimes \left(\sum_{|\nu| = j} c_{\mu\nu} u_\nu \right) + w_{m\mu}.$$

But

$$\frac{1}{d} \sum_{\beta \in \mathbb{Z}^s} (\overline{t_m a(\beta)}) = \overline{t_m A(0)} = \sum_{n=1}^{r} \overline{\eta_{mn} t_n}.$$

Therefore, for $m \in \{1, \ldots, r\}$ and $|\mu| = j$ we have

$$S_b(\overline{t_m} \otimes u_\mu) = \sum_{n=1}^{r} \sum_{|\nu| = j} \big(\overline{\eta_{mn}} c_{\mu\nu}\big)(\overline{t_n} \otimes u_\nu) + w_{m\mu}. \tag{1.40}$$

Let Δ_j denote the index set $\{(m,\mu) : m = 1,\dots,r, |\mu| = j\}$. With an appropriate ordering, the matrix

$$\left(\overline{\eta_{mn}}\, c_{\mu\nu}\right)_{(m,\mu)\in\Delta_j,(n,\nu)\in\Delta_j}$$

can be viewed as the Kronecker product of the matrices $(\overline{\eta_{mn}})_{1\le m,n\le r}$ and $(c_{\mu\nu})_{|\mu|=j,|\nu|=j}$. Hence, its spectrum is

$$\{\overline{\eta_m}\,\sigma^{-\mu} : m = 1,\dots,r,\ |\mu| = j\}.$$

An analogous argument shows that, for $|\mu| = j$ and $m \in \{2,\dots,r\}$,

$$S_b(u'_\mu \otimes t_m) = \sum_{n=2}^{r} \sum_{|\nu|=j} (\eta_{mn}\,\overline{c_{\mu\nu}})(u'_\nu \otimes t_n) + w'_{m\mu}, \tag{1.41}$$

where $w'_{m\mu} \in U'_j$. Note that the spectrum of the matrix $(\eta_{mn})_{2\le m,n\le r}$ is $\{\eta_2,\dots,\eta_r\}$.

For $j = 1,\dots,k$, let $\tilde{S}_b^{(j)}$ denote the quotient linear operator induced by S_b on the quotient space U'_j/U'_{j-1}. Then (1.40) and (1.41) tell us that

$$\mathrm{spec}\,(\tilde{S}_b^{(j)}) = \{\overline{\eta_m}\sigma^{-\mu} : m = 1,\dots,r, |\mu| = j-1\} \cup \{\eta_m\overline{\sigma^{-\mu}} : m = 2,\dots,r, |\mu| = j-1\}.$$

Since

$$\mathrm{spec}\,(S_b|_{U'_k}) = \cup_{j=1}^{k}\mathrm{spec}\,(\tilde{S}_b^{(j)}),$$

the proof of the lemma is complete. $\square$

By Lemma 6.3, we have $U_k = U'_k + \mathrm{span}\{\tilde{u}_\mu : k \le |\mu| < 2k\}$, where $\tilde{u}_\mu$ ($|\mu| < 2k$) are given by (1.36). As was done in §3, it can be easily proved that U_k is the direct sum of U'_k and $\mathrm{span}\{\tilde{u}_\mu : k \le |\mu| < 2k\}$. Also, the set $\{\tilde{u}_\mu : k \le |\mu| < 2k\}$ is linearly independent. For $j = k, k+1,\dots,2k$, let

$$U''_j := U'_k + \mathrm{span}\{\tilde{u}_\mu : k \le |\mu| < j\},$$

In particular, $U''_k = U'_k$ and $U''_{2k} = U_k$.

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$$\{\overline{\eta_m}\sigma^{-\mu}, \eta_m\overline{\sigma^{-\mu}} : m = 2,\dots,r, |\mu| < k\} \cup \{\sigma^{-\mu} : |\mu| < 2k\}. \tag{1.42}$$

Proof. Suppose $|\mu| = j \in \{k,\dots,2k-1\}$. Since U_k is invariant under S_b, there exist complex numbers $c_{\mu\nu}$ ($k \le |\nu| < 2k$) and an element $w_\mu \in U'_k$ such that

$$S_b\tilde{u}_\mu = \sum_{k\le|\nu|<2k} c_{\mu\nu}\tilde{u}_\nu + w_\mu.$$

Since $S_b(\nabla_\gamma \tilde{u}_\mu) = \nabla_{M\gamma}(S_b\tilde{u}_\mu)$ for $\gamma \in \mathbb{Z}^s$, it follows that

$$S_b(\nabla^\tau \tilde{u}_\mu) = \sum_{k\le|\nu|<2k} c_{\mu\nu}(\nabla_{Me_1}^{\tau_1}\cdots\nabla_{Me_s}^{\tau_s})\tilde{u}_\nu + (\nabla_{Me_1}^{\tau_1}\cdots\nabla_{Me_s}^{\tau_s})w_\mu, \quad \tau \in \mathbb{N}_0^s.$$

We claim that $c_{\mu\nu} = 0$ for $|\nu| > j$. If this is not the case, then $N := \max\{|\nu| : c_{\mu\nu} \neq 0\} > j$. For $|\tau| = N$, we have $\nabla^\tau \tilde{u}_\mu = 0$ and $(\nabla^{\tau_1}_{Me_1} \cdots \nabla^{\tau_s}_{Me_s}) w_\mu = 0$. Moreover, by (1.23) we have

$$(\nabla^{\tau_1}_{Me_1} \cdots \nabla^{\tau_s}_{Me_s}) \tilde{u}_\nu = b_{\tau\nu} \tilde{u}_0 \quad \text{for } |\tau| = |\nu| = N,$$

where the matrix $(b_{\tau\nu})_{|\tau|=N,|\nu|=N}$ has $\{\sigma^\mu : |\mu| = N\}$ as its spectrum. Consequently,

$$\sum_{|\nu|=N} c_{\mu\nu} b_{\tau\nu} = 0 \quad \forall\, |\tau| = N. \tag{1.43}$$

Since the matrix $(b_{\tau\nu})_{|\tau|=N,|\nu|=N}$ is invertible, we obtain $c_{\mu\nu} = 0$ for all $|\nu| = N$. This contradiction justifies our claim. Therefore,

$$S_b \tilde{u}_\mu = \sum_{|\nu|=j} c_{\mu\nu} \tilde{u}_\nu + w'_\mu, \tag{1.44}$$

where $w'_\mu \in U''_j$. For $|\tau| = j$, we deduce from (1.44) that

$$\delta_{\mu\tau} \tilde{u}_0 = S_b(\nabla^\tau \tilde{u}_\mu) = \sum_{|\nu|=j} c_{\mu\nu} (\nabla^{\tau_1}_{Me_1} \cdots \nabla^{\tau_s}_{Me_s}) \tilde{u}_\nu = \sum_{|\nu|=j} c_{\mu\nu} b_{\tau\nu} \tilde{u}_0.$$

Hence, the spectrum of the matrix $(c_{\mu\nu})_{|\mu|=j,|\nu|=j}$ is $\{\sigma^{-\mu} : |\mu| = j\}$.

For $j = k+1, \ldots, 2k$, let $\tilde{S}^{(j)}_b$ denote the quotient linear operator induced by S_b on the quotient space U''_j / U''_{j-1}. Then (1.44) tells us that

$$\operatorname{spec}(\tilde{S}^{(j)}_b) = \{\sigma^{-\mu} : |\mu| = j-1\}.$$

Since

$$\operatorname{spec}(S_b|_{U_k}) = \operatorname{spec}(S_b|_{U''_{2k}}) = \operatorname{spec}(S_b|_{U'_k}) \cup \Big(\cup_{j=k+1}^{2k} \operatorname{spec}(\tilde{S}^{(j)}_b) \Big),$$

we conclude that the set in (1.42) is indeed the spectrum of S_b restricted to U_k. $\square$

By Lemma 7.3 and Lemma 2.3 we have the following formula:

$$\rho(T_b|_{W_k}) = \max\Big\{ |\nu| : \nu \in \operatorname{spec}\big(b(M\alpha - \beta)\big)_{\alpha,\beta \in K} \setminus E_k \Big\}.$$

where

$$E_k := \{\eta_j \overline{\sigma^{-\mu}}, \overline{\eta_j} \sigma^{-\mu} : |\mu| < k, j = 2, \ldots, r\} \cup \{\sigma^{-\mu} : |\mu| < 2k\}.$$

This together with Theorem 5.3 verifies Theorem 1.1.

Let B be the matrix $\big(b(M\alpha - \beta)\big)_{\alpha,\beta \in K}$. We say that B satisfies condition E, if 1 is a simple eigenvalue of B and other eigenvalues of B are less than 1 in modulus. Suppose a satisfies the basic sum rule. Then W_1 is invariant under T_b and

$$\rho(T_b|_{W_1}) = \max\Big\{ |\nu| : \nu \in \operatorname{spec}(B) \setminus \{1, \eta_2, \ldots, \eta_r, \overline{\eta_2}, \ldots, \overline{\eta_r}\} \Big\}.$$

Thus, if B satisfies condition E, then $\rho(T_b|_{W_1}) < 1$, and hence the refinement equation (1.1) has a compactly supported solution $\Phi \in (L_2(\mathbb{R}^s))^r$, by Theorem 5.2. Conversely, if $\Phi \in (L_2(\mathbb{R}^s))^r$ is a compactly supported solution to the refinement equation (1.1), and if Φ is stable, then W_1 is invariant under T_b and $\rho(T_b|_{W_1}) < 1$. But, in this case, $|\eta_j| < 1$ for $j = 2, \ldots, r$ (see [9] and [4]). Therefore, the matrix B satisfies condition E. When the matrix M is 2 times the $s \times s$ identity matrix, this result was established in [42].

8. Examples. In this section we give three examples to illustrate the general theory. Our first example, taken from [14], is concerned with orthogonal multi-wavelets.

Example 8.1. Let $r = 2$, $s = 1$, and $M = (2)$. Suppose $a \in \ell_0^2(\mathbb{Z})$ is supported on $\{0, 1, 2, 3\}$. Moreover,

$$a(0) = \frac{1}{10}\begin{bmatrix} 6 & 8\sqrt{2} \\ \frac{-1}{\sqrt{2}} & -3 \end{bmatrix}, \qquad a(1) = \frac{1}{10}\begin{bmatrix} 6 & 0 \\ \frac{9}{\sqrt{2}} & 10 \end{bmatrix},$$

$$a(2) = \frac{1}{10}\begin{bmatrix} 0 & 0 \\ \frac{9}{\sqrt{2}} & -3 \end{bmatrix}, \qquad a(3) = \frac{1}{10}\begin{bmatrix} 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}.$$

We have

$$A(0) = [a(0) + a(1) + a(2) + a(3)]/2 = \frac{1}{10}\begin{bmatrix} 6 & 4\sqrt{2} \\ 4\sqrt{2} & 2 \end{bmatrix}.$$

The eigenvalues of $A(0)$ are $\eta_1 = 1$ and $\eta_2 = -1/5$. It can be easily verified that a satisfies the sum rules of order 2, but a does not satisfy the sum rules of order 3. Let b be the element in $\ell_0^4(\mathbb{Z})$ given by

$$b(\alpha) = \sum_{\beta \in \mathbb{Z}} a(\beta) \otimes a(\alpha + \beta)/2, \quad \alpha \in \mathbb{Z}.$$

Then b is supported on $\mathbb{Z}^2 \cap [-3, 3]$. Let B be the 28×28 matrix $(b(2\alpha - \beta))_{-3 \le \alpha, \beta \le 3}$. The nonzero eigenvalues of B are

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{10}, -\frac{1}{10}, -\frac{1}{20}, -\frac{1}{20}, -\frac{1}{20}, -\frac{1}{20}, -\frac{1}{25}, -\frac{1}{50}, -\frac{1}{50}.$$

Thus, there exists a unique compactly supported solution $\Phi = (\phi_1, \phi_2)^T \in (L_2(\mathbb{R}))^2$ to the refinement equation

$$\Phi = \sum_{\alpha=0}^{3} a(\alpha)\Phi(2\cdot - \alpha)$$

subject to the condition $[\sqrt{2}, 1]\hat{\Phi}(0) = 1$. The shifts of ϕ_1 and ϕ_2 are orthogonal (see [14]). Consequently, Φ is stable. Hence, we may apply Theorem 1.1 to obtain

$$\lambda(\Phi) = -(\log_2 \rho_2)/2,$$

where $\rho_2 = \max\{|\nu| : \nu \in \operatorname{spec}(B) \setminus E_2\}$ and

$$E_2 = \{1, 1/2, 1/4, 1/8, -1/5, -1/5, -1/10, -1/10\}.$$

Therefore, $\rho_2 = 1/8$ and $\lambda(\Phi) = -(\log_2 \rho_2)/2 = 3/2$. Note that

$$\max\big\{|\nu| : \nu \in \mathrm{spec}\,(B) \setminus \{(1/2)^{\mu} : \mu < 4\}\big\} = 1/5.$$

But we have $\lambda(\Phi) = 3/2 > -(\log_2 1/5)/2$. □

Our second example is motivated by the study given in [37] on norm bounds for iterated transfer operators related to numerical solutions to partial differential equations.

Example 8.2. Let $r = 2$, $s = 2$, and $M = 2I_2$, where I_2 denotes the 2×2 identity matrix. Suppose $a \in \ell_0^2(\mathbb{Z}^2)$ is supported on $(\mathbb{Z}^2 \cap [0,5]^2) \setminus \{(4,0),(5,0),(4,1),(5,1),(0,5)\}$. Moreover, $a(0,0), a(1,0), a(2,0)$ are given by

$$\frac{1}{8}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

$a(0,1), a(1,1), a(2,1)$ are given by

$$\frac{1}{8}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & 0 \\ 5 & 1 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$a(0,2), a(1,2), a(2,2), a(3,2), a(4,2)$ are given by

$$\frac{1}{8}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 1 & -1 \\ 5 & 8 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 8 & 1 \\ 1 & 8 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

$a(0,3), a(1,3), a(2,3), a(3,3), a(4,3)$ are given by

$$\frac{1}{8}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 8 & 5 \\ -1 & 1 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and $a(1,4), a(2,4), a(3,4), a(4,4)$ are given by

$$\frac{1}{8}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

We have

$$A(0) = \frac{1}{4}\sum_{\alpha \in \mathbb{Z}^2} a(\alpha) = \frac{1}{8}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

The eigenvalues of $A(0)$ are $\eta_1 = 1$ and $\eta_2 = 1/4$. Moreover, $[1,1]A(0) = [1,1]$. It can be verified that the optimal order of sum rules satisfied by a is $k = 2$. Let b be the element in $\ell_0^4(\mathbb{Z}^2)$ given by

$$b(\alpha) = \sum_{\beta \in \mathbb{Z}^2} a(\beta) \otimes a(\alpha + \beta)/4, \quad \alpha \in \mathbb{Z}^2.$$

Then b is supported in $[-5,5]^2$. Let B be the 484×484 matrix $(b(2\alpha-\beta))_{\alpha,\beta\in[-5,5]^2}$. The leading eigenvalues of B are

$$1,\ 1/2,\ 1/2,\ 1/4,\ 1/4,\ 1/4,\ 1/4,\ 1/4, 0.13129521, 0.13060779, \ldots.$$

Thus, there exists a unique compactly supported solution $\Phi \in (L_2(\mathbb{R}^2))^2$ to the refinement equation

$$\Phi = \sum_{\alpha\in\mathbb{Z}^2} a(\alpha)\Phi(2\cdot - \alpha)$$

subject to the condition $[1,1]\hat{\Phi}(0) = 1$. By using the method in [18] we can show that Φ is stable. Hence, we may apply Theorem 1.1 to obtain

$$\lambda(\Phi) = -\log_4 \rho_2,$$

where $\rho_2 = \max\{|\nu| : \nu \in \operatorname{spec}(B) \setminus E_2\}$ and

$$E_2 = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\} \cup \{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\}.$$

Therefore, $\rho_2 \approx 0.13129521$ and $\lambda(\Phi) = -\log_4 \rho_2 \approx 1.46436842$. $\square$

Our third example is a refinable vector of functions with Hermite interpolation properties (see [16]). Such refinable functions are useful in computer aided geometric design.

Example 8.3. Let $r = 3$, $s = 2$, and

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Clearly, the eigenvalues of M are $\sigma_1 = 1+i$ and $\sigma_2 = 1-i$, where i denotes the imaginary unit. Suppose $a \in \ell_0^3(\mathbb{Z}^2)$ is supported on $\{(0,0),(1,0),(0,1),(-1,0),(0,-1)\}$. Moreover, $a(0,0)$, $a(1,0)$, and $a(0,1)$ are given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 1/4 & -3/4 & 0 \\ 1/16 & -1/8 & 0 \\ -1/16 & 1/8 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1/4 & 0 & -3/4 \\ 1/16 & 0 & -1/8 \\ 1/16 & 0 & -1/8 \end{bmatrix},$$

and $a(-1,0)$, $a(0,-1)$ are given by

$$\begin{bmatrix} 1/4 & 3/4 & 0 \\ -1/16 & -1/8 & 0 \\ 1/16 & 1/8 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1/4 & 0 & 3/4 \\ -1/16 & 0 & -1/8 \\ -1/16 & 0 & -1/8 \end{bmatrix}.$$

We have

$$A(0) = \frac{1}{2}\sum_{\alpha\in\mathbb{Z}^2} a(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/8 & 1/8 \\ 0 & -1/8 & 1/8 \end{bmatrix}.$$

The eigenvalues of $A(0)$ are $\eta_1 = 1$, $\eta_2 = (1+i)/8$, and $\eta_3 = (1-i)/8$. Moreover, $[1,0,0]A(0) = [1,0,0]$. It can be verified that the optimal order of sum rules satisfied by a is $k = 4$ (see [16]). Let b be the element in $\ell_0^9(\mathbb{Z}^2)$ given by

$$b(\alpha) = \sum_{\beta\in\mathbb{Z}^2} a(\beta)\otimes a(\alpha+\beta)/2, \quad \alpha\in\mathbb{Z}^2.$$

Then b is supported on the set

$$\{(\alpha_1,\alpha_2)\in\mathbb{Z}^2 : -2\le\alpha_1-\alpha_2\le 2,\ -2\le\alpha_1+\alpha_2\le 2\}.$$

We observe that

$$\begin{aligned} K &:= \mathbb{Z}^2\cap\Big(\sum\nolimits_{n=1}^{\infty} M^{-n}(\operatorname{supp} b)\Big) \\ &= \{(\alpha_1,\alpha_2)\in\mathbb{Z}^2 : |\alpha_1|\le 6,\ |\alpha_2|\le 6,\ |\alpha_1-\alpha_2|\le 8,\ |\alpha_1+\alpha_2|\le 8\}. \end{aligned}$$

The set K has exactly 129 points. Let B be the 1161×1161 matrix $(b(2\alpha-\beta))_{\alpha,\beta\in K}$. The first 27 eigenvalues of B (in terms of their absolute values) are

$$1,\ (1+i)/2,\ (1-i)/2,\ 1/2,\ i/2,\ -i/2,\ (1+i)/4,\ (1-i)/4, -(1+i)/4,\ (-1+i)/4,$$

$$1/4,\ -1/4,\ -1/4,\ i/4,\ -i/4,\ (1+i)/8,\ (1+i)/8,\ (1+i)/8,\ (1-i)/8,\ (1-i)/8,\ (1-i)/8,$$

$$-(1+i)/8,\ -(1+i)/8,\ (-1+i)/8,\ (-1+i)/8,\ 0.149024,\ 0.148796.$$

Thus, there exists a unique compactly supported solution $\Phi\in(L_2(\mathbb{R}^2))^2$ to the refinement equation

$$\Phi = \sum_{\alpha\in\mathbb{Z}^2} a(\alpha)\Phi(M\cdot-\alpha)$$

subject to the condition $[1,0,0]\hat{\Phi}(0) = 1$. It is known that Φ is stable (see [16]). Hence, we may apply Theorem 1.1 to obtain

$$\lambda(\Phi) = -\log_2\rho_4,$$

where $\rho_4 = \max\{|\nu| : \nu\in\operatorname{spec}(B)\setminus E_4\}$ and

$$E_4 = \{\eta_2\overline{\sigma^{-\mu}}, \overline{\eta_2}\sigma^{-\mu}, \eta_3\overline{\sigma^{-\mu}}, \overline{\eta_3}\sigma^{-\mu} : |\mu|<4\}\cup\{\sigma^{-\mu} : |\mu|<8\}.$$

We see that

$$\rho_4 = \max\{|\nu| : \nu\in\operatorname{spec}(B)\setminus E_4\}\approx 0.149024.$$

Therefore, $\lambda(\Phi) = -\log_2\rho_4\approx 2.746387$. $\square$

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