A Second-order Synchrosqueezing Transform with a Simple Form of Phase Transformation*

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Abstract

To model a non-stationary signal as a superposition of amplitude and frequency-modulated Fourier-like oscillatory modes is important to extract information, such as the underlying dynamics, hidden in the signal. Recently, the synchrosqueezed wavelet transform (SST) and its variants have been developed to estimate instantaneous frequencies and separate the components of non-stationary multicomponent signals.

The short-time Fourier transform-based SST (FSST for short) reassigns the frequency variable to sharpen the time-frequency representation and to separate the components of a multicomponent non-stationary signal. However FSST works well only with multicomponent signals having slowly changing frequencies. To deal with multicomponent signals having fast-changing frequencies, the 2nd-order FSST (FSST2 for short) was proposed. The key point for FSST2 is to construct a phase transformation of a signal which is the instantaneous frequency when the signal has a linear chirp. In this paper we consider a phase transformation for FSST2 which has a simple expression than that used in the literature. In the study the theoretical analysis of FSST2 with this phase transformation, we observe that the proof for the error bounds for the instantaneous frequency estimation and component recovery is simpler than that with the conventional phase transformation. We also provide some experimental results which show that this FSST2 performs well in non-stationary multicomponent signal separation.

Keywords: Short-time Fourier transform, Second-order synchrosqueezing transform, Phase transformation, Instantaneous frequency estimation, Multicomponent signal separation

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1 Introduction

To model a non-stationary signal $x(t)$ as

$$x(t) = A_0(t) + \sum_{k=1}^{K} x_k(t), \qquad x_k(t) = A_k(t)e^{i2\pi\phi_k(t)}, \tag{1}$$

with $A_k(t), \phi_k'(t) > 0$ is important to extract information hidden in $x(t)$ since most real signals such as EEG and bearing signals can be formulated as (1), and its trend $A_0(t)$, instantaneous amplitudes $A_k(t)$ $(k \geq 1)$ and instantaneous frequencies $\phi_k'(t)$ can be used to describe the underlying dynamics of $x(t)$. Thus the representation (1) of non-stationary signals has been used in many applications including geophysics (seismic wave), atmospheric and climate studies, oceanographic studies, medical data analysis and speech recognition, see for example [10]. The empirical mode decomposition (EMD) [9] is a widely used method to separate a signal as a sum of finitely many intrinsic mode functions (IMFs) and represent the signal in the form of (1) by the Hilbert analysis. EMD is a data-driven decomposition algorithm and it was studied by many researchers and has been used in many applications, see e.g. [7, 17, 20, 27, 31, 34, 36, 37, 41]. However EMD hardly distinguishes two close IMFs and sometimes it leads to false components.

Recently the continuous wavelet transform-based synchrosqueezing transform (WSST) was developed in [6] as an alternative EMD-like tool to separate the components of a non-stationary multicomponent signal. In addition, the short-time Fourier transform-based SST (FSST) was also proposed in [30] and further studied in [23, 35] for this purpose. SST has been proved to be robust to noise and small perturbations [11, 21, 29]. However SST does not work well for multicomponent signals having fast changing frequencies.

To provide sharp representations for signals with significantly frequency changes, the 2nd-order FSST (FSST2) and the 2nd-order WSST (WSST2) were introduced in [24] and [22], and the theoretical analysis of them was carried out in [1] and [26] respectively. The 2nd-order SST improves the concentration of the time-frequency representation. The higher-order FSST is presented in [25] and [18], which aims to handle signals containing more general types. Other SST related methods include the generalized WSST [13], a hybrid empirical mode decomposition-SST computational scheme [5], the synchrosqueezed wave packet transform [38], the demodulation-transform based SST [12, 32, 33], signal separation operator [4], vertical synchrosqueezing [8] and empirical signal separation algorithm [16]. In addition, the synchrosqueezed curvelet transform for two-dimensional mode decomposition was introduced in [40] and the statistical analysis of synchrosqueezing transforms has been studied in [39]. Furthermore, the SST with a window function having a changing parameter was proposed in [28] and the FSST with the window function containing time and frequency parameters was studied in [2]. Very recently the authors of

[14, 15] considered the 2nd-order adaptive FSST and WSST with a time-varying parameter. They obtained the well-separated condition for multicomponent signals using the linear frequency modulation signals to approximate a non-stationary signal at any local time. The theoretical analysis of 2nd-order adaptive FSST and WSST was studied in [3] and [19] respectively. In this paper we conside an FSST2 with a phase transformation which has a simpler expression than that used in the literature. In the study the theoretical analysis of FSST2 with this phase transformation, we find that the proof is simpler. Our experimental results show that the performance of this FSST2 in instantaneous frequency estimation and component recovery is comparable with, and even better in some cases than, that of conventional FSST2.

The rest of this paper is organized as follows. In Section 2 we first briefly review FSST and FSST2. After that we introduce a phase transformation and the associated FSST2. In Sections 3 and 4, we consider the theoretical analysis of the FSST2 with this phase transformation. We establish an error bound for instantaneous frequency estimation in Section 3, and error bounds for component recovery in Section 4. We provide some experimental results in Section 5.

2 The second-order FSST with a simple phase transformation

In this section we first provide a brief review of FSST and FSST2. After that we propose a simple phase transformation for the 2nd-order FSST.

The (modified) STFT of $x(t) \in L_2(\mathbb{R})$ with a window function $g(t) \in L_2(\mathbb{R})$ is defined by

$$V_x(t,\eta) = V_x^g(t,\eta) := \int_{\mathbb{R}} x(\tau) g(\tau - t) e^{-i2\pi\eta(\tau-t)} d\tau, \tag{2}$$

where t and η are the time variable and the frequency variable respectively. For simplicity, we drop g in $V_x^g(t,\eta)$. Thus, unless otherwise stated, in this paper $V_x(t,\eta)$ denotes the STFT of $x(t)$ with window function $g(t)$.

If $g(0) \neq 0$, then the original signal $x(t)$ can be recovered back from its STFT:

$$x(t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_x(t,\eta) d\eta. \tag{3}$$

In addition, if the window function $g(t) \in L_2(\mathbb{R})$ is real, then for a real-valued $x(t) \in L_2(\mathbb{R})$, we have

$$x(t) = \frac{2}{g(0)} \mathrm{Re}\Big(\int_0^\infty V_x(t,\eta) d\eta \Big). \tag{4}$$

Here we remark that if the window function $g(t)$ has certain smoothness and certain decaying order as $t \to \infty$, then STFT $V_x(t,\eta)$ of a slowly growing $x(t)$ with $g(t)$ is well defined. Furthermore, the above formulas still hold. In this following, we always assume a window function $g(t)$ has certain smoothness and decaying properties, and a signal $x(t)$ is a slowly growing function. In addition, in this paper we assume $g(0) \neq 0$. For a signal $x(t)$, its Fourier transform $\widehat{x}(\xi)$ (maybe in the distribution sense) is defined by

$$\widehat{x}(\xi) := \int_{\mathbb{R}} x(t)e^{-i2\pi\xi t}dt.$$

The idea of the STFT-based synchrosqueezing transform (FSST) is to reassign the frequency variable [30]. More precisely, we first look at STFT of $x(t) = Ae^{i2\pi\omega_0 t}$, where A, ω_0 are constants with $\omega_0 > 0$. We have

$$V_x(t,\eta) = \int_{\mathbb{R}} Ae^{i2\pi\omega_0(t+\tau)}g(\tau)e^{-i2\pi\eta\tau}d\tau = Ae^{i2\pi t\omega_0}\widehat{g}(\eta - \omega_0). \tag{5}$$

Thus, the instantaneous frequency (IF) ω_0 of $x(t)$ can be obtained by

$$\frac{\partial_t V_x(t,\eta)}{2\pi i V_x(t,\eta)} = \omega_0.$$

Hence, for a signal $x(t)$, at (t,η) for which $V_x(t,\eta) \neq 0$, the quantity $\omega_x(t,\eta)$ defined by

$$\omega_x(t,\eta) := \mathrm{Re}\Big(\frac{\partial_t V_x(t,\eta)}{2\pi i V_x(t,\eta)}\Big) \tag{6}$$

should be a good candidate for the IF of $x(t)$, and it is called the "phase transformation" in [6] or "instantaneous frequency information" in [30].

With $\omega_x(t,\eta)$ defined by (6), FSST is to reassign the frequency variable η by transforming STFT $V_x(t,\eta)$ of $x(t)$ to a quantity, denoted by $R^{\lambda}_{x,\gamma}(t,\xi)$, on the time-frequency plane:

$$R^{\lambda}_{x,\gamma}(t,\xi) := \int_{|V_x(t,\eta)|>\gamma} V_x(t,\eta)\frac{1}{\lambda}h\Big(\frac{\xi - \omega_x(t,\eta)}{\lambda}\Big)d\eta, \tag{7}$$

where $\gamma > 0, \lambda > 0$, and throughout this paper $h(t)$ is a compactly supported function with certain smoothness and $\int_{\mathbb{R}} h(t)dt = 1$. In addition, in this paper $\int_{|V_x(t,\eta)|>\gamma}$ means the integral $\int_{\{\eta:\ |V_x(t,\eta)|>\gamma\}}$ with η over the set $\{\eta:\ |V_x(t,\eta)| > \gamma\}$.

To define and analyze the 2nd-order FSST (FSST2), we denote

$$g_1(\tau) := \tau g(\tau),\ g_2(\tau) := \tau^2 g(\tau),\ g_3(\tau) := \tau g'(\tau).$$

We use $V^{g_j}_x(t,\eta)$ and $V^{g'}_x(t,\eta)$ to denote the STFTs defined by (2) with g replaced by g_j and g' respectively, where $1 \le j \le 3$.

[24] and [1] introduced different phase transformations for the 2nd-order FSST, one of which is given by

$$\omega^{\text{2nd}}_x(t,\eta) = \begin{cases} \mathrm{Re}\big\{\frac{\partial_t V_x(t,\eta)}{2\pi i V_x(t,\eta)}\big\} + \mathrm{Re}\big\{\widetilde{q}(t,\eta)\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)}\big\}, \\ \qquad\qquad\qquad \text{if } \partial_t\big(\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)}\big) \neq i2\pi \text{ and } V_x(t,\eta) \neq 0; \\ \omega_x(t,\eta), \quad \text{if } \partial_t\big(\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)}\big) = i2\pi \text{ and } V_x(t,\eta) \neq 0, \end{cases} \tag{8}$$

where

$$\widetilde{q}(t,\eta) := \frac{\partial_t\Big(\frac{\partial_t V_x(t,\eta)}{V_x(t,\eta)}\Big)}{i2\pi - \partial_t\Big(\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)}\Big)}.$$

The quantity $\omega_x^{2\mathrm{nd}}(t,\eta)$ is defined in such a way that when $x(t)$ is a linera chrip, then $\omega_x^{2\mathrm{nd}}(t,\eta)$ will be the IF of $x(t)$. A linear chirp (also called a linear frequency modulation signal) considered here is a signal of

$$x(t) = Ae^{pt+\frac{q}{2}t^2}e^{i2\pi\phi(t)} = Ae^{pt+\frac{q}{2}t^2}e^{i2\pi(ct+\frac{1}{2}rt^2)}, \tag{9}$$

with IF $\phi'(t) = c+rt$, where A, p, q, c, r are real numbers and the chirp rate $r \neq 0$. Next proposition shows that for (t,η) satisfying $\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big) \neq 0$ and $V_x^{g_1}(t,\eta) \neq 0$, the quantity defined by

$$\mathrm{Re}\Big\{\frac{1}{i2\pi\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)}\partial_\eta\big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\Big\} \tag{10}$$

is also $c+rt$, the IF of linear chirp $x(t)$.

Proposition 1. *Let $x(t)$ be the linear chirp defined by (9). Then for (t,η) satisfying $\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big) \neq 0$ and $V_x^{g_1}(t,\eta) \neq 0$, the quantity defined by (10) is $c+rt$.*

The proof of Proposition 1 is postponed to the end of this section. Thus for a general signal $x(t)$, we may define a phase transformation for FSST2 as the real part of

$$u_x^{2\mathrm{nd,c}}(t,\eta) := \begin{cases} \frac{1}{i2\pi\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)}\partial_\eta\big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big), & \text{if } \partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big) \neq 0 \text{ and } V_x^{g_1}(t,\eta) \neq 0; \\ \\ \frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}, & \text{if } \partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big) = 0 \text{ and } V_x(t,\eta) \neq 0. \end{cases} \tag{11}$$

We denote

$$u_x^{2\mathrm{nd}}(t,\eta) := \mathrm{Re}\big\{u_x^{2\mathrm{nd,c}}(t,\eta)\big\}. \tag{12}$$

Here and below, the letter c in $u_x^{2\mathrm{nd,c}}(t,\eta)$ denotes the complex-valued version of the phase transformation. Comparing $u_x^{2\mathrm{nd}}(t,\eta)$ with $\omega_x^{2\mathrm{nd}}(t,\eta)$ in (8), we know that $u_x^{2\mathrm{nd}}(t,\eta)$ is simpler.

Next we define the 2nd-order FSST with this phase transformation $u_x^{2\mathrm{nd}}(t,\eta)$, where $V_x(t,\eta) \neq 0$, $V_x^{g_1}(t,\eta) \neq 0$ and $\partial_\eta\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) \neq 0$ are described by thresholds $\gamma_0 > 0, \gamma_1 > 0, \gamma_2 > 0$. More precisely, we define

$$u_{x,\gamma_0,\gamma_1,\gamma_2}^{2\mathrm{nd}}(t,\eta) := \begin{cases} \mathrm{Re}\Big\{\frac{1}{i2\pi\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)}\partial_\eta\big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\Big\}, & \text{if } \big|\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\big| > \gamma_2 \text{ and } \Big|V_x^{g_1}(t,\eta)\Big| > \gamma_1; \\ \\ \mathrm{Re}\Big\{\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big\}, & \text{if } \big|\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\big| \leq \gamma_2 \text{ or } \Big|V_x^{g_1}(t,\eta)\Big| \leq \gamma_1; \\ & \text{and } \Big|V_x(t,\eta)\Big| > \gamma_0. \end{cases} \tag{13}$$

Let $h(t)$ be a compactly supported function with certain smoothness and $\int_{\mathbb{R}} h(t)dt = 1$. We define the 2nd-order FSST $S^{2\mathrm{nd},\lambda}_{x,\gamma_1,\gamma_2}$ with phase transformation $u^{2\mathrm{nd}}_{x,\gamma_1,\gamma_2}(t,\eta)$ by

$$S^{2\mathrm{nd},\lambda}_{x,\gamma_0,\gamma_1,\gamma_2}(t,\xi) := \int_{\left\{\eta:\, |V_x(t,\eta)|>\gamma_0\right\}} V_x(t,\eta)\frac{1}{\lambda}h\Big(\frac{\xi - u^{2\mathrm{nd}}_{x,\gamma_0,\gamma_1,\gamma_2}(t,\eta)}{\lambda}\Big)d\eta. \tag{14}$$

The phase transformation $u^{2\mathrm{nd}}_x(t,\eta)$ will be used to approximate IFs $\phi_k'(t)$ and $S^{2\mathrm{nd},\lambda}_{x,\gamma_0,\gamma_1,\gamma_2}(t,\xi)$ will be employed to recover components $x_k(t)$, which are the topics to be studied in the next two sections. Before moving on to the next section, we give the proof of Proposition 1.

Proof of Proposition 1. Let $x(t)$ be the linear chirp given by (9). From

$$x'(t) = \big(p + qt + i2\pi(c + rt)\big)x(t)$$

and

$$V_x(t,\eta) = \int_{-\infty}^{\infty} x(t+\tau)g(\tau)e^{-i2\pi\eta\tau}d\tau,$$

we have

$$\begin{aligned}
\partial_t V_x(t,\eta) &= \int_{-\infty}^{\infty} x'(t+\tau)\; g(\tau)e^{-i2\pi\eta\tau}d\tau \\
&= \big(p + qt + i2\pi(c + rt)\big)V_x(t,\eta) + (q + i2\pi r)\int_{-\infty}^{\infty} \tau x(t+\tau)\; g(\tau)e^{-i2\pi\eta\tau}d\tau \\
&= \big(p + qt + i2\pi(c + rt)\big)V_x(t,\eta) + (q + i2\pi r)V^{g_1}_x(t,\eta).
\end{aligned}$$

Thus, if $V^{g_1}_s(t,\eta) \neq 0$, we have

$$\frac{\partial_t V_x(t,\eta)}{V^{g_1}_x(t,\eta)} = \big(p + qt + i2\pi(c + rt)\big)\;\frac{V_x(t,\eta)}{V^{g_1}_x(t,\eta)} + q + i2\pi r. \tag{15}$$

Taking partial derivative ∂_η to the both sides of (15),

$$\partial_\eta\Big(\frac{\partial_t V_x(t,\eta)}{V^{g_1}_x(t,\eta)}\Big) = \big(p + qt + i2\pi(c + rt)\big)\;\partial_\eta\Big(\frac{V_x(t,\eta)}{V^{g_1}_x(t,\eta)}\Big).$$

Therefore, if in addition, $\partial_\eta\Big(\frac{V_x(t,\eta)}{V^{g_1}_x(t,\eta)}\Big) \neq 0$, then

$$\frac{p + qt}{i2\pi} + c + rt = \frac{1}{i2\pi\partial_\eta\Big(\frac{V_x(t,\eta)}{V^{g_1}_x(t,\eta)}\Big)}\partial_\eta\Big(\frac{\partial_t V_x(t,\eta)}{V^{g_1}_x(t,\eta)}\Big).$$

Thus,

$$c + rt = \mathrm{Re}\Big\{\frac{1}{i2\pi\partial_\eta\Big(\frac{V_x(t,\eta)}{V^{g_1}_x(t,\eta)}\Big)}\partial_\eta\Big(\frac{\partial_t V_x(t,\eta)}{V^{g_1}_x(t,\eta)}\Big)\Big\}.$$

This completes the proof of Proposition 1. ■

3 Instantaneous frequency estimation

We consider multicomponent signals $x(t)$ given by (1) with the trend $A_0(t)$ of $x(t)$ removed, namely,

$$x(t) = \sum_{k=1}^{K} x_k(t) = \sum_{k=1}^{K} A_k(t) e^{i2\pi\phi_k(t)}. \tag{16}$$

One may refer to [5] about how to remove $A_0(t)$. We assume

$$A_k(t) \in C^1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \phi_k(t) \in C^3(\mathbb{R}), \phi_k''(t) \in L_\infty(\mathbb{R}), \tag{17}$$

$$A_k(t) > 0, \ \inf_{t\in R} \phi_k'(t) > 0, \ \sup_{t\in R} \phi_k'(t) < \infty. \tag{18}$$

For a given t, we use $G_k(\xi)$ to denote the Fourier transform of $e^{i\pi\phi_k''(t)\tau^2} g(\tau)$, namely,

$$G_k(\xi) := \int_{\mathbb{R}} e^{i\pi\phi_k''(t)\tau^2} g(\tau) e^{-i2\pi\xi\tau} d\tau. \tag{19}$$

Note that $G_k(\xi)$ depends on t also if $\phi_k''(t) \neq 0$. We drop t in G_k for simplicity.

In the following we assume that each component $x_k(t)$ of $x(t)$ is well approximated locally by linear chirp signals of (9) with $A_k'(t)$ and $\phi_k^{(3)}(t)$ small:

$$|A_k'(t)| \le \varepsilon_1, \ |\phi_k^{(3)}(t)| \le \varepsilon_3, \ t \in \mathbb{R}, \ 1 \le k \le K, \tag{20}$$

for some small positive numbers $\varepsilon_1, \varepsilon_3$.

Let $x(t)$ be a multicomponent signal of (16) satisfying (17), (18) and (20). Write $x(t+\tau)$ as

$$x(t+\tau) = x_{\mathrm{m}}(t,\tau) + x_{\mathrm{r}}(t,\tau),$$

where

$$\begin{aligned} x_{\mathrm{m}}(t,\tau) &:= \sum_{k=1}^{K} x_k(t) e^{i2\pi(\phi_k'(t)\tau + \frac{1}{2}\phi_k''(t)\tau^2)}, \\ x_{\mathrm{r}}(t,\tau) &:= \sum_{k=1}^{K} \Big\{ (A_k(t+\tau) - A_k(t)) e^{i2\pi\phi_k(t+\tau)} \\ &\quad + x_k(t) e^{i2\pi(\phi_k'(t)\tau + \frac{1}{2}\phi_k''(t)\tau^2)} \big(e^{i2\pi(\phi_k(t+\tau) - \phi_k(t) - \phi_k'(t)\tau - \frac{1}{2}\phi_k''(t)\tau^2)} - 1 \big) \Big\}. \end{aligned}$$

Then we have

$$\begin{aligned} V_x(t,\eta) &= \sum_{k=1}^{K} \int_{\mathbb{R}} x_k(t+\tau) g(\tau) e^{-i2\pi\eta\tau} d\tau \\ &= \sum_{k=1}^{K} \int_{\mathbb{R}} x_k(t) e^{i2\pi(\phi_k'(t)\tau + \frac{1}{2}\phi_k''(t)\tau^2)} g(\tau) e^{-i2\pi\eta\tau} d\tau + \mathrm{res}_0 \\ &= \sum_{k=1}^{K} x_k(t) G_k\big(\eta - \phi_k'(t)\big) + \mathrm{res}_0, \end{aligned} \tag{21}$$

where

$$\mathrm{res}_0 := \sum_{k=1}^{K} \int_{\mathbb{R}} x_{\mathrm{r}}(t,\tau) g(\tau) e^{-i2\pi\eta\tau} d\tau. \tag{22}$$

The conditions in (20) imply that $|A_k(t+\tau) - A_k(t)| \le \varepsilon_1 |\tau|$ and

$$|e^{i2\pi(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau-\frac{1}{2}\phi_k''(t)\tau^2)} - 1| \le \frac{\pi}{3}\varepsilon_3|\tau|^3.$$

Thus we have

$$\begin{aligned} |\mathrm{res}_0| &\le \sum_{k=1}^{K} \int_{\mathbb{R}} \varepsilon_1 |\tau||g(\tau)| d\tau + M(t) \int_{\mathbb{R}} \frac{\pi}{3}\varepsilon_3 |\tau|^3 |g(\tau)| d\tau \\ &= K\varepsilon_1 I_1 + \frac{\pi}{3}\varepsilon_3 I_3 M(t) =: \Pi_0(t), \end{aligned} \tag{23}$$

where $M(t)$ and I_n are defined by

$$M(t) := \sum_{k=1}^{K} A_k(t), \quad I_n := \int_{\mathbb{R}} |\tau^n g(\tau)| d\tau, \ n = 1, 2, \cdots. \tag{24}$$

Observe that if $\varepsilon_1, \varepsilon_3$ are small, then $|\mathrm{res}_0|$ could be negligible. Thus $G_k\big(\eta - \phi_k'(t)\big)$ governs the time-frequency zone in which $V_{x_k}(t,\eta)$ lies. Let $0 < \tau_0 < 1$ be a given small number as the threshold for 0. Denote

$$O_k := \{(t,\eta) : |G_k\big(\eta - \phi_k'(t)\big)| > \tau_0, t \in \mathbb{R}\}. \tag{25}$$

In this paper we assume $|G_k(\xi)|$ is even and decreasing for $\xi \ge 0$. In this case O_k can be written as

$$O_k = \{(t,\eta) : |\eta - \phi_k'(t)| < \alpha_k, t \in \mathbb{R}\}$$

with $\alpha_k = \alpha_k(t) = \xi_k(t)$, where $\xi_k(t) > 0$ is the root of $|G_k(\xi)| = \tau_0$. As in [14], we could let O_k be a larger zone by selecting an $\alpha_k(t) \ge \xi_k(t)$. In either case, we have that

$$|G_k\big(\eta - \phi_k'(t)\big)| \le \tau_0, \text{ for } (t,\eta) \notin O_k. \tag{26}$$

In the following we assume that the multicomponent signal $x(t)$ is well-separated in the sense that

$$O_k \cap O_\ell = \emptyset, \quad k \ne \ell. \tag{27}$$

In the following we let $\mathcal{D}_{\varepsilon_1,\varepsilon_3}$ denote the set of multicomponent signals of (16) satisfying (17), (18), (20) and (27).

Let res_1, res_2, res_0', and res_1' be the residuals defined as res_0 in (22) with $g(\tau)$ replaced respectively by $g_1(\tau)$, $g_2(\tau)$, $g'(\tau)$, and $g_3(\tau) = \tau g'(\tau)$. Then we have the estimates for these residuals similar to (23):

$$|\text{res}_1| \le \Pi_1(t),\ |\text{res}_2| \le \Pi_2(t), \quad |\text{res}_0'| \le \widetilde{\Pi}_0(t),\ |\text{res}_1'| \le \widetilde{\Pi}_1(t), \tag{28}$$

where

$$\Pi_1(t) := K\varepsilon_1 I_2 + \frac{\pi}{3}\varepsilon_3 I_4 M(t), \quad \Pi_2(t) := K\varepsilon_1 I_3 + \frac{\pi}{3}\varepsilon_3 I_5 M(t),$$
$$\widetilde{\Pi}_0(t) := K\varepsilon_1 \widetilde{I}_1 + \frac{\pi}{3}\varepsilon_3 \widetilde{I}_3 M(t), \quad \widetilde{\Pi}_1(t) := K\varepsilon_1 \widetilde{I}_2 + \frac{\pi}{3}\varepsilon_3 \widetilde{I}_4 M(t),$$

with I_n defined by (24) and $\widetilde{I}_n$ given by

$$\widetilde{I}_n := \int_{\mathbb{R}} |\tau^n g'(\tau)| d\tau,\ n = 1, 2, \cdots. \tag{29}$$

Next we introduce more notations to describe our main theorems on the 2nd-order FSST. For $j \ge 0$, denote

$$\begin{aligned} G_{j,k}(t,\eta) &:= \int_{\mathbb{R}} e^{i2\pi(\phi_k'(t)\tau + \frac{1}{2}\phi_k''(t)\tau^2)} \tau^j g(\tau) e^{-i2\pi\eta\tau} d\tau \\ &= \mathcal{F}\Big(e^{i\pi\phi_k''(t)\tau^2} \tau^j g(\tau)\Big)(\eta - \phi_k'(t)). \end{aligned} \tag{30}$$

Then we have

$$G_{0,k}(t,\eta) = G_k(\eta - \phi_k'(t)),$$

and for $j \ge 1$,

$$G_{j,k}(t,\eta) = \frac{1}{-i2\pi} \frac{\partial}{\partial\eta} G_{(j-1),k}(t,\eta) = \frac{1}{(-i2\pi)^j} G_k^{(j)}(\eta - \phi_k'(t)). \tag{31}$$

In addition, we denote

$$\text{Res}_1 := i2\pi B_k(t,\eta) + i2\pi D_k(t,\eta) + i2\pi\big(\eta - \phi_k'(t)\big)\text{res}_0 - \text{res}_0' - i2\pi\phi_k''(t)\text{res}_1, \tag{32}$$

where

$$B_k(t,\eta) := \sum_{\ell \ne k} x_\ell(t)\big(\phi_\ell'(t) - \phi_k'(t)\big) G_{0,\ell}(t,\eta), \quad D_k(t,\eta) := \sum_{\ell \ne k} x_\ell(t)\big(\phi_\ell''(t) - \phi_k''(t)\big) G_{1,\ell}(t,\eta).$$

Throughout this paper, $\sum_{\ell \ne k}$ denotes $\sum_{\{\ell:\ \ell \ne k, 1 \le k \le K\}}$.

Let $\text{Res}_2 := \frac{\partial}{\partial\eta}\big(\text{Res1}\big)$. Then by direct calculations, we have

$$\begin{aligned} \text{Res}_2 = {} & 4\pi^2 E_k(t,\eta) + 4\pi^2 F_k(t,\eta) \\ & + i2\pi\ \text{res}_0 + 4\pi^2\big(\eta - \phi_k'(t)\big)\ \text{res}_1 + i2\pi\ \text{res}_1' - 4\pi^2\phi_k''(t)\ \text{res}_2, \end{aligned} \tag{33}$$

where

$$E_k(t,\eta) := \sum_{\ell\neq k} x_\ell(t)\big(\phi'_\ell(t)-\phi'_k(t)\big)G_{1,\ell}(t,\eta), \quad F_k(t,\eta) := \sum_{\ell\neq k} x_\ell(t)\big(\phi''_\ell(t)-\phi''_k(t)\big)G_{2,\ell}(t,\eta).$$

To present the estimate of IFs $\phi'_k(t)$ using $u_x^{\rm 2nd}(t,\eta)$, first we have the following lemma.

Lemma 1. *Let* Res_1 *be the quantity defined by* (32)*. Then*

$$\partial_t V_x(t,\eta) = i2\pi\phi'_k(t)V_x(t,\eta) + i2\pi\phi''_k(t)V_x^{g_1}(t,\eta) + \mathrm{Res}_1. \tag{34}$$

One can obtain (34) by direct calculations (also refer to Lemma 1 in [3]).

Theorem 1. *Suppose* $x(t)\in\mathcal{D}_{\varepsilon_1,\varepsilon_3}$ *for some* $\varepsilon_1,\varepsilon_3>0$*. Suppose the window function* g *is in* $\mathcal{S}$*. Then we have the following.*

(i) *For* (t,η) *satisfies* $|V_x^{g_1}(t,\eta)|\neq 0$, $|\partial_\eta\big(V_x(t,\eta)/V_x^{g_1}(t,\eta)\big)|\neq 0$*, we have*

$$u_x^{\rm 2nd,c}(t,\eta)-\phi'_k(t) = \mathrm{Res}_4, \tag{35}$$

where

$$\mathrm{Res}_4 := \frac{1}{i2\pi\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)(V_x^{g_1}(t,\eta))^2}\Big(\mathrm{Res}_2\, V_x^{g_1}(t,\eta) + i2\pi\mathrm{Res}_1\, V_x^{g_2}(t,\eta)\Big), \tag{36}$$

with Res1 *and* Res2 *defined by* (32) *and* (33) *respectively.*

(ii) *For* $(t,\eta)\in O_k$ *satisfies* $|V_x^{g_1}(t,\eta)|>\widetilde{\varepsilon}_1$, $|\partial_\eta\big(V_x(t,\eta)/V_x^{g_1}(t,\eta)\big)|>\widetilde{\varepsilon}_2$*, we have*

$$|u_x^{\rm 2nd}(t,\eta)-\phi'_k(t)| < \mathrm{Bd}_k, \tag{37}$$

where

$$\mathrm{Bd}_k := \frac{1}{2\pi\widetilde{\varepsilon}_1\widetilde{\varepsilon}_2}\sup_{\{\eta:(t,\eta)\in O_k\}}\Big\{|\mathrm{Res}_2| + \frac{2\pi}{\widetilde{\varepsilon}_1}|\mathrm{Res}_1||V_x^{g_2}(t,\eta)|\Big\}. \tag{38}$$

Proof By (34), we have, if $V_x^{g_1}(t,\eta)\neq 0$, that

$$\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)} = i2\pi\phi'_k(t)\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)} + i2\pi\phi''_k(t) + \frac{\mathrm{Res}_1}{V_x^{g_1}(t,\eta)}.$$

Taking the partial derivative with respect to η to both sides of the above equation, we have

$$\partial_\eta\big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big) = i2\pi\phi'_k(t)\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big) + \partial_\eta\big(\frac{\mathrm{Res}_1}{V_x^{g_1}(t,\eta)}\big).$$

Thus if in addition, $\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\neq 0$, then the above equation leads to

$$\frac{1}{i2\pi\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)}\partial_\eta\Big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big)=\phi_k'(t)+\frac{1}{i2\pi\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)}\partial_\eta\Big(\frac{\mathrm{Res}_1}{V_x^{g_1}(t,\eta)}\Big).$$

Observe that the left-hand side of the above equation is $u_x^{\mathrm{2nd,c}}(t,\eta)$, while the facts that $\partial_\eta(\mathrm{Res}_1)=\mathrm{Res}_2$ and $\partial_\eta V_x^{g_1}(t,\eta)=-i2\pi V_x^{g_2}(t,\eta)$ lead to that the second term in the right-hand side of the above equation is Res_4. Hence, (35) holds true. This shows (i) of Theorem 1.

Using $|V_x^{g_1}(t,\eta)|>\widetilde{\varepsilon}_1$ and $|\partial_\eta\big(V_x(t,\eta)/V_x^{g_1}(t,\eta)\big)|>\widetilde{\varepsilon}_2$, we have from (35), that for any η with $(t,\eta)\in O_k$,

$$\begin{aligned}&|u_x^{\mathrm{2nd}}(t,\eta)-\phi_k'(t)|\le|u_x^{\mathrm{2nd,c}}(t,\eta)-\phi_k'(t)|\\&=|\mathrm{Res}_4|<\frac{1}{2\pi\widetilde{\varepsilon}_1\widetilde{\varepsilon}_2}\Big\{|\mathrm{Res}_2|+\frac{2\pi}{\widetilde{\varepsilon}_1}|\mathrm{Res}_1||V_x^{g_2}(t,\eta)|\Big\}\le\mathrm{Bd}_k.\end{aligned}$$

Therefore, (37) holds true. This completes the proof of Theorem 1. ■

We conclude this section by looking at the estimate error bounds Bd_k when $g(t)$ is the Gaussian function given:

$$g(t)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}},\tag{39}$$

where $\sigma>0$.

First we look at the bounds for $\mathrm{Res}_1,\mathrm{Res}_2$. From (32) and (33), we have for $(t,\eta)\in O_k$,

$$\begin{aligned}&|\mathrm{Res}_1|\le 2\pi|B_k(t,\eta)|+2\pi|D_k(t,\eta)|+2\pi\alpha_k\,\Pi_0(t)+\widetilde{\Pi}_0(t)+2\pi|\phi_k''(t)|\Pi_1(t),\\&|\mathrm{Res}_2|\le 4\pi^2|E_k(t,\eta)|+4\pi^2|F_k(t,\eta)|\\&\qquad+2\pi\Pi_0(t)+4\pi^2\alpha_k\,\Pi_1(t)+2\pi\widetilde{\Pi}_1(t)+4\pi^2|\phi_k''(t)|\,\Pi_2(t).\end{aligned}$$

Thus we need to look at the estimates for $B_k(t,\eta),D_k(t,\eta),E_k(t,\eta),F_k(t,\eta)$, which are determined by $G_{j,\ell}(t,\eta)$, for $(t,\eta)\in O_k$.

When $g(t)$ is given by (39), then one can obtain (see [14])

$$G_k(\xi)=\frac{1}{\sqrt{1-i2\pi\sigma^2\phi_k''(t)}}\;e^{-\frac{2\pi^2\sigma^2\xi^2}{1+4\pi^2\sigma^4\phi_k''(t)^2}(1+i2\pi\sigma^2\phi_k''(t))},\tag{40}$$

where $\sqrt{1-i2\pi\sigma^2\phi_k''(t)}$ denotes one of the square-roots of $1-i2\pi\sigma^2\phi_k''(t)$ which lies in the same quadrant as $1-i2\pi\sigma^2\phi_k''(t)$. Thus

$$|G_k(\xi)|=\frac{1}{\big(1+4\pi^2\sigma^4\phi_k''(t)^2\big)^{\frac14}}\;e^{-\frac{2\pi^2\sigma^2\xi^2}{1+4\pi^2\sigma^4\phi_k''(t)^2}}.\tag{41}$$

For $(t,\eta)\in O_k$, we have

$$|\eta-\phi'_\ell(t)|\geq|\phi'_k(t)-\phi'_\ell(t)|-|\eta-\phi'_k(t)|>|\phi'_k(t)-\phi'_\ell(t)|-\alpha_k.$$

Therefore,

$$B_k(t,\eta)\leq\sum_{\ell\neq k}A_\ell(t)\frac{\left|\phi'_\ell(t)-\phi'_k(t)\right|}{(1+4\pi^2\sigma^4\phi''_\ell(t)^2)^{\frac{1}{4}}}\ e^{-\frac{2\pi^2\sigma^2}{1+4\pi^2\sigma^4\phi''_\ell(t)^2}\left(|\phi'_k(t)-\phi'_\ell(t)|-\alpha_k\right)^2}.$$

In other words, $B_k(t,\eta)$ is essentially bounded by

$$A_{k\pm1}(t)\frac{\left|\phi'_{k\pm1}(t)-\phi'_k(t)\right|}{(1+4\pi^2\sigma^4\phi''_{k\pm1}(t)^2)^{\frac{1}{4}}}\ e^{-\frac{2\pi^2\sigma^2}{1+4\pi^2\sigma^4\phi''_{k\pm1}(t)^2}\left(|\phi'_k(t)-\phi'_{k\pm1}(t)|-\alpha_k\right)^2}.$$

For $D_k(t,\eta)$ and $E_k(t,\eta)$, we need to estimate $G_{1,\ell}(t,\eta)$. By (31), we have

$$G_{1,\ell}(t,\eta)=\frac{1}{-i2\pi}G'_\ell(\eta-\phi'_\ell(t)).$$

Thus when g is the Gaussian function, we have from (40)

$$G'_\ell(\xi)=-\frac{1}{\sqrt{1-i2\pi\sigma^2\phi''_\ell(t)}}\ \frac{4\pi^2\sigma^2\xi}{1+4\pi^2\sigma^4\phi''_\ell(t)^2}(1+i2\pi\sigma^2\phi''_\ell(t))e^{-\frac{2\pi^2\sigma^2\xi^2}{1+4\pi^2\sigma^4\phi''_\ell(t)^2}(1+i2\pi\sigma^2\phi''_\ell(t))}. \quad (42)$$

Hence,

$$|G_{1,\ell}(t,\eta)|=\frac{1}{2\pi}|G'_\ell(\eta-\phi'_\ell(t))|\leq\frac{2\pi\sigma^2|\eta-\phi'_\ell(t)|}{\left(1+4\pi^2\sigma^4\phi''_\ell(t)^2\right)^{\frac{3}{4}}}\ e^{-\frac{2\pi^2\sigma^2}{1+4\pi^2\sigma^4\phi''_\ell(t)^2}(\eta-\phi'_\ell(t))^2}.$$

This estimate leads to that for $(t,\eta)\in O_k$,

$$D_k(t,\eta)\leq 2\pi\sigma^2\sum_{\ell\neq k}A_\ell(t)\frac{\left|\phi''_\ell(t)-\phi''_k(t)\right|}{(1+4\pi^2\sigma^4\phi''_\ell(t)^2)^{\frac{3}{4}}}\left(|\phi'_k(t)-\phi'_\ell(t)|+\alpha_k\right)\ e^{-\frac{2\pi^2\sigma^2}{1+4\pi^2\sigma^4\phi''_\ell(t)^2}\left(|\phi'_k(t)-\phi'_\ell(t)|-\alpha_k\right)^2},$$

$$E_k(t,\eta)\leq 2\pi\sigma^2\sum_{\ell\neq k}A_\ell(t)\frac{\left|\phi'_\ell(t)-\phi'_k(t)\right|}{(1+4\pi^2\sigma^4\phi''_\ell(t)^2)^{\frac{3}{4}}}\left(|\phi'_k(t)-\phi'_\ell(t)|+\alpha_k\right)\ e^{-\frac{2\pi^2\sigma^2}{1+4\pi^2\sigma^4\phi''_\ell(t)^2}\left(|\phi'_k(t)-\phi'_\ell(t)|-\alpha_k\right)^2}.$$

Using the fact $G_{2,\ell}(t,\eta)=\frac{1}{(-i2\pi)^2}G''_\ell(\eta-\phi'_\ell(t))$, we can obtain the estimate for $F_k(t,\eta)$ similarly. Here we omit the details.

By the above discussions, we can conclude that for $(t,\eta)\in O_k$, $B_k(t,\eta), D_k(t,\eta), E_k(t,\eta)$ and $F_k(t,\eta)$ are all quite small if $\phi''_\ell(t)$ is not too large and α_k is reasonable large. Thus, if addition ε_1 and ε_3 are small, then Bd_k is small.

When $g(t)$ is given by (39), form (41), assuming $\tau_0\big(1+(2\pi\phi''_k(t)\sigma^2)^2\big)^{\frac{1}{4}}\leq 1$, one case get the positive root of $|G_k(\xi)|=\tau_0$ is

$$\alpha_k=\frac{1}{\sqrt{2}\pi\sigma}\sqrt{1+\left(2\pi\phi''_k(t)\sigma^2\right)^2}\ \sqrt{\ln\left(\frac{1}{\tau_0}\right)-\frac{1}{4}\ln\left(1+\left(2\pi\phi''_k(t)\sigma^2\right)^2\right)}. \quad (43)$$

4 Component recovery estimation

In this section, we consider the component recovery analysis. Denote

$$B_t := \big\{\eta:\ (t,\eta)\in O_k, |V_x(t,\eta)|>\widetilde{\varepsilon}_0, |V_x^{g_1}(t,\eta)|>\widetilde{\varepsilon}_1, \big|\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\big|>\widetilde{\varepsilon}_2\big\}, \tag{44}$$

$$\widetilde{B}_t := \big\{\eta:\ (t,\eta)\in O_k, |V_x(t,\eta)|>\widetilde{\varepsilon}_0\big\}\cap\big\{\eta: |V_x^{g_1}(t,\eta)|\le\widetilde{\varepsilon}_1 \text{ or } \big|\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\big|\le\widetilde{\varepsilon}_2\big\}. \tag{45}$$

Recall from Theorem 1 that we obtained an error bound of $\phi_k'(t)-u_x^{\rm 2nd}(t,\eta)$ for $\eta\in B_t$ as shown in (37). In the next lemma, we provide an error bound for $\phi_k'(t)-\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}$ when $\eta\in\widetilde{B}_t$.

Lemma 2. *For (t,η) satisfies $|V_x(t,\eta)|\neq 0$,*

$$\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}-\phi_k'(t)=\phi_k''(t)\frac{V_x^{g_1}(t,\eta)}{V_x(t,\eta)}+\frac{{\rm Res}_1}{i2\pi V_x(t,\eta)}, \tag{46}$$

where ${\rm Res}_1$ is defined by (32). Hence for $\eta\in\widetilde{B}_t$,

$$\Big|\phi_k'(t)-\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big|\le\widetilde{\rm Bd}_k:=\frac{1}{\widetilde{\varepsilon}_0}\sup_{\{\eta:(t,\eta)\in\widetilde{B}_t\}}\Big\{|\phi_k''(t)||V_x^{g_1}(t,\eta)|+\frac{1}{2\pi}|{\rm Res}_1|\Big\}. \tag{47}$$

Proof Following Lemma 1, we have (34). Dividing both sides of (34) by $i2\pi V_x(t,\eta)$ leads to (46). (47) follows directly from (46) and the assumption $|V_x(t,\eta)|>\widetilde{\varepsilon}_0$. ■

Next we provide the component recovery estimate by using $S^{\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)$.

Theorem 2. *Suppose $x(t)\in\mathcal{D}_{\varepsilon_1,\varepsilon_3}$ for some small $\varepsilon_1,\varepsilon_3>0$. Let g be a window function in $\mathcal{S}$. Let ${\rm Bd}_k,\widetilde{\rm Bd}_k$ be defined by (38) and (47) respectively. Then we have the following.*

(i) *Suppose $\widetilde{\varepsilon}_0$ satisfies $\widetilde{\varepsilon}_0\ge\Pi_0(t)+\tau_0M(t)$. Then for (t,η) with $|V_x(t,\eta)|>\widetilde{\varepsilon}_0$, there exists $k\in\{1,2,\cdots,K\}$ such that $(t,\eta)\in O_k$.*

(ii) *Suppose that $\widetilde{\varepsilon}_0$ satisfies the condition in part* (i) *and $\max\limits_{1\le k\le K}\{{\rm Bd}_k,\widetilde{\rm Bd}_k\}\le\frac12 L_k$, where*

$$L_k=L_k(t):=\min\{\alpha_k+\alpha_{k-1},\alpha_k+\alpha_{k+1}\}. \tag{48}$$

Then for any $\widetilde{\varepsilon}_3=\widetilde{\varepsilon}_3(t)>0$ satisfying $\max\limits_{1\le k\le K}\{{\rm Bd}_k,\widetilde{\rm Bd}_k\}\le\widetilde{\varepsilon}_3\le\frac12 L_k(t)$,

$$\Big|\lim_{\lambda\to0}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon}_3}S^{\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)d\xi-g(0)x_k(t)\Big|\le C_k, \tag{49}$$

where

$$C_k:=2\alpha_k\left(\widetilde{\varepsilon}_0+\Pi_0(t)\right)+A_k(t)\ \Big|\int_{|\xi|\ge\alpha_k}G_k(\xi)d\xi\Big|+\sum_{\ell\neq k}A_\ell(t)M_{\ell,k}(t), \tag{50}$$

with $\Pi_0(t)$ defined by (23), and

$$M_{\ell,k}(t):=\int_{|\xi|<\alpha_k}|G_\ell\big(\xi+\phi_k'(t)-\phi_\ell'(t)\big)|d\xi. \tag{51}$$

Proof of Theorem 2 (i) Suppose t, η satisfy $|V_x(t,\eta)| > \widetilde{\varepsilon}_0$. Assume $(t,\eta) \notin \cup_{k=1}^K O_k$. Then for any k, by the definition of O_k in (25), we have (26), which together with (21) and (23), implies

$$\begin{aligned}|V_x(t,\eta)| &\le |\mathrm{res}_0| + \sum_{k=1}^K |x_k(t) G_k(\eta - \phi_k'(t))| \\ &\le \Pi_0(t) + \sum_{k=1}^K \tau_0 A_k(t) = \Pi_0(t) + \tau_0 M(t) \le \widetilde{\varepsilon}_0,\end{aligned}$$

a contradiction to the assumption $|V_x(t,\eta)| > \widetilde{\varepsilon}_0$. Hence the statement in (i) holds. ■

Proof of Theorem 2 (ii) First we have the following result which can be derived as that on p.254 in [6]:

$$\lim_{\lambda\to 0} \int_{|\xi - \phi_k'(t)| < \widetilde{\varepsilon}_3} S^{\mathrm{2nd},\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi) d\xi = \int_{\left\{\eta:\ |V_x(t,\eta)| > \widetilde{\varepsilon}_0,\ |\phi_k'(t) - u^{\mathrm{2nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)| < \widetilde{\varepsilon}_3\right\}} V_x(t,\eta) d\eta. \tag{52}$$

Denote

$$\begin{aligned}A_t := \Big\{\eta :\ &|V_x(t,\eta)| > \widetilde{\varepsilon}_0,\ |V_x^{g_1}(t,\eta)| > \widetilde{\varepsilon}_1,\ \big|\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\big| > \widetilde{\varepsilon}_2, \text{ and} \\ &\big|\phi_k'(t) - u^{\mathrm{2nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)\big| < \widetilde{\varepsilon}_3\Big\},\end{aligned} \tag{53}$$

$$\begin{aligned}\widetilde{A}_t := \Big\{\eta :\ &|V_x(t,\eta)| > \widetilde{\varepsilon}_0,\ \big|\phi_k'(t) - \mathrm{Re}\big\{\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\big\}\big| < \widetilde{\varepsilon}_3\Big\} \cap \\ &\Big\{\eta :\ |V_x^{g_1}(t,\eta)| > \widetilde{\varepsilon}_1 \text{ or } \big|\partial_\eta\big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\big)\big| \le \widetilde{\varepsilon}_2\Big\}.\end{aligned} \tag{54}$$

Then

$$\{\eta :\ |V_x(t,\eta)| > \widetilde{\varepsilon}_0,\ |\phi_k'(t) - u^{\mathrm{2nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)| < \widetilde{\varepsilon}_3\} = A_t \cup \widetilde{A}_t. \tag{55}$$

Let B_t and $\widetilde{B}_t$ be the sets defined by (44) and (45) respectively. Next we show that $B_t = A_t$, $\widetilde{B}_t = \widetilde{A}_t$.

Suppose $\eta \in B_t$. Then by Theorem 1 (ii), we have $\big|\phi_k'(t) - u^{\mathrm{2nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)\big| < \mathrm{Bd}_k \le \widetilde{\varepsilon}_3$. Thus $\eta \in A_t$. This concludes $B_t \subseteq A_t$.

To show $A_t \subseteq B_t$, let us consider $\eta \in A_t$. By Theorem 2 (i), the assumption $|V_x(t,\eta)| > \widetilde{\varepsilon}_0$ implies $(t,\eta) \in O_\ell$ for an ℓ in $\{1, 2, \cdots, K\}$. Thus, by Theorem 1 (ii), we have

$$\big|\phi_\ell'(t) - u^{\mathrm{2nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)\big| < \mathrm{Bd}_\ell \le \widetilde{\varepsilon}_3.$$

If $\ell \neq k$, then

$$\begin{aligned}\big|\phi_k'(t) - u^{\mathrm{2nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)\big| &\ge |\phi_k'(t) - \phi_\ell'(t)| - \big|\phi_\ell'(t) - u^{\mathrm{2nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)\big| \\ &> L_k - \widetilde{\varepsilon}_3 \ge \widetilde{\varepsilon}_3.\end{aligned}$$

This contradicts to the assumption $\eta \in A_t$ with $\left|\phi_k'(t) - u^{2\mathrm{nd}}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\eta)\right| < \widetilde{\varepsilon}_3$. Therefore, $\ell = k$ and hence, $\eta \in B_t$. This shows $A_t = B_t$, as desired.

The proof for $\widetilde{A}_t = \widetilde{B}_t$ is similar to that for $A_t = B_t$. In this case, one needs to apply (47) in Lemma 2 to conclude that for $(t,\eta) \in O_\ell$,

$$\left|\phi_\ell'(t) - \frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\right| \le \widetilde{\mathrm{Bd}}_\ell \le \widetilde{\varepsilon}_3.$$

The other details are omitted here.

From (52) and (55), we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon}_3} S^{2\mathrm{nd},\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)d\xi = \int_{A_t\cup \widetilde{A}_t} V_x(t,\eta)d\eta \\
&= \int_{B_t\cup\widetilde{B}_t} V_x(t,\eta)d\eta = \int_{\{|V_x(t,\eta)|>\widetilde{\varepsilon}_0\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta \\
&= \int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - \int_{\{|V_x(t,\eta)|\le\widetilde{\varepsilon}_0\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta. \qquad (56)
\end{aligned}$$

Clearly,

$$\Big|\int_{\{|V_x(t,\eta)|\le\widetilde{\varepsilon}_0\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta\Big| \le \int_{\{|V_x(t,\eta)|\le\widetilde{\varepsilon}_0\}\cap\{\eta:(t,\eta)\in O_k\}} \widetilde{\varepsilon}_0 d\eta \le 2\widetilde{\varepsilon}_0\alpha_k. \qquad (57)$$

In addition,

$$\begin{aligned}
&\Big|\int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - g(0)x_k(t)\Big| \qquad (58)\\
&= \Big|\int_{\{\eta:(t,\eta)\in O_k\}} \big(\sum_{\ell=1}^{K} x_\ell(t)G_{0,\ell}(t,\eta) + \mathrm{res}_0\big)d\eta - g(0)x_k(t)\Big| \\
&\le 2\alpha_k\,\Pi_0(t) + \big|x_k(t)\int_{|\xi|<\alpha_k} G_k(\xi)d\xi - g(0)x_k(t)\big| + \sum_{\ell\neq k} A_\ell(t)\Big|\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,\ell}(t,\eta)d\eta\Big| \\
&\le 2\alpha_k\,\Pi_0(t) + \big|x_k(t)\int_{\mathbb{R}} G_k(\xi)d\xi - g(0)x_k(t) - x_k(t)\int_{|\xi|\ge\alpha_k} G_k(\xi)d\xi\big| + \sum_{\ell\neq k} A_\ell(t)M_{\ell,k}(t) \\
&= 2\alpha_k\,\Pi_0(t) + \big|x_k(t)g(0) - g(0)x_k(t) - x_k(t)\int_{|\xi|\ge\alpha_k} G_k(\xi)d\xi\big| + \sum_{\ell\neq k} A_\ell(t)M_{\ell,k}(t).
\end{aligned}$$

In the above we have used the facts

$$\int_{\{\eta:\,(t,\eta)\in O_k\}} |G_{0,\ell}(t,\eta)|d\eta = \int_{|\xi|<\alpha_k} \big|G_\ell\big(\xi+\phi_k'(t)-\phi_\ell'(t)\big)\big|d\xi = M_{\ell,k}(t),$$

and

$$\int_{\mathbb{R}} G_k(\xi)d\xi = \int_{\mathbb{R}} \mathcal{F}\big\{e^{i\pi\phi_k''(t)\tau^2}g(\tau)\big\}(\xi)d\xi = \big(e^{i\pi\phi_k''(t)\tau^2}g(\tau)\big)\Big|_{\tau=0} = g(0), \qquad (59)$$

where $\mathcal{F}$ denotes the Fourier transform of function $e^{i\pi\phi_k''(t)\tau^2}g(\tau)$ with indepedent variable τ. Thus we have

$$\Big|\int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - g(0)x_k(t)\Big| \le 2\alpha_k\,\Pi_0(t) + A_k(t)\,\Big|\int_{|\xi|\ge\alpha_k} G_k(\xi)d\xi\Big| + \sum_{\ell\neq k} A_\ell(t)M_{\ell,k}(t). \tag{60}$$

This estimate, together with (56) and (57), leads to (49). This completes the proof of Theorem 2 (ii). ■

Denote

$$g_k := \int_{|\xi|<\alpha_k} G_k(\xi)d\xi. \tag{61}$$

If we replace $g(0)$ in (58) by g_k, then as the proof of (60), we have

$$\Big|\int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - g_k x_k(t)\Big| \le 2\alpha_k\,\Pi_0(t) + \sum_{\ell\neq k} A_\ell(t)M_{\ell,k}(t).$$

This and the proof of Theorem 2 lead to the following corollary.

Corollary 1. *Suppose the conditions in Theorem 2 hold. Then we have*

$$\Big|\lim_{\lambda\to 0}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon}_3} S^{\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)d\xi - g_k x_k(t)\Big| \le 2\alpha_k\,\big(\widetilde{\varepsilon}_0 + \Pi_0(t)\big) + \sum_{\ell\neq k} A_\ell(t)M_{\ell,k}(t), \tag{62}$$

where g_k is defined by (61)*.*

Next we consider another type of the 2nd-order FSST, which is defined as:

$$R^{2\mathrm{nd},\lambda}_{x,\gamma_0,\gamma_1,\gamma_2}(t,\xi) = \int_{\Big\{\eta:\,|V_x(t,\eta)|>\gamma_0,\,|V_x^{g_1}(t,\eta)|>\gamma_1,\,\Big|\partial\eta\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big)\Big|>\gamma_2\Big\}} V_x(t,\eta)\frac{1}{\lambda}h\Big(\frac{\xi-u^{2\mathrm{nd}}_{x,\gamma_0,\gamma_1,\gamma_2}(t,\eta)}{\lambda}\Big)d\eta. \tag{63}$$

Theorem 3. *Let $x(t)\in\mathcal{D}_{\varepsilon_1,\varepsilon_3}$ for some small $\varepsilon_1,\varepsilon_3>0$. Suppose that $\widetilde{\varepsilon}_0$ satisfies the condition in Theorem 2 part* (i) *and $\max\limits_{1\le k\le K}\{\mathrm{Bd}_k\}\le\frac{1}{2}L_k$. Then for any $\widetilde{\varepsilon}_3=\widetilde{\varepsilon}_3(t)>0$ satisfying $\max\limits_{1\le k\le K}\{\mathrm{Bd}_k\}\le\widetilde{\varepsilon}_3\le\frac{1}{2}L_k$, we have*

$$\Big|\lim_{\lambda\to 0}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon}_3} R^{2\mathrm{nd},\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)d\xi - g(0)x_k(t)\Big| \le C_k + C_k', \tag{64}$$

where C_k is defined by (50) *and*

$$C_k' := 2\alpha_k\,\Pi_0(t)\big) + A_k(t)\,\|g\|_1|\widetilde{B}_t| + \textstyle\sum_{\ell\neq k} A_\ell(t)M_{\ell,k}(t), \tag{65}$$

with $|\widetilde{B}_t|$ denoting the Lebesgue measure of the set $\widetilde{B}_t$ defined by (45)*.*

Proof of Theorem 3 One can show that

$$\lim_{\lambda\to 0}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon}_3} R^{2\mathrm{nd},\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)d\xi=\int_{A_t} V_x(t,\eta)d\eta, \tag{66}$$

where A_t is defined by (53). Let B_t and $\widetilde{B}_t$ be the sets defined by (44) and (45) respectively. In the proof of Theorem 2 (ii) we have shown $B_t=A_t$. This fact, together with $B_t\cap\widetilde{B}_t=\emptyset$ and (66), leads to that

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon}_3} R^{2\mathrm{nd},\lambda}_{x,\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)d\xi=\int_{B_t} V_x(t,\eta)d\eta\\
&=\int_{B_t\cup\widetilde{B}_t} V_x(t,\eta)d\eta-\int_{\widetilde{B}_t} V_x(t,\eta)d\eta\\
&=\int_{\{|V_x(t,\eta)|>\widetilde{\varepsilon}_0\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta-\int_{\widetilde{B}_t} V_x(t,\eta)d\eta\\
&=\int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta-\int_{\{|V_x(t,\eta)|\le\widetilde{\varepsilon}_0\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta-\int_{\widetilde{B}_t} V_x(t,\eta)d\eta.
\end{aligned}\tag{67}$$

In addition,

$$\begin{aligned}
&\Big|\int_{\widetilde{B}_t} V_x(t,\eta)d\eta\Big|=\Big|\int_{\widetilde{B}_t}\Big(\sum_{\ell=1}^K x_\ell(t)G_{0,\ell}(t,\eta)+\mathrm{res}_0\Big)d\eta\Big|\\
&\le 2\alpha_k\,\Pi_0(t)+A_k(t)\sup_{\eta\in\widetilde{B}_t}\big|G_k\big((\eta-\phi_k'(t)\big)\big|\,|\widetilde{B}_t|+\sum_{\ell\ne k}A_\ell(t)\Big|\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,\ell}(t,\eta)d\eta\Big|\\
&\le 2\alpha_k\,\Pi_0(t)+A_k(t)\|g\|_1\,|\widetilde{B}_t|+\sum_{\ell\ne k}A_\ell(t)M_{\ell,k}(t).
\end{aligned}$$

The above estimate, together with (67), (57) and (60), leads to (64). This completes the proof of Theorem 3. ■

5 Experimental results

In this section, we present some experimental results with the second-order FSST with a phase transformation $u_x^{2\mathrm{nd}}$ introduced in this paper. We denote this FSST as FSST2s. We will compare the performance of FSST2s in instantaneous frequency estimation and component recovery with that by the conventional second-order FSST, denoted by FSST2, with $\omega_x^{2\mathrm{nd}}$ given by (8) [1]. One can verify that when the window function g is the Gaussian function, then the quantity in the first line of (8) is that in (10). Thus in this case FSST2s is essentially FSST2. In the following we consider the window function function to be $h(t)$ defined by

$$\widehat{h}(\xi):=e^{-2\pi^2\sigma^2|\xi|^{5/2}}. \tag{68}$$

We use the relative root mean square error (RMSE) to evaluate the signal separation performances of FSST2 and FSST2s with this window function $h(t)$. RMSE is defined by

$$\text{RMSE} := \frac{1}{K}\sum_{k=1}^{K}\frac{\|x_k - \hat{x}_k\|_2}{\|x_k\|_2}, \tag{69}$$

where $\hat{x}_k$ is the reconstructed (recovered) x_k and K is the number of components. For a component $x_k(t)$ defined by $[a,\ b]$, due to the big errors of FSST near the end points, which is caused by the boundary issue, we calculate RMSE on $[a+\frac{1}{50}(b-a), b-\frac{1}{50}(b-a)]$. In addition, here we remark that the errors displayed in the following pictures are the errors on these intervals excluding the end points a, b.

We proceed with two signals. The first one is a three-component non-stationary signal given by $x(t) = x_1(t) + x_2(t) + x_3(t)$, where

$$\begin{aligned}
x_1(t) &= \ln(10+t)\cos\left(118\pi(t-\tfrac{1}{2}) + 100\pi(t-\tfrac{1}{2})^2\right)\mathbb{1}_{[\frac{1}{2},1]},\\
x_2(t) &= e^{-0.2t}\cos\left(94\pi t + 110\pi t^2 + 13\cos(4\pi t - \tfrac{\pi}{2})\right),\ t\in[0,1],\\
x_3(t) &= \ln(1.5+t^2)\cos\left(194\pi t + 112\pi t^2\right)\mathbb{1}_{[0,\frac{3}{4}]}.
\end{aligned} \tag{70}$$

Signal $x(t)$ and its components $x_1(t)$, $x_2(t)$ and $x_3(t)$ are shown in Fig.1.

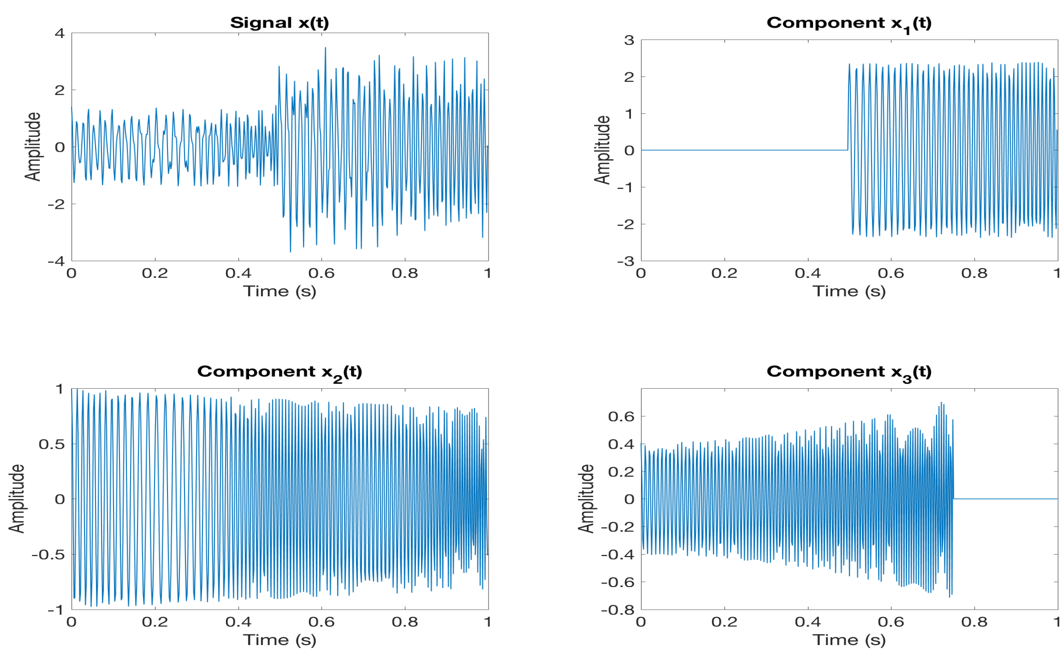

Figure 1: Signal $x(t)$ and its components $x_1(t)$, $x_2(t)$ and $x_3(t)$.

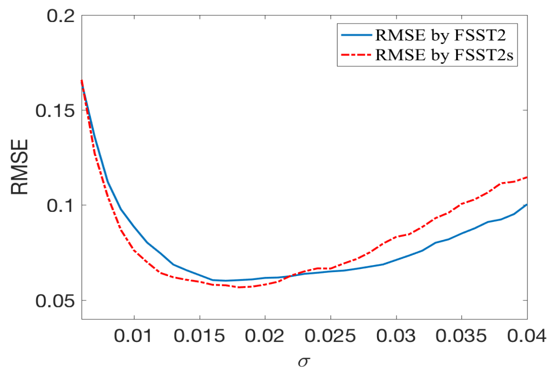

Figure 2: RMSEs for $x(t)$ by FSST2 and FSST2s with $\sigma = 0.006 + 0.001j, 0 \le j \le 34$.

σ	0.014	0.015	0.016	0.017	0.018	0.019	0.020	0.021
RMSE with FSST2	0.0657	0.0631	0.0605	**0.0602**	0.0605	0.0609	0.0616	0.0618
RMSE with FSST2s	0.0607	0.0596	0.0580	0.0577	**0.0567**	0.0570	0.0581	0.0596

Table 1: RMSEs by FSST2 and FSST2s with some σ near 0.018.

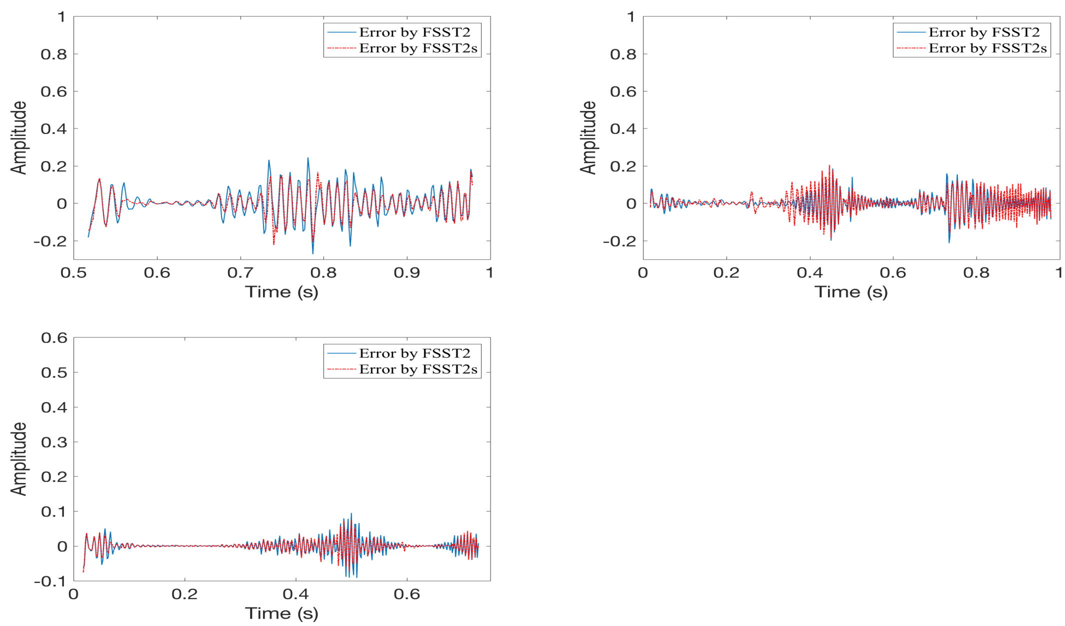

Figure 3: Errors between reconstructed components and original components by FSST2 and FSST2s: $x_1(t)$ (top-left), $x_2(t)$ (top-right), $x_3(t)$ (bottom row).

Here, we test some parameters of σ from 0.006 to 0.04 for component recovery. In Fig.2, we provide the RMSEs for signal $x(t)$ with FSST2 and FSST2s for $\sigma \in [0.006, 0.04]$. Observe that when σ is near 0.017, the resulting RMSEs gain their minima. In Table 1, we provide the RMSEs for some σ near 0.017. In Fig.3, we show the difference between reconstructed component and original component by FSST2 and FSST2s with $\sigma = 0.017$. As we can see from Fig.2, Table 1 and Fig.3, the performances of FSST2s and FSST2 are essentially similar, with FSST2s resulting in a little smaller recovery error. In Fig.4, we show the original IFs of $x_j(t), j = 1, 2, 3$ and the reconstructed IFs by FSST2 with $\sigma = 0.017$ and by FSST2s with $\sigma = 0.018$. Overall, both methods perform well in IF estimation.

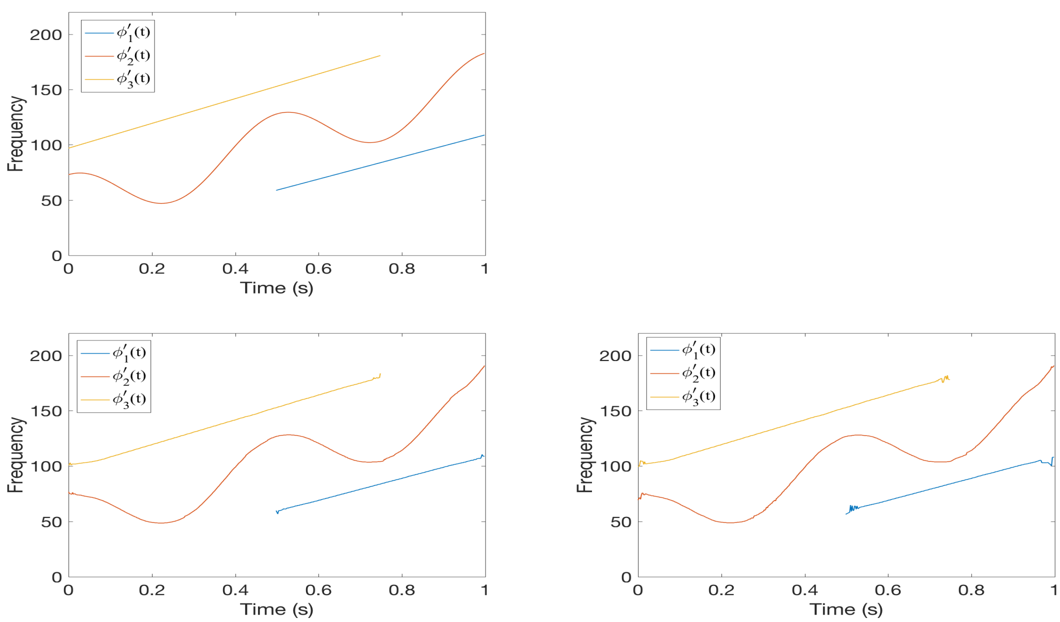

Figure 4: IFs of the components of $x(t)$ (top row); Estimated IFs by FSST2 (bottom-left) and by FSST2s (bottom-right).

The second signal we consider is a two-component signal given by

$$y(t) = y_1(t) + y_2(t),\ y_1(t) = e^{0.2t} \cos(2\pi(12t + 25t^2)),\ y_2(t) = \ln(2 + \sqrt{t}) \cos(2\pi(34t + 32t^2)), \quad (71)$$

for $t \in [0, 1]$. In Fig.5, we provide the RMSEs for signal $y(t)$ with FSST2 and FSST2s for $\sigma \in [0.006, 0.09]$. Observe that for $\sigma \in [0.015, 0.06]$, the recovery errors for either FSST2 or FSST2s are quite small. In Table 2, we provide the RMSEs for some σ near 0.02. For $y(t)$, FSST2 and FSST2s perform similarly in component recovery and IF estimation. In Fig.6, we provide the pictures of the reconstructed components of $y_1(t)$, $y_2(t)$ by FSST2 (with $\sigma = 0.022$) and FSST2s (with $\sigma = 0.02$).

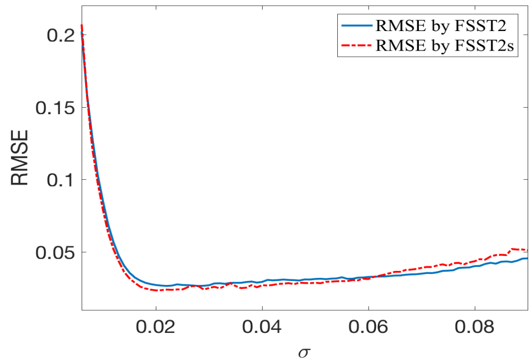

Figure 5: RMSEs for $y(t)$ by FSST2 and FSST2s with $\sigma \in [0.006, 0.09]$

σ	0.018	0.019	0.020	0.021	0.022	0.023	0.024	0.025
RMSE with FSST2	0.0287	0.0278	0.0272	0.0268	**0.0266**	0.0268	0.0276	0.0272
RMSE with FSST2s	0.0248	0.0240	**0.0235**	0.0236	0.0241	0.0239	0.0242	0.0242

Table 2: RMSEs by FSST2 and FSST2s with some σ near 0.02.

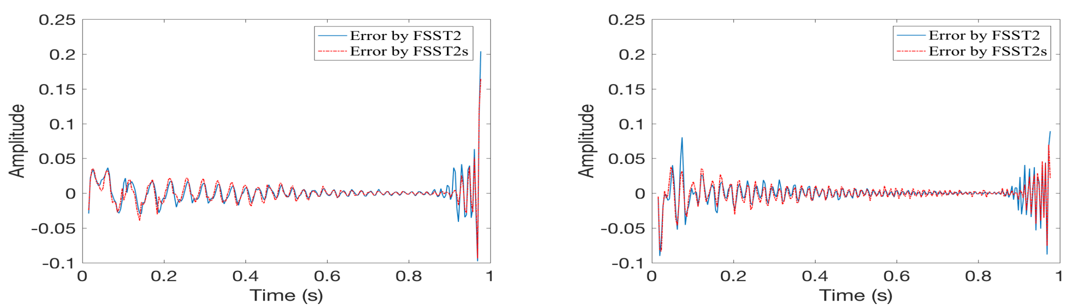

Figure 6: Errors between reconstructed components and original components by FSST2 and FSST2s: $y_1(t)$ (left) and $y_2(t)$ (right).

From the above two examples, we see FSST2s performs similiarly to or even better in some cases than the conventional FSST2 in instantaneous frequency estimation and component recovery.

6 Conclusion

In this paper we consider a second-order STFT-based synchrosqueezing transform (FSST2s). This transform has a phase transformation which has a simpler expresion than that used in the literature. We study the theoretical analysis of FSST2s. We establish an error bound for instantaneous frequency estimation and error bounds for component recovery with FSST2s. We also present more accurate component recovery formulas. Our experimental results show that the performance of FSST2s in instantaneous frequency estimation and component recovery is comparable with, and even better in some cases than, that of conventional second-order STFT-based synchrosqueezing transform.

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