ON THE REGULARITY OF MATRIX REFINABLE FUNCTIONS*

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Abstract. It is shown that the transition operator $\mathbf{T_P}$ associated with the matrix refinement mask $\mathbf{P}(\omega) = 2^{-d}\sum_{\alpha\in[0,N]^d}\mathbf{P}_\alpha\exp(-i\alpha\omega)$ is equivalent to the matrix $(2^{-d}\mathcal{A}_{2i-j})_{i,j}$ with $\mathcal{A}_j = \sum_{\kappa\in[0,N]^d}\mathbf{P}_{\kappa-j}\otimes\mathbf{P}_\kappa$ and $\mathbf{P}_{\kappa-j}\otimes\mathbf{P}_\kappa$ denoting the Kronecker product of matrices $\mathbf{P}_{\kappa-j}$, $\mathbf{P}_\kappa$. Some spectral properties of $\mathbf{T_P}$ are studied and a complete characterization of the matrix refinable functions in the Sobolev space $W^n(\mathbf{R}^d)$ for nonnegative integers n is provided. The Sobolev regularity estimate of the matrix refinable function is given in terms of the spectral radius of a restricted transition operator. These estimates are analyzed in some examples.

Key words. matrix refinable function, transition operator, regularity

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1. Introduction.

Let $\{\mathbf{P}_\alpha\}$ be a real $r\times r$ matrix sequence with finite elements nonzero. The vectors Φ, r-dimensional column functions, used in this paper are solutions to functional equations of the type

$$\Phi = \sum_{\alpha\in\mathbf{Z}^d}\mathbf{P}_\alpha\Phi(2\cdot-\alpha). \tag{1.1}$$

Define

$$\mathbf{P}(\omega) := 2^{-d}\sum_{\alpha\in\mathbf{Z}^d}\mathbf{P}_\alpha\exp(-i\alpha\omega),$$

then in the Fourier domain, functional equations (1.1) can be written as

$$\widehat{\Phi}(\omega) = \mathbf{P}(\omega/2)\widehat{\Phi}(\omega/2). \tag{1.2}$$

Equations of the type (1.1) or (1.2) are called *matrix* (*vector*) *refinement equations*; $\mathbf{P}$ ($\{\mathbf{P}_\alpha\}$) is called the (*matrix*) *refinement mask* and any solution Φ of (1.1) is called a *matrix refinable function* (or *refinable vector*). Equations (1.1) are considered in the area of wavelets for the construction of multiwavelets and there are many papers on the existence of the solutions of equations (1.1), the constructions of multiwavelets and related topics, see e.g. [1], [3], [7], [8], [11] to [16], [21] to [23], [25] to [27] and [29] to [31]. The present paper considers the Sobolev regularity of the matrix refinable functions.

For the case $r = 1, d = 1$, compactly supported refinable functions are solutions of the two-scale equation

$$\phi(x) = \sum_{j=0}^{J}h_j\phi(2x-j).$$

Over the years, several techniques have been developed to determine the regularity of refinable functions, see [5], [9], [32], [6], [10], [17] and [2] (in [17] and [2], the refinement

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mask $\{h_j\}$ is not necessarily finitely supported). One of the main results is following (see [32], [9]): assume that the refinement mask

$$m_0(\omega) = \frac{1}{2}\sum_{j=0}^{J} h_j e^{-ij\omega} \tag{1.3}$$

can be factorized as

$$m_0(\omega) = (\frac{1+e^{-i\omega}}{2})^K q(\omega), \tag{1.4}$$

where $q(\omega)$ is a trigonometric polynomial. Then the Sobolev exponent $s(\phi) := \sup\{s \geq 0 : \int (1+|\omega|^2)^s |\widehat{\phi}(\omega)|^2 d\omega < +\infty\}$ satisfies

$$s(\phi) \geq K - \log_4 \rho(\mathbf{T}_q),$$

where $\mathbf{T}_q$ is the transition operator associated with q and $\rho(\mathbf{T}_q)$ is the spectral radius of $\mathbf{T}_q$. For a trigonometric polynomial $p(\omega) = \sum_{l=0}^{L} p_l e^{-il\omega}$, the transition operator associated with p is defined by

$$\mathbf{T}_p f(\omega) := |p(\frac{\omega}{2})|^2 f(\frac{\omega}{2}) + |p(\frac{\omega}{2}+\pi)|^2 f(\frac{\omega}{2}+\pi), \quad f \in V_L,$$

where V_L denotes the vector space of trigonometric polynomials defined by

$$V_L := \{\sum_{l=-L}^{L} f_l e^{-il\omega} : \quad f_l \in \mathbf{C}\}.$$

Further, if refinable function ϕ is stable and $q(\pi) \neq 0$, then above regularity estimate is optimal, i.e.

$$s(\phi) = K - \log_4 \rho(\mathbf{T}_q).$$

There is another method to give regularity estimates of refinable functions. Let ϕ be a compactly supported refinable function with corresponding mask $m_0(\omega)$ given by (1.3) for some positive integer J. Assume that $m_0(\omega)$ satisfies the vanishing moment conditions of order $K+1$, i.e. $\frac{d^\alpha}{d\omega^\alpha} m_0(\omega)|_{\omega=\pi} = 0, 0 \leq \alpha \leq K$. Equivalently, $m_0(\omega)$ can be written in the form of (1.4). Let V_J^0 denote the subspace of V_J defined by

$$V_J^0 := \{f \in V_J : \quad \sum_{j=-J}^{J} j^n f_j = 0, \quad n = 0, \cdots, 2K-1\}. \tag{1.5}$$

Then V_J^0 is invariant under $\mathbf{T}_{m_0}$. Let $\mathbf{T}_{m_0}|_{V_J^0}$ denotes the restriction of $\mathbf{T}_{m_0}$ to V_J^0. If $\rho(\mathbf{T}_{m_0}|_{V_J^0}) < 1$, then

$$s(\phi) \geq -\log_4 \rho(\mathbf{T}_{m_0}|_{V_J^0}).$$

In fact above two methods are completely equivalent, see [6]. The first method relies upon the factorization of the refinement mask $m_0(\omega)$. However in the higher dimension case, the refinement mask is often irreducible. The second method was successfully used by Riemenschneider and Shen to estimate the regularities of two dimension refinable functions constructed in [28]. Further studies on the problem of the regularity in higher dimensions with dilation matrices were carried out in [20] and [4].

The regularity of the matrix refinable function Φ (for the case $d=1$) was first studied by Cohen, Daubechies and Plonka [3] based on the factorization of the matrix refinement mask $\mathbf{P}(\omega)$. However such estimates of regularity are usually hard to compute. There is another approach (essentially the second method for the scalar case) to the regularity estimate of refinable vector Φ carried out by Shen in [29], and such estimates are provided in terms of the spectral radius of a restricted transition operator. More precisely, letting $\mathbf{P}(\omega)=2^{-d}\sum_{\alpha\in[0,N]^d}\mathbf{P}_\alpha e^{-i\alpha\omega}$ be the corresponding matrix refinement mask, the *transition operator* $\mathbf{T_P}$ associated with $\mathbf{P}$ is defined by

$$\mathbf{T_P}H(\omega):=\sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d}\mathbf{P}(\frac{\omega}{2}+\pi\nu)H(\frac{\omega}{2}+\pi\nu)\mathbf{P}^*(\frac{\omega}{2}+\pi\nu),\quad H\in\mathbf{H}_N. \tag{1.6}$$

Throughout this paper $\mathbf{H}_N$ denotes the space of all $r\times r$ matrices with each entry a trigonometric polynomial whose Fourier coefficients are supported in $[-N,N]^d$, M^* and M^T denote the Hermitian adjoint and the transpose of a matrix M, respectively. The transition operator $\mathbf{T_P}$ leaves $\mathbf{H}_N$ invariant. In [29], the regularity of Φ was given in terms of the spectral radius of $\mathbf{T_P}|_{\mathbf{H}_N^0}$, the restricted operator of $\mathbf{T_P}$ to an invariant subspace $\mathbf{H}_N^0$ of $\mathbf{H}_N$ under $\mathbf{T_P}$. The smaller is the invariant subspace $\mathbf{H}_N^0$, the smaller will be $\rho(\mathbf{T_P}|_{\mathbf{H}_N^0})$ and hence the better the estimate on the regularity of Φ. Thus a small $\mathbf{T_P}$ invariant subspace of $\mathbf{H}_N$ is required.

For the case $r=1,d=1$, let m_0 be a given refinement mask defined by (1.3) for some positive integer J, then the transition operator $\mathbf{T}_{m_0}$ is equivalent under the basis $\{e^{-ij\omega}\}_{j=-J}^J$ of V_J to the matrix

$$\mathcal{T}_{m_0}=(2^{-1}a_{2i-j})_{-N\le i,j\le N},$$

where a_j is the autocorrelation of $\{c_\kappa\}$ defined by $a_j:=\sum_\kappa c_{\kappa-j}\overline{c_\kappa}$, see [24], [6]. We note that the invariant subspace V_J^0 defined by (1.5) can be written as

$$V_J^0=\{f\in V_J:\quad v_n(f_{-J},\cdots,f_J)^T=0,\quad n=0,\cdots,2K-1\},$$

where

$$v_n:=((-J)^n,\cdots,J^n),\quad n=0,\cdots,2K-1. \tag{1.7}$$

The row vector v_n is a generalized left 2^{-n}-eigenvector of the matrix $\mathcal{T}_{m_0}$ (see [6]). Thus to give the regularity estimates of refinable vectors, we at first change equivalently the transition operator $\mathbf{T_P}$ into its representing matrix $\mathcal{T}_\mathbf{P}$, then find left 2^{-n}-eigenvectors of the matrix $\mathcal{T}_\mathbf{P}$. Using these left eigenvectors, we construct the invariant subspace $\mathbf{H}_N^0$ and then provide the regularity estimates in terms of the spectral radius of the restricted transition operator $\mathbf{T_P}|_{\mathbf{H}_N^0}$.

This paper is organized as follows. In §2 we show that the transition operator $\mathbf{T_P}$ is equivalent to the matrix $\mathcal{T}_\mathbf{P}=(2^{-d}\mathcal{A}_{2i-j})_{i,j\in[-N,N]^d}$, where $\mathcal{A}_j$ is the $r^2\times r^2$ matrix given by

$$\mathcal{A}_j=\sum_{\kappa\in[0,N]^d}\mathbf{P}_{\kappa-j}\otimes\mathbf{P}_\kappa,$$

and $\mathbf{P}_{\kappa-j}\otimes\mathbf{P}_\kappa$ is the Kronecker product of $\mathbf{P}_{\kappa-j}$ and $\mathbf{P}_\kappa$. In §2, we also find left eigenvectors of $\mathcal{T}_\mathbf{P}$ which will be used for the regularity estimate of refinable vectors.

In the first part of §3, we will give a characterization of refinable vectors in the Sobolev space $W^n(\mathbf{R}^d), n \in \mathbf{Z}_+$. In the second part of §3, we provide a $\mathbf{T_P}$ invariant subspace $\mathbf{H}_N^0$ of $\mathbf{H}_N$ and give the regularity estimate of refinable vectors in terms of the spectral radius of the restricted transition operator $\mathbf{T_P}|_{\mathbf{H}_N^0}$. In the last part of this paper, §4, we will give the estimates on the smoothness of some matrix refinable functions. About the B-splines defined by knots $0, 0, 1, 1$ and $0, 1, 1, 2$, and the GHM-orthogonal scaling functions, our estimates on their regularities are optimal.

Before going to the next section, we introduce some notations used in this paper. Let $\mathbf{Z}_+$ denote the set of all nonnegative integers and $\mathbf{Z}_+^d$ denote the set of all d-tuples of nonnegative integers. We shall adopt the multi-index notations

$$\omega^\beta := \omega^{\beta_1} \cdots \omega_d^{\beta_d}, \quad \beta! := \beta_1! \cdots \beta_d!, \quad |\beta| := \beta_1 + \cdots + \beta_d$$

for $\omega = (\omega_1, \cdots, \omega_d)^T \in \mathbf{R}^d, \beta = (\beta_1, \cdots, \beta_d)^T \in \mathbf{Z}_+^d$. If $\alpha, \beta \in \mathbf{Z}^d$ satisfy $\beta - \alpha \in \mathbf{Z}_+^d$, we shall write $\alpha \le \beta$ and denote

$$\binom{\beta}{\alpha} := \frac{\beta!}{\alpha!(\beta - \alpha)!}.$$

For $\beta = (\beta_1, \cdots, \beta_d)^T \in \mathbf{Z}_+^d$, denote

$$D^\beta := \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}},$$

where $\partial_j = \frac{\partial}{\partial x_j}$ is the partial derivative operator with respect to the jth coordinate, $1 \le j \le d$. For $\omega, \zeta \in \mathbf{R}^d$, we use $\zeta\omega$ to denote their scalar product.

For $j = 1, \cdots, r$, let $\mathbf{e}_j := (\delta_j(k))_{k=1}^r$ denote the standard unit vectors in $\mathbf{R}^r$. In this paper, for an $r \times 1$ vector function $f = (f_1, \cdots, f_r)^T$, f is in a space on $\mathbf{R}^d$ means that every component f_i of f is in this space, and we will use the notation $|f| := (\sum_{i=1}^r |f_i|^2)^{\frac{1}{2}}$.

For a matrix or an operator A, we say$A$ satisfies *Condition E* if the spectral radius of $A \le 1$, 1 is the unique eigenvalue of A on the unit circle and 1 is simple. For two matrices A, B, $A \le B$ should be understood as that $B - A$ is positive semidefinite.

For a finitely supported sequence s on $\mathbf{Z}^d$, its support is defined by $\mathrm{supp} s := \{\beta \in \mathbf{Z}^d : s(\beta) \neq 0\}$, and for a finitely supported $r \times r$ matrix sequence S on $\mathbf{Z}^d$, its support is defined by $\mathrm{supp} S := \cup \mathrm{supp} s_{ij}$, where s_{ij} is the (i, j)-entry of S. Throughout this paper, we assume that the matrix refinement mask $\mathbf{P}$ satisfying $\mathrm{supp}\{\mathbf{P}_\alpha\} \subset [0, N]^d$ for some positive integer N, and we use c to denote the universal constant which may be different at different occurrences.

2. Transition operator. In this section, we first show that the transition operator $\mathbf{T_P}$ defined by (1.6) is equivalent to a matrix, then we study some spectral properties of $\mathbf{T_P}$.

For any $H = \sum_{j \in [-N,N]^d} H_j e^{-ij\omega} \in \mathbf{H}_N$,

$$\begin{aligned}
\mathbf{P}(\omega)H(\omega)\mathbf{P}(\omega)^* &= 2^{-2d} \sum_{\ell \in [0,N]^d} \sum_{\kappa \in [0,N]^d} \mathbf{P}_\kappa e^{-i\omega\kappa} H(\omega) \mathbf{P}_\ell^T e^{i\omega\ell} \\
&= 2^{-2d} \sum_{\ell \in [0,N]^d} \sum_{\kappa \in [0,N]^d} \mathbf{P}_\kappa H(\omega) \mathbf{P}_\ell^T e^{-i\omega(\kappa - \ell)} \\
&= 2^{-2d} \sum_{\kappa \in [0,N]^d} \sum_{n \in [-N,N]^d} \mathbf{P}_\kappa H(\omega) \mathbf{P}_{\kappa - n}^T e^{-i\omega n}
\end{aligned}$$

$$= 2^{-2d} \sum_{j\in[-N,N]^d} \sum_{\kappa\in[0,N]^d} \sum_{n\in[-N,N]^d} \mathbf{P}_\kappa H_j \mathbf{P}_{\kappa-n}^T e^{-i\omega(n+j)}.$$

Thus

$$\mathbf{T}_\mathbf{P} H(\omega) = 2^{-2d} \sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d} \sum_{j\in[-N,N]^d} \sum_{n\in[-N,N]^d} \sum_{\kappa\in[0,N]^d} \mathbf{P}_\kappa H_j \mathbf{P}_{\kappa-n}^T (-1)^{\nu(n+j)} e^{-i\frac{\omega}{2}(n+j)}.$$

For any $n \in [-N,N]^d, j \in [-N,N]^d$, write $n+j = 2\ell+\mu$ for some $\ell \in [-N,N]^d$ and $\mu \in \mathbf{Z}^d/2\mathbf{Z}^d$. By the fact that $\sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d}(-1)^{\nu\mu} = 2^d\delta_\mu$,

$$\sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d} (-1)^{\nu(n+j)} = 2^d\delta_\mu.$$

Hence

$$\begin{aligned}\mathbf{T}_\mathbf{P} H(\omega) &= 2^{-d} \sum_{j\in[-N,N]^d} \sum_{\ell\in[-N,N]^d} \sum_{\kappa\in[0,N]^d} \mathbf{P}_\kappa H_j \mathbf{P}_{\kappa-(2\ell-j)}^T e^{-i\omega\ell} \\ &= \sum_{\ell\in[-N,N]^d} (2^{-d} \sum_{j\in[-N,N]^d} \sum_{\kappa\in[0,N]^d} \mathbf{P}_\kappa H_j \mathbf{P}_{\kappa-(2\ell-j)}^T) e^{-i\omega\ell}\end{aligned} \tag{2.1}$$

That is $\mathbf{T}_\mathbf{P}$ changes sequence $\{H_j\}_{j\in[-N,N]^d}$ into another sequence

$$\{2^{-d} \sum_{j\in[-N,N]^d} \sum_{\kappa\in[0,N]^d} \mathbf{P}_\kappa H_j \mathbf{P}_{\kappa-(2\ell-j)}^T\}_{\ell\in[-N,N]^d}.$$

Let M be an $r\times r$ matrix with $M(j)$ the jth column of M, define the $r^2\times 1$ vector $\mathrm{vec}(M)$ by

$$\mathrm{vec}(M) := (M(1)^T, \cdots, M(r)^T)^T.$$

For $H = \sum_{j\in[-N,N]^d} H_j e^{-i\omega j} \in \mathbf{H}_N$, let $\mathrm{vec}(H)$ be the $(r^2(2N+1)^d)\times 1$ vectors defined by

$$\mathrm{vec}(H) := \left((\mathrm{vec}(H_j))^T|_{j=(-N,\cdots,-N)}, \cdots, (\mathrm{vec}(H_j))^T|_{j=(N,\cdots,N)}\right)^T. \tag{2.2}$$

For the matrices of the form $\mathbf{P}_\ell H_j \mathbf{P}_\kappa^T$, we have (see [19])

$$\mathrm{vec}(\mathbf{P}_\ell H_j \mathbf{P}_\kappa^T) = (\mathbf{P}_\kappa \otimes \mathbf{P}_\ell)\mathrm{vec}(H_j), \tag{2.3}$$

where $(\mathbf{P}_\kappa \otimes \mathbf{P}_\ell)$ denotes the Kronecker product of matrices $\mathbf{P}_\kappa$ and $\mathbf{P}_\ell$:

$$\mathbf{P}_\kappa \otimes \mathbf{P}_\ell = (p_\kappa(\tau,i)\mathbf{P}_\ell)_{1\le\tau,i\le r}, \quad \mathbf{P}_\kappa = (p_\kappa(\tau,i))_{1\le\tau,i\le r}.$$

For $j \in \mathbf{Z}^d$, define the $r^2\times r^2$ matrices

$$\mathcal{A}_j := \sum_{\ell\in[0,N]^d} \mathbf{P}_{\ell-j} \otimes \mathbf{P}_\ell,$$

and define the $(r^2(2N+1)^d)\times(r^2(2N+1)^d)$ matrix

$$\mathcal{T}_\mathbf{P} := \left(2^{-d}\mathcal{A}_{2i-j}\right)_{i,j\in[-N,N]^d}. \tag{2.4}$$

Then from (2.1), (2.3), for any $\kappa \in [-N,N]^d$,

$$\begin{aligned}\mathrm{vec}((\mathbf{T_P}H)_\kappa) &= 2^{-d}\sum_{j\in[-N,N]^d}\sum_{\ell\in[0,N]^d}\mathrm{vec}(\mathbf{P}_\ell H_j \mathbf{P}^T_{\ell-(2\kappa-j)})\\ &= 2^{-d}\sum_{j\in[-N,N]^d}\sum_{\ell\in[0,N]^d}(\mathbf{P}_{\ell-(2\kappa-j)}\otimes\mathbf{P}_\ell)\mathrm{vec}(H_j)\\ &= \sum_{j\in[-N,N]^d} 2^{-d}\mathcal{A}_{2\kappa-j}\mathrm{vec}(H_j) = (\mathcal{T}_\mathbf{P}\mathrm{vec}(H))(\kappa).\end{aligned}$$

Hence we have

THEOREM 2.1. *The transition operator $\mathbf{T_P}$ is equivalent to the matrix $\mathcal{T}_\mathbf{P}$ defined by* (2.4) *under the basis $\{e^{-i\omega\ell}\}_{\ell\in[-N,N]^d}$ of $\mathbf{H}_N$ and for any $H\in\mathbf{H}_N$,*

$$\mathrm{vec}(\mathbf{T_P}H) = \mathcal{T}_\mathbf{P}\mathrm{vec}(H), \tag{2.5}$$

where $\mathrm{vec}(H)$ *is the vector defined by* (2.2).

In the rest of this section, we will find some left eigenvectors of $\mathcal{T}_\mathbf{P}$. These eigenvectors are associated with the vanishing moment conditions of the matrix refinement mask $\mathbf{P}$. We say that mask $\mathbf{P}(\omega)$ satisfies the *vanishing moment conditions* of order $m\in\mathbf{Z}_+$ if there exist $1\times r$ real vectors $\mathbf{l}_0^\beta$ with $\mathbf{l}_0^0\neq 0$, $\beta\in\mathbf{Z}_+^d, |\beta|\le m-1$, such that

$$\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}(2i)^{|\alpha-\beta|}\mathbf{l}_0^\alpha(D^{\beta-\alpha}\mathbf{P})(\nu\pi) = \delta_\nu 2^{-|\beta|}\mathbf{l}_0^\beta, \quad \nu\in\mathbf{Z}^d/2\mathbf{Z}^d. \tag{2.6}$$

Assume that $\Phi = (\phi_l)_{l=1}^r \in L^2(\mathbf{R}^d)$ is a compactly supported matrix refinable function with corresponding mask $\mathbf{P}$. Under the assumption that $\phi_l(x-j)$, $1\le l\le r$, $j\in\mathbf{Z}^d$, are linearly independent, (2.6) is equivalent to that ϕ_l, $1\le l\le r$, provide approximation of order m, see [15], [27] for $d=1$; and for $d=1$ (2.6) implies a matrix factorization of $\mathbf{P}(\omega)$ under the assumption that Φ is stable (see [27], [3]). It is shown in [23] that if $\det G_\Phi(\nu\pi)\neq 0, \nu\in\mathbf{Z}^d/2\mathbf{Z}^d$, then $\mathbf{P}(0)$ satisfies Condition E and $\mathbf{P}$ satisfies the vanishing moment conditions of order at least 1, where

$$G_\Phi(\omega) := \sum_{\kappa\in\mathbf{Z}^d}\widehat{\Phi}(\omega+2\pi\kappa)\widehat{\Phi}^*(\omega+2\pi\kappa).$$

Thus in the sequel we will assume that $\mathbf{P}(0)$ satisfies Condition E and $m\ge 1$ in (2.6). In this case, if Φ is a compactly supported nontrivial refinable vector, then $\widehat{\Phi}(0)=c\mathbf{r}$ for some nonzero constant c, where $\mathbf{r}$ is the normalized right 1-eigenvector of $\mathbf{P}(0)$.

Let $m_0\in\mathbf{Z}_+, m_0\le m$ be the largest integer such that there exist row vectors $\mathbf{l}_0^\beta\in\mathbf{R}^r$, $\beta\in\mathbf{Z}_+^d, m\le|\beta|\le m+m_0-1$ satisfying

$$\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}(2i)^{|\alpha-\beta|}\mathbf{l}_0^\alpha(D^{\beta-\alpha}\mathbf{P})(0) = 2^{-|\beta|}\mathbf{l}_0^\beta. \tag{2.7}$$

Equations (2.7) can be written as

$$\mathbf{l}_0^\beta\left(2^{-|\beta|}\mathbf{I}_r - \mathbf{P}(0)\right) = \sum_{0\le\alpha<\beta}\binom{\beta}{\alpha}(2i)^{|\alpha-\beta|}\mathbf{l}_0^\alpha(D^{\beta-\alpha}\mathbf{P})(0), \tag{2.8}$$

where $\mathbf{I}_r$ is the $r\times r$ identity matrix. Thus if each of all numbers of $2^{-m}, 2^{-m-1}, \cdots$, 2^{-m-m_0} is not an eigenvalue of $\mathbf{P}(0)$ for some $m_0\in\mathbf{Z}_+$, then vectors $\mathbf{l}_0^\beta\in\mathbf{R}^r$,

$\beta \in \mathbf{Z}_+^d, m \le |\beta| \le m + m_0 - 1$ can be chosen iteratively by (2.8). Since in the examples which are analyzed below $m_0 = m$, in the following we will assume that $m_0 = m$. For the case $r = 1$, since $\mathbf{P}(0) = 1$, such assumption is not needed.

Let $B(\omega) = \sum_{\kappa \in \mathbf{Z}_+^d, |\kappa| \le 2m-1} B_\kappa e^{i\kappa\omega}$ be the vector trigonometric polynomial satisfying

$$D^\beta B(0) = i^{|\beta|} \mathbf{l}_0^\beta, \quad \beta \in \mathbf{Z}_+^d, |\beta| \le 2m - 1. \tag{2.9}$$

The coefficients B_κ, $1 \times r$ vectors, can be found by the following equations

$$\sum_{|\kappa| \le 2m-1} \kappa^\beta B_\kappa = \mathbf{l}_0^\beta, \quad \beta \in \mathbf{Z}_+^d, |\beta| \le 2m - 1.$$

One can check that the vanishing moment conditions (2.6) and (2.7) can be written equivalently in the form

$$D^\beta (B(2\omega)\mathbf{P}(\omega))|_{\omega=0} = D^\beta B(0), \quad \forall \beta \in \mathbf{Z}_+^d, |\beta| \le 2m - 1, \tag{2.10}$$

and

$$D^\beta \left(B(2\omega)\, \mathbf{P}(\omega)\right)|_{\omega=\nu\pi} = 0, \quad \forall \beta \in \mathbf{Z}_+^d, |\beta| \le m - 1, \nu \in \mathbf{Z}^d / 2\mathbf{Z}^d \backslash \{0\}. \tag{2.11}$$

Let $\mathbf{l}_0^\beta$, $\beta \in \mathbf{Z}_+^d, |\beta| \le 2m - 1$ be the row vectors satisfying (2.6) and (2.7). For $\kappa \in \mathbf{Z}^d$, define row vectors $\mathbf{l}_\kappa^\beta$ by

$$\mathbf{l}_\kappa^\beta := \sum_{0 \le \alpha \le \beta} \binom{\beta}{\alpha} \kappa^{\beta-\alpha} \mathbf{l}_0^\alpha, \quad \text{for } \beta \in \mathbf{Z}_+^d, |\beta| \le 2m - 1, \tag{2.12}$$

and then define the $1 \times (r^2(2N+1)^d)$ vectors $\mathbf{L}_N^\beta$ by

$$\mathbf{L}_N^\beta := (\mathbf{l}^\beta(\kappa)|_{\kappa=(-N,\cdots,-N)}, \cdots, \mathbf{l}^\beta(\kappa)|_{\kappa=(N,\cdots,N)}) \tag{2.13}$$

with

$$\mathbf{l}^\beta(\kappa) := \sum_{0 \le \alpha \le \beta} (-1)^\alpha \binom{\beta}{\alpha} \mathbf{l}_\kappa^\alpha \otimes \mathbf{l}_0^{\beta-\alpha}, \quad \kappa \in \mathbf{Z}^d.$$

For the case $d = 1$, $\mathbf{l}_\kappa^\beta, \kappa \in \mathbf{Z}^d$, are the coefficients for the reproduction of polynomials by the integer translates of Φ, see [15].

For two $1 \times r$ vectors $\mathbf{v}, \mathbf{u}$ and $r \times r$ matrix M, we have (see [19])

$$(\mathbf{v} \otimes \mathbf{u})\mathrm{vec}(M) = \mathbf{u} M \mathbf{v}^T. \tag{2.14}$$

Lemma 2.1. *Assume that the refinement mask* $\mathbf{P}$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta$, $|\beta| \le 2m - 1$, *and* B *is the vector trigonometric polynomial satisfying* (2.9). *Let* $\mathbf{L}_N^\beta$ *be the vectors defined by* (2.13), *then for any* $H \in \mathbf{H}_N$

$$\mathbf{L}_N^\beta \mathrm{vec}(H) = (-i)^{|\beta|} D^\beta \left(B(\omega) H(\omega) B^*(\omega)\right)|_{\omega=0}, \quad \beta \in \mathbf{Z}_+^d, |\beta| \le 2m - 1,$$

where $\mathrm{vec}(H)$ *is the vector defined by* (2.2).

Proof. By (2.14), for any $\beta\in\mathbf{Z}_+^d, |\beta|\le 2m-1$, and any $H\in\mathbf{H}_N$

$$
\begin{aligned}
&\mathbf{L}_N^\beta \mathrm{vec}(H)=\sum_\kappa \mathbf{L}_N^\beta(\kappa)\mathrm{vec}(H_\kappa)=\sum_\kappa\sum_{0\le\alpha\le\beta}(-1)^{|\alpha|}\binom{\beta}{\alpha}\mathbf{l}_0^{\beta-\alpha}H(\kappa)(\mathbf{l}_\kappa^\alpha)^T\\
&=\sum_\kappa\sum_{0\le\alpha\le\beta}(-1)^{|\alpha|}\binom{\beta}{\alpha}\mathbf{l}_0^{\beta-\alpha}H(\kappa)\sum_{0\le\gamma\le\alpha}\kappa^\gamma\binom{\alpha}{\gamma}(\mathbf{l}_0^{\alpha-\gamma})^T\\
&=\sum_\kappa\sum_{0\le\alpha\le\beta}\sum_{0\le\gamma\le\alpha}(-1)^{|\alpha|}\binom{\beta}{\alpha}\kappa^\gamma\binom{\alpha}{\gamma}(-i)^{|\beta-\alpha|}D^{\beta-\alpha}B(0)H(\kappa)i^{|\alpha-\gamma|}D^{\alpha-\gamma}B^*(0)\\
&=(-i)^{|\beta|}\sum_{0\le\alpha\le\beta}\sum_{0\le\gamma\le\alpha}\binom{\beta}{\alpha}\binom{\alpha}{\gamma}D^{\beta-\alpha}B(0)\sum_\kappa(-i\kappa)^\gamma H(\kappa)D^{\alpha-\gamma}B^*(0)\\
&=(-i)^{|\beta|}\sum_{0\le\alpha\le\beta}\sum_{0\le\gamma\le\alpha}\binom{\beta}{\alpha}\binom{\alpha}{\gamma}D^{\beta-\alpha}B(0)D^\gamma H(0)D^{\alpha-\gamma}B^*(0)\\
&=(-i)^{|\beta|}D^\beta\left(B(\omega)H(\omega)B^*(\omega)\right)|_{\omega=0}.
\end{aligned}
$$

□

THEOREM 2.2. *Assume that the refinement mask* $\mathbf{P}$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta$, $|\beta|\le 2m-1$, *and* B *is the vector trigonometric polynomial satisfying* (2.9). *Let* $\mathbf{L}_N^\beta$ *be the vectors defined by* (2.13), *then*

$$\mathbf{L}_N^\beta\mathcal{T}_\mathbf{P}=2^{-|\beta|}\mathbf{L}_N^\beta,\quad \beta\in\mathbf{Z}_+^d,|\beta|\le 2m-1.$$

Proof. We need only to show that for any $H\in\mathbf{H}_N$, $\mathbf{L}_N^\beta\mathcal{T}_\mathbf{P}\mathrm{vec}(H)=2^{-|\beta|}\mathbf{L}_N^\beta\mathrm{vec}(H)$. In fact by (2.5) and Lemma 2.1,

$$
\begin{aligned}
(2i)^{|\beta|}\mathbf{L}_N^\beta\mathcal{T}_\mathbf{P}\mathrm{vec}(H)&=(2i)^{|\beta|}\mathbf{L}_N^\beta\mathrm{vec}(\mathbf{T_P}H)=D^\beta\left(B(2\omega)\mathbf{T_P}H(2\omega)B^*(2\omega)\right)|_{\omega=0}\\
&=\sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d}D^\beta(B(2\omega)\mathbf{P}(\omega+\nu\pi)H(\omega+\nu\pi)\mathbf{P}(\omega+\nu\pi)^*B^*(2\omega))|_{\omega=0}\\
&=\sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d}\sum_{0\le\alpha\le\beta}\sum_{0\le\gamma\le\alpha}\binom{\beta}{\alpha}\binom{\alpha}{\gamma}D^\alpha\left(B(2\omega)\mathbf{P}(\omega)\right)|_{\omega=\nu\pi}\cdot\\
&\qquad\cdot D^\gamma H(\omega)|_{\omega=\nu\pi}D^{\beta-\alpha-\gamma}\left(B(2\omega)\mathbf{P}(\omega)\right)^*|_{\omega=\nu\pi}.
\end{aligned}
$$

Since for any $\beta,\alpha,\gamma\in\mathbf{Z}_+^d$ with $|\beta|\le 2m-1$ and $\gamma\le\alpha\le\beta$, $\min(|\alpha|,|\beta-\alpha-\gamma|)\le m-1$, thus by (2.10) and (2.11),

$$
\begin{aligned}
&(2i)^{|\beta|}\mathbf{L}_N^\beta\mathcal{T}_\mathbf{P}\mathrm{vec}(H)\\
&=\sum_{0\le\alpha\le\beta}\sum_{0\le\gamma\le\alpha}\binom{\beta}{\alpha}\binom{\alpha}{\gamma}D^\alpha(B(2\omega)\mathbf{P}(\omega))|_{\omega=0}D^\gamma H(\omega)|_{\omega=0}D^{\beta-\alpha-\gamma}\left(B(2\omega)\mathbf{P}(\omega)\right)^*|_{\omega=0}\\
&=\sum_{0\le\alpha\le\beta}\sum_{0\le\gamma\le\alpha}\binom{\beta}{\alpha}\binom{\alpha}{\gamma}D^\alpha B(0)D^\gamma H(0)D^{\beta-\alpha-\gamma}B^*(0)\\
&=D^\beta\left(B(\omega)H(\omega)B^*(\omega)\right)|_{\omega=0}=i^{|\beta|}\mathbf{L}_N^\beta\mathrm{vec}(H).
\end{aligned}
$$

Therefore $\mathbf{L}_N^\beta\mathcal{T}_\mathbf{P}\mathrm{vec}(H)=2^{-|\beta|}\mathbf{L}_N^\beta\mathrm{vec}(H)$ and the proof of Theorem 2.2 is completed.
□

Since $\mathbf{L}_N^0 = (\mathbf{l}_0^0 \otimes \mathbf{l}_0^0, \cdots, \mathbf{l}_0^0 \otimes \mathbf{l}_0^0) \neq 0$, 1 is an eigenvalue of $\mathbf{T_P}$. In the case $r = 1, d = 1$, for any $n \in \mathbf{Z}_+$, $n \leq 2m-1$, the vector v_n defined by (1.7) is a generalized left eigenvector of eigenvalue 2^{-n} of $\mathcal{T}_\mathbf{P}$ (see page 228 in [6]), and hence $2^{-n}, 0 \leq n \leq 2m-1$ are eigenvalues of $\mathbf{T_P}$. Theorem 2.2 says that for $n \in \mathbf{Z}_+, n \leq 2m-1$, if there exists $\beta \in \mathbf{Z}_+^d, |\beta| = n$ and $\mathbf{L}_N^\beta \neq 0$, then 2^{-n} is an eigenvalue of $\mathcal{T}_\mathbf{P}$ (also $\mathbf{T_P}$) with $\mathbf{L}_N^\beta$ being a corresponding left eigenvector. As the vectors v_n do for the case $r = 1, d = 1$, vectors $\mathbf{L}_N^\beta$ also play an important role in the estimate of the Sobolev regularity of refinable vector Φ, which will be shown in the next section.

3. Sobolev regularity estimates. In this section we will consider the Sobolev regularity of the matrix refinable function Φ of (1.1). For $s \geq 0$, we say $f \in W^s(\mathbf{R}^d)$ if $(1+|\omega|^2)^{\frac{s}{2}} \widehat{f}(\omega) \in L^2(\mathbf{R}^d)$. In the first part of this section, we will provide a characterization of Φ in $W^n(\mathbf{R}^d)$ for $n \in \mathbf{Z}_+$. We need a lemma.

LEMMA 3.1. *Assume that* $\mathbf{P}(\omega)$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta$, $|\beta| \leq 2m-1$, *and* B *is the vector trigonometric polynomial satisfying* (2.9), *then for any compactly supported solution* Φ *of* (1.1),

$$D^\beta \left(B(\omega)\widehat{\Phi}(\omega)\right)|_{\omega=2\pi\ell} = 0, \quad \textit{for any } \ell \in \mathbf{Z}^d\backslash\{0\}, \beta \in \mathbf{Z}_+^d, |\beta| \leq m-1.$$

Proof. Since Φ is compactly supported, $\widehat{\Phi}(\omega)$ is analytic. For any $\ell \in \mathbf{Z}^d\backslash\{0\}$, write ℓ in the form of $\ell = 2^n\nu + 2^{n+1}\kappa$, for some $n \in \mathbf{Z}_+, \nu \in \mathbf{Z}^d/2\mathbf{Z}^d\backslash\{0\}, \kappa \in \mathbf{Z}^d$, then

$$\begin{aligned}\widehat{\Phi}(2\pi\ell+\omega) &= \mathbf{P}(\frac{2\pi\ell+\omega}{2})\cdots\mathbf{P}(\frac{2\pi\ell+\omega}{2^n})\mathbf{P}(\frac{2\pi\ell+\omega}{2^{n+1}})\widehat{\Phi}(\frac{2\pi\ell+\omega}{2^{n+1}})\\ &= \mathbf{P}(\frac{\omega}{2})\cdots\mathbf{P}(\frac{\omega}{2^n})\mathbf{P}(\frac{\omega}{2^{n+1}}+\nu\pi)\widehat{\Phi}(\frac{2\pi\ell+\omega}{2^{n+1}}).\end{aligned}$$

Thus by (2.6) and (2.7), or by its equivalent forms (2.10) and (2.11)

$$\begin{aligned}&D^\beta\left(B(\omega)\widehat{\Phi}(\omega)\right)|_{\omega=2\pi\ell} = D^\beta\left(B(\omega)\widehat{\Phi}(2\pi\ell+\omega)\right)|_{\omega=0}\\ &= \sum_{0\leq\alpha\leq\beta}\binom{\beta}{\alpha}D^\alpha(B(\omega)\mathbf{P}(\frac{\omega}{2}))|_{\omega=0}D^{\beta-\alpha}(\mathbf{P}(\frac{\omega}{2^2})\cdots\mathbf{P}(\frac{\omega}{2^n})\mathbf{P}(\frac{\omega}{2^{n+1}}+\nu\pi)\widehat{\Phi}(\frac{2\pi\ell+\omega}{2^{n+1}}))|_{\omega=0}\\ &= \sum_{0\leq\alpha\leq\beta}\binom{\beta}{\alpha}D^\alpha B(\frac{\omega}{2})|_{\omega=0}D^{\beta-\alpha}\left(\mathbf{P}(\frac{\omega}{2^2})\cdots\mathbf{P}(\frac{\omega}{2^n})\mathbf{P}(\frac{\omega}{2^{n+1}}+\nu\pi)\widehat{\Phi}(\frac{2\pi\ell+\omega}{2^{n+1}})\right)|_{\omega=0}\\ &= D^\beta\left(B(\frac{\omega}{2})\mathbf{P}(\frac{\omega}{2^2})\cdots\mathbf{P}(\frac{\omega}{2^n})\mathbf{P}(\frac{\omega}{2^{n+1}}+\nu\pi)\widehat{\Phi}(\frac{2\pi\ell+\omega}{2^{n+1}})\right)|_{\omega=0} = \cdots\\ &= D^\beta\left(B(\frac{\omega}{2^n})\mathbf{P}(\frac{\omega}{2^{n+1}}+\nu\pi)\widehat{\Phi}(\frac{2\pi\ell+\omega}{2^{n+1}})\right)|_{\omega=0} = 0\end{aligned}$$

since $D^\alpha\left(B(\frac{\omega}{2^n})\mathbf{P}(\frac{\omega}{2^{n+1}}+\nu\pi)\right)|_{\omega=0} = 0$ for any $\alpha \leq \beta$. $\square$

If a refinable vector Φ is contained in $W^{\frac{n}{2}}(\mathbf{R}^d)$ for some $n \in \mathbf{Z}_+$, then

$$\int_{\mathbf{R}^d}|\omega|^n|\widehat{\Phi}(\omega)|^2d\omega < \infty. \tag{3.1}$$

For any $\beta_0 \in \mathbf{Z}_+^d$, $|\beta_0| = n$, define

$$H_{\beta_0}(\kappa) := \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}(i\omega)^{\beta_0}\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)e^{i\kappa\omega}d\omega, \quad \kappa \in \mathbf{Z}_+^d.$$

Let F be the matrix function defined by

$$\widehat{F}(\omega) := (i\omega)^{\beta_0}\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega).$$

The finiteness of the integral in (3.1) implies that every entry of F is continuous and hence $H_{\beta_0}(\kappa) = F(\kappa)$. (3.1) also implies the existence of $D^{\beta_0}(\mathrm{auto}(\Phi))(= F)$, where

$$\mathrm{auto}(\Phi)(y) := \int_{\mathbf{R}^d} \Phi(x)\Phi^*(x-y)dx.$$

Since Φ is compactly supported on $[0,N]^d$, the support of F is contained in $[-N,N]^d$. Therefore $H_{\beta_0}(\kappa) = 0$ for $\kappa \notin [-N,N]^d$. Define

$$G^{(\beta_0)}(\omega) := \sum_{\kappa} H_{\beta_0}(\kappa)e^{-i\kappa\omega},$$

then $G^{(\beta_0)}(\omega) \in \mathbf{H}_N$ for any $|\beta_0| = n$.

PROPOSITION 3.1. *Assume that the refinement mask* $\mathbf{P}$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta$, $|\beta| \le 2m-1$. *Suppose there exists a refinable vector* Φ *contained in* $W^{\frac{n}{2}}(\mathbf{R}^d)$ *for some* $n \in \mathbf{Z}_+$ *with* $n \le 2m-1$, *then for any* $\beta_0 \in \mathbf{Z}_+^d$, $|\beta_0| = n$,

$$\mathcal{T}_{\mathbf{P}}\mathrm{vec}(G^{(\beta_0)}) = 2^{-n}\mathrm{vec}(G^{(\beta_0)}),$$

and for any $\beta \in \mathbf{Z}_+^d, \beta \le \beta_0$,

$$\mathbf{L}_N^\beta \mathrm{vec}(G^{(\beta_0)}) = \beta_0!\delta_{\beta_0}(\beta)|\mathbf{l}_0^0\widehat{\Phi}(0)|^2 = \begin{cases} 0, & \beta < \beta_0, \\ \beta_0!|\mathbf{l}_0^0\widehat{\Phi}(0)|^2, & \beta = \beta_0. \end{cases}$$

Proof. By the Poisson summation formula,

$$G^{(\beta_0)}(\omega) = \sum_{\ell \in \mathbf{Z}^d} (i\omega + i2\pi\ell)^{\beta_0}\widehat{\Phi}(\omega + 2\pi\ell)\widehat{\Phi}^*(\omega + 2\pi\ell).$$

By the definition of $\mathbf{T}_{\mathbf{P}}$,

$$\begin{aligned} \mathbf{T}_{\mathbf{P}}G^{(\beta_0)}(\omega) = &\sum_{\nu \in \mathbf{Z}^d/2\mathbf{Z}^d}\sum_{\ell \in \mathbf{Z}^d} (i\omega/2 + 2\ell\pi i + \nu\pi i)^{\beta_0}\mathbf{P}(\omega/2 + \nu\pi)\cdot \\ &\cdot\widehat{\Phi}(\omega/2 + 2\ell\pi + \nu\pi)\widehat{\Phi}^*(\omega/2 + 2\ell\pi + \nu\pi)\mathbf{P}^*(\omega/2 + \nu\pi) \\ = \frac{1}{2^n} &\sum_{\nu \in \mathbf{Z}^d/2\mathbf{Z}^d}\sum_{\ell \in \mathbf{Z}^d} (i\omega + 4\ell\pi i + 2\nu\pi i)^{\beta_0}\widehat{\Phi}(\omega + 4\ell\pi + 2\nu\pi)\widehat{\Phi}^*(\omega + 4\ell\pi + 2\nu\pi) \\ = \frac{1}{2^n} &G^{(\beta_0)}(\omega), \end{aligned}$$

and hence $\mathcal{T}_{\mathbf{P}}\mathrm{vec}(G^{(\beta_0)}) = 2^{-n}\mathrm{vec}(G^{(\beta_0)})$ by (2.5).

By Lemma 3.1, for any $\alpha \in \mathbf{Z}_+^d, |\alpha| < 2m-1$ and $\ell \in \mathbf{Z}^d \backslash \{0\}$,

$$D^\alpha\left(B(\omega)\widehat{\Phi}(\omega + 2\ell\pi)\widehat{\Phi}^*(\omega + 2\ell\pi)B^*(\omega)\right)|_{\omega=0} = 0.$$

Therefore by Lemma 2.1,

$$\begin{aligned}
&\mathbf{L}_N^{\beta}\mathrm{vec}(G^{(\beta_0)}) = (-i)^{|\beta|}D^{\beta}(B(\omega)G^{(\beta_0)}(\omega)B^*(\omega))|_{\omega=0}\\
&= (-i)^{|\beta|}D^{\beta}((i\omega)^{\beta_0}B(\omega)\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)B^*(\omega))|_{\omega=0} +\\
&\quad (-i)^{|\beta|}D^{\beta}(\sum_{\ell\in\mathbf{Z}^d\setminus\{0\}}(i\omega+i2\ell\pi)^{\beta_0}B(\omega)\widehat{\Phi}(\omega+2\ell\pi)\widehat{\Phi}^*(\omega+2\ell\pi)B^*(\omega))|_{\omega=0}\\
&= \beta_0!\delta_{\beta_0}(\beta)|\mathbf{l}_0^0\widehat{\Phi}(0)|^2.
\end{aligned}$$

□

We note that if λ is a simple eigenvalue of a matrix, then the product of the corresponding left row eigenvector and right column eigenvector is not zero (see Lemma 6.3.10 in [18]). Thus $\mathbf{l}_0^0\widehat{\Phi}(0) \neq 0$ since $\mathbf{P}(0)$ satisfies Condition E and $\widehat{\Phi}(0)$ is a right 1-eigenvector of $\mathbf{P}(0)$. By the fact that $\Phi \in W^{s_1}(\mathbf{R}^d)$ if $\Phi \in W^{s}(\mathbf{R}^d)$ and $s_1 < s$, Proposition 3.1 leads to the following corollary.

COROLLARY 3.1. *Assume that the refinement mask* $\mathbf{P}$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^{\beta}$, $|\beta| \leq 2m-1$. *Suppose there exists a nontrivial refinable vector* Φ *contained in* $W^{\frac{n}{2}}(\mathbf{R}^d)$ *for some* $n \in \mathbf{Z}_+$ *with* $n \leq 2m-1$, *then for any* $\beta \in \mathbf{Z}_+^d, |\beta| \leq n$, $\mathbf{L}_N^{\beta} \neq 0$, *and* $1, 2^{-1}, \cdots, 2^{-n}$ *are eigenvalues of* $\mathbf{T}_{\mathbf{P}}$.

The next theorem will give a characterization of the refinable vector Φ in the Sobolev space $W^n(\mathbf{R}^d)$, $n \in \mathbf{Z}_+$. But first, we need another lemma. For $j \in \mathbf{Z}_+$, denote

$$\Pi_j(\omega) := \chi_{2^j\mathbf{T}^d}(\omega)\Pi_{i=1}^{j}\mathbf{P}(2^{-i}\omega).$$

LEMMA 3.2. *For any* $H_1(\omega), H_2(\omega) \in \mathbf{H}_N$,

$$\int_{\mathbf{T}^d} H_1(\omega)(\mathbf{T}_{\mathbf{P}}^{j}H_2)(\omega)d\omega = \int_{\mathbf{R}^d} H_1(\omega)\Pi_j(\omega)H_2(2^{-j}\omega)\Pi_j(\omega)^*d\omega. \tag{3.2}$$

Proof. The proof of (3.2) can be found in [26]. In fact for $j=1$,

$$\begin{aligned}
&\int_{\mathbf{R}^d} H_1(\omega)\Pi_1(\omega)H_2(\frac{\omega}{2})\Pi_1(\omega)^*d\omega\\
&= \sum_{\beta\in\mathbf{Z}^d}\int_{\mathbf{T}^d} H_1(\omega)\mathbf{P}(\frac{\omega}{2}+\beta\pi)H_2(\frac{\omega}{2}+\beta\pi)\mathbf{P}^*(\frac{\omega}{2}+\beta\pi)\chi_{\mathbf{T}^d}(\frac{\omega}{2}+\beta\pi)d\omega\\
&= \sum_{\alpha\in\mathbf{Z}^d}\sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d}\int_{\mathbf{T}^d} H_1(\omega)\mathbf{P}(\frac{\omega}{2}+\nu\pi)H_2(\frac{\omega}{2}+\nu\pi)\mathbf{P}^*(\frac{\omega}{2}+\nu\pi)\chi_{\mathbf{T}^d}(\frac{\omega}{2}+2\alpha\pi+\nu\pi)d\omega\\
&= \sum_{\nu\in\mathbf{Z}^d/2\mathbf{Z}^d}\int_{\mathbf{T}^d} H_1(\omega)\mathbf{P}(\frac{\omega}{2}+\nu\pi)H_2(\frac{\omega}{2}+\nu\pi)\mathbf{P}^*(\frac{\omega}{2}+\nu\pi)d\omega\\
&= \int_{\mathbf{T}^d} H_1(\omega)\mathbf{T}_{\mathbf{P}}H_2(\omega)d\omega.
\end{aligned}$$

For general j, this formula can be found by induction. □

THEOREM 3.1. *Assume that the refinement mask* $\mathbf{P}$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^{\beta}$, $|\beta| \leq 2m-1$, *then a refinable vector* Φ *is contained in* $W^n(\mathbf{R}^d)$ *for some* $n \in \mathbf{Z}_+$ *with* $n \leq m-1$, *if and only if there exists a positive semidefinite* $H \in \mathbf{H}_N$ *satisfying the following conditions:*

(i) $\mathbf{T_P} H = 4^{-n} H$;
(ii) *there exist constants* $c_0, \delta > 0$ *such that*

$$H(\omega) \geq c_0 |\omega|^{2n} \mathbf{r}\mathbf{r}^T, \quad \text{for } \omega \in [-\delta, \delta]^d,$$

where $\mathbf{r}$ *is the normalized right* 1*-eigenvector of* $\mathbf{P}(0)$.

Proof. "$\Longrightarrow$" If $\Phi \in W^n(\mathbf{R}^d)$, let

$$H(\omega) = (-1)^n \sum_{|\beta_0|=n} G^{(2\beta_0)}(\omega) \geq 0. \tag{3.3}$$

Then Proposition 3.1 leads to $\mathbf{T_P} H = 4^{-n} H$.

Since $\mathbf{P}(0)$ satisfies Condition E, $\widehat{\Phi}(\omega) \to c\mathbf{r}$ with $c \neq 0$ as $\omega \to 0$ (see [14], [23] and [26]). Thus there exists a constant $\delta > 0$ such that

$$\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega) \geq \frac{c^2}{2}\mathbf{r}\mathbf{r}^T, \quad \text{for } \omega \in [-\delta, \delta]^d.$$

Therefore

$$\begin{aligned} H(\omega) &= (-1)^n \sum_{|\beta_0|=n} \sum_{\ell \in \mathbf{Z}^d} (i\omega + i2\ell\pi)^{2\beta_0} \widehat{\Phi}(\omega + 2\ell\pi)\widehat{\Phi}^*(\omega + 2\ell\pi) \\ &\geq \sum_{|\beta_0|=n} \omega^{2\beta_0} \widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega) \geq \frac{c^2}{2}\mathbf{r}\mathbf{r}^T \sum_{|\beta_0|=n} \omega^{2\beta_0} = \frac{c^2|\omega|^{2n}}{2}\mathbf{r}\mathbf{r}^T. \end{aligned}$$

"$\Longleftarrow$" Denote $g_j(\omega) := 4^{nj}\Pi_j(\omega)H(2^{-j}\omega)\Pi_j(\omega)^*$. Then

$$\begin{aligned} g_j(\omega) &\geq c_0 4^{nj} \chi_{[-\delta,\delta]^d}(\frac{\omega}{2^j})\Pi_j(\omega)(\frac{|\omega|}{2^j})^{2n}\mathbf{r}\mathbf{r}^T\Pi_j(\omega)^* \\ &= c_0|\omega|^{2n}\chi_{[-\delta,\delta]^d}(\frac{\omega}{2^j})\Pi_j(\omega)\mathbf{r}(\Pi_j(\omega)\mathbf{r})^*. \end{aligned}$$

Thus by the fact that $\widehat{\Phi}(\omega) = \lim_{j\to\infty} \chi_{[-\delta,\delta]^d}(\frac{\omega}{2^j})\Pi_j(\omega)\mathbf{r}$ and Fatou lemma,

$$\begin{aligned} \int_{\mathbf{R}^d} |\omega|^{2n}|\widehat{\Phi}(\omega)|^2 d\omega &= c\int_{\mathbf{R}^d} \sum_{i=1}^r \mathbf{e}_i^T \liminf_{j\to\infty} |\omega|^{2n}\chi_{[-\delta,\delta]^d}(\frac{\omega}{2^j})\Pi_j(\omega)\mathbf{r}(\Pi_j(\omega)\mathbf{r})^*\mathbf{e}_i d\omega \\ &\leq c\sum_{i=1}^r \mathbf{e}_i^T \liminf_{j\to\infty} \int_{\mathbf{R}^d} |\omega|^{2n}\chi_{[-\delta,\delta]^d}(\frac{\omega}{2^j})\Pi_j(\omega)\mathbf{r}(\Pi_j(\omega)\mathbf{r})^* d\omega \mathbf{e}_i \\ &\leq c\sum_{i=1}^r \mathbf{e}_i^T \liminf_{j\to\infty} \int_{\mathbf{R}^d} g_j(\omega) d\omega \mathbf{e}_i < \infty, \end{aligned}$$

where the last inequality follows from

$$\int_{\mathbf{R}^d} g_j(\omega)d\omega = 4^{jn}\int_{\mathbf{T}^d} \mathbf{T}_{\mathbf{P}}^j H(\omega)d\omega = \int_{\mathbf{T}^d} H(\omega)d\omega < \infty.$$

By the continuity of $\widehat{\Phi}$, this leads to $\Phi \in W^n(\mathbf{R}^d)$. ∎

For $n = 0$, $W^0(\mathbf{R}^d) = L^2(\mathbf{R}^d)$. In fact the characterization of $\Phi \in L^2(\mathbf{R}^d)$ can be given in a more easy checking way. In [23], it was shown that under assumption that $\mathbf{P}(0)$ satisfies condition E, $\Phi \in L^2(\mathbf{R}^d)$ if and only if there exists a positive semidefinite $H \in \mathbf{H}_N$ satisfying $\mathbf{T_P} H = H$ and $\mathbf{l}_0^0 H(0)(\mathbf{l}_0^0)^T > 0$.

If $\Phi \in W^n(\mathbf{R}^d)$, $n \le m-1$, where $H \in \mathbf{H}_N$ is defined by (3.3), then Proposition 3.1 implies that there exists a positive semidefinite H satisfying $\mathbf{T_P} H = 4^{-n} H$ and

$$\mathbf{L}_N^\beta \mathrm{vec}(H) = c\beta! \sum_{|\beta_0|=n} \delta_{2\beta_0}(\beta), \tag{3.4}$$

for any $\beta \in \mathbf{Z}_+^d, |\beta| \le 2n$, where c is a nonzero constant independent of β. In the case $r=1$, the existence of such positive semidefinite H is also sufficient for $\Phi \in W^n(\mathbf{R}^d)$. In fact by Lemma 2.1, (3.4) is equivalent to that for any $\beta \in \mathbf{Z}_+^d, |\beta| \le 2n$,

$$D^\beta \left(|B(\omega)|^2 H(\omega)\right)|_{\omega=0} = c \sum_{|\beta_0|=n} \delta_{2\beta_0}(\beta), \tag{3.5}$$

which implies that $D^\beta H(0) = c(\mathbf{l}_0^0)^{-2} \sum_{|\beta_0|=n} \delta_{2\beta_0}(\beta)$ (in this case $\mathbf{l}_0^0$ is a nonzero real number). Thus $H(\omega) = c|\omega|^{2n} + \mathrm{o}(|\omega|^{2n})$ (as $\omega \to 0$) and hence $H(\omega)$ satisfies condition (ii) of Theorem 3.1. For $r=1, d=1$, such results were given in [32].

Theorem 3.1 gives the characterization of refinable vectors $\Phi \in W^s(\mathbf{R}^d)$ with s being nonnegative integers. In the following, we will give an estimate of the Sobolev regularity of Φ in terms of the spectral radius of $\mathbf{T_P}|_{\mathbf{H}_N^0}$, the restricted operator of $\mathbf{T_P}$ to an invariant subspace $\mathbf{H}_N^0$ of $\mathbf{H}_N$.

For $j \in \mathbf{Z}_+, 1 \le j \le r$ and $\alpha \in \mathbf{Z}_+^d, |\alpha| \le m-1$, let ${}_j\mathbf{l}_N^\alpha$, ${}_j\mathbf{r}_N^\alpha$ be the $1 \times (r^2(2N+1)^d)$ vectors defined by

$$\begin{aligned} {}_j\mathbf{l}_N^\alpha &:= \left({}_j\mathbf{l}^\alpha(\kappa)|_{\kappa=(-N,\cdots,-N)}, \cdots, {}_j\mathbf{l}^\alpha(\kappa)|_{\kappa=(N,\cdots,N)}\right), \\ {}_j\mathbf{r}_N^\alpha &:= \left({}_j\mathbf{r}^\alpha(\kappa)|_{\kappa=(-N,\cdots,-N)}, \cdots, {}_j\mathbf{r}^\alpha(\kappa)|_{\kappa=(N,\cdots,N)}\right), \end{aligned} \tag{3.6}$$

with

$${}_j\mathbf{l}^\alpha(\kappa) := \mathbf{e}_j^T \otimes \mathbf{l}_{-\kappa}^\alpha, \quad {}_j\mathbf{r}^\alpha(\kappa) := \mathbf{l}_\kappa^\alpha \otimes \mathbf{e}_j^T, \quad \kappa \in \mathbf{Z}^d,$$

where $\mathbf{l}_\kappa^\alpha$ are the vectors defined by (2.12).

LEMMA 3.3. *Assume that the refinement mask* $\mathbf{P}$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta$, $|\beta| \le 2m-1$, *and* B *is the vector trigonometric polynomial satisfying* (2.9). *For* $1 \le j \le r$ *and* $\alpha \in \mathbf{Z}_+^d, |\alpha| \le m-1$, *let* ${}_j\mathbf{l}_N^\alpha$ *and* ${}_j\mathbf{r}_N^\alpha$ *be the row vectors defined by* (3.6), *then for any* $H \in \mathbf{H}_N$,

$${}_j\mathbf{l}_N^\alpha \mathrm{vec}(H) = i^\alpha D^\alpha \left(B(\omega)H(\omega)\mathbf{e}_j\right)|_{\omega=0}, {}_j\mathbf{r}_N^\alpha \mathrm{vec}(H) = (-i)^\alpha D_V^\alpha \left(\mathbf{e}_j^T H(\omega) B^*(\omega)\right)|_{\omega=0},$$

where $\mathrm{vec}(H)$ *is the vector defined by* (2.2).

Proof. For any $H \in \mathbf{H}_N$, $H(\omega) = \sum_{\kappa \in [-N,N]^d} H_\kappa e^{-i\kappa\omega}$,

$$\begin{aligned} D^\alpha \left(B(\omega)H(\omega)\mathbf{e}_j\right)|_{\omega=0} &= \sum_{0 \le \gamma \le \alpha} \binom{\alpha}{\gamma} D^\gamma B(0) D^{\alpha-\gamma} H(0)\mathbf{e}_j \\ &= i^\alpha \sum_\kappa \sum_{0 \le \gamma \le \alpha} \binom{\alpha}{\gamma} (-\kappa)^{\alpha-\gamma} \mathbf{l}_0^\gamma H_\kappa \mathbf{e}_j = i^\alpha \sum_\kappa \mathbf{l}_{-\kappa}^\alpha H_\kappa \mathbf{e}_j \\ &= i^\alpha \sum_\kappa (\mathbf{e}_j^T \otimes \mathbf{l}_{-\kappa}^\alpha) \mathrm{vec}(H_\kappa) = i^\alpha {}_j\mathbf{l}_N^\alpha \mathrm{vec}(H). \end{aligned}$$

The proof of the second formula is similar and details are omitted here. □

Let $\mathbf{H}_N^0$ be the subspace of $\mathbf{H}_N$ defined by

$$\begin{aligned}(3.7)\quad \mathbf{H}_N^0 := \{H \in \mathbf{H}_N : \quad & \mathbf{L}_N^\beta \mathrm{vec}(H) = 0, \quad {}_j\mathbf{l}_N^\alpha \mathrm{vec}(H) = 0 \text{ and} \\ & {}_j\mathbf{r}_N^\alpha \mathrm{vec}(H) = 0, \forall \beta, \alpha \in \mathbf{Z}_+^d, |\beta| \le 2m-1, |\alpha| \le m-1, 1 \le j \le r\}.\end{aligned}$$

PROPOSITION 3.2. *Assume that the refinement mask* $\mathbf{P}$ *satisfies* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta$, $|\beta| \le 2m-1$. *Let* $\mathbf{H}_N^0$ *be the subspace of* $\mathbf{H}_N$ *defined by* (3.7), *then* $\mathbf{H}_N^0$ *is invariant under* $\mathbf{T}_\mathbf{P}$.

Proof. By Theorem 2.2, for any $H \in \mathbf{H}_N^0$ and $\beta \in \mathbf{Z}_+^d, |\beta| \le 2m-1$,

$$\mathbf{L}_N^\beta \mathrm{vec}(\mathbf{T}_\mathbf{P} H) = \mathbf{L}_N^\beta \mathcal{T}_\mathbf{P} \mathrm{vec}(H) = 2^{-|\beta|} \mathbf{L}_N^\beta \mathrm{vec}(H) = 0.$$

Let B be the vector trigonometric polynomial satisfying (2.9). By Lemma 3.3, for any $\alpha \in \mathbf{Z}_+^d, |\alpha| < m$, ${}_j\mathbf{l}_N^\alpha \mathrm{vec}(H) = 0$ and ${}_j\mathbf{r}_N^\alpha \mathrm{vec}(H) = 0$ for all $j, 1 \le j \le r$, are equivalent to $D^\alpha (B(\omega)H(\omega))|_{\omega=0} = 0$ and $D^\alpha (H(\omega)B^*(\omega))|_{\omega=0} = 0$, respectively. One can check by (2.10) and (2.11), $D^\alpha (B(\omega)\mathbf{T}_\mathbf{P} H(\omega))|_{\omega=0} = 0$ ($D_V^\alpha (\mathbf{T}_\mathbf{P} H(\omega)B^*(\omega))|_{\omega=0} = 0$ resp.) for all $\alpha \in \mathbf{Z}_+^d, |\alpha| < m$ if $D^\alpha (B(\omega)H(\omega))|_{\omega=0} = 0$ ($D_V^\alpha (H(\omega)B^*(\omega))|_{\omega=0} = 0$ resp.) for any $\alpha \in \mathbf{Z}_+^d, |\alpha| < m$. Thus $\mathbf{H}_N^0$ is invariant under $\mathbf{T}_\mathbf{P}$. □

Let $\mathbf{T}_\mathbf{P}|_{\mathbf{H}_N^0}$ denote the restriction of $\mathbf{T}_\mathbf{P}$ to $\mathbf{H}_N^0$. By the fact that the product of the left and right eigenvectors of a simple eigenvalue of a matrix is not zero again, Theorem 2.2 leads to the following corollary,

COROLLARY 3.2. *If* 2^{-n}, $0 \le n \le 2m-1$, *is a simple eigenvalue of* $\mathbf{T}_\mathbf{P}$ *and there exists* $\beta \in \mathbf{Z}_+^d$ *such that* $|\beta| = n$, $\mathbf{L}_N^\beta \ne 0$, *then* 2^{-n} *is not an eigenvalue of* $\mathbf{T}_\mathbf{P}|_{\mathbf{H}_N^0}$.

For the next proposition, we need to consider the transition operators on other spaces. Let $\mathbf{P}$ ($\{\mathbf{P}_\kappa\}$) be a given matrix mask satisfying (2.6) and (2.7) for some row vectors $\mathbf{l}_0^\beta$, $|\beta| \le 2m-1$, and $\mathrm{supp}\{\mathbf{P}_\kappa\} \subset [0, N]^d$. Denote $\mathcal{N} := \max(N, 2m)$. Let $\mathbf{H}_\mathcal{N}$ denote the space of all $r \times r$ matrices with each entry a trigonometric polynomial whose Fourier coefficients are supported in $[-\mathcal{N}, \mathcal{N}]^d$ and $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ denote the transition operator on $\mathbf{H}_\mathcal{N}$ defined by

$$\mathbf{T}_{\mathbf{P},\mathcal{N}} H(\omega) := \sum_{\nu \in \mathbf{Z}^d / 2\mathbf{Z}^d} \mathbf{P}(\frac{\omega}{2} + \pi\nu) H(\frac{\omega}{2} + \pi\nu) \mathbf{P}^*(\frac{\omega}{2} + \pi\nu), \quad H \in \mathbf{H}_\mathcal{N}.$$

Then $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ is a linear operator on $\mathbf{H}_\mathcal{N}$ leaving $\mathbf{H}_\mathcal{N}$ and $\mathbf{H}_N$ invariant and $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ is equivalent to the matrix

$$\mathcal{T}_{\mathbf{P},\mathcal{N}} := (2^{-d} \mathcal{A}_{2i-j})_{i,j \in [-\mathcal{N}, \mathcal{N}]^d},$$

where $\mathcal{A}_j = \sum_{\ell \in [0,N]^d} \mathbf{P}_{\ell - j} \otimes \mathbf{P}_\ell$.

Let $\mathbf{H}_\mathcal{N}^0$ be the subspace of $\mathbf{H}_\mathcal{N}$ defined as follows: $H \in \mathbf{H}_\mathcal{N}^0$ if and only if $\mathbf{L}_\mathcal{N}^\beta \mathrm{vec}(H) = 0, {}_j\mathbf{l}_\mathcal{N}^\alpha \mathrm{vec}(H) = 0$ and ${}_j\mathbf{r}_\mathcal{N}^\alpha \mathrm{vec}(H) = 0$ for all $\beta, \alpha \in \mathbf{Z}_+^d, |\beta| \le 2m - 1, |\alpha| \le m-1, 1 \le j \le r$. In this case $\mathbf{L}_\mathcal{N}^\beta$, ${}_j\mathbf{l}_\mathcal{N}^\alpha$ and ${}_j\mathbf{r}_\mathcal{N}^\alpha$ are $1 \times (r^2(2\mathcal{N}+1)^d)$ vectors defined by (2.12) and (3.6) respectively with $\mathcal{N}$ instead of N. It can be shown similarly that $\mathbf{H}_\mathcal{N}^0$ is invariant under $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ and let $\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}_\mathcal{N}^0}$ denote the restriction of $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ to $\mathbf{H}_\mathcal{N}^0$. Let $H_0 \in \mathbf{H}_\mathcal{N}$ defined by

$$(3.8)\qquad H_0(\omega) := \sum_{i=1}^d (1 - \cos\omega_i)^{2m} \mathbf{I}_r, \quad \omega = (\omega_1, \cdots, \omega_d)^T \in \mathbf{T}^d,$$

then $H_0(\omega) \in \mathbf{H}^0_{\mathcal{N}}$.

We note that the transition operator $\mathbf{T_P}$ defined by (1.6) is the restriction of $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ to the subspace $\mathbf{H}_N$ of $\mathbf{H}_{\mathcal{N}}$ and $\mathcal{T}_\mathbf{P}$ defined by (2.4) is a submatrix of $\mathcal{T}_{\mathbf{P},\mathcal{N}}$. In fact, if $\mathcal{N} > N$, then $\mathcal{T}_{\mathbf{P},\mathcal{N}}$ can be written as

$$\mathcal{T}_{\mathbf{P},\mathcal{N}} = \begin{bmatrix} M_1 & \mathbf{0} & \mathbf{0} \\ * & \mathcal{T}_\mathbf{P} & * \\ \mathbf{0} & \mathbf{0} & M_2 \end{bmatrix},$$

where M_1 (M_2 resp.) is a strictly lower (upper resp.) triangular matrix. Thus $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ ($\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}}$ resp.) and $\mathbf{T_P}$ ($\mathbf{T_P}|_{\mathbf{H}^0_N}$ resp.) have the same nonzero eigenvalues and the eigenvectors of $\mathbf{T}_{\mathbf{P},\mathcal{N}}$ are in $\mathbf{H}_N$. Hence $\rho(\mathbf{T_P}|_{\mathbf{H}^0_N}) = \rho(\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}})$, where $\rho(\mathbf{T_P}|_{\mathbf{H}^0_N})$ and $\rho(\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}})$ denote the spectral radii of $\mathbf{T_P}|_{\mathbf{H}^0_N}$ and $\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}}$, respectively.

Choose a vector norm on space $\mathbf{H}^0_{\mathcal{N}}$ and define the operator (matrix) norm $\|\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}}\|$ with respect to this vector norm, then

$$\lim_{n\to\infty} \|(\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}})^n\|^{1/n} = \rho(\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}}) = \rho(\mathbf{T_P}|_{\mathbf{H}^0_N}).$$

PROPOSITION 3.3. *Assume that* $\mathbf{P}$ *satisfies conditions* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta, |\beta| \le 2m-1$. *Let* $\mathbf{H}^0_N$ *be the subspace of* $\mathbf{H}_N$ *defined by* (3.7) *and* $\rho(\mathbf{T_P}|_{\mathbf{H}^0_N})$ *the spectral radius of* $\mathbf{T_P}|_{\mathbf{H}^0_N}$. *Then for any* $\epsilon > 0$, *for the corresponding refinable function* Φ, *there exists a constant* c *independent of* n *such that*

$$\int_{\mathbf{\Omega}_n} \left|\widehat{\Phi}(w)\right|^2 dw \le c\left(\rho(\mathbf{T_P}|_{\mathbf{H}^0_N}) + \epsilon\right)^n,$$

where $\mathbf{\Omega}_n := 2^n\mathbf{T}^d \backslash 2^{n-1}\mathbf{T}^d$, $n \in \mathbf{Z}_+$.

This proposition together with the usual Littlewood-Paley technique leads to the following Sobolev estimate of refinable vector Φ.

THEOREM 3.2. *Assume that* $\mathbf{P}$ *satisfies conditions* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}_0^\beta, |\beta| \le 2m-1$. *Let* $\mathbf{H}^0_N$ *be the subspace of* $\mathbf{H}_N$ *defined by* (3.7) *and* $\rho(\mathbf{T_P}|_{\mathbf{H}^0_N})$ *the spectral radius of* $\mathbf{T_P}|_{\mathbf{H}^0_N}$. *Then the matrix refinable function* Φ *is in* $W^s(\mathbf{R}^d)$ *for any* $s < s_0 := -\log_4 \rho(\mathbf{T_P}|_{\mathbf{H}^0_N})$.

The proof of Proposition 3.3 and Theorem 3.2 can be carried out by modifying the proofs of Proposition 4.4 and Theorem 4.5 in [29]. For completeness, we give them here.

Proof of Proposition 3.3. Let $H_0(\omega) \in \mathbf{H}^0_{\mathcal{N}}$ defined by (3.8). Note that $H_0(\omega) \ge \mathbf{I}_r$ for $\omega \in \mathbf{T}^d \backslash (\frac{1}{2}\mathbf{T}^d)$, and $\widehat{\Phi}$ is continuous on $\mathbf{T}^d$, thus for any positive integer n,

$$\begin{aligned}\int_{\mathbf{\Omega}_n} \widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)d\omega &= \int_{\mathbf{\Omega}_n} \Pi_n(\omega)\widehat{\Phi}(2^{-n}\omega)\widehat{\Phi}^*(2^{-n}\omega)\Pi_n^*(\omega)d\omega \\ &\le c\int_{\mathbf{\Omega}_n} \Pi_n(\omega)\Pi_n^*(\omega)d\omega \le c\int_{\mathbf{\Omega}_n} \Pi_n(\omega)H_0(2^{-n}\omega)\Pi_n^*(\omega)d\omega \\ &= c\int_{\mathbf{T}^d} (\mathbf{T}^n_{\mathbf{P},\mathcal{N}}H_0)(\omega)d\omega,\end{aligned}$$

where last equation can be shown similarly to (3.2). Since the Hilbert-Schmidt norm $\|M\|_2 = \sqrt{\mathrm{Tr}(MM^*)}$ is an equivalent norm for finite matrices, by applying the trace operation, we obtain

$$\int_{\mathbf{\Omega}_n} |\widehat{\Phi}(\omega)|^2 d\omega = \int_{\mathbf{\Omega}_n} \mathrm{Tr}\left(\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)\right) d\omega$$
$$\leq c_\epsilon \left(\rho(\mathbf{T}_{\mathbf{P},\mathcal{N}}|_{\mathbf{H}^0_{\mathcal{N}}}) + \epsilon\right)^n = c_\epsilon \left(\rho(\mathbf{T}_{\mathbf{P}}|_{\mathbf{H}^0_N}) + \epsilon\right)^n$$

with c_ϵ independent of n. ∎

Proof of Theorem 3.2. For $s < s_0$, let $\epsilon > 0$ be a constant satisfying $s < -\log_4(\rho(\mathbf{T}_{\mathbf{P}}|_{\mathbf{H}^0_N}) + \epsilon)$. Since

$$\int_{\mathbf{\Omega}_n} |\widehat{\Phi}(w)|^2 d\omega \leq c(\epsilon + \rho(\mathbf{T}_{\mathbf{P}}|_{\mathbf{H}^0_N}))^n,$$

for some constant c independent of n and $\widehat{\Phi}$ is continuous on $\mathbf{T}^d$, thus

$$\int_{\mathbf{R}^d} (1+|\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega = \int_{\mathbf{T}^d} (1+|\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega + \sum_{n=1}^{\infty} \int_{\mathbf{\Omega}_n} (1+|\omega|^2)^s |\widehat{\Phi}(\omega)|^2 d\omega$$
$$\leq c + c\sum_{n=1}^{\infty} 2^{2ns} \left(\epsilon + \rho(\mathbf{T}_{\mathbf{P}}|_{\mathbf{H}^0_N})\right)^n < \infty.$$

Therefore $\Phi \in W^s(\mathbf{R}^d)$. ∎

Let $C^\gamma(\mathbf{R}^d)$ denote the space defined in the following way: if $\gamma = n + \gamma_1$ with $n \in \mathbf{Z}_+$ and $0 \leq \gamma_1 < 1$, then $f \in C^\gamma(\mathbf{R}^d)$ if and only if $f \in C^{(n)}(\mathbf{R}^d)$ and $f^{(n)}$ is uniformly Hölder continuous with exponent γ_1, i.e.

$$|D^\beta f(x+y) - D^\beta f(x)| \leq c|y|^{\gamma_1}, \text{for any } \beta \in \mathbf{Z}^d_+, |\beta| = n,$$

for some constant c independent of $x, y \in \mathbf{R}^d$. With the well-known inclusion

$$W^s(\mathbf{R}^d) \subset C^\gamma(\mathbf{R}^d), \quad \text{for } s > \gamma + \frac{d}{2},$$

Theorem 3.2 leads to the following corollary.

COROLLARY 3.3. *Suppose* $\mathbf{P}$ *satisfies conditions* (2.6) *and* (2.7) *for some row vectors* $\mathbf{l}^\beta_0, |\beta| \leq 2m-1$. *Let* $\mathbf{H}^0_N$ *be the subspace of* $\mathbf{H}_N$ *defined by* (3.7) *and* $\rho(\mathbf{T}_{\mathbf{P}}|_{\mathbf{H}^0_N})$ *the spectral radius of* $\mathbf{T}_{\mathbf{P}}|_{\mathbf{H}^0_N}$, *then refinable vector* $\Phi \in C^\gamma(\mathbf{R}^d)$ *for any* $\gamma < -\log_4 \rho(\mathbf{T}_{\mathbf{P}}|_{\mathbf{H}^0_N}) - \frac{d}{2}$.

4. Examples. In this section, we will give the Sobolev regularity estimates of some refinable vectors Φ. Before doing this, we shall decide if $\Phi = (\phi_l)_{l=1}^r$ is stable or orthogonal. It was shown (see [7], [13], [21] and [26]) that Φ is stable if and only if there exists a positive constant c such that $G_\Phi(\omega) \geq c\mathbf{I}_r$ for all $\omega \in \mathbf{T}^d$; and that Φ is orthogonal if and only if $G_\Phi(\omega) = \mathbf{I}_r$ for all $\omega \in \mathbf{T}^d$ and the matrix mask $\mathbf{P}$ is a *CQF* (*Conjugate Quadrature Filter*), i.e. $\mathbf{P}$ satisfies

$$\sum_{\nu \in \mathbf{Z}^d/2\mathbf{Z}^d} \mathbf{P}(\omega + \nu\pi)\mathbf{P}^*(\omega + \nu\pi) = \mathbf{I}_r.$$

Assume that $\mathbf{P}$ satisfies the vanishing moment conditions of order at least one, and $\mathbf{P}(0)$ satisfies Condition E. By Theorem 2.2, 1 is an eigenvalue of $\mathbf{T}_{\mathbf{P}}$. If the 1-eigenmatrix of $\mathbf{T}_{\mathbf{P}}$ is positive (or negative) definite on $\mathbf{T}^d$, then there exists a nontrivial refinable vector Φ in $L^2(\mathbf{R}^d)$ by Theorem 3.1, and $G_\Phi(\omega)$ is also a 1-eigenmatrix of $\mathbf{T_P}$. Therefore if eigenvalue 1 is simple, then $G_\Phi(\omega)$ is the unique (up to a constant) 1-eigenmatrix of $\mathbf{T_P}$ and hence Φ is stable. If $\mathbf{P}$ is a CQF, then $\mathbf{I}_r$ is a 1-eigenmatrix of $\mathbf{T_P}$. Thus if 1 is a simple eigenvalue of $\mathbf{T_P}$, then $\Phi \in L^2(\mathbf{R}^d)$ and $G_\Phi(\omega) = c\mathbf{I}_r$ for some nonzero constant c. Hence Φ is orthogonal, i.e. the integer shifts of ϕ_l, $1 \le l \le r$, form an orthogonal basis of their closed linear span in $L^2(\mathbf{R}^d)$. Therefore to decide if the refinable vector Φ is stable (or orthogonal), we need only to check that if 1 is a simple eigenvalue of $\mathbf{T_P}$ and the corresponding eigenmatrix is positive (or negative) definite on $\mathbf{T}^d$. In fact the stability of Φ implies that $\mathbf{T_P}$ satisfies Condition E and the 1-eigenmatrix of $\mathbf{T_P}$ is positive (or negative) definite on $\mathbf{T}^d$, see [29].

Assume that Φ is a compactly supported refinable vector with refinement mask $\mathbf{P}$ satisfying (2.6) and (2.7) for some row vectors $\mathbf{l}_0^\beta$, $|\beta| \le 2m-1$. To estimate the regularity of Φ by Theorem 3.2, we need to find $\rho(\mathbf{T_P}|_{\mathbf{H}_N^0})$. We note that λ is an eigenvalue of $\mathbf{T_P}|_{\mathbf{H}_N^0}$ if and only if there exists a right eigenvector $\mathbf{v}$ of eigenvalue λ of $\mathcal{T}_\mathbf{P}$ satisfying that for any $\beta, \alpha \in \mathbf{Z}_+^d, |\beta| \le 2m-1, |\alpha| \le m-1, 1 \le j \le r$

$$\mathbf{L}_N^\beta \mathbf{v} = 0, \quad {}_j\mathbf{l}_N^\alpha \mathbf{v} = 0 \text{ and } {}_j\mathbf{r}_N^\alpha \mathbf{v} = 0, \tag{4.1}$$

where $\mathbf{L}_N^\beta$, ${}_j\mathbf{l}_N^\alpha$ and ${}_j\mathbf{r}_N^\alpha$ are the vectors defined by (2.13) and (3.6), respectively. Let $H_0 \in \mathbf{H}_N$ be the unique matrix function such that $\mathrm{vec}(H_0) = \mathbf{v}$, then H_0 is a λ-eigenmatrix of $\mathbf{T_P}|_{\mathbf{H}_N^0}$. Thus $\rho(\mathbf{T_P}|_{\mathbf{H}_N^0})$ is the largest modulus of all such eigenvalues of $\mathcal{T}_\mathbf{P}$ that have corresponding right eigenvectors satisfying (4.1).

We say that the Sobolev regularity estimate s_0 is optimal if $\Phi \in W^s(\mathbf{R}^d)$ if and only if $s < s_0$.

Example 4.1. Let ϕ_1 and ϕ_2 be the B-splines defined by the knots $0,0,1,1$ and $0,1,1,2$, respectively, i.e. $\phi_1(x) = 2x(1-x)\chi_{[0,1]}(x)$ and $\phi_2(x) = x^2\chi_{[0,1)}(x) + (2-x)^2\chi_{[1,2]}(x)$. Then $\Phi = (\phi_1, \phi_2)^T$ satisfies the matrix refinement equation (1.1) with mask

$$\mathbf{P}(\omega) := \frac{1}{4}\begin{bmatrix} 1+e^{-i\omega} & 1 \\ e^{-i\omega}+e^{-2i\omega} & \frac{1}{2}+2e^{-i\omega}+\frac{1}{2}e^{-2i\omega} \end{bmatrix}.$$

Mask $\mathbf{P}$ satisfies the vanishing moment conditions of order 3 with $\mathbf{l}_0^0 = (1,1)$, $\mathbf{l}_0^1 = (\frac{1}{2}, 1)$ and $\mathbf{l}_0^2 = (0,1)$, see [27]. The eigenvalues of $\mathbf{P}(0)$ are $1, \frac{1}{4}$. We can find vectors $\mathbf{l}_0^3 = (-\frac{1}{4}, 1)$, $\mathbf{l}_0^4 = (-1/10, 9/10)$ and $\mathbf{l}_0^5 = (\frac{1}{4}, \frac{1}{2})$ satisfying (2.7). In this case, $\mathcal{T}_\mathbf{P}$ is a 20×20 matrix. For $0 \le \beta \le 5$, $\mathbf{L}_2^\beta \ne 0$. Thus $2^{-\beta}$, $0 \le \beta \le 5$, are eigenvalues of $\mathcal{T}_\mathbf{P}$. In fact the eigenvalues of $\mathcal{T}_\mathbf{P}$ or $\mathbf{T_P}$ are $1, \frac{1}{2}, \frac{1}{4}(3), \frac{1}{8}(4), \frac{1}{16}(3), \frac{1}{32}(2), 0(4)$. Here for an eigenvalue λ, the notation $\lambda(\ell)$ means that the algebraic multiplicity of λ is ℓ. Thus $\mathbf{T_P}$ satisfies Condition E. We can find a right 1-eigenvector $\mathbf{v}$ of $\mathcal{T}_\mathbf{P}$:

$$\mathbf{v} = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 3 \;\; 1 \;\; 4 \;\; 3 \;\; 3 \;\; 12 \;\; 0 \;\; 3 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0)^T.$$

That is

$$H(\omega) = \begin{bmatrix} 4 & 3+3e^{i\omega} \\ 3+3e^{-i\omega} & 12+e^{i\omega}+e^{-i\omega} \end{bmatrix}$$

is a 1-eigenmatrix of $\mathbf{T_P}$. Checking directly, $H(\omega) \ge 2\mathbf{I}_2$ for all $\omega \in \mathbf{T}^d$, thus Φ is stable since $\mathbf{T_P}$ satisfies Condition E.

To estimate the regularity by our method, we need to find the largest eigenvalue module of $\mathbf{T_P}|_{\mathbf{H}_N^0}$. By Corollary 3.2, 1, $\frac{1}{2}$ are not eigenvalues of $\mathbf{T_P}|_{\mathbf{H}_N^0}$. We find $\frac{1}{8}$

is the largest eigenvalue module of $\mathbf{T_P}|_{\mathbf{H}_N^0}$ with a corresponding eigenmatrix

$$\begin{bmatrix} (e^{-i\omega}+e^{i\omega})/2 & -1-e^{i\omega} \\ -1-e^{-i\omega} & 2 \end{bmatrix}.$$

Therefore $\Phi \in W^{\frac{3}{2}-\epsilon}(\mathbf{R})$ or $\Phi \in C^{1-\epsilon}(\mathbf{R})$ for any $\epsilon > 0$, and our estimate is optimal from the definition of Φ.

Example 4.2. Let $\Phi = (\phi_1, \phi_2)^T$ be the refinable vectors treated in [11]. The mask of Φ is given by

$$\mathbf{P}(\omega) := \frac{1}{20}\begin{bmatrix} 6+6e^{-i\omega} & 8\sqrt{2} \\ (-1+9e^{-i\omega}+9e^{-2i\omega}-e^{-3i\omega})/\sqrt{2} & -3+10e^{-i\omega}-3e^{-2i\omega} \end{bmatrix}.$$

Mask $\mathbf{P}$ is a CQF and satisfies the vanishing moment conditions of order 2 with $\mathbf{l}_0^0 = (1.4142, 1)$ and $\mathbf{l}_0^1 = (.7071, 1)$, see [27]. The eigenvalues of $\mathbf{P}(0)$ are $1, -.2$ and we can find vectors $\mathbf{l}_0^2 = (.4714, .8333)$ and $\mathbf{l}_0^3 = (.3536, .5)$ satisfying (2.7). For $0 \le \beta \le 3$, vectors $\mathbf{L}_3^\beta \ne 0$, thus $2^{-\beta}$ are eigenvalues of $\mathcal{T}_\mathbf{P}$. The eigenvalues of $\mathcal{T}_\mathbf{P}$ or $\mathbf{T_P}$ are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}(2), -.2(2), .2(2), -.1(2), -.05(4), .04$ and $0(12)$. Thus $\mathbf{T_P}$ satisfies Condition E and hence Φ is orthogonal.

By Corollary 3.2, $1, \frac{1}{2}$ and $\frac{1}{4}$ are not eigenvalues of $\mathbf{T_P}|_{\mathbf{H}_N^0}$. We find that the largest eigenvalue module of $\mathbf{T_P}|_{\mathbf{H}_N^0}$ is $\frac{1}{8}$ with a corresponding eigenmatrix $H(\omega) = \sum_{k=-3}^{3} H_k e^{-ik\omega}$ given by

$$H_0 = \begin{bmatrix} -.0875 & .0674 \\ .0674 & -.1085 \end{bmatrix}, \quad H_1 = H_{-1}^T = \begin{bmatrix} -.0042 & .0004 \\ .0674 & -.0417 \end{bmatrix}$$

and

$$H_2 = H_{-2}^T = \begin{bmatrix} 0 & 0 \\ .0004 & 0 \end{bmatrix}, \quad H_3 = H_{-3} = \mathbf{0}.$$

Thus $\Phi \in W^{\frac{3}{2}-\epsilon}(\mathbf{R})$ or $\Phi \in C^{1-\epsilon}(\mathbf{R})$ for any $\epsilon > 0$. It was shown in [11] that Φ is in the Lip space, i.e. $|\Phi(x) - \Phi(y)| \le c|x-y|$ for some constant c independent of $x, y \in \mathbf{R}$. However $\Phi \notin C^1(\mathbf{R})$ since $\frac{1}{\sqrt{2}}(\phi_1(x) + \phi_1(x-1)) + \phi_2(x)$ is the hat function $x\chi_{[0,1]}(x) + (2-x)\chi_{(1,2]}(x)$ (see [31]), thus our estimate is optimal.

At last we will analyze two refinable vectors from [1].

Example 4.3. Let $\Phi = (\phi_1, \phi_2)^T$ be the refinable vector treated in [1]. The mask of Φ is given by

$$\mathbf{P}(\omega) := \frac{1}{8}\begin{bmatrix} 2+4e^{-i\omega}+2e^{-2i\omega} & 2-2e^{-2i\omega} \\ -\sqrt{7}+\sqrt{7}e^{-2i\omega} & -\sqrt{7}+2e^{-i\omega}-\sqrt{7}e^{-2i\omega} \end{bmatrix}.$$

Mask $\mathbf{P}$ is a CQF and satisfies the vanishing moment conditions of order 2 with $\mathbf{l}_0^0 = (1, 0)$ and $\mathbf{l}_0^1 = (1, .2743)$, see [1]. The eigenvalues of $\mathbf{P}(0)$ are $1, -.4114$ and we can find vectors $\mathbf{l}_0^2 = (1.0752, .5486)$ and $\mathbf{l}_0^3 = (1.2257, .7909)$ satisfying (2.7). For $0 \le \beta \le 3$, vectors $\mathbf{L}_2^\beta \ne 0$, thus $2^{-\beta}, 0 \le \beta \le 3$ are eigenvalues of $\mathcal{T}_\mathbf{P}$. The eigenvalues of $\mathbf{T_P}$ or $\mathcal{T}_\mathbf{P}$ are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, -.4114(2), .2318, -.2057(3), .0130(2)$ and $0(8)$. Thus $\mathbf{T_P}$ satisfies Condition E and Φ is orthogonal.

By Corollary 3.2, $1, \frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ are not eigenvalues of $\mathbf{T_P}|_{\mathbf{H}_N^0}$. We find the largest eigenvalue module of $\mathbf{T_P}|_{\mathbf{H}_N^0}$ is .2318 with a corresponding eigenmatrix $H(\omega) =$

$\sum_{k=-2}^{2} H_k e^{-ik\omega}$ given by

$$H_0 = \begin{bmatrix} .2117 & 0 \\ 0 & .7564 \end{bmatrix}, \quad H_1 = H_{-1}^T = \begin{bmatrix} -.1059 & .1930 \\ -.1930 & .3253 \end{bmatrix}$$

and $H_2 = H_{-2} = \mathbf{0}$. Thus $\Phi \in W^{1.0545-\epsilon}(\mathbf{R})$ or $\Phi \in C^{.5545-\epsilon}(\mathbf{R})$ for any $\epsilon > 0$.

Example 4.4. Let $\Phi = (\phi_1, \phi_2)^T$ be another refinable vector treated in [1]. The mask $\mathbf{P}(\omega) := \frac{1}{2}\sum_{k=0}^{3} \mathbf{P}_k e^{-ik\omega}$ of Φ is given by

$$\mathbf{P}_0 = \frac{1}{40}\begin{bmatrix} 10-3\sqrt{10} & 5\sqrt{6}-2\sqrt{15} \\ 5\sqrt{6}-3\sqrt{15} & 5-3\sqrt{10} \end{bmatrix}, \mathbf{P}_1 = \frac{1}{40}\begin{bmatrix} 30+3\sqrt{10} & 5\sqrt{6}-2\sqrt{15} \\ -5\sqrt{6}-7\sqrt{15} & 5-3\sqrt{10} \end{bmatrix}$$

and $\mathbf{P}_2 = S_0\mathbf{P}_1 S_0, \mathbf{P}_3 = S_0 \mathbf{P}_0 S_0$, where $S_0 = \text{diag}(1,-1)$. Mask $\mathbf{P}$ is a CQF and satisfies the vanishing moment conditions of order 3 with $\mathbf{l}_0^0 = (1,0)$, $\mathbf{l}_0^1 = (1.5, .2372)$ and $\mathbf{l}_0^2 = (2.3063, .7117)$, see [1]. The eigenvalues of $\mathbf{P}(0)$ are $1, .0257$ and we can find vectors $\mathbf{l}_0^3 = (3.6283, 1.8980)$, $\mathbf{l}_0^4 = (6.0943, 4.9822)$ and $\mathbf{l}_0^5 = (11.5329, 13.4836)$ satisfying (2.7). Vectors $\mathbf{L}_3^\beta \neq 0$, thus $2^{-\beta}$, $0 \leq \beta \leq 5$, are eigenvalues of $\mathbf{T_P}$. The eigenvalues of $\mathbf{T_P}$ or $\mathcal{T}_\mathbf{P}$ are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, .1357, -.0625, -.0576, .0257(2)$, $.0128(2), .0078(2), .0064(4), -.0016(4), .0032(2), .0007$ and $.0003(2)$. Thus $\mathbf{T_P}$ satisfies Condition E and Φ is orthogonal.

By Corollary 3.2, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ and $\frac{1}{32}$ are not eigenvalues of $\mathbf{T_P}|_{\mathbf{H}_N^0}$. We find the largest eigenvalue module of $\mathbf{T_P}|_{\mathbf{H}_N^0}$ is .1357 with a corresponding eigenmatrix $H(\omega) = \sum_{k=-3}^{3} H_k e^{-ik\omega}$ given by

$$H_0 = \begin{bmatrix} .1180 & 0 \\ 0 & .8072 \end{bmatrix}, \quad H_1 = H_{-1}^T = \begin{bmatrix} -.0506 & .1602 \\ -.1602 & .3362 \end{bmatrix}$$

and

$$H_2 = H_{-2}^T = \begin{bmatrix} -.0084 & .0087 \\ -.0087 & .0086 \end{bmatrix}, \quad H_3 = H_{-3} = \mathbf{0}.$$

Thus $\Phi \in W^{1.4408-\epsilon}(\mathbf{R})$ or $\Phi \in C^{.9408-\epsilon}(\mathbf{R})$ for any $\epsilon > 0$.

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