Refinable bivariate quartic and quintic C^2-splines for quadrilateral subdivisions

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Abstract

Refinable compactly supported bivariate C^2 quartic and quintic spline function vectors on the four-directional mesh are introduced in this paper to generate matrix-valued templates for approximation and Hermite interpolatory surface subdivision schemes, respectively, for both the $\sqrt{2}$ and 1-to-4 split quadrilateral topological rules. These splines have their full local polynomial preservation orders. In addition, we extend our study to parametric approach and use the symmetric properties of our refinable quintic spline components as a guideline to reduce the number of free parameters in constructing second order C^2 Hermite interpolatory quadrilateral subdivision schemes with precisely six components.

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1. Introduction

There are, basically, two standard approaches to introducing compactly supported bivariate (polynomial) splines on some pre-assigned grid partition, namely: (i) convolution along certain direction sets with an initial (trivial) piecewise polynomial function, and (ii) smoothing the polynomial pieces across the given grid partition with the zero polynomial outside the designated supports. (See [2] for an in-depth discussion of these two approaches.)

It is clear that the convolution approach only applies to regular grid partitions with the direction sets strictly dictated by the grid lines. For example, by adopting the direction set

$$X_{mm} := \{\underbrace{\mathbf{e}_1, \cdots, \mathbf{e}_1}_{m}, \underbrace{\mathbf{e}_2, \cdots, \mathbf{e}_2}_{m}\},$$

with $\mathbf{e}_1 = (1,\ 0), \mathbf{e}_2 = (0,\ 1)$, and the characteristic function B_{11} of the unit square $[0,1]^2$ as the initial function, we obtain the C^{m-2} tensor-product cardinal B-spline B_{mm} of degree $2(m-1)$. More generally, by introducing other directions, such as $\mathbf{e}_3 = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_4 = \mathbf{e}_1 - \mathbf{e}_2$, while keeping the same initial function B_{11}, we have the three-directional and four-directional box-splines $B_{k\ell m}$ and $B_{k\ell mn}$, with direction sets

$$X_{k\ell m} := \{\underbrace{\mathbf{e}_1, \cdots, \mathbf{e}_1}_{k}, \underbrace{\mathbf{e}_2, \cdots, \mathbf{e}_2}_{\ell}, \underbrace{\mathbf{e}_3, \cdots, \mathbf{e}_3}_{m}\},$$

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and

$$X_{k\ell mn} := \{\underbrace{\mathbf{e}_1, \cdots, \mathbf{e}_1}_{k}, \underbrace{\mathbf{e}_2, \cdots, \mathbf{e}_2}_{\ell}, \underbrace{\mathbf{e}_3, \cdots, \mathbf{e}_3}_{m}, \underbrace{\mathbf{e}_4, \cdots, \mathbf{e}_4}_{n}\},$$

respectively. Since the convolution operation for generating box splines from the initial refinable characteristic function B_{11} of $[0, 1]^2$ preserves the property of refinability, and since this operation also increases the smoothness orders (while increasing the polynomial degrees), the refinement masks of box splines constitute a fundamental tool for designing surface subdivision schemes. The best known schemes are the Catmull-Clark scheme [1] and the Loop scheme [16], based on B_{44} and B_{222} for the 1-to-4 split topological rule, for the quadrilateral and triangular subdivisions, respectively.

A drawback of box splines is, however, their unnecessarily larger support sizes. Since templates with one single ring is somewhat essential for designing weighted averages to take care of extraordinary vertices, we follow the smoothing approach to introducing bivariate splines with the desirable order of smoothness and "smallest" support sizes. To achieve the refinability property, however, we usually require more than one compactly supported spline function. That is, we must extend scalar subdivision to vector subdivision and to introduce matrix-valued templates. In our earlier work [5, 6, 7], we considered triangular subdivisions. In particular, in [5, 6], C^1-quadratic, C^2-cubic, and C^2-quartic splines on the six-directional mesh were introduced for generating matrix-valued templates for both approximation and Hermite interpolatory triangular subdivisions for both the $\sqrt{3}$ and 1-to-4 topological rules.

In the present paper, we will focus on quadrilateral subdivisions. Since we will engage both the $\sqrt{2}$ and 1-to-4 topological rules, we consider the four-directional mesh instead. A refinable function vector of C^2-quartic splines will be introduced for generating approximation quadrilateral subdivisions, and that of C^2-quintic splines will be constructed for generating a second order Hermite interpolatory quadrilateral subdivision.

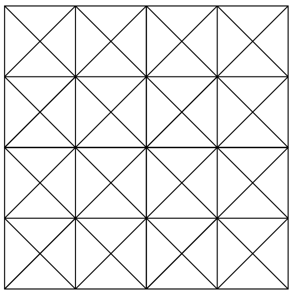

Figure 1: *Four-directional mesh* $\triangle^2$

To be more precise, let $\triangle^2$ denote the four-directional mesh with a truncated portion shown in Fig.1. For integers d and r, with $0 \le r < d$, let $S_d^r(\triangle^2)$ be the collection of all (real-valued) functions in $C^r(\mathbb{R}^2)$ whose restrictions on each triangle of the triangulation $\triangle^2$ are bivariate polynomials of total degree $\le d$. Each function ϕ in $S_d^r(\triangle^2)$ is called a bivariate C^r-spline of degree d on $\triangle^2$. In addition, if $\phi_0, \phi_1, \cdots, \phi_{n-1}$ are compactly supported functions in $S_d^r(\triangle^2)$ such that the column vector $\Phi := [\phi_0, \cdots, \phi_{n-1}]^T$ satisfies a refinement equation

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}} P_{\mathbf{k}} \Phi(A\mathbf{x} - \mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{1.1}$$

for some $n\times n$ matrices $P_{\mathbf{k}}$ with constant entries and an expansive integer matrix A, Φ is called a refinable function vector with subdivision (refinement) mask $\{P_{\mathbf{k}}\}$. For the refinable function vector Φ to be useful for surface subdivisions, it must satisfy the condition of "generalized partition of unity":

$$\sum_{\mathbf{k}\in\mathbf{Z}^2} \mathbf{w}\Phi(\mathbf{x}-\mathbf{k}) \equiv 1, \quad \mathbf{x}\in\mathbb{R}^2, \tag{1.2}$$

for some constant n-vector $\mathbf{w} = [w_0, w_1, \cdots, w_{n-1}]$. By changing the order of the ϕ_ℓ and multiplying them with some constant, if necessary, we may and will, assume that

$$w_0 = 1. \tag{1.3}$$

Analogous to (scalar-valued) refinable functions, a refinable function vector Φ with the subdivision mask $\{P_{\mathbf{k}}\}$ gives the local averaging rule

$$\mathbf{v}_{\mathbf{k}}^{m+1} = \sum_{\mathbf{j}} \mathbf{v}_{\mathbf{j}}^{m} P_{\mathbf{k}-A\mathbf{j}}, \quad m = 0, 1, \cdots, \tag{1.4}$$

where

$$\mathbf{v}_{\mathbf{k}}^{m} =: [v_{\mathbf{k}}^{m}, s_{\mathbf{k},1}^{m} \cdots, s_{\mathbf{k},n-1}^{m}] \tag{1.5}$$

are "row-vectors" with n components of points $v_{\mathbf{k}}^m, s_{\mathbf{k},i}^m$, $i = 1, \cdots, n-1$, in $\mathbb{R}^3$. We will use the first components $v_{\mathbf{k}}^m$ of $\mathbf{v}_{\mathbf{k}}^m$ as the vertices of the (non-planar) quadrilateral mesh for the m^{th} iteration. Therefore, for sufficiently large values of m, the vertices $v_{\mathbf{k}}^m$ provide an accurate discrete approximation of the target subdivision surface. In [7], we have shown that this subdivision surface is precisely given by the series representation:

$$F(\mathbf{x}) = \sum_{\mathbf{k}} v_{\mathbf{k}}^0 \phi_0(\mathbf{x}-\mathbf{k}) + \sum_{\mathbf{k}} \left(s_{\mathbf{k},1}^0 \phi_1(\mathbf{x}-\mathbf{k}) + \cdots + s_{\mathbf{k},n-1}^0 \phi_{n-1}(\mathbf{x}-\mathbf{k}) \right),$$

with $[v_{\mathbf{k}}^0, s_{\mathbf{k},1}^0, \cdots, s_{\mathbf{k},n-1}^0]$ as coefficients, provided that the iteration process converges. Of course, the assumption (1.3) is essential for the first components to be used as vertices. As in [5, 6, 7], we will call the initial row vectors $\mathbf{v}_{\mathbf{k}}^0$, "control vectors", their first components $v_{\mathbf{k}}^0$, "control vertices", and the other components $s_{\mathbf{k},1}^0, \cdots, s_{\mathbf{k},n-1}^0$, "shape-control parameters". The first components $v_{\mathbf{k}}^0$ of the control vectors $\mathbf{v}_{\mathbf{k}}^0$ are used to control the positions of the (limiting) subdivision surface. In particular, for interpolatory subdivision schemes, $v_{\mathbf{k}}^m, m = 0, 1, \cdots$, lie on the subdivision surface. On the other hand, as demonstrated in an independent paper [7], variation of the other components of $\mathbf{v}_{\mathbf{k}}^0$, namely the shape control parameters, could change the shape of subdivision surfaces dramatically. A preliminary result concerning the choices of shape control parameters is given in [7].

To describe the 1-to-4 split and $\sqrt{2}$ topological rules that govern how new vertices are chosen and how they are connected to yield a finer quadrilateral subdivided surface in $\mathbb{R}^3$, we use a two-dimensional regular quadrilateral mesh as guideline. That is, each (non-planar) quadrilateral of the subdivided surface in $\mathbb{R}^3$ is represented by a quadrilateral cell $\square$. The 1-to-4 split (dyadic) topological rule for the quadrilateral subdivision is quite simple. See Fig. 2, where each quadrilateral $\square$ in the parametric domain is split into four sub-quadrilaterals. The dilation matrix A in (1.1) corresponding to the 1-to-4 split topological rule is just $2I_2$, where I_2 is the 2×2 identity matrix.

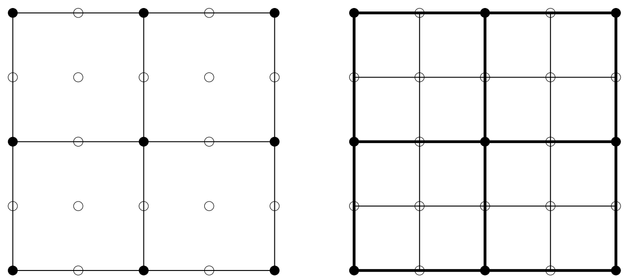

Figure 2: *Topological rule of* 1*-to-*4 *split quadrilateral subdivision*

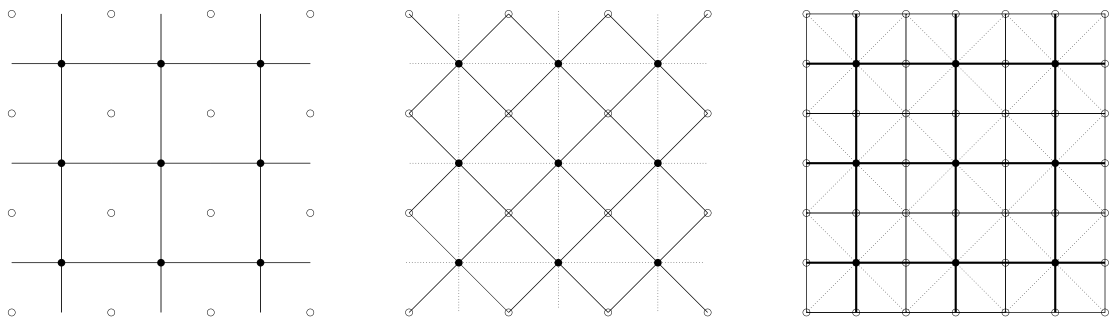

Figure 3: *Topological rule of* $\sqrt{2}$*-subdivision*

The $\sqrt{2}$-subdivision we consider in this paper for the quadrilateral mesh is analogous to the $\sqrt{3}$-subdivision for the triangular mesh in [14, 15]. It can be viewed as the 4-8 subdivision in [19, 20], corresponding to the quincunx dilation matrix B defined by

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{1.6}$$

To describe the $\sqrt{2}$ topological rule, the new vertices are represented by the centers of the quadrilateral cells, while the new edges are obtained by joining the center of each quadrilateral cell to its four adjacent (old) vertices. To complete describing this topological rule, the old edges are to be removed (see the graphs on the left and in the middle of Fig.3). Hence, if the regular quadrilateral mesh is the mesh of $\mathbb{R}^2$ generated by grid lines $x = i, y = j$, where $i, j, \in \mathbb{Z}$, as shown on the left of Fig.3, then before removing the old edges as dictated by the $\sqrt{2}$ topological rule, we have the four-directional mesh $\triangle^2$ as shown in the middle of Fig.3. Observe that if the topological rule is applied for a second time, then we arrive at the 2-dilated quadrilateral mesh shown on the right of Fig.3 of the original quadrilateral mesh shown on the left of Fig.3. That is why it is called $\sqrt{2}$-subdivision. Note that the quincunx dilation matrix B has the property that $\triangle^2 \subset B^{-1}\triangle^2$.

In Sections 2 and 3, we will construct a C^2 quartic-spline function vector and a C^2 Hermite interpolatory quintic-spline function vector, respectively, that are refinable with both dilation matrices B and $2I_2$. The polynomial preservation of these splines and the sum rule orders of the corresponding masks will be studied. In Section 4, we will use the parametric approach to construct second order C^2 Hermite interpolatory quadrilateral 1-to-4 split and $\sqrt{2}$ subdivision schemes with precisely six components. Here we remark that in our companion work [8], an innovative concept of interpolatory vector subdivision was introduced. By modifying the refinable C^2-quartic splines constructed in this paper, we are able to construct C^2 interpolatory vector quadrilateral 1-to-4 split and $\sqrt{2}$ subdivision schemes with C^2-quartic splines as the basis functions (see [8] for details). In this paper, we only consider the schemes for regular vertices (with valence 4) for the quadrilateral mesh. The matrix-valued schemes for extraordinary vertices could be designed so that the smoothness conditions for the vector subdivision surfaces developed in our paper [7] are satisfied.

Before moving on to the next section, let us recall some relevant results concerning sum rule orders of subdivision masks. For $n \geq 2$, let $\Phi = [\phi_0, \cdots, \phi_{n-1}]^T$, with bounded supp$\phi_\ell$ and $\phi_\ell \in L^2(\mathbb{R}^s)$ (ϕ_ℓ not necessarily piecewise polynomial functions), $\ell = 0, \ldots, n-1$, satisfy the refinement equation (1.1) for some finite sequence $\{P_{\mathbf{k}}\}$ of $r \times r$ matrices. Let

$$P(\mathbf{z}) := |\det A|^{-1} \sum_{\mathbf{k}} P_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$$

be the two-scale (matrix Laurent polynomial) symbol of Φ. Then $P(\mathbf{z})$ is said to possess the property of sum rule of order m, if there exist row n-vectors $\mathbf{y}_\alpha$, $|\alpha| < m$ with $\mathbf{y}_{0,0} \neq \mathbf{0}$, such that the trigonometric polynomial vector $\mathbf{t}(\omega) = \sum_{\beta \in \mathbf{Z}_+^2, |\beta| < m} \mathbf{t}_\beta e^{-i\beta\omega}$ defined by

$$(-iD)^\alpha \mathbf{t}(0) = \mathbf{y}_\alpha, \quad |\alpha| < m,$$

satisfies

$$D^\beta(\mathbf{t}(A^T\omega)P(e^{-i\omega}))|_{\omega = 2\pi A^{-T}\omega_h} = \delta_{h,0} D^\beta \mathbf{t}(0), \quad |\beta| < m, \tag{1.7}$$

where ω_h, with $\omega_0 = 0$ and $0 \leq h \leq |\det A| - 1$, are the representers of $\mathbb{Z}^2/A^T\mathbb{Z}^2$. See [12, 10] for the method of computation of $\mathbf{y}_\alpha$. It is well-known (see, for example [12]) that if $P(\mathbf{z})$ satisfies the sum rule of order m, then Φ has the polynomial reproduction property of order m (or total degree $m-1$), and in fact, the vectors $\mathbf{y}_\alpha$ can be used to give the following explicit polynomial reproduction formula:

$$\mathbf{x}^\alpha = \sum_{\mathbf{k}} \left\{ \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathbf{k}^{\alpha - \beta} \, \mathbf{y}_\beta \right\} \Phi(\mathbf{x} - \mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^2, \ |\alpha| < m. \tag{1.8}$$

Here and throughout, we use the following standard notations: for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and multi-indices $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$, $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$; D^α stands for the differential operator $D_1^{\alpha_1} D_2^{\alpha_2}$; $|\alpha| = \alpha_1 + \alpha_2$; $\beta \leq \alpha$ means that $\beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2$; and for $\beta \leq \alpha$, $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}$. For a point $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ or a multi-index $\alpha = (\alpha_1, \alpha_2)$, and a 2×2 matrix M, the notations $M\mathbf{x}$ and $M\alpha$ should be understood as $M\mathbf{x}^T$ and $M\alpha^T$.

2. Refinable C^2-quartic splines

It is not difficult to construct the (unique) bivariate C^2 quartic spline ϕ_0 with minimum support, by applying the C^2-smoothing formula (see [2, Theorem 5.1]), with uniqueness being governed by the normalization condition $\phi_0(0) = 1$, as shown in Fig.4, where Bézier coefficients that are obviously equal to zero are not displayed. This function is, however, not refinable. We therefore need other compactly supported basis functions to formulate refinable function

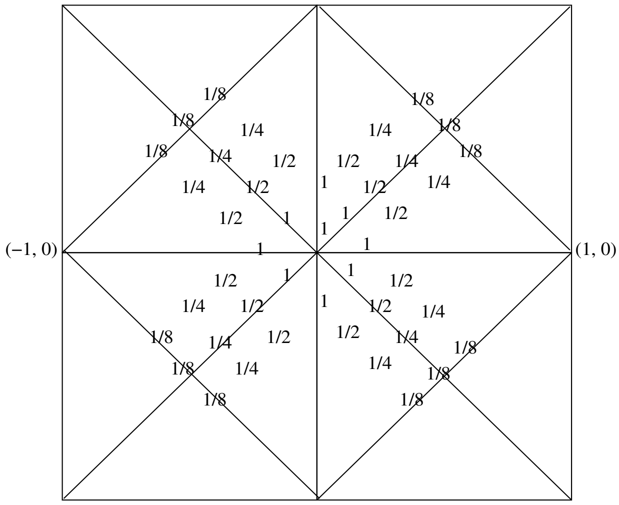

Figure 4: *Support and Bézier coefficients of* $\phi_0 \in S_4^2(\triangle^2)$

vector. To do so, we introduce:

$$\phi_1 := \phi_0(B^{-1}\cdot), \tag{2.1}$$

$$\begin{aligned}\phi_2 := {} & \phi_0(\tfrac{\cdot}{2}) - \frac{1}{8}\{\phi_0(\cdot - (1,0)) + \phi_0(\cdot + (1,0)) + \phi_0(\cdot - (0,1)) + \phi_0(\cdot + (0,1))\} \\ & -\frac{1}{16}\{\phi_0(\cdot - (1,1)) + \phi_0(\cdot + (1,1)) + \phi_0(\cdot - (1,-1)) + \phi_0(\cdot + (1,-1))\},\end{aligned} \tag{2.2}$$

where B is the quincunx matrix in (1.6). See Fig.5 and Fig.6 for the supports and Bézier coefficients of ϕ_1 and ϕ_2, respectively, where only portions of the Bézier coefficients are displayed, since the remaining non-zero Bézier coefficients of ϕ_1 and ϕ_2 follow from the symmetry.

Let $\Phi^d := [\phi_0, \phi_1, \phi_2]^T$. Then we will see that the function vector Φ^d is refinable with the quincunx dilation matrix B. The nonzero matrix coefficients $L_{\mathbf{k}}$ of the corresponding subdivision mask $L := \{L_{\mathbf{k}}\}$ are given by

$$\begin{aligned} & L_{0,0} = \frac{1}{4}\begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}, \quad L_{1,0} = L_{0,1} = L_{-1,0} = L_{0,-1} = H, \\ & L_{1,1} = L_{1,-1} = L_{-1,1} = L_{-1,-1} = I, \end{aligned} \tag{2.3}$$

where

$$H := \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad I := \frac{1}{32}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.4}$$

The subdivision mask $L = \{L_{\mathbf{k}}\}$ immediately yields the matrix-valued templates for the local averaging rule of the C^2 approximating $\sqrt{2}$-subdivision scheme, as shown in Fig.7. It is easy to verify that the so-called "transition operator" T_L associated with this subdivision mask $L = \{L_{\mathbf{k}}\}$ above satisfies Condition E, namely, 1 is a simple eigenvalue of T_L and other eigenvalues of T_L are smaller than 1 in modulus. Therefore, this scheme is convergent in L_2 norm. (The reader is referred to [18, 4] for the convergence discussion, to [12] for the matrix

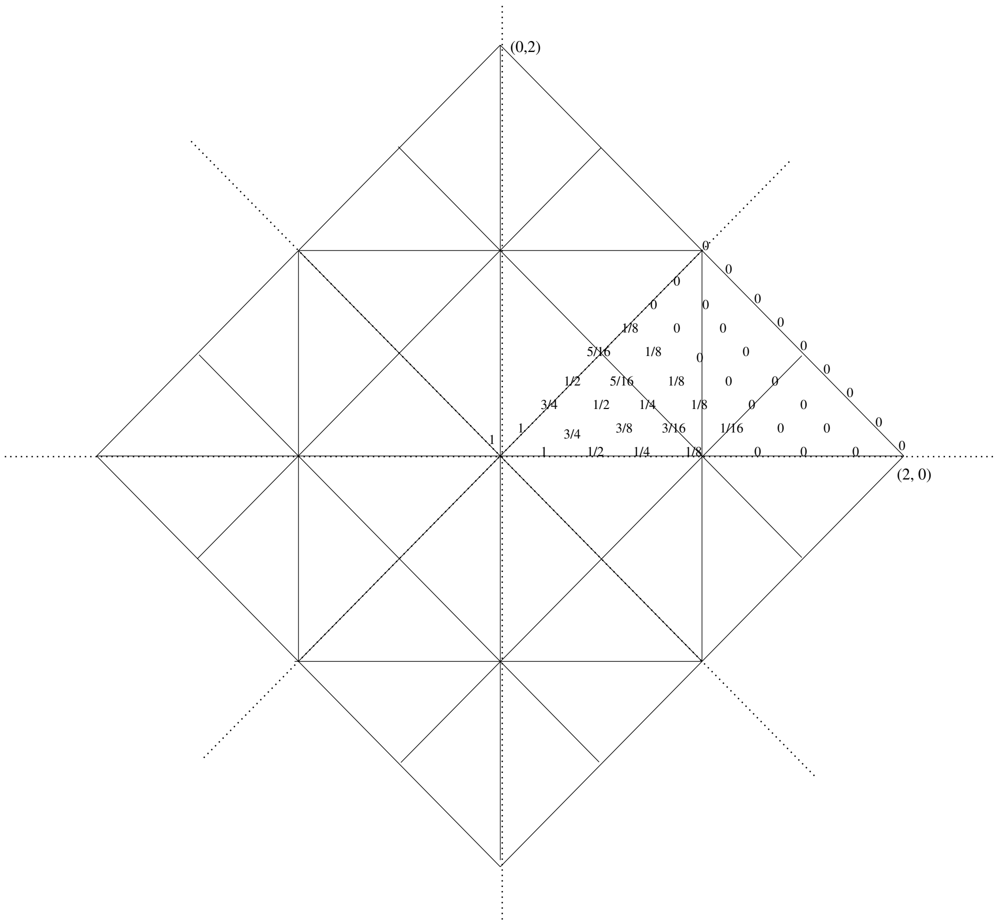

Figure 5: *Support and Bézier coefficients of* $\phi_1 \in S_4^2(\triangle^2)$

representation of the transition operator, and to [13] for the Matlab routines for the calculation of the matrix representation of T_L.)

Remark 1. *It turns out that* ϕ_0, ϕ_1 *are the two splines (up to normalization constants) with the minimal support and quasi-minimal support constructed in [17] and [3], respectively. In [17], another spline with minimal support, denoted as* f_4^0 *in [2], was constructed. See Fig.8 for the support and Bézier coefficients of* f_4^0*, where, again, only a portion of Bézier coefficients are displayed due to the symmetry of* f_4^0*. These three splines* ϕ_0, ϕ_1, f_4^0*, generate the space* $S_4^2(\triangle^2)$ *(see e.g, [2]). Since* $\triangle^2 \subset B^{-1}\triangle^2$ *and* $\triangle^2 \subset (2I_2)^{-1}\triangle^2$*,* $[\phi_0, \phi_1, f_4^0]^T$ *is refinable with both the quincunx matrix* B *and* $2I_2$ *as dilation matrices, and the corresponding subdivision masks will provide the subdivision schemes. The reason we prefer using* ϕ_2 *over* f_4^0 *is that it has better symmetric property around the origin* 0*, and therefore the mask of* $[\phi_0, \phi_1, \phi_2]^T$ *yields more symmetric templates for subdivision than that associated with the mask of* $[\phi_0, \phi_1, f_4^0]^T$.

We have the following result.

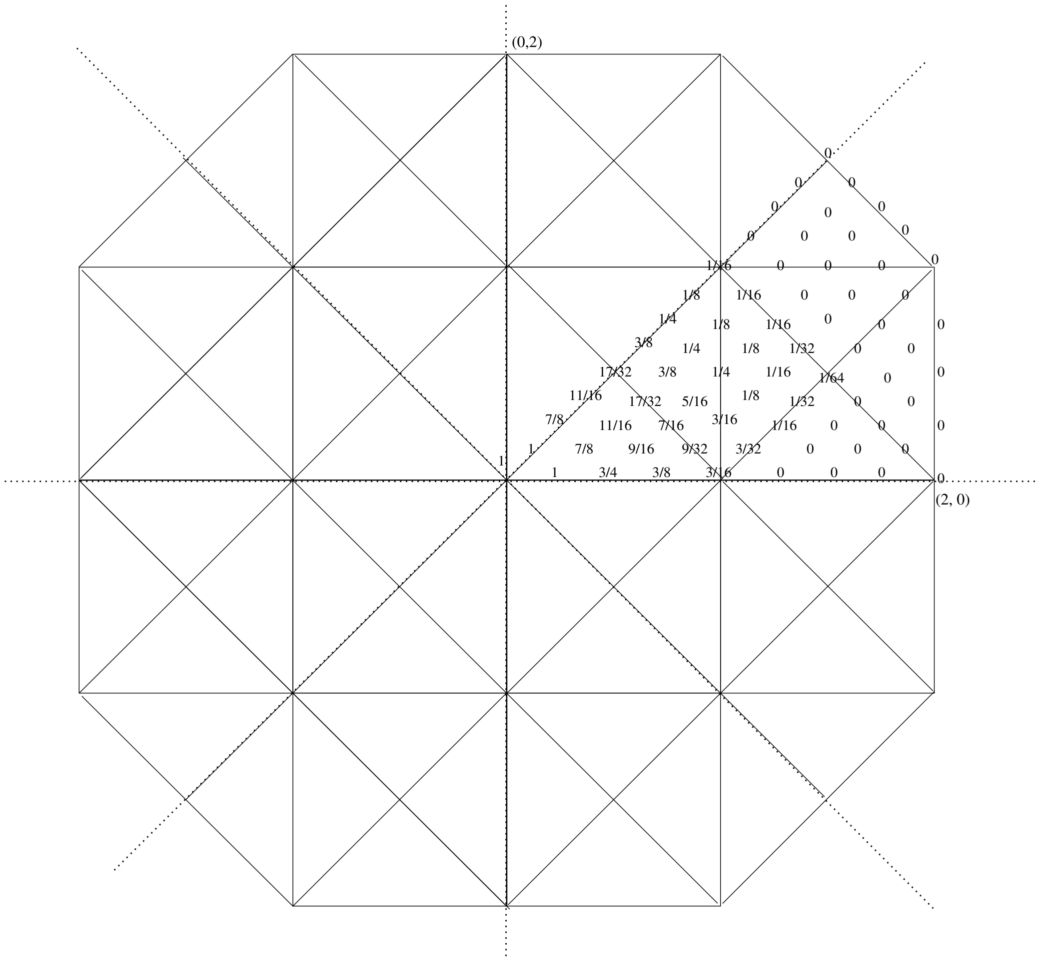

Figure 6: *Support and Bézier coefficients of* $\phi_2 \in S_4^2(\triangle^2)$

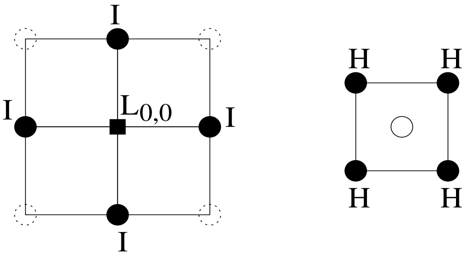

Figure 7: *Matrix-valued templates for C^2 approximating $\sqrt{2}$-subdivision scheme*

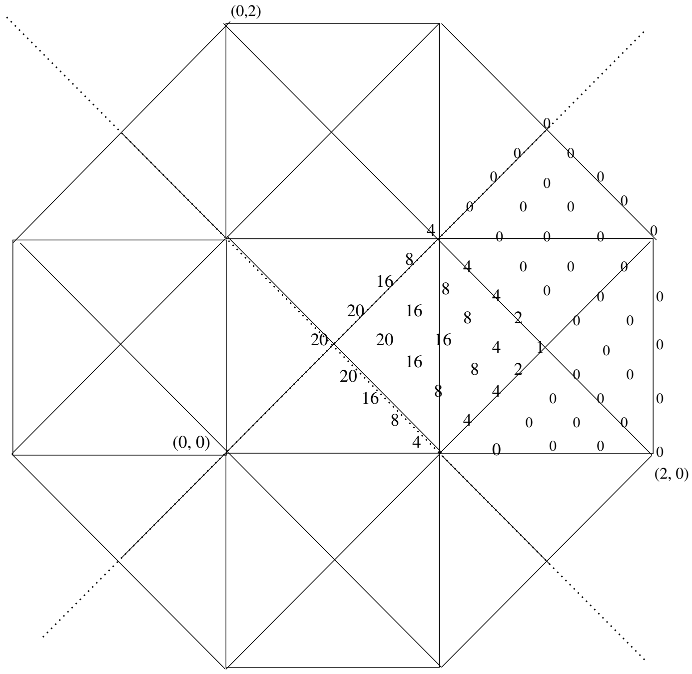

Figure 8: *Support and Bézier coefficients of $48f_4^0 \in S_4^2(\triangle^2)$*

Theorem 1. *For $\phi_0 \in S_4^2(\triangle^2)$, with Bézier coefficients shown in Fig.4, and ϕ_1, ϕ_2 defined by (2.1) and (2.2) respectively, the following statements hold.*

(i) $\Phi^d = [\phi_0, \phi_1, \phi_2]^T$ is refinable with the quincunx dilation matrix B, with nonzero matrix coefficients $L_{\mathbf{k}}$ of the corresponding subdivision mask $\{L_{\mathbf{k}}\}$ given by (2.3);

(ii) The two-scale symbol $L(\mathbf{z})$ of Φ^d satisfies the sum rule of order 3, with the vectors for (1.7) given by

$$\begin{aligned} \mathbf{y}_{0,0} &= \frac{1}{12}[1,2,4], \quad \mathbf{y}_{1,0} = \mathbf{y}_{0,1} = [0,0,0], \\ \mathbf{y}_{2,0} &= \mathbf{y}_{0,2} = \frac{1}{18}[1,-1,-2]; \quad \mathbf{y}_{1,1} = [0,0,0]; \end{aligned} \tag{2.5}$$

(iii) Φ^d *locally reproduces all bivariate quartic polynomials, with*

$$
\begin{aligned}
\mathbf{x}^\alpha &= \sum_{\mathbf{k}} \left\{ \sum_{\beta\le\alpha} \binom{\alpha}{\beta}\ \mathbf{k}^{\alpha-\beta}\ \mathbf{y}_\beta \right\} \Phi^d(\mathbf{x}-\mathbf{k}), |\alpha|\le 3, \alpha=(3,1), \alpha=(1,3),\\
x^4 &= \sum_{k_1,k_2} \{k_1^4\mathbf{y}_{0,0} + 6k_1^2\widetilde{\mathbf{y}}_2 + \widetilde{\mathbf{y}}_4\}\ \Phi^d(x-k_1, y-k_2), \quad (x,y)\in\mathbb{R}^2,\\
y^4 &= \sum_{k_1,k_2} \{k_2^4\mathbf{y}_{0,0} + 6k_2^2\widetilde{\mathbf{y}}_2 + \widetilde{\mathbf{y}}_4\}\ \Phi^d(x-k_1, y-k_2), \quad (x,y)\in\mathbb{R}^2,\\
x^2y^2 &= \sum_{\mathbf{k}} \left\{ \sum_{0\le\beta\le(2,2)} \binom{(2,2)}{\beta}\ \mathbf{k}^{(2,2)-\beta}\ \widetilde{\mathbf{y}}_\beta \right\} \Phi^d(\mathbf{x}-\mathbf{k}), \mathbf{x}=(x,y)\in\mathbb{R}^2,
\end{aligned}
$$

where $\mathbf{y}_\beta$*,* $|\beta|\le 2$*, are given by (2.5), and*

$$
\begin{aligned}
&\mathbf{y}_{3,0} = \mathbf{y}_{2,1} = \mathbf{y}_{1,2} = \mathbf{y}_{0,3} = [0,0,0],\\
&\widetilde{\mathbf{y}}_2 = \frac{1}{144}[1,2,-20], \quad \widetilde{\mathbf{y}}_4 = \frac{1}{2}[-1,1,0],\\
&\widetilde{\mathbf{y}}_\beta = \mathbf{y}_\beta, \quad |\beta|\le 1, \beta=(1,1), \beta=(1,2), \beta=(2,1);\\
&\widetilde{\mathbf{y}}_{2,0} = \widetilde{\mathbf{y}}_{0,2} = \frac{1}{48}[5,-6,-4], \quad \widetilde{\mathbf{y}}_{2,2} = \frac{1}{24}[-1,2,0].
\end{aligned} \tag{2.6}
$$

Remark 2. *The vectors* $\mathbf{y}_\alpha$ *in Theorem 1 (ii) above for the sum rule order of* $L(\mathbf{z})$ *are not unique due to the linear dependence of* ϕ_0, ϕ_1, ϕ_2 governed by

$$\sum_{\mathbf{k}} (7\phi_0 - 10\phi_1 + 4\phi_2)(\mathbf{x}-\mathbf{k}) = 0.$$

The reader is referred to [12, 10] for the discussion on the issue of uniqueness of $\mathbf{y}_\alpha$*.*

Proof of Theorem 1. Statement (i) follows from direct calculations. More precisely, we compute the Bézier representation of $\phi_2(B^{-1}\cdot)$ by applying the C^4-smoothing formula in [2, Theorem 5.1], and then write down the linear equation of $\phi_2(B^{-1}\cdot)$, formulated as (finite) linear combinations of $\phi_n(\cdot-\mathbf{k})$, $\mathbf{k}\in\mathbb{Z}^2$, at the Bézier points for $0\le n\le 2$. The (unique) solution, arranged in 1×3 vector formulation, gives the third rows of the mask $\{L_\mathbf{k}\}$ in (i). The first and second rows of the mask $\{L_\mathbf{k}\}$ follow from the following refinement relation derived from the definitions of ϕ_1 and ϕ_2, namely:

$$
\begin{aligned}
&\phi_0(B^{-1}\cdot) = \phi_1,\\
&\phi_1(B^{-1}\cdot) = \phi_0(\frac{\cdot}{2}) = \phi_2 + \frac{1}{8}\{\phi_0(\cdot-(1,0)) + \phi_0(\cdot+(1,0)) + \phi_0(\cdot-(0,1)) + \phi_0(\cdot+(0,1))\}\\
&\qquad + \frac{1}{16}\{\phi_0(\cdot-(1,1)) + \phi_0(\cdot+(1,1)) + \phi_0(\cdot-(1,-1)) + \phi_0(\cdot+(1,-1))\}.
\end{aligned}
$$

We obtain (ii) by solving the equations from (1.7) for $\mathbf{y}_\alpha$ by using Maple.

The reproduction of $\mathbf{x}^\alpha$ for $|\alpha|\le 2$ in (iii) follows from the sum rule order 3 of the $L(\mathbf{z})$, while that of $\mathbf{x}^\alpha$ for $|\alpha| = 3, 4$ in (iii) follows from direct calculations using Maple. $\diamond$

Clearly, $\Phi^d = [\phi_0, \phi_1, \phi_2]^T$ is also refinable with dilation matrix $2I_2$. In this case the refinement symbol, denoted by $R(\mathbf{z})$, is given by $R(\mathbf{z}) = L(e^{-iB\omega})L(\mathbf{z})$, where $L(\mathbf{z})$ is the refinement symbol of Φ^d with the quincunx dilation matrix B and

$$e^{-iB\omega} := (e^{-i(\omega_1+\omega_2)}, e^{-i(\omega_1-\omega_2)}).$$

One can easily obtain the following nonzero matrix coefficients $R_{\mathbf{k}}$ of $R(\mathbf{z})$.

$$\begin{aligned}
&R_{0,0} = \frac{1}{16}\begin{bmatrix} 0 & 0 & 16 \\ 0 & 4 & 8 \\ 1 & 2 & 8 \end{bmatrix}, \quad R_{1,0} = R_{0,1} = R_{-1,0} = R_{0,-1} = J, \\
&R_{1,1} = R_{-1,1} = R_{1,-1} = R_{-1,-1} = K, \quad R_{2,0} = R_{0,2} = R_{0,-2} = R_{-2,0} = L, \\
&R_{2,1} = R_{1,2} = R_{-1,-2} = R_{-2,-1} = R_{2,-1} = R_{-2,1} = R_{-1,2} = R_{1,-2} = M, \\
&R_{2,2} = R_{-2,-2} = R_{-2,2} = R_{2,-2} = N,
\end{aligned} \tag{2.7}$$

where

$$J := \frac{1}{16}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}, \quad K := \frac{1}{32}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 2 & 8 \end{bmatrix},$$

$$L := \frac{1}{32}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad M := \frac{1}{64}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}, \quad N := \frac{1}{64}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is easy to verify with Maple that $R(\mathbf{z})$ satisfies sum rule of order 4 with the vectors $\mathbf{y}_\alpha$, $|\alpha| \leq 3$, for (1.7) given by (2.5) and (2.6). In conclusion, we have the following theorem.

Theorem 2. *For $\phi_0 \in S_4^2(\triangle^2)$, with Bézier coefficients shown in Fig.4, and ϕ_1 and ϕ_2 defined by (2.1) and (2.2) respectively, the following statements hold.*

(i) $\Phi^d = [\phi_0, \phi_1, \phi_2]^T$ is refinable with dilation matrix $2I_2$, with nonzero matrix coefficients $R_{\mathbf{k}}$ of the corresponding subdivision mask $\{R_{\mathbf{k}}\}$ given by (2.7);

(ii) The two-scale symbol $R(\mathbf{z})$ of Φ^d with dilation matrix $2I_2$ satisfies the sum rule of order 4, with the vectors $\mathbf{y}_\alpha$, $|\alpha| \leq 3$, for (1.7) given by (2.5) and (2.6);

(iii) Φ^d locally reproduces all bivariate quartic polynomials as shown in Theorem 1 (iii).

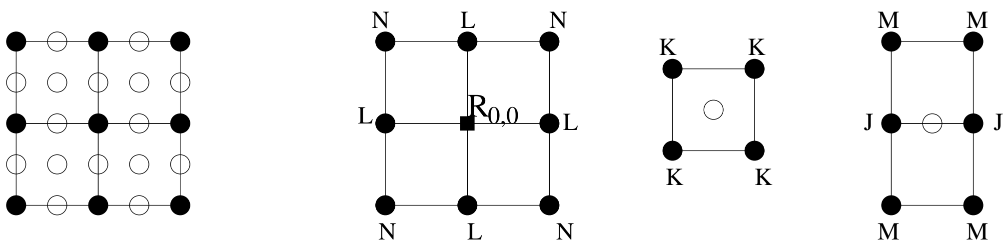

Figure 9: *Matrix-valued templates for C^2 approximating* 1*-to-*4 *split subdivision scheme*

Again, the subdivision mask $\{R_{\mathbf{k}}\}$ immediately yields the matrix-valued templates for the local averaging rule of a C^2 approximating 1-to-4 split subdivision scheme, as shown in Fig.9. By applying the Matlab routines in [13], it is easy to verify that the transition operator associated with the subdivision mask $\{R_{\mathbf{k}}\}$ satisfies Condition E. Therefore, this scheme is convergent in L_2 norm.

3. Refinable Hermite Interpolatory C^2-quintic splines

In this section we will construct $\varphi_0, \cdots, \varphi_{n-1} \in S_5^2(\triangle^2)$ with $n = 9$ to be discussed later such that $\Phi^e := [\varphi_0, \cdots, \varphi_{n-1}]^T$ is refinable and satisfies the second order Hermite interpolatory properties:

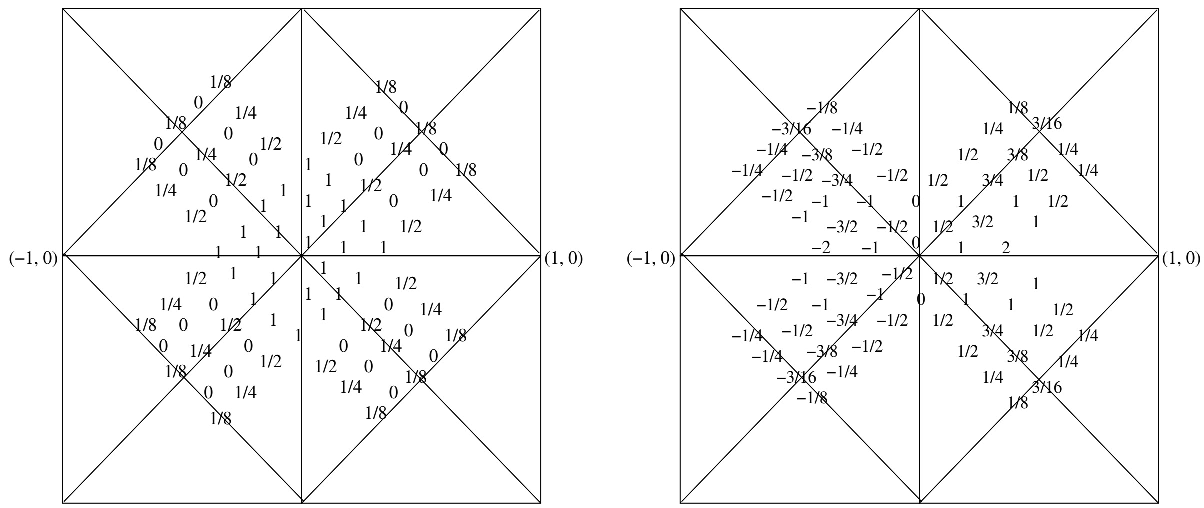

Figure 10: *Supports and Bézier coefficients of φ_0 (left) and $5\varphi_1$ (right), with $\varphi_2(x, y) := \varphi_1(y, x)$*

$$\left[\Phi^e \quad \frac{\partial}{\partial x}\Phi^e \quad \frac{\partial}{\partial y}\Phi^e \quad \frac{\partial^2}{\partial x^2}\Phi^e \quad \frac{\partial^2}{\partial x \partial y}\Phi^e \quad \frac{\partial^2}{\partial y^2}\Phi^e\right](\mathbf{k}) = \delta_{\mathbf{k},\mathbf{0}} \begin{bmatrix} I_6 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{k} \in \mathbb{Z}^2. \tag{3.1}$$

Let k, l, m, n be arbitrary positive integers and set $\Omega := [k, k+m+1] \times [l, l+n+1]$. Let $S_5^2(\Omega \cap \triangle^2)$ denote the restriction of $S_5^2(\triangle^2)$ on Ω. Then by applying the dimension formula in [2, Theorem 4.3], we have

$$\dim S_5^2(\Omega \cap \triangle^2) = 9mn + 18(m+n) + 33. \tag{3.2}$$

Since the coefficient of mn is 9, we investigate the existence of 9 compactly supported basis functions whose integer translates restricted to Ω span all of $S_5^2(\Omega \cap \triangle^3)$. In search of these functions, we first find $\varphi_0, \cdots, \varphi_5$ that have the second order Hermite interpolatory properties, namely, $\Phi^f := [\varphi_0, \cdots, \varphi_5]^T$ satisfies

$$\left[\Phi^f \quad \frac{\partial}{\partial x}\Phi^f \quad \frac{\partial}{\partial y}\Phi^f \quad \frac{\partial^2}{\partial x^2}\Phi^f \quad \frac{\partial^2}{\partial x \partial y}\Phi^f \quad \frac{\partial^2}{\partial y^2}\Phi^f\right](\mathbf{k}) = \delta_{\mathbf{k},\mathbf{0}} I_6, \ \mathbf{k} \in \mathbb{Z}^2. \tag{3.3}$$

The Bézier coefficients and supports of these six components $\varphi_0, \cdots, \varphi_5$ of Φ^f are shown in Fig.10 and Fig.11, where those that are obviously equal to zero are not displayed. Unfortunately, Φ^f is not refinable with either the quincunx B or $2I_2$ as dilation matrix. We therefore need three additional components, $\varphi_6, \varphi_7, \varphi_8$. Their Bézier coefficients and supports are shown in Fig.12. Consequently, we have a total of 9 spline components that constitute the refinable function vector

$$\Phi^e := [\varphi_0, \cdots, \varphi_8]^T.$$

It is clear that Φ^e has the second Hermite interpolatory property (3.1). We will see that Φ^e is refinable with the quincunx matrix B (hence, also refinable with $2I_2$) as dilation matrix as shown in Theorem 3, and $\{\varphi_\ell(\cdot - \mathbf{k})|_\Omega : \mathbf{k} \in \mathbb{Z}^2, 0 \le \ell \le 8\}$ indeed spans $S_5^2(\Omega \cap \triangle^2)$, which will be seen in Proposition 2 below. However, we will also see that $\varphi_0, \cdots, \varphi_8$, are linearly dependent, as follows.

Proposition 1. *$\varphi_0, \cdots, \varphi_8$ are linearly dependent, with*

$$\sum_{\mathbf{k}} (\varphi_6 - \varphi_7)(\mathbf{x} - \mathbf{k}) = 0, \quad \mathbf{x} \in \mathbb{R}^2. \tag{3.4}$$

Furthermore, this dependency identity describes the only linearly dependency relationship among $\varphi_0, \cdots, \varphi_8$.

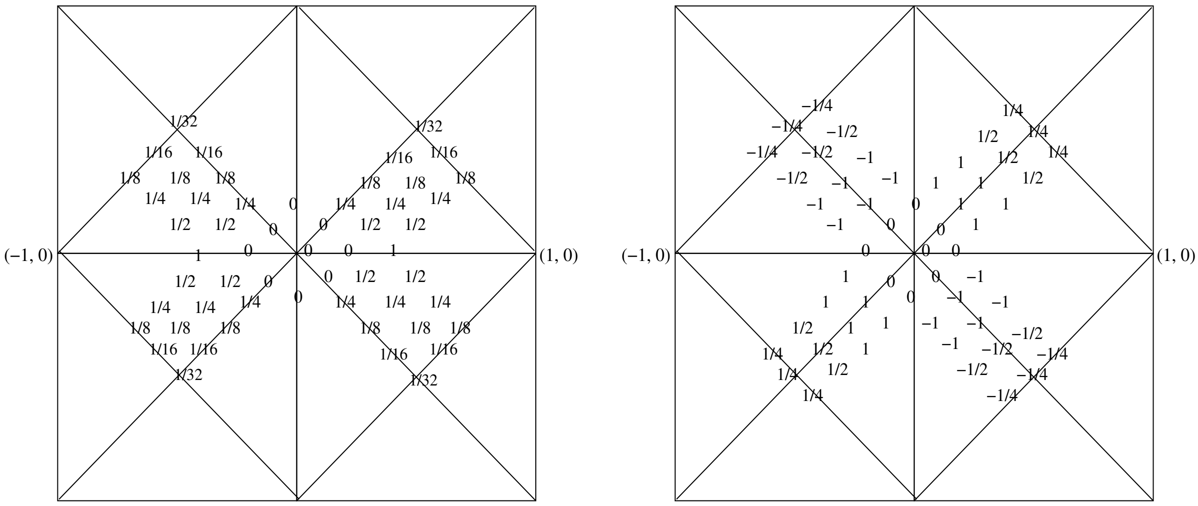

Figure 11: *Supports and Bézier coefficients of* $20\varphi_3$ *(left) and* $40\varphi_4$ *(right), also* $\varphi_5(x,y) := \varphi_3(y,x)$

Proof. Let $F := \sum_{\mathbf{k}} (\varphi_6 - \varphi_7)(\cdot - \mathbf{k})$. Since $F(\cdot - \mathbf{j}) = F$ for any $\mathbf{j} \in \mathbb{Z}^2$, it is enough to show that $F(\mathbf{x}) = 0$ for $\mathbf{x} \in [0,1] \times [0,1]$, which is easily verified by evaluating F at each Bézier point in $[0,1] \times [0,1]$.

To show that (3.4) governs the only linearly dependency relationship, we set

$$\sum_{\ell=0}^{8} \sum_{\mathbf{k}} c_{\mathbf{k}}^\ell \varphi_\ell(\mathbf{x} - \mathbf{k}) = 0.$$

For an arbitrary $\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2$, consider $\mathbf{x}$ in $[j_1, j_1 + 1] \times [j_2, j_2 + 1]$. The Hermite interpolatory properties (3.1) of Φ^e leads to that $c_{\mathbf{j}}^\ell = 0, \ell = 0, \cdots, 5$. By evaluating the above equation at the Bézier points, we have $c_{\mathbf{j}}^8 = 0$, and

$$\begin{aligned} c_{\mathbf{j}}^6 &= c_{\mathbf{j}+(1,0)}^6 = c_{\mathbf{j}+((0,1)}^6 = c_{\mathbf{j}+(-1,0)}^6 = c_{\mathbf{j}+((0,-1)}^6 \\ &= -c_{\mathbf{j}+((0,0)}^7 = -c_{\mathbf{j}+((1,0)}^7 = -c_{\mathbf{j}+((0,1)}^7 = -c_{\mathbf{j}+((1,1)}^7. \end{aligned}$$

Therefore the linearly dependency relationship among $\varphi_0, \cdots, \varphi_8$ is governed by (3.4). $\diamond$

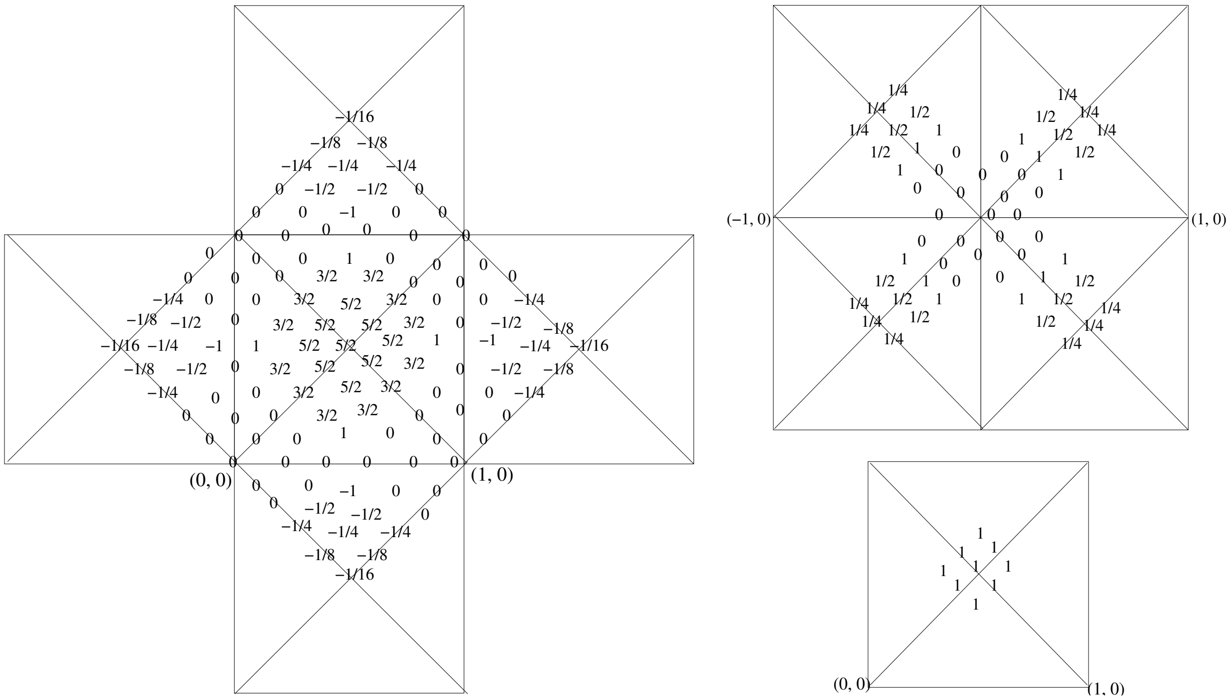

Figure 12: *Supports and Bézier coefficients of* $20\varphi_6$ *(left),* $20\varphi_7$ *(right top), and* $20\varphi_8$ *(right bottom)*

Proposition 2. Let $G = [k, k+m+1] \times [l, l+n+1]$. Then $\mathrm{span}\{\varphi_\ell(\cdot - \mathbf{k}), 0 \le \ell \le 8, \mathbf{k} \in G \cap \mathbb{Z}^2\} = S_5^2(G \cap \triangle^2)$.

Proof. One can easily verify that for each $0 \le \ell \le 5$ and $\ell = 7$, the number of $\phi_\ell(\cdot - \mathbf{k})$ whose supports overlap with Ω is equal to $(m+2)(n+2)$, while the cardinalities of $\phi_6(\cdot - \mathbf{k})$ and $\phi_8(\cdot - \mathbf{k})$ whose supports overlap with Ω are equal to $(m+3)(n+3)-4$, and $(m+1)(n+1)$, respectively. Hence, the total number of $\phi_\ell(\cdot - \mathbf{k}), 0 \le \ell \le 8$, whose supports overlap with Ω is given by

$$\begin{aligned} &7(m+2)(n+2) + \{(m+3)(n+3) - 4\} + (m+1)(n+1) \\ &= 9mn + 18(m+n) + 34, \end{aligned}$$

which is exactly 1 plus $\dim S_5^2(\Omega \cap \triangle^2)$ in (3.2). This fact, along with the linear dependency property in Proposition 1, assures the validity of Proposition 2. $\diamond$

To show that Φ^e is refinable with dilation matrix B, we follow the proof of Theorem 1 (i) to compute the Bézier representations of $\varphi_\ell(B^{-1}\cdot)$, $\ell = 0, \cdots, 8$, by applying the C^5-smoothing formula in [2, Theorem 5.1], and to write down the linear equations of $\varphi_\ell(B^{-1}\cdot)$, formulated as (finite) linear combinations of $\varphi_n(\cdot - \mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^2$, at the Bézier points for $0 \le n \le 8$. The (unique) solution for these equations, arranged in 9×9 matrix formulation, gives the subdivision mask $\{P_{\mathbf{k}}\}$ of Φ. The nonzero matrix coefficients of $\{P_{\mathbf{k}}\}$ are given by

$$P_{0,0} = \frac{1}{16}\begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 20 & 260 & 70 \\ 0 & 8 & 8 & 0 & 0 & 0 & 8 & 0 & 16 \\ 0 & 8 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 4 & 1 & 1 & 2 \\ 0 & 0 & 0 & 8 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -4 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.5}$$

$$P_{0,1} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 320 & 0 & 0 & 0 & 0 & 0 \\ 24/5 & 8 & -24 & -32 & -32 & 96 & 0 & 72 & 0 \\ -24/5 & 8 & 24 & 32 & -32 & -96 & 0 & -72 & 0 \\ 1/5 & 1 & -1 & 4 & -4 & 4 & 0 & 3 & 0 \\ -4/5 & 0 & 4 & 16 & 0 & -16 & 0 & -12 & 0 \\ 1/5 & -1 & -1 & 4 & 4 & 4 & 0 & 3 & 0 \\ -2/5 & -2 & 2 & -8 & 8 & -8 & 0 & -6 & 0 \\ 8/5 & 0 & -8 & -32 & 0 & 32 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.6}$$

$$P_{0,-1} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 320 & 0 & 0 & 160 & 0 & 560 \\ -24/5 & 8 & -24 & 32 & 32 & -96 & 0 & -72 & 0 \\ 24/5 & 8 & 24 & -32 & 32 & 96 & 64 & 72 & 128 \\ 1/5 & -1 & 1 & 4 & -4 & 4 & 0 & 3 & 0 \\ -4/5 & 0 & -4 & 16 & 0 & -16 & 0 & -12 & 0 \\ 1/5 & 1 & 1 & 4 & 4 & 4 & 8 & 3 & 16 \\ -2/5 & -2 & -2 & -8 & -8 & -8 & -16 & -6 & -16 \\ 8/5 & 0 & 8 & -32 & 0 & 32 & 0 & 24 & -16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.7}$$

$$P_{1,0} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 320 & 0 & 0 & 0 \\ 24/5 & -24 & 8 & 96 & -32 & -32 & 0 & 72 & 0 \\ 24/5 & -24 & -8 & 96 & 32 & -32 & 0 & 72 & 0 \\ 1/5 & -1 & 1 & 4 & -4 & 4 & 0 & 3 & 0 \\ 4/5 & -4 & 0 & 16 & 0 & -16 & 0 & 12 & 0 \\ 1/5 & -1 & -1 & 4 & 4 & 4 & 0 & 3 & 0 \\ 8 & 0 & 0 & 32 & 0 & 32 & -16 & 152 & -16 \\ 8/5 & -8 & 0 & 32 & 0 & -32 & 0 & 24 & 0 \\ 32/5 & 0 & 0 & -128 & 0 & -128 & 0 & 32 & 0 \end{bmatrix}, \tag{3.8}$$

$$P_{-1,0} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 320 & 160 & 0 & 560 \\ -24/5 & -24 & 8 & -96 & 32 & 32 & 0 & -72 & 0 \\ -24/5 & -24 & -8 & -96 & -32 & 32 & -64 & -72 & -128 \\ 1/5 & 1 & -1 & 4 & -4 & 4 & 0 & 3 & 0 \\ 4/5 & 4 & 0 & 16 & 0 & -16 & 0 & 12 & 0 \\ 1/5 & 1 & 1 & 4 & 4 & 4 & 8 & 3 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8/5 & 8 & 0 & 32 & 0 & -32 & 0 & 24 & -16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.9}$$

$$P_{-1,-1} = \Big[\mathbf{0}_{9\times 6},\ \mathbf{u}_1,\ \mathbf{0}_{9\times 1},\ \mathbf{u}_2\Big], \tag{3.10}$$

$$P_{1,-1} = \begin{bmatrix} \mathbf{0}_{6\times 9} \\ \mathbf{u}_3 \\ \mathbf{0}_{2\times 9} \end{bmatrix},\ P_{2,-1} = \begin{bmatrix} \mathbf{0}_{6\times 9} \\ \mathbf{u}_4 \\ \mathbf{0}_{2\times 9} \end{bmatrix},\ P_{2,1} = \begin{bmatrix} \mathbf{0}_{6\times 9} \\ \mathbf{u}_5 \\ \mathbf{0}_{2\times 9} \end{bmatrix} \tag{3.11}$$

where

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{16}[20,\ -8,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0]^T, \\ \mathbf{u}_2 &= \frac{1}{8}[35,\ -8,\ 0,\ 1,\ 0,\ 0,\ 0,\ -1,\ 0]^T \\ \mathbf{u}_3 &= \frac{1}{8}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ -1], \\ \mathbf{u}_4 &= \frac{1}{64}\left[-1/5,\ 1,\ -1,\ -4,\ 4,\ -4,\ 0,\ -3,\ 0\right], \\ \mathbf{u}_5 &= \frac{1}{64}\left[-1/5,\ 1,\ 1,\ -4,\ -4,\ -4,\ 0,\ -3,\ 0\right]. \end{aligned}$$

Let us summarize the above discussions in the following theorem.

Theorem 3. *For $\varphi_\ell \in S_5^2(\triangle^2), 0 \le \ell \le 8$, with Bézier coefficients shown in Figs.10-12, the following statements hold.*

(i) *$\Phi^e = [\varphi_0, \cdots, \varphi_8]^T$ satisfies the second-order Hermite interpolatory condition (3.1);*

(ii) *$\Phi^e = [\varphi_0, \cdots, \varphi_8]^T$ is refinable with the quincunx dilation matrix B, with nonzero matrix coefficients $P_{\mathbf{k}}$ of the corresponding subdivision mask $\{P_{\mathbf{k}}\}$ given above;*

(iii) *The two-scale symbol $P(\mathbf{z})$ of Φ^e with dilation matrix B satisfies the sum rule of order 4, with the vectors for (1.7) given by*

$$\begin{aligned} &\mathbf{y}_{0,0} = [1,0,0,0,0,0,2,13,5], \quad \mathbf{y}_{1,0} = [0,1,0,0,0,0,1,0,\tfrac{5}{2}], \\ &\mathbf{y}_{0,1} = [0,0,1,0,0,0,1,0,\tfrac{5}{2}], \quad \mathbf{y}_{2,0} = [0,0,0,2,0,0,\tfrac{1}{3},\tfrac{1}{6},1], \\ &\mathbf{y}_{1,1} = [0,0,0,0,1,0,\tfrac{1}{2},0,\tfrac{5}{4}], \quad \mathbf{y}_{0,2} = [0,0,0,0,0,2,\tfrac{1}{3},\tfrac{1}{6},1], \\ &\mathbf{y}_{3,0} = \mathbf{y}_{0,3} = \frac{1}{4}[0,0,0,0,0,0,0,0,1], \\ &\mathbf{y}_{2,1} = \mathbf{y}_{1,2} = \frac{1}{6}[0,0,0,0,0,0,1,0,3]; \end{aligned} \tag{3.12}$$

(iv) Φ^e locally reproduces all bivariate quintic polynomials, with

$$\mathbf{x}^\alpha = \sum_{\mathbf{k}} \left\{ \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathbf{k}^{\alpha-\beta} \mathbf{y}_\beta \right\} \Phi^e(\mathbf{x}-\mathbf{k}), |\alpha| \le 4, \alpha = (4,1), \alpha = (1,4),$$

$$x^5 = \sum_{k_1,k_2} \left\{ \sum_{0 \le j \le 4} \binom{5}{j} k_1^{5-j} \widetilde{\mathbf{y}_j} \right\} \Phi^e(x-k_1, y-k_2), \quad (x,y) \in \mathbb{R}^2,$$

$$y^5 = \sum_{k_1,k_2} \left\{ \sum_{0 \le j \le 4} \binom{5}{j} k_2^{5-j} \widetilde{\mathbf{y}}_j \right\} \Phi^e(x-k_1, y-k_2), \quad (x,y) \in \mathbb{R}^2,$$

$$x^3y^2 = \sum_{\mathbf{k}} \left\{ \sum_{0 \le \beta < (3,2)} \binom{(3,2)}{\beta} \mathbf{k}^{(3,2)-\beta} \widetilde{\mathbf{y}}_\beta \right\} \Phi^e(\mathbf{x}-\mathbf{k}), \quad \mathbf{x} = (x,y) \in \mathbb{R}^2,$$

$$x^2y^3 = \sum_{\mathbf{k}} \left\{ \sum_{0 \le \beta < (2,3)} \binom{(2,3)}{\beta} \mathbf{k}^{(2,3)-\beta} \widetilde{\mathbf{y}}_\beta \right\} \Phi^e(\mathbf{x}-\mathbf{k}), \quad \mathbf{x} = (x,y) \in \mathbb{R}^2,$$

where $\mathbf{y}_\beta$, $|\beta| \le 3$, are given by (3.12), and

$$\mathbf{y}_{4,0} = \mathbf{y}_{0,4} = \mathbf{0}, \quad \mathbf{y}_{3,1} = \mathbf{y}_{1,3} = \frac{1}{8}[0,0,0,0,0,0,0,0,1], \tag{3.13}$$

$$\mathbf{y}_{2,2} = \frac{1}{6}[0,0,0,0,0,0,0,1,0]; \tag{3.14}$$

$$\widetilde{\mathbf{y}}_j = \mathbf{y}_{j,0}, \quad 0 \le j \le 3,$$

$$\widetilde{\mathbf{y}}_4 = \frac{1}{15}[0,0,0,0,0,0,-1,1,0],$$

$$\widetilde{\mathbf{y}}_\beta = \mathbf{y}_\beta, \quad |\beta| \le 3,$$

$$\widetilde{\mathbf{y}}_{3,1} = \widetilde{\mathbf{y}}_{1,3} = \mathbf{0}, \quad \widetilde{\mathbf{y}}_{2,2} = \frac{1}{18}[0,0,0,0,0,0,-1,1,3].$$

We obtain (iii) in the above theorem by solving the equations from (1.7) for $\mathbf{y}_\alpha$ by using Maple. The polynomial reproduction formulas of $\mathbf{x}^\alpha$ for $|\alpha| \le 3$ in (iv) follow from the sum rule order 4 of the $P(\mathbf{z})$, while those of $\mathbf{x}^\alpha$ for $|\alpha| = 4, 5$ in (iv) follow again from direct calculations by using Maple.

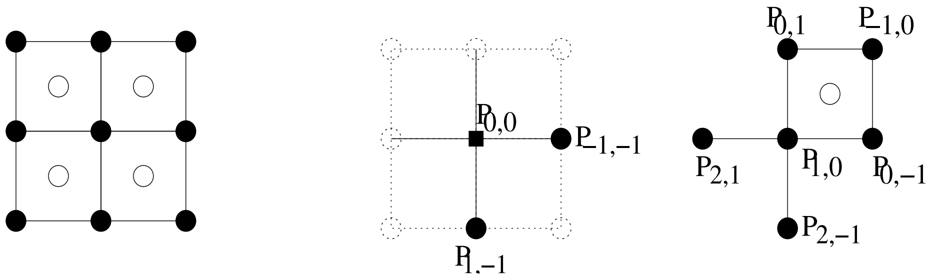

Figure 13: *Matrix-valued templates for C^2 Hermite interpolatory $\sqrt{2}$-subdivision scheme*

The subdivision mask $\{P_\mathbf{k}\}$ immediately gives the matrix-valued templates for the local averaging rule of a C^2 $\sqrt{2}$-subdivision scheme, as shown in Fig.13. Again, by applying the

Matlab routines in [13], it is easy to verify that the transition operator associated with this subdivision mask satisfies Condition E. Hence, this scheme is convergent in L_2 norm.

From template in the middle of Fig.13, it might seem to the reader that this scheme is not interpolatory. The fact that it is interpolatory is a result of the special structures of $P_{0,0}, P_{-1,-1}$, and $P_{1,-1}$. More precisely, let $\mathbf{v}_0^m = [v_0^m, s_{0,1}^m, \cdots, s_{0,8}^m]$ be a specific control vector associated with the vertex v_0^m in $\mathbb{R}^3$ after m^{th} iterations (recall that we use the first column, a row 3-vector, of a control vector as the position of the vertex in $\mathbb{R}^3$). Let $\mathbf{v}_j^m = [v_j^m, s_{j,1}^m, \cdots, s_{j,8}^m]$, $j = 1, \cdots 4$, be the control vectors associated with the vertices v_j^m adjacent to v_0^m. Then by the matrix-valued template of local averaging rule in the middle of Fig.13, the control vector $\mathbf{v}_0^{m+1}$ after $(m+1)^{\text{th}}$ iterations is given by

$$\begin{aligned}\mathbf{v}_0^{m+1} = [v_0^m,\ &\frac{1}{2}(s_{0,1}^m + s_{0,2}^m),\ \frac{1}{2}(s_{0,1}^m - s_{0,2}^m),\ \frac{1}{4}(s_{0,3}^m + 2s_{0,4}^m + s_{0,5}^m),\\ &\frac{1}{4}(s_{0,3}^m - s_{0,5}^m),\ \frac{1}{4}(s_{0,3}^m - 2s_{0,4}^m + s_{0,5}^m),\ *,\ *,\ *].\end{aligned} \tag{3.15}$$

We remark that that the first column v_0^{m+1} of $\mathbf{v}_0^{m+1}$ is still v_0^m, i.e., the positions of old vertices are never changed in the course of the iteration process. So indeed the templates in Fig.13 give an interpolatory scheme. In fact from (3.15), we also find that the second to the sixth columns of $\mathbf{v}_0^{m+1}$, which are related to the normals and curvatures of the surfaces, are independent of the control vectors associated with the adjacent vertices.

More generally, without the Hermite interpolatory constraint, a vector subdivision scheme is said to be interpolatory, if the vertices in the coarse net, the net before the subdivision is carried out, are also vertices in the finer net. In other word, $v_{A\mathbf{k}}^m = v_{\mathbf{k}}^{m-1}$ for all $m = 1, 2, \cdots$. This definition is perhaps the most natural extension to vector subdivision. In an independent work [8], it is shown that if a subdivision mask $\{P_{\mathbf{k}}\}$ with dilation matrix A satisfies

$$P_{0,0} = \begin{bmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}, \quad P_{A\mathbf{j}} = \begin{bmatrix} 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}, \quad \mathbf{j} \in \mathbb{Z}^2 \backslash \{(0,0)\}, \tag{3.16}$$

then this subdivision scheme is interpolatory. Observe that the particular matrices $P_{0,0}, P_{-1,-1}, P_{1,-1}$ given by (3.5), (3.10), (3.11) satisfy (3.16) with $A = B$. Therefore, the scheme with the template shown in Fig.13 is interpolatory.

Again, $\Phi^e = [\varphi_0, \cdots, \varphi_8]^T$ is also refinable with dilation matrix $2I_2$, and its refinement symbol, denoted by $G(\mathbf{z})$, is given by $G(\mathbf{z}) = P(e^{-iB\omega})P(\mathbf{z})$, where P is the refinement symbol of Φ^e with the quincunx dilation matrix B. By direct calculations by using Maple, one can easily obtain the nonzero matrix coefficients $G_{\mathbf{k}}$ of $G(\mathbf{z}\ )$ given by

$$G_{-1,-1} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 80 & -80 & 80 & 120 & 20 & 260 \\ -24/5 & -8 & -16 & 0 & -32 & -32 & -48 & -64 & -104 \\ -24/5 & -16 & -8 & -32 & -32 & 0 & -48 & -64 & -104 \\ 1/5 & 0 & 1 & 0 & 0 & 4 & 2 & 3 & 4 \\ 4/5 & 2 & 2 & 0 & 8 & 0 & 8 & 10 & 16 \\ 1/5 & 1 & 0 & 4 & 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8/5 & 4 & 4 & 0 & 16 & 0 & 8 & 20 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{-1,0} = \frac{1}{64}\begin{bmatrix} 32 & 60 & 0 & 0 & 0 & -360 & 60 & 360 & 130 \\ -10 & -14 & 0 & 24 & 0 & 24 & -24 & -126 & -52 \\ 0 & 0 & 10 & 0 & 24 & 0 & 24 & 0 & 52 \\ 1 & 1 & 0 & -4 & 0 & -6 & 1 & 12 & 2 \\ 0 & 0 & -2 & 0 & -4 & 0 & -4 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 24 & 4 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{-1,1} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 80 & 80 & 80 & 0 & 20 & 20 \\ -24/5 & -8 & 16 & 0 & 32 & -32 & 0 & -64 & 0 \\ 24/5 & 16 & -8 & 32 & -32 & 0 & 0 & 64 & 8 \\ 1/5 & 0 & -1 & 0 & 0 & 4 & 0 & 3 & 0 \\ -4/5 & -2 & 2 & 0 & 8 & 0 & 0 & -10 & 0 \\ 1/5 & 1 & 0 & 4 & 0 & 0 & 0 & 3 & 1 \\ -2/5 & -2 & 0 & -8 & 0 & 0 & 0 & -6 & -2 \\ 8/5 & 4 & -4 & 0 & -16 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{0,-1} = \frac{1}{64}\begin{bmatrix} 32 & 0 & 60 & -360 & 0 & 0 & 60 & 360 & 130 \\ 0 & 10 & 0 & 0 & 24 & 0 & 24 & 0 & 52 \\ -10 & 0 & -14 & 24 & 0 & 24 & -24 & -126 & -52 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 2 \\ 0 & -2 & 0 & 0 & -4 & 0 & -4 & 0 & -8 \\ 1 & 0 & 1 & -6 & 0 & -4 & 1 & 12 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 24 & 0 & 0 & 4 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{0,0} = \frac{1}{64}\begin{bmatrix} 64 & 0 & 0 & 0 & 0 & 0 & 60 & 780 & 130 \\ 0 & 32 & 0 & 0 & 0 & 0 & 24 & 0 & 52 \\ 0 & 0 & 32 & 0 & 0 & 0 & 24 & 0 & 52 \\ 0 & 0 & 0 & 16 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 16 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 16 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$G_{0,1} = \frac{1}{128}\begin{bmatrix} 64 & 0 & -120 & -720 & 0 & 0 & 0 & 720 & 20 \\ 0 & 20 & 0 & 0 & -48 & 0 & 0 & 0 & 0 \\ 20 & 0 & -28 & -48 & 0 & -48 & 0 & 252 & 8 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & -8 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & -12 & 0 & -8 & 0 & 24 & 1 \\ 0 & 12 & 0 & 48 & 0 & 0 & 16 & 4 & 24 \\ 0 & 0 & 0 & 48 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix},$$

$$G_{1,-1} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 80 & 80 & 80 & 0 & 20 & 20 \\ 24/5 & -8 & 16 & 0 & -32 & 32 & 0 & 64 & 8 \\ -24/5 & 16 & -8 & -32 & 32 & 0 & 0 & -64 & 0 \\ 1/5 & 0 & 1 & 0 & 0 & 4 & 0 & 3 & 1 \\ -4/5 & 2 & -2 & 0 & 8 & 0 & 0 & -10 & 0 \\ 1/5 & -1 & 0 & 4 & 0 & 0 & 0 & 3 & 0 \\ -2/5 & 0 & -2 & 0 & 0 & -8 & 0 & -6 & -2 \\ 8/5 & -4 & 4 & 0 & -16 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{1,0} = \frac{1}{128}\begin{bmatrix} 64 & -120 & 0 & 0 & 0 & -720 & 0 & 720 & 20 \\ 20 & -28 & 0 & -48 & 0 & -48 & 0 & 252 & 8 \\ 0 & 0 & 20 & 0 & -48 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & -8 & 0 & -12 & 0 & 24 & 1 \\ 0 & 0 & 4 & 0 & -8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 48 & 16 & 4 & 24 \\ 0 & 0 & 0 & 0 & 0 & 48 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix},$$

$$G_{1,1} = \frac{1}{128}\begin{bmatrix} 0 & 0 & 0 & 80 & -80 & 80 & 0 & 20 & 0 \\ 24/5 & -8 & -16 & 0 & 32 & 32 & 0 & 64 & 0 \\ 24/5 & -16 & -8 & 32 & 32 & 0 & 0 & 64 & 0 \\ 1/5 & 0 & -1 & 0 & 0 & 4 & 0 & 3 & 0 \\ 4/5 & -2 & -2 & 0 & 8 & 0 & 0 & 10 & 0 \\ 1/5 & -1 & 0 & 4 & 0 & 0 & 0 & 3 & 0 \\ 8 & 0 & 0 & 16 & 0 & 16 & 16 & 96 & 24 \\ 8/5 & -4 & -4 & 0 & 16 & 0 & 0 & 20 & 0 \\ 32/5 & 0 & 0 & -64 & 0 & -64 & 0 & 80 & 8 \end{bmatrix},$$

$$G_{-2,-1} = G_{-2,0} = \Big[\mathbf{0}_{9\times 8},\ \mathbf{w}_1\Big],\ G_{-1,-2} = G_{0,-2} = \Big[\mathbf{0}_{9\times 8},\ \mathbf{w}_2\Big],$$

$$G_{0,2} = G_{2,0} = \Big[\mathbf{0}_{9\times 8},\ \mathbf{w}_3\Big],$$

$$G_{1,2} = \begin{bmatrix} \mathbf{0}_{6\times 9} \\ \mathbf{w}_4 \\ \mathbf{0}_{2\times 9} \end{bmatrix},\ G_{1,3} = \begin{bmatrix} \mathbf{0}_{6\times 9} \\ \mathbf{w}_5 \\ \mathbf{0}_{2\times 9} \end{bmatrix},\ G_{2,1} = \begin{bmatrix} \mathbf{0}_{6\times 9} \\ \mathbf{w}_6 \\ \mathbf{0}_{2\times 9} \end{bmatrix},\ G_{3,1} = \begin{bmatrix} \mathbf{0}_{6\times 9} \\ \mathbf{w}_7 \\ \mathbf{0}_{2\times 9} \end{bmatrix},$$

where

$$\begin{aligned}
\mathbf{w}_1 &= \frac{1}{128}\,[20,\ -8,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0]^T\,,\\
\mathbf{w}_2 &= \frac{1}{128}\,[20,\ 0,\ -8,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0]^T\,,\\
\mathbf{w}_3 &= \frac{1}{64}\,[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 0]^T\,,\\
\mathbf{w}_4 &= \frac{1}{64}\,[0,\ 0,\ -6,\ 0,\ 0,\ 24,\ 0,\ 2,\ -1]\,,\\
\mathbf{w}_5 &= \frac{1}{64}\,[-1/5,\ 0,\ 1,\ 0,\ 0,\ -4,\ 0,\ -3,\ 0]\,,\\
\mathbf{w}_6 &= \frac{1}{64}\,[0,\ -6,\ 0,\ 24,\ 0,\ 0,\ 0,\ 2,\ -1]\,,\\
\mathbf{w}_7 &= \frac{1}{64}\,[-1/5,\ 1,\ 0,\ -4,\ 0,\ 0,\ 0,\ -3,\ 0]\,.
\end{aligned}$$

It is easy to verify by using Maple that $G(\mathbf{z})$ satisfies the sum rule of order 5 with the vectors $\mathbf{y}_\alpha$, $|\alpha| \le 4$, for (1.7) given by (3.12), (3.13), and (3.14). Therefore, we have the following theorem.

Theorem 4. *For $\phi_\ell \in S_5^2(\triangle^2), 0 \le \ell \le 8$, with Bézier coefficients shown in Figs.10-12, the following statements hold.*

(i) $\Phi^e = [\varphi_0, \cdots, \varphi_8]^T$ is refinable with dilation matrix $2I_2$, with nonzero matrix coefficients $G_\mathbf{k}$ of the corresponding subdivision mask $\{G_\mathbf{k}\}$ given above;

(ii) The two-scale symbol $G(\mathbf{z})$ of Φ^e with dilation $2I_2$ satisfies the sum rule of order 5, with the vectors $\mathbf{y}_\alpha$, $|\alpha| \le 4$ for (1.7) given by (3.12), (3.13) and (3.14);

(iii) Φ^e locally reproduces all bivariate quintic polynomials as shown in Theorem 3 (iv).

Again, the subdivision mask $\{G_\mathbf{k}\}$ immediately yields the matrix-valued templates for the local averaging rule of a C^2 1-to-4 split subdivision scheme, as shown in Fig.14. It is easy to verify by applying the Matlab routines in [13] that the transition operator associated with this subdivision mask satisfies Condition E. Therefore, this scheme is convergent in L_2 norm. As discussed above, this is also an interpolatory scheme due to the special structures of $G_{0,0}, G_{2,0}, G_{0,2}, G_{-2,0}$ and $G_{0,-2}$, namely, $G_{2\mathbf{k}}$ satisfy (3.16) with $A = 2I_2$.

Remark 3. *For fixed-point computer implementation, one should use the subdivision masks corresponding to the refinable spline vector*

$$[\varphi_0, 5\varphi_1, 5\varphi_2, 20\varphi_3, 40\varphi_4, 20\varphi_5, 20\varphi_6, 20\varphi_7, 20\varphi_8],$$

which are $\{UP_\mathbf{k}U^{-1}\}$ (with quincunx dilation matrix B) and $\{UG_\mathbf{k}U^{-1}\}$ (with dilation matrix $2I_2$), where

$$U := diag\ (1, 5, 5, 20, 40, 20, 20, 20, 20).$$

In this regard, all the entries of $2^7UP_\mathbf{k}U^{-1}$ and $2^7UG_\mathbf{k}U^{-1}$ are integers.

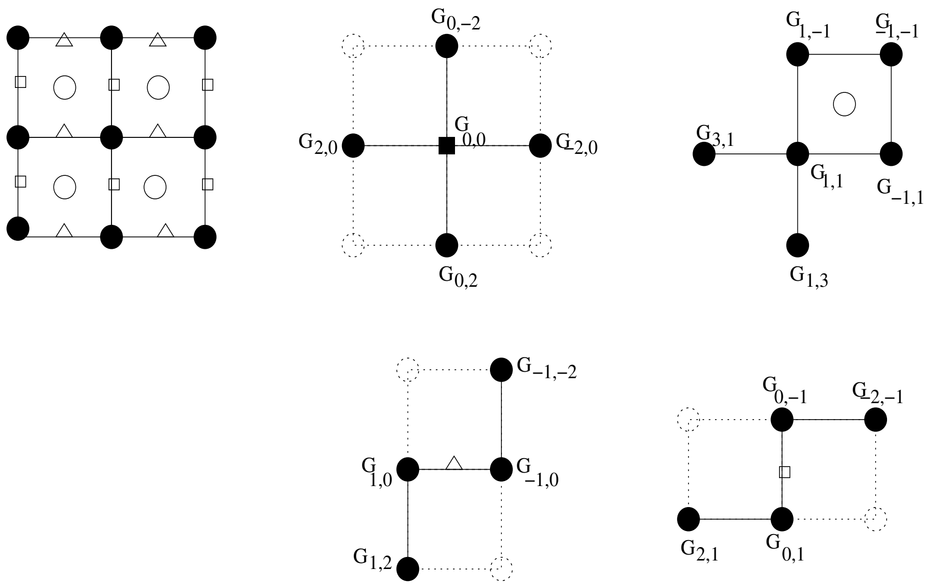

Figure 14: *Matrix-valued templates for C^2 Hermite interpolatory 1-to-4-subdivision scheme*

4. Two/four-point matrix-valued templates

The matrix-valued templates in Fig.13 or Fig.14 for C^2 surface display are not 4-point templates. In the following, we give a second-order Hermite interpolatory $\sqrt{2}$-subdivision scheme with the 4-point template as shown in Fig.15, and give a second-order Hermite interpolatory 1-to-4 split subdivision scheme with the 4-point template for the new vertices in the faces and 2-point templates for the new vertices in the edges as shown in Fig.16. The square matrices $P^{\#}_{\mathbf{k}}$ and $G^{\#}_{\mathbf{k}}$ in Fig.15 and Fig.16, respectively, have dimension 6.

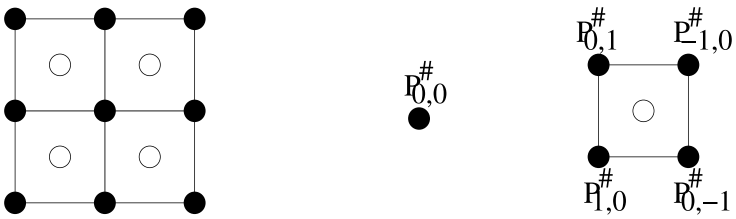

Figure 15: *Matrix-valued templates for C^2 Hermite interpolatory $\sqrt{2}$-subdivision scheme*

For the $\sqrt{2}$-subdivision, to compute the (possibly nonzero) matrices $P^{\#}_{0,0}, P^{\#}_{1,0}, P^{\#}_{0,-1}, P^{\#}_{-1,0}, P^{\#}_{1,0}$, we impose the sum rule (1.7) of order 4 to the two-scale symbol (denoted by $P^{\#}(\mathbf{z}) = \frac{1}{2}\sum_{\mathbf{k}} P^{\#}_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}$) along with

$$\begin{bmatrix} \mathbf{y}_{0,0}^T & \mathbf{y}_{1,0}^T & \mathbf{y}_{0,1}^T & \frac{1}{2}\mathbf{y}_{2,0}^T & \mathbf{y}_{1,1}^T & \frac{1}{2}\mathbf{y}_{0,2}^T \end{bmatrix} = I_6 \tag{4.1}$$

(which is a necessary condition for second-order Hermite interpolation). Hence, by following the symmetric properties of the C^2-quintic basis functions $\varphi_\ell, \ell = 0, \cdots, 5$, namely, those of the Bézier coefficients of φ_0, φ_1 (in Fig.10) and φ_4, φ_5 (in Fig.11), as well as the properties

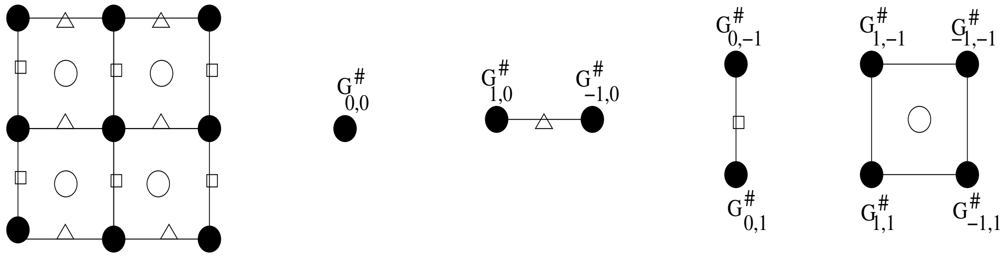

Figure 16: *Matrix-valued templates for C^2 Hermite interpolatory 1-to-4 split subdivision scheme*

of $\varphi_2(x,y) = \varphi_1(y,x)$ and $\varphi_5(x,y) = \varphi_3(y,x)$, the mask $\{P^\#_{\mathbf{k}}\}$ is reduced to a 13-parameter family (refer also to [9] for the discussion on the symmetry of the refinable function vectors). By enforcing the symbol to satisfy the sum rule of order 6, or equivalently (1.7) for $m = 6$, the mask is further reduced to the following 2-parameter family. We carried out this computation by using Maple.

$$P^\#_{0,0} = \frac{1}{4}\text{diag}\ (4, \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}),$$

$$P^\#_{1,0} = \begin{bmatrix} \frac{1}{4} & -\frac{15}{16} & 0 & -4(6t_1+t_2) & 0 & 4(6t_1+t_2) \\ \frac{5}{64} & -\frac{7}{32} & 0 & -(6t_1+2t_2+\frac{3}{8}) & -\frac{3}{16} & 6t_1+2t_2 \\ \frac{5}{64} & -\frac{7}{32} & 0 & -(6t_1+2t_2+\frac{3}{8}) & \frac{3}{16} & 6t_1+2t_2 \\ \frac{1}{128} & -\frac{1}{64} & 0 & -(t_1+\frac{1}{16}) & -\frac{1}{32} & t_1 \\ \frac{1}{64} & -\frac{1}{32} & 0 & -(t_2+\frac{1}{8}) & 0 & t_2 \\ \frac{1}{128} & -\frac{1}{64} & 0 & -(t_1+\frac{1}{16}) & \frac{1}{32} & t_1 \end{bmatrix},$$

$$P^\#_{-1,0} = M_1 P^\#_{1,0} M_1,\ P^\#_{0,1} = M_2 P^\#_{1,0} M_3,\ P^\#_{0,-1} = M_1 M_2 P^\#_{1,0} M_1 M_3,$$

where

$$M_1 = \text{diag}\ (1,\ -1,\ -1,\ 1,\ 1,\ 1), \quad M_2 = \text{diag}\ (1,\ 1,\ -1,\ 1,\ -1,\ 1), \tag{4.2}$$

$$M_3 = \text{diag}\ (1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}). \tag{4.3}$$

By applying the Matlab routines in [13] for the Sobolev exponent formula for $\Phi^\#$ derived in [11], free parameters t_1, t_2 can be adjusted to achieve certain desirable smoothness. For example, by selecting

$$t_1 = -0.04991030661181,\ t_2 = -0.10095791152136,$$

the corresponding refinable function vector $\Phi^\#$ is assured to be in the Sobolev space $W^{3.61979}(\mathbb{R}^2)$. For fixed-point computer implementation, one may even choose

$$t_1 = -1/8,\ t_2 = -1/16,$$

for which $\Phi^\#$ is in $W^{3.60875}(\mathbb{R}^2)$. In either case, the corresponding refinable function vector $\Phi^\#$ is in C^2 and it has the second order Hermite interpolatory property (3.3).

For the 1-to-4 split, to compute the (possibly nonzero) matrices $G^{\#}_{0,0}, G^{\#}_{1,0}, G^{\#}_{-1,0}, G^{\#}_{0,1}, G^{\#}_{0,-1}, G^{\#}_{1,1}, G^{\#}_{-1,1}, G^{\#}_{-1,-1}, G^{\#}_{1,-1}$, we impose the sum rule (1.7) of order 4 to the two-scale symbol (denoted by $G^{\#}(\mathbf{z}) = \frac{1}{4}\sum_{\mathbf{k}} G^{\#}_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}$) along with $\mathbf{y}_\alpha, |\alpha| < 3$ given by (4.1) and

$$\mathbf{y}_{3,0} = \mathbf{y}_{2,1} = \mathbf{y}_{1,2} = \mathbf{y}_{0,3} = [0, \cdots, 0].$$

With the symmetric properties discussed above, the mask $\{G^{\#}_{\mathbf{k}}\}$ is reduced to a 16-parameter family, given by

$$G^{\#}_{0,0} = \text{diag}\ (1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4}),$$

$$G^{\#}_{1,0} = \begin{bmatrix} \frac{1}{2} & 12s_2 - \frac{3}{4} & 0 & 0 & 0 & 0 \\ 2s_1 + \frac{1}{8} & 6s_2 - \frac{1}{8} & 0 & 2s_3 - \frac{1}{4} & 0 & 2s_4 \\ 0 & 0 & \frac{1}{4} & 0 & 2s_5 - \frac{1}{4} & 0 \\ s_1 & s_2 & 0 & s_3 & 0 & s_4 \\ 0 & 0 & \frac{1}{16} & 0 & s_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \end{bmatrix},$$

$$G^{\#}_{1,1} = \begin{bmatrix} \frac{1}{4} & 12s_7 - \frac{3}{8} & 12s_7 - \frac{3}{8} & 0 & 4(s_{16} - s_{15}) + \frac{1}{4} & 0 \\ 2s_6 + \frac{1}{16} & 6s_7 - \frac{1}{16} & 2(s_8 + s_{13}) - \frac{1}{16} & 2s_9 - \frac{1}{8} & s_{16} & 2s_{11} \\ 2s_6 + \frac{1}{16} & 2(s_8 + s_{13}) - \frac{1}{16} & 6s_7 - \frac{1}{16} & 2s_{11} & s_{16} & 2s_9 - \frac{1}{8} \\ s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} \\ s_{12} & s_{13} & s_{13} & s_{14} & s_{15} & s_{14} \\ s_6 & s_8 & s_7 & s_{11} & s_{10} & s_9 \end{bmatrix},$$

$$G^{\#}_{-1,0} = M_1 G^{\#}_{1,0} M_1,\ G^{\#}_{0,1} = M_3 G^{\#}_{1,0} M_3,\ G^{\#}_{0,-1} = M_1 M_3 G^{\#}_{1,0} M_1 M_3,$$
$$G^{\#}_{-1,-1} = M_1 G^{\#}_{1,1} M_1,\ G^{\#}_{1,-1} = M_2 G^{\#}_{1,0} M_2,\ G^{\#}_{-1,1} = M_1 M_2 G^{\#}_{1,0} M_1 M_2,$$

where M_1, M_2, M_3 are the matrices in (4.2).

Again, the free parameters s_j can be adjusted to achieve certain desirable properties. For example, one may choose

$$[s_1, s_2, \cdots, s_{16}] = -\frac{1}{256}[-4,\ 4,\ 16,\ 4,\ 16,\ -2,\ 2,\ 4,\ 8,\ -1,\ 2,\ -4,\ 4,\ 8,\ 4,\ -8],$$

to assure that the corresponding refinable function vector is in the Sobolev space $W^{3.5-\epsilon}(\mathbb{R}^2)$ for any $\epsilon > 0$. Hence, the corresponding refinable function vector is C^2 and satisfies the second order Hermite interpolatory property (3.3). Again, we used the Matlab routines in [13] to find the Sobolev smoothness exponent.

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