PARAMETRIZATIONS OF SYMMETRIC ORTHOGONAL MULTIFILTER BANKS WITH DIFFERENT FILTER LENGTHS

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Abstract. This paper is devoted to a study of parametrizations of symmetric orthogonal multifilter banks with different fileter lengths. To construct symmetric orthogonal multifilter banks $\{\mathbf{H},\mathbf{G}\}$ which generate balanced multiwavelets of multiplicity 2, the fileter lengths of the rows of $\mathbf{H}$, regarded as the scalar filters, must be different. In this paper complete factorizations of symmetric orthogonal multifilter banks with different fileter lengths are obtained. Based on these factorizations, construction of balanced multiwavelets with good approximation and smoothness properties are discussed.

1. Introduction

A column vector of functions $\Psi = (\psi_1, \psi_2)^T$ is called an **orthonormal multiwavelet of multiplicity** 2 if $\psi_1(2^j x - k), \psi_2(2^j x - k), j, k \in \mathbb{Z}$, form an orthonormal basis of $L^2(\mathbb{R})$. The construction of multiwavelets is associated with the construction of scaling functions. A column vector of functions $\Phi = (\phi_1, \phi_2)^T$ is called an **orthonormal scaling function** if the integer shifts $\phi_1(\cdot - k), \phi_2(\cdot - k), k \in \mathbb{Z}$, form an orthonormal basis of their closed linear span in $L^2(\mathbb{R})$ and Φ is refinable, i.e., Φ satisfies the refinement equation

$$\Phi(x) = 2\sum_{k\in\mathbb{Z}} \mathbf{h}_k \Phi(2x-k), \tag{1.1}$$

for some 2×2 matrices $\mathbf{h}_k$. A necessary condition for Φ to be an orthonormal scaling function is that $\mathbf{H}(\omega) = \sum_{k\in\mathbb{Z}} \mathbf{h}_k e^{-ik\omega}$ is a matrix **Conjugate Quadrature Filter (CQF)** (see e.g., [3]), i.e.,

$$\mathbf{H}(\omega)\mathbf{H}^*(\omega) + \mathbf{H}(\omega+\pi)\mathbf{H}^*(\omega+\pi) = \mathbf{I}_2, \quad \omega \in [-\pi, \pi), \tag{1.2}$$

where $\mathbf{H}^*$ denotes the Hermitian adjoint of $\mathbf{H}$ and $\mathbf{I}_2$ denotes the 2×2 identity matrix. We use $\mathbf{0}_2$ to denote the 2×2 zero matrix. If Φ is an orthonormal scaling function with matrix filter $\mathbf{H}$, and $\mathbf{G} = \sum_{k\in\mathbb{Z}} \mathbf{g}_k e^{-ik\omega}$ is a matrix filter satisfying

$$\mathbf{H}(\omega)\mathbf{G}^*(\omega) + \mathbf{H}(\omega+\pi)\mathbf{G}^*(\omega+\pi) = \mathbf{0}_2, \quad \omega \in [-\pi, \pi), \tag{1.3}$$

$$\mathbf{G}(\omega)\mathbf{G}^*(\omega) + \mathbf{G}(\omega+\pi)\mathbf{G}^*(\omega+\pi) = \mathbf{I}_2, \quad \omega \in [-\pi, \pi), \tag{1.4}$$

then Ψ defined by

$$\Psi(x) := 2\sum_{k\in\mathbb{Z}} \mathbf{g}_k \Phi(2x-k), \tag{1.5}$$

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is an orthonormal multiwavelet (see [6]). In this case, we say $\mathbf{H}, \mathbf{G}$ generate the scaling function Φ and multiwavelet Ψ. The pair $\{\mathbf{H}, \mathbf{G}\}$ is called a multifilter bank. For matrix filters $\mathbf{H}, \mathbf{G}$, if they satisfy (1.2), (1.3) and (1.4), then $\{\mathbf{H}, \mathbf{G}\}$ is said to be orthogonal. For a matrix filter $\mathbf{H}$, it is called **casual** if $\mathbf{h}_k = \mathbf{0}, k < 0$, and is a **finite impulse response** (**FIR**) filter if there exists a positive integer N such that $\mathbf{h}_k = \mathbf{0}, |k| > N$.

Let $\mathbf{H}, \mathbf{G}$ be two FIR matrix filters. Suppose $\mathbf{H}$ is also casual, i.e., the coefficients $\mathbf{h}_k$ of $\mathbf{H}$ satisfy $\mathbf{h}_k = \mathbf{0}$ if $k < 0$ or $k > N$ for some positive integer N. Let $\mathcal{T}_{\mathbf{H}}$ be the matrix defined by

$$\mathcal{T}_{\mathbf{H}} := (2\mathcal{A}_{2i-j})_{1-N \le i,j \le N-1}, \tag{1.6}$$

where $\mathcal{A}_j$ is the 4×4 matrix defined by

$$\mathcal{A}_j := \sum_{\kappa=0}^{N} \mathbf{h}_{\kappa-j} \otimes \mathbf{h}_\kappa,$$

and $\mathbf{h}_{\kappa-j} \otimes \mathbf{h}_\kappa$ denotes the Kronecker product of $\mathbf{h}_{\kappa-j}, \mathbf{h}_\kappa$. Then the solution Φ of (1.1) is an orthonormal scaling function and Ψ defined by (1.5) is an orthonormal multiwavelet if and only if $\{\mathbf{H}, \mathbf{G}\}$ is orthogonal and $\mathbf{H}(0)$ and $\mathcal{T}_{\mathbf{H}}$ satisfy Condition E (see [9]). A matrix $\mathbf{B}$ is said to satisfy **Condition E** if the spectral radius of $\mathbf{B}$ is 1, 1 is the only eigenvalue on the unit circle and 1 is simple. Therefore to construct orthonormal multiwavelets, we need only to find orthogonal matrix filters $\mathbf{H}, \mathbf{G}$ with $\mathbf{H}(0)$ and $\mathcal{T}_{\mathbf{H}}$ satisfying Condition E.

The general procedure to construct multiwavelets with good approximation and smoothness properties is: first we construct a matrix filter $\mathbf{H}$ such that $\mathbf{H}$ is a matrix CQF with $\mathbf{H}(0)$ and $\mathcal{T}_{\mathbf{H}}$ satisfying Condition E and the corresponding scaling function Φ having good approximation and smoothness properties, then we construct the matrix filter $\mathbf{G}$ such that $\mathbf{G}$ satisfies (1.3) and (1.4). Then Ψ defined by (1.5) is an orthonormal multiwavelet with good approximation and smoothness. In practice, in the design of filter banks with some special properties, the parametrization of the FIR orthogonal systems are of fundamental importance (see [18], [17] and references therein). Based on the lattice structures of $M \times M$ casual FIR orthogonal systems, a parametric expression for casual FIR orthogonal multifilter banks was obtained in [9]. In [13] and [9], the explicit expressions for a group of symmetric casual FIR orthogonal multifilter banks were presented, and in [10] the completeness of the M–channel symmetric orthogonal multifilter banks was discussed.

Symmetric (linear phase) property of filters are very important in image applications. For symmetric filters, symmetric extension transforms of the finite length signals can be carried out, which will improve the rate–distortion performance in image compression (see e.g., [16], [19]). For a multifilter bank, since the input are vector signals, it is also required that the corresponding multiwavelet be balanced (see [11]). A multiwavelet Ψ is said to be balanced if the corresponding scaling function Φ satisfies $\widehat{\Phi}(0) = (1,1)^T/\sqrt{2}$. (In this case, we also say that Φ is balanced.) The symmetric multifilter banks provided in [13] and [9] will generate scaling functions Φ and multiwavelets Ψ with all components of Φ, Ψ having the same symmetric center and the first components of Φ, Ψ symmetric and the other components antisymmetric. Thus $\widehat{\Phi}(0) = (1,0)^T$ and Ψ cannot be balanced. It was shown in [10] that there is no orthonormal scaling function $\Phi = (\phi_1, \phi_2)^T$ and orthonormal multiwavelet $\Psi = (\psi_1, \psi_2)^T$ such that ϕ_j, ψ_j have the same symmetric center $n + \frac{1}{2}$ for some integer n and that both ϕ_1 and ϕ_2 have the same symmetry. Thus to construct symmetric and balanced $\Phi = (\phi_1, \phi_2)^T$, ϕ_1, ϕ_2 must have different symmetry centers. The purpose of this paper is to give a parametrization of orthogonal filters for such types of orthonormal scaling functions and multiwavelets.

In Section 2, we will discuss symmetric orthogonal multifilter banks with corresponding scaling function Φ and multiwavelet Ψ such that ϕ_1 and ϕ_2 are symmetric at $\gamma - 1/2$ and γ respectively, and ψ_1, ψ_2 are symmetric/antisymmetric at γ for some $\gamma \in \mathbb{Z}_+ \backslash \{0\}$. Equivalently we discuss casual FIR orthogonal filters ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ satisfying (see e.g., [10])

(1.7)
$$z^{-(2\gamma+1)} \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} {}_\gamma\mathbf{H}(-\omega) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} = {}_\gamma\mathbf{H}(\omega), \quad z^{-(2\gamma+1)} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} {}_\gamma\mathbf{G}(-\omega) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} = {}_\gamma\mathbf{G}(\omega),$$

where $z = e^{i\omega}, s_1 = \pm 1, s_2 = \pm 1$. We provide in Section 2 a complete factorization of the casual FIR orthogonal filters ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ with properties (1.7).

For a matrix filter ${}_\gamma\mathbf{H}$, it is easily verified that ${}_\gamma\mathbf{H}$ satisfies (1.7) if and only if the coefficients $\mathbf{h}_j$ of ${}_\gamma\mathbf{H}$ have the form of

(1.8)
$$[\mathbf{h}_0, \mathbf{h}_1, \cdots, \mathbf{h}_{2\gamma-1}, \mathbf{h}_{2\gamma}, \mathbf{h}_{2\gamma+1}] =$$
$$\left[\begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}, \cdots, \begin{bmatrix} a_2 & a_1 \\ b_6 & b_5 \end{bmatrix}, \begin{bmatrix} a_0 & 0 \\ b_4 & b_3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b_2 & b_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b_0 & 0 \end{bmatrix} \right],$$

for some $a_j, b_j \in \mathbb{R}$ and $\mathbf{h}_j = 0, j < 0, j > 2\gamma + 1$. Note that when we regard the rows of $[\mathbf{h}_0, \cdots, \mathbf{h}_{2\gamma+1}]$ as two scalar filters, they are symmetric and their lengths are both odd integers but not the same ($4\gamma - 3$ and $4\gamma + 1$, respectively). We find that when we construct the scaling functions and multiwavelets based on these filters, we cannot get smooth scaling functions and multiwavelets with small supports. For these reasons, we also discuss orthogonal filters ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ with the coefficients $\mathbf{h}_j$ of ${}_\gamma\mathbf{H}$ having the form

(1.9)
$$[\mathbf{h}_0, \mathbf{h}_1, \cdots, \mathbf{h}_{2\gamma-1}, \mathbf{h}_{2\gamma}, \mathbf{h}_{2\gamma+1}] =$$
$$\left[\begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}, \cdots, \begin{bmatrix} a_1 & a_0 \\ b_5 & b_4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b_3 & b_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b_1 & b_0 \end{bmatrix} \right],$$

for some $a_j, b_j \in \mathbb{R}$, $\mathbf{h}_j = 0, j < 0, j > 2\gamma + 1$, and the coefficients $\mathbf{g}_j$ of ${}_\gamma\mathbf{G}$ having a similar form. The corresponding scaling functions Φ and multiwavelets Ψ are not symmetric or antisymmetric. However as scalar filters, the rows of $[\mathbf{h}_0, \mathbf{h}_1, \cdots, \mathbf{h}_{2\gamma-1}, \mathbf{h}_{2\gamma}, \mathbf{h}_{2\gamma+1}]$ and $[\mathbf{g}_0, \mathbf{g}_1, \cdots, \mathbf{g}_{2\gamma-1}, \mathbf{g}_{2\gamma}, \mathbf{g}_{2\gamma+1}]$ are symmetric and antisymmetric (linear phase filters) respectively, which is more important than the symmetry of Φ, Ψ in image applications. This type of multifilter banks was introduced in [14] and some examples are constructed there. Clearly the filter lengths of the rows of $[\mathbf{h}_0, \mathbf{h}_1, \cdots, \mathbf{h}_{2\gamma}, \mathbf{h}_{2\gamma+1}]$ are both even integers but not the same. In Section 3 we provide a complete factorization of these orthogonal filters. In Section 4, we construct some scaling functions and multiwavelets based on the parametric expressions of the multifilter banks provided in Sections 2, 3. We find we can construct smooth scaling functions and multiwavelets with small supports using the filters provided in Section 3.

In this paper, we use $O(2)$ to denote the set consisting of all 2×2 orthogonal matrices. Any element in $O(2)$ can be written as $\mathbf{r}_\theta$ or $-\mathbf{r}_\theta\mathbf{D}_0$ for some $\theta\in\mathbb{R}$, where

$$\mathbf{r}_\theta := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{D}_0 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.10}$$

For a positive integer n, let $\mathbf{I}_n$ denote $n\times n$ identity matrx, and let $\mathbf{J}_n$ denote the $n\times n$ exchange matrix with ones on the anti–diagonal. For $s\ge 0$, we use $W^s(\mathbb{R})$ to denote the Sobolev space consisting of all functions f with $(1+|\omega|^2)^{\frac{s}{2}}\widehat{f}(\omega)\in L^2(\mathbb{R})$. In this paper all scaling functions, multiwavelets, and the filter coefficients of multifilter banks discussed are real.

2. Symmetric orthogonal multifilter banks with odd filter lengths

In this section we discuss the parametrization of the orthogonal filters ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ satisfying (1.7). In (1.7), the choice of $s_j=+1$ (or $s_j=-1$) means that ψ_j is symmetric (or antisymmetric) at $(\gamma+1)/2$. The next proposition shows that we can only construct Ψ with one component symmetric and the other component antisymmetric.

Proposition 2.1. *Assume that the orthogonal multifilter bank $\{{}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}\}$ generates the scaling function Φ and multiwavelet Ψ. If ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ satisfy (1.7), then $s_1s_2=-1$, i.e., one component of Ψ is symmetric and other component is antisymmetric.*

Proof. By (1.7),

$$\begin{bmatrix} \mathbf{I}_2 & 0 & 0 \\ 0 & s_1 & 0 \\ 0 & 0 & s_2 \end{bmatrix} \begin{bmatrix} {}_\gamma\mathbf{H}(0) & -{}_\gamma\mathbf{H}(\pi) \\ {}_\gamma\mathbf{G}(0) & -{}_\gamma\mathbf{G}(\pi) \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 & 0 \\ 0 & \mathbf{D}_0 \end{bmatrix} = \begin{bmatrix} {}_\gamma\mathbf{H}(0) & {}_\gamma\mathbf{H}(\pi) \\ {}_\gamma\mathbf{G}(0) & {}_\gamma\mathbf{G}(\pi) \end{bmatrix}.$$

Since $\{{}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}\}$ is orthogonal, the matrix in the left side of the above equation is unitary. Hence it is invertible. Thus we have

$$\begin{bmatrix} \mathbf{I}_2 & 0 & 0 \\ 0 & s_1 & 0 \\ 0 & 0 & s_2 \end{bmatrix} = \begin{bmatrix} {}_\gamma\mathbf{H}(0) & {}_\gamma\mathbf{H}(\pi) \\ {}_\gamma\mathbf{G}(0) & {}_\gamma\mathbf{G}(\pi) \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{D}_0 \end{bmatrix} \begin{bmatrix} {}_\gamma\mathbf{H}(0) & {}_\gamma\mathbf{H}(\pi) \\ {}_\gamma\mathbf{G}(0) & {}_\gamma\mathbf{G}(\pi) \end{bmatrix}^{-1}.$$

This implies that the trace of $\mathrm{diag}(\mathbf{I}_2, s_1, s_2)$ equals that of $\mathrm{diag}(\mathbf{I}_2, -\mathbf{D}_0)$. Therefore $2+s_1+s_2=2$. That is $s_1s_2=-1$. □

In the following we choose $s_1=1, s_2=-1$. In this case one can check directly that ${}_\gamma\mathbf{G}$ satisfies (1.7) if and only if the coefficients $\mathbf{g}_j$ of ${}_\gamma\mathbf{G}$ have the form

$$[\mathbf{g}_0, \mathbf{g}_1, \cdots, \mathbf{g}_{2\gamma}, \mathbf{g}_{2\gamma+1}] = \left[\begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix}, \begin{bmatrix} c_2 & c_3 \\ d_2 & d_3 \end{bmatrix}, \cdots, \begin{bmatrix} c_2 & c_1 \\ -d_2 & -d_1 \end{bmatrix}, \begin{bmatrix} c_0 & 0 \\ -d_0 & 0 \end{bmatrix}\right], \tag{2.1}$$

for some $c_j, d_j\in\mathbb{R}$ and $\mathbf{g}_j=0, j<0, j>2\gamma+1$. Thus the rows of $[\mathbf{g}_0, \mathbf{g}_1, \cdots, \mathbf{g}_{2\gamma-1}, \mathbf{g}_{2\gamma}, \mathbf{g}_{2\gamma+1}]$ are symmetric and antisymmetric respectively.

As the $M\times M$ casual FIR orthogonal systems, it is expected that ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ can also be factorized as

$$\begin{bmatrix} {}_\gamma\mathbf{H}(\omega) \\ {}_\gamma\mathbf{G}(\omega) \end{bmatrix} = (\mathbf{I}_4 - \mathbf{B} + \mathbf{B}z^{-2}) \begin{bmatrix} {}_{\gamma-1}\mathbf{H}^\vee(\omega) \\ {}_{\gamma-1}\mathbf{G}^\vee(\omega) \end{bmatrix}, \quad z=e^{i\omega}, \tag{2.2}$$

where $\{{}_{\gamma-1}\mathbf{H}^\vee, {}_{\gamma-1}\mathbf{G}^\vee\}$ is a casual FIR orthogonal multifilter bank, and $\mathbf{B}$ is a 4×4 projection matrix, i.e., $\mathbf{B}$ satisfies

$$\mathbf{B}^T = \mathbf{B}, \quad \mathbf{B}^2 = \mathbf{B}.$$

If ${}_\gamma\mathbf{H}$ satisfies (1.7), or equivalently ${}_\gamma\mathbf{H}$ has the form (1.8), then the first row of ${}_\gamma\mathbf{H}$ is a polynomial of $e^{-i\omega}$ of degree not greater than $2r-1$. The entries of ${}_{\gamma-1}\mathbf{H}^\vee, {}_{\gamma-1}\mathbf{G}^\vee$ are polynomials of $e^{-i\omega}$ of degree possible not smaller than $2r-1$. Thus in order that ${}_\gamma\mathbf{H}$ have the form (1.8), $\mathbf{B}$ shall have the form of

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{b} \end{bmatrix},$$

where $\mathbf{b}$ is a 3×3 projection matrix.

Denote

$$\mathbf{b}_\theta := \mathbf{v}_\theta^\pm(\mathbf{v}_\theta^\pm)^T, \quad \text{where } \mathbf{v}_\theta^\pm := \tfrac{\sqrt{2}}{2}(\sin\theta, \cos\theta, \pm 1)^T. \tag{2.3}$$

That is $\mathbf{b}_\theta$ defined by (2.3) is either $\mathbf{v}_\theta^+(\mathbf{v}_\theta^+)^T$ or $\mathbf{v}_\theta^-(\mathbf{v}_\theta^-)^T$. By a direct calculation, one has the following lemma.

Lemma 2.1. *Let $\mathbf{b}_\theta$ and $\mathbf{r}_\theta$ be the matrices defined by (2.3) and (1.10) respectively. Then*

$$\big((\mathbf{I}_3 - \mathbf{b}_\theta)z^{-1} + \mathbf{b}_\theta\big)\, diag(1,1,-1)\, \big((\mathbf{I}_3 - \mathbf{b}_\theta)z^{-1} + \mathbf{b}_\theta\big) = z^{-1} diag(\mathbf{r}_\theta, 1)\, diag(z^{-1},1,1)\, diag(\mathbf{r}_\theta^T, -1).$$

Denote

$$\mathbf{B}_\theta := \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{b}_\theta \end{bmatrix}, \quad \mathbf{R}_\theta := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{r}_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{J}(z) := \begin{bmatrix} 0 & 1 & 0 \\ z^{-1} & 0 & 0 \\ 0 & 0 & \mathbf{I}_2 \end{bmatrix}.$$

Lemma 2.2. *Let $\mathbf{V}_\theta(z)$ be the matrix polynomial of z^{-1} defined by*

$$\mathbf{V}_\theta(z) := (\mathbf{I}_4 - \mathbf{B}_\theta + \mathbf{B}_\theta z^{-1})\mathbf{R}_\theta \mathbf{J}(z). \tag{2.4}$$

Then

(i) $\mathbf{V}_\theta(z)\mathbf{V}_\theta(z^{-1})^T = \mathbf{I}_4$;

(ii) $z^{-1} diag(z, 1, \mathbf{D}_0)\mathbf{V}_\theta(z^{-1}) diag(z^{-1}, 1, \mathbf{D}_0) = \mathbf{V}_\theta(z)$.

Proof. (i) follows from the fact that $\mathbf{B}_\theta$ is a projection matrix and $\mathbf{R}_\theta$ is a unitary matrix; and (ii) follows from Lemma 2.1 and a direction calculation. ∎

If casual FIR orthogonal filters ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ can be written as

$$\begin{bmatrix} {}_\gamma\mathbf{H}(\omega) \\ {}_\gamma\mathbf{G}(\omega) \end{bmatrix} = \mathbf{V}_{\theta_\gamma}(z^2) \begin{bmatrix} {}_{\gamma-1}\mathbf{H}(\omega) \\ {}_{\gamma-1}\mathbf{G}(\omega) \end{bmatrix}, \quad z = e^{i\omega}, \tag{2.5}$$

for some casual FIR orthogonal filter bank $\{{}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}\}$, then by Lemma 2.2, ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ satisfy (1.7) if and only if ${}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}$ satisfy (1.7) for $\gamma - 1$. For casual FIR orthogonal filters ${}_1\mathbf{H}, {}_1\mathbf{G}$, if they satisfy (1.7) for $\gamma = 1$, then the first row of ${}_1\mathbf{H}$ is a polynomial of $e^{-i\omega}$ of degree not greater than 1. Thus ${}_1\mathbf{H}, {}_1\mathbf{G}$ shall not be written in the form of (2.5). Instead, we write them in the following form

$$\begin{bmatrix} {}_1\mathbf{H}(\omega) \\ {}_1\mathbf{G}(\omega) \end{bmatrix} = (\mathbf{I}_4 - \mathbf{B}_{\theta_1} + \mathbf{B}_{\theta_1} z^{-2})\mathbf{R}_{\theta_1} \begin{bmatrix} {}_0\mathbf{H}(\omega) \\ {}_0\mathbf{G}(\omega) \end{bmatrix} \tag{2.6}$$

for some casual FIR orthogonal filters ${}_0\mathbf{H}, {}_0\mathbf{G}$. One can show that ${}_1\mathbf{H}, {}_1\mathbf{G}$ satisfy (1.7) for $\gamma = 1$ if and only if ${}_0\mathbf{H}, {}_0\mathbf{G}$ satisfy

$$z^{-1}\mathrm{diag}(1, z^{-2}, 1, -1)\begin{bmatrix} {}_0\mathbf{H}(-\omega) \\ {}_0\mathbf{G}(-\omega) \end{bmatrix}\mathrm{diag}(1, z) = \begin{bmatrix} {}_0\mathbf{H}(\omega) \\ {}_0\mathbf{G}(\omega) \end{bmatrix}. \tag{2.7}$$

In this way, it is reasonable to expect that ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ can be factorized as

$$\begin{bmatrix} {}_\gamma\mathbf{H}(\omega) \\ {}_\gamma\mathbf{G}(\omega) \end{bmatrix} = \mathbf{V}_{\theta_\gamma}(z^2)\mathbf{V}_{\theta_{\gamma-1}}(z^2)\cdots\mathbf{V}_{\theta_2}(z^2)(\mathbf{I}_4 - \mathbf{B}_{\theta_1} + \mathbf{B}_{\theta_1}z^{-2})\mathbf{R}_{\theta_1}\begin{bmatrix} {}_0\mathbf{H}(\omega) \\ {}_0\mathbf{G}(\omega) \end{bmatrix}, \tag{2.8}$$

where $z = e^{i\omega}$, and ${}_0\mathbf{H}, {}_0\mathbf{G}$ are casual FIR orthogonal filters satisfying (2.7).

The next lemma gives the casual FIR orthogonal filters ${}_0\mathbf{H}, {}_0\mathbf{G}$ satisfying (2.7).

Lemma 2.3. *A casual FIR filter bank $\{{}_0\mathbf{H}, {}_0\mathbf{G}\}$ is orthogonal and satisfies (2.7) if and only if it is given by*

$$ {}_0\mathbf{H}(\omega) = \frac{1}{2}\begin{bmatrix} \alpha_0\cos\theta_0(1+z^{-1}) & \sqrt{2}\sin\theta_0 \\ 0 & \alpha_1\sqrt{2}z^{-1} \end{bmatrix}, {}_0\mathbf{G}(\omega) = \frac{1}{2}\begin{bmatrix} -\alpha_0\sin\theta_0(1+z^{-1}) & \sqrt{2}\cos\theta_0 \\ \alpha_2(1-z^{-1}) & 0 \end{bmatrix}, \tag{2.9}$$

where $z = e^{i\omega}, \alpha_j = \pm 1, 0 \le j \le 2$, $\theta_0 \in [-\pi, \pi)$.

Proof. If ${}_0\mathbf{H}, {}_0\mathbf{G}$ are casual and FIR, then ${}_0\mathbf{H}, {}_0\mathbf{G}$ satisfy (2.7) if and only if

$${}_0\mathbf{H}(\omega) = \begin{bmatrix} a(1+z^{-1}) & b \\ 0 & cz^{-1} \end{bmatrix}, {}_0\mathbf{G}(\omega) = \begin{bmatrix} d(1+z^{-1}) & e \\ f(1-z^{-1}) & 0 \end{bmatrix},$$

for some $a, b, c, d, e, f \in \mathbb{R}$. The orthogonality of ${}_0\mathbf{H}, {}_0\mathbf{G}$ is equivalent to $f = \pm\frac{1}{2}, c = \pm\frac{\sqrt{2}}{2}$ and $\begin{bmatrix} 2a & \sqrt{2}b \\ 2d & \sqrt{2}e \end{bmatrix}$ being a 2×2 orthogonal matrix, i.e.,

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \pm\cos\theta_0 & \sin\theta_0 \\ \mp\sin\theta_0 & \cos\theta_0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \quad \theta_0 \in [-\pi, \pi).$$

Therefore ${}_0\mathbf{H}, {}_0\mathbf{G}$ are given by (2.9). □

Clearly if ${}_0\mathbf{H}, {}_0\mathbf{G}$ are casual FIR orthogonal filters given by (2.9), then ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ defined by (2.8) are casual, FIR and orthogonal, and satisfy (1.7). The next theorem shows that any such a multifilter bank can be factorized in the form of (2.8).

Theorem 2.1. *A casual FIR multifilter bank $\{{}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}\}$ is orthogonal and satisfies (1.7) with $s_1 = 1, s_2 = -1$ if and only if it can be factorized into the product of (2.8) for some $\theta_k, 1 \le k \le r$ with ${}_0\mathbf{H}, {}_0\mathbf{G}$ given by (2.9) for some θ_0.*

Proof. "⇐" The direct part follows from Lemmas 2.2, 2.3.

"⇒" For the proof of the converse part, we show the above factorization by induction on the order γ. We show that any casual FIR orthogonal multifilter bank $\{{}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}\}$ satisfying (1.7) for $\gamma \ge 2$ can be factorized as (2.5) for some casual FIR orthogonal filter bank $\{{}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}\}$ satisfying (1.7) for $\gamma - 1$, and also that any casual FIR orthogonal bank $\{{}_1\mathbf{H}, {}_1\mathbf{G}\}$ satisfying (1.7) for $\gamma = 1$ can be factorized as (2.6) for some ${}_0\mathbf{H}, {}_0\mathbf{G}$ given by (2.9).

For $\gamma = 1$, it is easily verified that

$$z^{-1}\mathrm{diag}(z,1,\mathbf{D}_0)(\mathbf{I}_4 - \mathbf{B}_{\theta_1} + \mathbf{B}_{\theta_1} z)\mathbf{R}_{\theta_1}\mathrm{diag}(1,z,\mathbf{D}_0) = (\mathbf{I}_4 - \mathbf{B}_{\theta_1} + \mathbf{B}_{\theta_1} z^{-1})\mathbf{R}_{\theta_1}.$$

This together with the facts that $\mathbf{B}_{\theta_1}$ is a projection matrix and that $\mathbf{R}_{\theta_1}$ is a unitary matrix implies that ${}_0\mathbf{H}, {}_0\mathbf{G}$ defined by

$$\begin{aligned}\begin{bmatrix} {}_0\mathbf{H}(\omega) \\ {}_0\mathbf{G}(\omega) \end{bmatrix} &= \left((\mathbf{I}_4 - \mathbf{B}_{\theta_1} + \mathbf{B}_{\theta_1} z^{-2})\mathbf{R}_{\theta_1}\right)^{-1} \begin{bmatrix} {}_1\mathbf{H}(\omega) \\ {}_1\mathbf{G}(\omega) \end{bmatrix} \\ &= (\mathbf{R}_{\theta_1}^T - \mathbf{R}_{\theta_1}^T\mathbf{B}_{\theta_1}) \begin{bmatrix} {}_1\mathbf{H}(\omega) \\ {}_1\mathbf{G}(\omega) \end{bmatrix} + \mathbf{R}_{\theta_1}^T\mathbf{B}_{\theta_1} z^2 \begin{bmatrix} {}_1\mathbf{H}(\omega) \\ {}_1\mathbf{G}(\omega) \end{bmatrix}, \quad z = e^{i\omega},\end{aligned} \tag{2.10}$$

are orthogonal and satisfy (2.7). Thus we need only to show that ${}_0\mathbf{H}, {}_0\mathbf{G}$ given by (2.10) are casual for some θ_1. The second term on the right-hand side of the second equation is responsible for any possible noncasuality. In particular, the noncasual part of the second term is given by

$$\mathbf{R}_{\theta_1}^T\mathbf{B}_{\theta_1} \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix},$$

where $\mathbf{h}_j, \mathbf{g}_j$ are the matrix coefficients of ${}_1\mathbf{H}, {}_1\mathbf{G}$. Note that

$$\mathbf{R}_{\theta_1}^T\mathbf{B}_{\theta_1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{l}_{\theta_1}^{\pm} \\ 0 & \pm\mathbf{l}_{\theta_1}^{\pm} \end{bmatrix}, \quad \mathbf{l}_\theta^{\pm} := (\sin\theta, \cos\theta, \pm 1),$$

where the choice of $\mathbf{l}_\theta^+$ or $\mathbf{l}_\theta^-$ depends on the choice of $\mathbf{v}_\theta^+$ or $\mathbf{v}_\theta^-$ in the definition of $\mathbf{b}_\theta$. Thus we need only to show that there exists a θ_1 such that

$$[0, \mathbf{l}_{\theta_1}^+] \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0 \quad \text{or} \quad [0, \mathbf{l}_{\theta_1}^-] \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0.$$

For the case $\gamma \geq 2$, since $\mathbf{V}_{\theta_\gamma}(z)$ satisfies (i) and (ii) in Lemma 2.2, ${}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}$ defined by

$$\begin{bmatrix} {}_{\gamma-1}\mathbf{H}(\omega) \\ {}_{\gamma-1}\mathbf{G}(\omega) \end{bmatrix} = \mathbf{V}_{\theta_\gamma}(z^2)^{-1} \begin{bmatrix} {}_\gamma\mathbf{H}(\omega) \\ {}_\gamma\mathbf{G}(\omega) \end{bmatrix}, \quad z = e^{i\omega}, \tag{2.11}$$

are orthogonal and satisfy (1.7) for $\gamma - 1$. Thus we need only to show that ${}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}$ defined by (2.11) are casual for some θ_γ. Note that

$$\mathbf{V}_\theta(z)^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & \sin\theta & \cos\theta & \mp 1 \\ 0 & \mp\sin\theta & \mp\cos\theta & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2\cos\theta & -2\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sin\theta & \cos\theta & \pm 1 \\ 0 & \pm\sin\theta & \pm\cos\theta & 1 \end{bmatrix} z.$$

Thus the terms in ${}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}$ responsible for the noncasuality are

$$\begin{bmatrix} 0 & 2\cos\theta & -2\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sin\theta & \cos\theta & \pm 1 \\ 0 & \pm\sin\theta & \pm\cos\theta & 1 \end{bmatrix} \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix}.$$

Therefore to show that ${}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}$ are casual, we need only to show that there is a θ_γ such that

$$[0, \mathbf{L}^+_{\theta_\gamma}] \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0 \quad \text{or} \quad [0, \mathbf{L}^-_{\theta_\gamma}] \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0$$

where

$$\mathbf{L}^\pm_\theta := \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & \pm 1 \end{bmatrix}.$$

Write

$$\begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} =: \begin{bmatrix} * & * \\ \mathbf{A} & \mathbf{v}_3 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{h}_2 & \mathbf{h}_3 \\ \mathbf{g}_2 & \mathbf{g}_3 \end{bmatrix} =: \begin{bmatrix} * & * \\ \mathbf{C} & * \end{bmatrix},$$

where $\mathbf{A}, \mathbf{C}$ are 3×3 matrices and $\mathbf{v}_3$ is a 3×1 vector. To complete the proof, it is enough to show for the case $\gamma = 1$, that there exists a θ_1 such that

$$\mathbf{l}^+_{\theta_1}\mathbf{A} = 0, \quad \mathbf{l}^+_{\theta_1}\mathbf{v}_3 = 0 \quad \text{or} \quad \mathbf{l}^-_{\theta_1}\mathbf{A} = 0, \quad \mathbf{l}^-_{\theta_1}\mathbf{v}_3 = 0, \tag{2.12}$$

and for the case $\gamma \geq 2$, that there exists a θ_γ such that

$$\mathbf{L}^+_{\theta_\gamma}\mathbf{A} = 0, \quad \mathbf{L}^+_{\theta_\gamma}\mathbf{v}_3 = 0 \quad \text{or} \quad \mathbf{L}^-_{\theta_\gamma}\mathbf{A} = 0, \quad \mathbf{L}^-_{\theta_\gamma}\mathbf{v}_3 = 0. \tag{2.13}$$

Since ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ have the forms (1.8), (2.1), one has

(2.14)
$$\begin{bmatrix} \mathbf{I}_2 & 0 \\ 0 & \mathbf{D}_0 \end{bmatrix} \begin{bmatrix} \mathbf{h}_{2\gamma} & \mathbf{h}_{2\gamma+1} \\ \mathbf{g}_{2\gamma} & \mathbf{g}_{2\gamma+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathbf{A}\mathbf{J}_3 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{I}_2 & 0 \\ 0 & \mathbf{D}_0 \end{bmatrix} \begin{bmatrix} \mathbf{h}_{2\gamma-2} & \mathbf{h}_{2\gamma-1} \\ \mathbf{g}_{2\gamma-2} & \mathbf{g}_{2\gamma-1} \end{bmatrix} = \begin{bmatrix} * & 0 \\ \mathbf{C}\mathbf{J}_3 & \mathbf{v}_3 \end{bmatrix},$$

where $\mathbf{J}_3$ is the 3×3 exchange matrix defined in the introduction. By the orthogonality,

$$\begin{bmatrix} \mathbf{h}_{2\gamma} & \mathbf{h}_{2\gamma+1} \\ \mathbf{g}_{2\gamma} & \mathbf{g}_{2\gamma+1} \end{bmatrix}^T \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0. \tag{2.15}$$

This equation and (2.14) imply that

$$\mathbf{A}^T \mathrm{diag}(1, 1, -1)\mathbf{A} = 0. \tag{2.16}$$

Since rank($\mathbf{A}^T \mathrm{diag}(1, 1, -1)$)=rank($\mathbf{A}$), (2.16) implies that rank$(\mathbf{A}) \leq 1$. Thus $\mathbf{A} = \mathbf{u}\mathbf{v}^T$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ with $\mathbf{v} \neq 0$. By (2.16) again, $\mathbf{u}^T \mathrm{diag}(1, 1, -1)\mathbf{u} = 0$. Thus $\mathbf{u} = u(\sin\theta, \cos\theta, \pm 1)^T$ for some $u, \theta \in \mathbb{R}$. Therefore we have

$$\mathbf{L}^\mp_\theta \mathbf{A} = 0,$$

and in particular $\mathbf{l}^\mp_\theta \mathbf{A} = 0$.

Finally we need to prove that $\mathbf{l}^\mp_\theta \mathbf{v}_3 = 0$ (for $\gamma = 1$) and $\mathbf{L}^\mp_\theta \mathbf{v}_3 = 0$ (for $\gamma \geq 2$). We first consider the case $\gamma = 1$. In this case (2.15) is

$$\begin{bmatrix} 0 & \mathbf{J}_3\mathbf{A}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 & 0 \\ 0 & \mathbf{D}_0 \end{bmatrix} \begin{bmatrix} * & * \\ \mathbf{A} & \mathbf{v}_3 \end{bmatrix} = 0.$$

Thus we have $\mathbf{J}_3\mathbf{A}^T \mathrm{diag}(1, \mathbf{D}_0)\mathbf{v}_3 = 0$. Note that for the case $\gamma = 1$, the third component of $\mathbf{v}_3$ is 0. Therefore we have $\mathbf{A}^T\mathbf{v}_3 = 0$. That is $u(\sin\theta, \cos\theta, \pm 1)\mathbf{v}_3 = 0$. Clearly in this case $u \neq 0$. Thus we have $\mathbf{l}^\mp_\theta \mathbf{v}_3 = \mathbf{l}^\pm_\theta \mathbf{v}_3 = 0$.

Finally, let us consider the case $\gamma \geq 2$. Again by the orthogonality of the filter bank,

$$\begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} \begin{bmatrix} \mathbf{h}_{2\gamma-2} & \mathbf{h}_{2\gamma-1} \\ \mathbf{g}_{2\gamma-2} & \mathbf{g}_{2\gamma-1} \end{bmatrix}^T + \begin{bmatrix} \mathbf{h}_2 & \mathbf{h}_3 \\ \mathbf{g}_2 & \mathbf{g}_3 \end{bmatrix} \begin{bmatrix} \mathbf{h}_{2\gamma} & \mathbf{h}_{2\gamma+1} \\ \mathbf{g}_{2\gamma} & \mathbf{g}_{2\gamma+1} \end{bmatrix}^T = 0.$$

That is, by (2.14),

$$\begin{bmatrix} * & * \\ \mathbf{A} & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} * & \mathbf{J}_3\mathbf{C}^T \\ * & \mathbf{v}_3^T \end{bmatrix} + \begin{bmatrix} * & * \\ \mathbf{C} & * \end{bmatrix} \begin{bmatrix} 0 & \mathbf{J}_3\mathbf{A}^T \\ 0 & 0 \end{bmatrix} = 0.$$

Thus we have

$$\mathbf{A}\mathbf{J}_3\mathbf{C}^T + \mathbf{v}_3\mathbf{v}_3^T + \mathbf{C}\mathbf{J}_3\mathbf{A}^T = 0. \tag{2.17}$$

Since we have proved $\mathbf{L}_\theta^\mp \mathbf{A} = 0$, by (2.17), we have $\mathbf{L}_\theta^\mp \mathbf{v}_3 = 0$. Thus we have (2.12) and (2.13), and the proof of Theorem 2.1 is complete. □

3. Symmetric orthogonal multifilter banks with even filter lengths

In this section we will discuss orthogonal multifilter banks $\{{}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}\}$ with ${}_\gamma\mathbf{H}$ having the form of (1.9) and ${}_\gamma\mathbf{G}$ having similar properties. One can obtain that (1.9) is equivalent to

$$z^{-(2\gamma+1)} \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} {}_\gamma\mathbf{H}(-\omega)\mathbf{J}_2 = {}_\gamma\mathbf{H}(\omega), \quad z = e^{i\omega}. \tag{3.1}$$

Suppose that ${}_\gamma\mathbf{G}$ satisfies

$$z^{-(2\gamma+1)} \begin{bmatrix} s_1 z^{2\ell} & 0 \\ 0 & s_2 \end{bmatrix} {}_\gamma\mathbf{G}(-\omega)\mathbf{J}_2 = {}_\gamma\mathbf{G}(\omega), \tag{3.2}$$

for some integer ℓ and that $s_1 = \pm 1, s_2 = \pm 1$. One can prove as in Proposition 2.1 that we only have the choices $s_1 = s_2 = -1$.

Proposition 3.1. *Assume that the multifilter bank $\{{}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}\}$ is orthogonal and that ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ satisfy (3.1), (3.2) respectively, then $s_1 = s_2 = -1$.*

In the following we discuss factorizations of $\{{}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}\}$ with ${}_\gamma\mathbf{H}$ and ${}_\gamma\mathbf{G}$ satisfying (3.1) and (3.2) respectively. Here we discuss the factorizations for the case $\ell = 1$ for simplicity. In this case the orthogonal filters satisfy

$$z^{-(2\gamma+1)} \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} \big({}_\gamma\mathbf{H}(-\omega), -{}_\gamma\mathbf{G}(-\omega)\big)\mathbf{J}_2 = \big({}_\gamma\mathbf{H}(\omega), {}_\gamma\mathbf{G}(\omega)\big), \quad z = e^{i\omega}. \tag{3.3}$$

One can check directly that if ${}_\gamma\mathbf{G}$ satisfies (3.3), then its coefficients $\mathbf{g}_j$ have the following form

$$\begin{aligned} &[\mathbf{g}_0, \mathbf{g}_1, \cdots, \mathbf{g}_{2\gamma}, \mathbf{g}_{2\gamma+1}] = \\ &\left[\begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix}, \begin{bmatrix} c_2 & c_3 \\ d_2 & d_3 \end{bmatrix}, \dots, \begin{bmatrix} -c_1 & -c_0 \\ -d_5 & -d_4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -d_3 & -d_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -d_1 & -d_0 \end{bmatrix} \right], \end{aligned} \tag{3.4}$$

for some $c_j, d_j \in \mathbb{R}$ and $\mathbf{g}_j = 0, j < 0, j > 2\gamma + 1$. Thus the rows of $[\mathbf{g}_0, \mathbf{g}_1, \cdots, \mathbf{g}_{2\gamma}, \mathbf{g}_{2\gamma+1}]$ are antisymmetric scalar filters.

It is expected ${}_{\gamma}\mathbf{H}, {}_{\gamma}\mathbf{G}$ be factorized as in (2.2) for some casual FIR orthogonal multifilter bank $\{{}_{\gamma-1}\mathbf{H}^{\vee}, {}_{\gamma-1}\mathbf{G}^{\vee}\}$ and some 4×4 projection matrix $\mathbf{B}$. Denote

$$\mathbf{b}_0 := \frac{1}{2}(1,0,\pm 1)^T(1,0,\pm 1), \quad \mathbf{B}_0 := \operatorname{diag}(0, \mathbf{b}_0). \tag{3.5}$$

Then one has

$$z^2\big((\mathbf{I}_4-\mathbf{B}_0)z^{-2}+\mathbf{B}_0\big)\operatorname{diag}(z^2,1,-z^2,-1)\big((\mathbf{I}_4-\mathbf{B}_0)z^{-2}+\mathbf{B}_0\big) = \operatorname{diag}(\mathbf{I}_2, -\mathbf{I}_2).$$

Thus ${}_{\gamma-1}\mathbf{H}^{\vee}, {}_{\gamma-1}\mathbf{G}^{\vee}$ defined by

$$\begin{bmatrix} {}_{\gamma-1}\mathbf{H}^{\vee}(\omega) \\ {}_{\gamma-1}\mathbf{G}^{\vee}(\omega) \end{bmatrix} = \big(\mathbf{I}_4 - \mathbf{B}_0 + \mathbf{B}_0 z^2\big) \begin{bmatrix} {}_{\gamma}\mathbf{H}(\omega) \\ {}_{\gamma}\mathbf{G}(\omega) \end{bmatrix}, \quad z = e^{i\omega}, \tag{3.6}$$

are orthogonal and satisfy

$$z^{-(2\gamma-1)}\big({}_{\gamma-1}\mathbf{H}^{\vee}(-\omega), -{}_{\gamma-1}\mathbf{G}^{\vee}(-\omega)\big)\mathbf{J}_2 = \big({}_{\gamma-1}\mathbf{H}^{\vee}(\omega), {}_{\gamma-1}\mathbf{G}^{\vee}(\omega)\big), \quad z = e^{i\omega}. \tag{3.7}$$

Define

$$\big({}_{\gamma-1}\mathbf{H}(\omega), {}_{\gamma-1}\mathbf{G}(\omega)\big) = \frac{\sqrt{2}}{2}\big({}_{\gamma-1}\mathbf{H}^{\vee}(\omega), {}_{\gamma-1}\mathbf{G}^{\vee}(\omega)\big)\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then ${}_{\gamma-1}\mathbf{H}^{\vee}, {}_{\gamma-1}\mathbf{G}^{\vee}$ satisfy (3.7) if and only if ${}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}$ satisfy

$$z^{-(2\gamma-1)}\big({}_{\gamma-1}\mathbf{H}(-\omega), -{}_{\gamma-1}\mathbf{G}(-\omega)\big)\mathbf{D}_0 = \big({}_{\gamma-1}\mathbf{H}(\omega), {}_{\gamma-1}\mathbf{G}(\omega)\big), \quad z = e^{i\omega}. \tag{3.8}$$

For a casual FIR orthogonal filter bank $\{{}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}\}$ satisfying (3.8), we have the following result about its factorization obtained in [10].

Theorem 3.1. (Jiang [10]) *A casual FIR filter bank $\{{}_{\gamma-1}\mathbf{H}, {}_{\gamma-1}\mathbf{G}\}$ is orthogonal and satisfies (3.8) if and only if it can be factorized as*

$$\begin{bmatrix} {}_{\gamma-1}\mathbf{H}(\omega) \\ {}_{\gamma-1}\mathbf{Q}(\omega) \end{bmatrix} = \frac{1}{2}\mathbf{U}_{\gamma-1}(z^2)\mathbf{U}_{\gamma-2}(z^2)\cdots\mathbf{U}_1(z^2)\begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_0\mathbf{D}_0 \\ \mathbf{v}_0 & -\mathbf{v}_0\mathbf{D}_0 \end{bmatrix}\begin{bmatrix} \mathbf{I}_2 \\ z^{-1}\mathbf{I}_2 \end{bmatrix}, \quad z = e^{i\omega},$$

where $\mathbf{w}_0, \mathbf{v}_0 \in O(2)$, and

$$\mathbf{U}_k(z) = \frac{1}{2}\begin{bmatrix} \mathbf{I}_2 & \mathbf{u}_k \\ \mathbf{u}_k^T & \mathbf{I}_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{I}_2 & -\mathbf{u}_k \\ -\mathbf{u}_k^T & \mathbf{I}_2 \end{bmatrix} z^{-1}, \quad \mathbf{u}_k \in O(2). \tag{3.9}$$

Therefore, by Theorem 3.1, if ${}_{\gamma-1}\mathbf{H}^{\vee}, {}_{\gamma-1}\mathbf{G}^{\vee}$ are casual, then ${}_{\gamma}\mathbf{H}, {}_{\gamma}\mathbf{G}$ can be factorized as

$$\begin{bmatrix} {}_{\gamma}\mathbf{H}(\omega) \\ {}_{\gamma}\mathbf{Q}(\omega) \end{bmatrix} = \frac{\sqrt{2}}{4}\big(\mathbf{I}_4 - \mathbf{B}_0 + \mathbf{B}_0 z^{-2}\big)\mathbf{U}_{\gamma-1}(z^2)\cdots\mathbf{U}_1(z^2)\begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_0\mathbf{D}_0 \\ \mathbf{v}_0 & -\mathbf{v}_0\mathbf{D}_0 \end{bmatrix}\begin{bmatrix} \mathbf{I}_2 \\ z^{-1}\mathbf{I}_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{3.10}$$

Theorem 3.2. *A casual FIR multifilter bank $\{{}_{\gamma}\mathbf{H}, {}_{\gamma}\mathbf{G}\}$ is orthogonal and satisfies (3.3) if and only if it can be factorized in the form of (3.10) with $\mathbf{B}_0$ defined by (3.5), $\mathbf{U}_k$ defined by (3.9) for some $\mathbf{u}_k \in O(2)$, and $\mathbf{w}_0, \mathbf{v}_0 \in O(2)$.*

Proof. The direct part follows from the above derivations. For the proof of the converse part, by Theorem 3.1 and the above derivations, we need only to show that ${}_{\gamma-1}\mathbf{H}^\vee, {}_{\gamma-1}\mathbf{G}^\vee$ defined by (3.6) are casual, i.e., to shown that

$$\mathbf{B}_0 z^2 \begin{bmatrix} {}_\gamma\mathbf{H}(\omega) \\ {}_\gamma\mathbf{G}(\omega) \end{bmatrix}$$

is casual, or equivalently to show that

$$(0,1,0,1)\begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0 \quad \text{or} \quad (0,1,0,-1)\begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0.$$

The sign choices in the definition of $\mathbf{b}_0$ provide the sign choices in the above equation. Here $\mathbf{h}_j, \mathbf{g}_j$ are the matrix coefficients of ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$. Let η_j denote the jth row of $\begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix}$. Then what we need to show is that

$$\eta_2 + \eta_4 = 0 \quad \text{or} \quad \eta_2 - \eta_4 = 0. \tag{3.11}$$

We may suppose

$$\mathbf{E} := (\eta_2^T, \eta_4^T) \neq 0,$$

for otherwise, (3.11) holds automatically. By the orthogonality,

$$\begin{bmatrix} \mathbf{h}_{2\gamma} & \mathbf{h}_{2\gamma+1} \\ \mathbf{g}_{2\gamma} & \mathbf{g}_{2\gamma+1} \end{bmatrix}^T \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 \\ \mathbf{g}_0 & \mathbf{g}_1 \end{bmatrix} = 0. \tag{3.12}$$

Note that in this case

$$\begin{bmatrix} \mathbf{h}_{2\gamma} & \mathbf{h}_{2\gamma+1} \\ \mathbf{g}_{2\gamma} & \mathbf{g}_{2\gamma+1} \end{bmatrix} = (0, \eta_2^T, 0, -\eta_4^T)^T \mathbf{J}_4.$$

Thus (3.12) is equivalent to

$$\mathbf{E}\mathrm{diag}(1,-1)\mathbf{E}^T = 0. \tag{3.13}$$

Therefore $\mathrm{rank}(\mathbf{E}) = 1$, and $\mathbf{E}$ can be written as $\mathbf{E} = \eta\mathbf{v}^T, \eta \in \mathbb{R}^4\backslash\{0\}, \mathbf{v} \in \mathbb{R}^2$ with $\|\mathbf{v}\|^2 = 2$. By (3.13) again, we have that $\mathbf{v}^T\mathrm{diag}(1,-1)\mathbf{v} = 0$. Thus $\mathbf{v} = (\pm 1, \pm 1)^T$. Therefore $\eta_2 = \pm\eta^T, \eta_4 = \pm\eta^T$ and (3.11) holds true. The proof of Theorem 3.2 is complete. ☐

4. Multiwavelets with good regularity

In this section, we construct multiwavelets with good approximation and smoothness properties based on the parametric expressions of the orthogonal filter banks provided above. For a given FIR matrix filter $\mathbf{H}$, if there exists a positive integer k and some $1 \times r$ vectors $\mathbf{y}_j, 0 \le j < k$ with $\mathbf{y}_0 \neq 0$, such that

$$\sum_{0 \le s \le j} \binom{j}{s} (i2)^{s-j}\mathbf{y}_s D^{j-s}\mathbf{H}(\ell\pi) = \delta(\ell)2^{-j}\mathbf{y}_j, \quad \ell = 0, 1, \tag{4.1}$$

for all $0 \le j < k$, we say that $\mathbf{H}$ has the **sum rules** of order k or $\mathbf{H}$ satisfies the **vanishing moment conditions** of order k. Here $D^j\mathbf{H}(\omega)$ denotes the matrix formed by the jth derivatives of the entries of $\mathbf{H}(\omega)$. For an FIR matrix filter $\mathbf{H}$, if $\mathbf{H}$ generates an orthonormal scaling function Φ, then Φ has accuracy of order k if and only if $\mathbf{H}$ has the sum rules of order k (see e.g., [4], [12], [8], and [5]). For a vector $\Phi = (\phi_1, \phi_2)^T$, we say Φ has accuracy of order k provided polynomials of degree up to $k - 1$

can be reproduced by the integer shifts $\phi_1(x-k), \phi_2(x-k), k \in \mathbb{Z}$. If Φ is orthonormal, then $\mathbf{y}_0^T$ is also a right 1–eigenvector of $\mathbf{H}(0)$ (see [9]). On the other hand, $\widehat{\Phi}(0)$ is also a right 1-eigenvector of $\mathbf{H}(0)$. Thus $\mathbf{y}_0^T = \widehat{\Phi}(0)$ (up to a nonzero constant). Therefore to construct a balanced multiwavelet, the corresponding $\mathbf{y}_0$ must be the vector $(1,1)$.

Example 4.1. Let ${}_1\mathbf{H} = (h_{ij}), {}_1\mathbf{G} = (g_{ij})$ be the orthogonal filters given by (2.8) with $\gamma = 1$ and the choice $+$ in $\mathbf{B}_{\theta_1}$. In this case h_{ij}, g_{ij} are given by

$$\begin{aligned}
h_{11}(z) &= \alpha_0 \cos\theta_0(1+z^{-1})/2, \quad h_{12}(z) = \sqrt{2}\sin\theta_0/2,\\
h_{21}(z) &= -\sin\theta_1(\alpha_0\sin\theta_0 + \alpha_2 + (\alpha_0\sin\theta_0 - \alpha_2)z^{-1} + (\alpha_0\sin\theta_0 - \alpha_2)z^{-2} + (\alpha_0\sin\theta_0 + \alpha_2)z^{-3})/4,\\
h_{22}(z) &= \sqrt{2}(\cos\theta_0\sin\theta_1 + 2\cos\theta_1\alpha_1 z^{-1} + \cos\theta_0\sin\theta_1 z^{-2})/4,\\
g_{11}(z) &= -\cos\theta_1(\alpha_0\sin\theta_0 + \alpha_2 + (\alpha_0\sin\theta_0 - \alpha_2)z^{-1} + (\alpha_0\sin\theta_0 - \alpha_2)z^{-2} + (\alpha_0\sin\theta_0 + \alpha_2)z^{-3})/4,\\
g_{12}(z) &= \sqrt{2}(\cos\theta_0\cos\theta_1 - 2\alpha_1\sin\theta_1 z^{-1} + \cos\theta_0\cos\theta_1 z^{-2})/4,\\
g_{21}(z) &= (\alpha_0\sin\theta_0 + \alpha_2 + (\alpha_0\sin\theta_0 - \alpha_2)z^{-1} - (\alpha_0\sin\theta_0 - \alpha_2)z^{-2} - (\alpha_0\sin\theta_0 + \alpha_2)z^{-3})/4,\\
g_{22}(z) &= \sqrt{2}\cos\theta_0(-1+z^{-2})/4.
\end{aligned}$$

For the choices of $\alpha_0 = \alpha_1 = 1, \alpha_2 = -1$, and $\theta_0 := \arcsin(4/5), \theta_1 := -\pi/4$, the corresponding Φ, Ψ are in $W^{1.5-\epsilon}(\mathbb{R})$ for any $\epsilon > 0$ and Φ has accuracy of order 2 (see [7]). Φ is the scaling function constructed in [2] and $\mathrm{diag}(-1,1)\Psi$ is the multiwavelet constructed in [1]. The supports of ϕ_1 and ϕ_2 are on $[0,1]$ and $[0,2]$ respectively (see [15] about the discussion on the supports of vector scaling functions). They are the most smooth scaling functions supported on $[0,1]$ and $[0,2]$ respectively. In this case $\mathbf{y}_0 = (\sqrt{2},1)$. If $\mathbf{y}_0 = (1,1)$, then the corresponding scaling functions constructed based on the above parametric expression are also supported on $[0,1]$ and $[0,2]$ respectively. However Φ has accuracy of order 1 and its smoothness is very poor. Thus we cannot construct balanced scaling functions ϕ_1, ϕ_2 which are supported on $[0,1]$ and $[0,2]$ respectively and have good regularities. $\square$

Let ${}_\gamma\mathbf{H}$ be the filter given by (2.8). There are $\gamma+1$ free parameters for ${}_\gamma\mathbf{H}$. However ${}_\gamma\mathbf{H}$ does not satisfy the sum rules of order 1. In order that ${}_\gamma\mathbf{H}$ satisfy the sum rules of order 1, we need to solve some equations and reduce some parameters. For example, for $\gamma = 2, 3$, there is only one free parameter left if ${}_2\mathbf{H}$, ${}_3\mathbf{H}$ have the sum rule of order 1 with $\mathbf{y}_0 = (1,1)$, and we cannot construct smooth balanced multiwavelets based on ${}_2\mathbf{H}$ and ${}_3\mathbf{H}$. Suppose ${}_\gamma\mathbf{H}, {}_\gamma\mathbf{G}$ are the orthogonal filters defined by (3.10). Then ${}_\gamma\mathbf{H}(0), {}_\gamma\mathbf{H}(\pi)$ are determined by $\mathbf{w}_0$ and it is easy to verify that ${}_\gamma\mathbf{H}$ has the sum rules of order 1 if and only if $\mathbf{w}_0$ is $\mathbf{r}_{-\frac{\pi}{4}}$ or $-\mathbf{r}_{\frac{3\pi}{4}}\mathbf{D}_0$ and $\mathbf{y}_0 = (1,1)$. Thus there are γ free parameters for ${}_\gamma\mathbf{H}$ with the sum rules of order 1. We find it is easier to construct balanced multiwavelets with high accuracy and good smoothness if we use these filters. In the following we focus on the construction of balanced multiwavelets based on the parametric expression given by (3.10).

Example 4.2. Let ${}_2\mathbf{H}, {}_2\mathbf{G}$ be the orthogonal filters defined by (3.10) with $\gamma = 2$. Then the matrix coefficients $\mathbf{h}_0, \cdots, \mathbf{h}_5, \mathbf{g}_0, \cdots, \mathbf{g}_5$ of ${}_2\mathbf{H}, {}_2\mathbf{G}$ have the forms of (1.9) and (3.4). Here we choose $\mathbf{w}_0$ to be $\mathbf{r}_{-\frac{\pi}{4}}$, $\mathbf{v}_0$ and $\mathbf{u}_1$ to be $\mathbf{r}_\beta$ and $\mathbf{r}_{\theta_1}$ respectively, and $\pm$ in $\mathbf{B}_0$ to be $+$. In this case a_j, b_j, c_j, d_j in (1.9) and

(3.4) are given by

$$
\begin{aligned}
a_0 &= \sqrt{2}(\sqrt{2} - \cos(\beta+\theta_1) + \sin(\beta+\theta_1))/8, \quad a_1 = -\sqrt{2}(\sin(\beta+\theta_1) + \cos(\beta+\theta_1))/8,\\
a_2 &= \sqrt{2}(\sin(\beta+\theta_1) + \cos(\beta+\theta_1))/8, \quad a_3 = \sqrt{2}(\cos(\beta+\theta_1) - \sin(\beta+\theta_1) + \sqrt{2})/8,\\
b_0 &= \sqrt{2}(\cos(\beta+\theta_1) + \cos\beta + \sin\beta + \sin(\beta+\theta_1) + \sqrt{2}\sin\theta_1/16,\\
b_1 &= \sqrt{2}(-\cos\beta + \sin\beta + \sqrt{2} + \sin(\beta+\theta_1) + \sqrt{2}\cos\theta_1 - \cos(\beta+\theta_1))/16,\\
b_2 &= \sqrt{2}(\sqrt{2}\cos\theta_1 - \sin(\beta+\theta_1) + \sqrt{2} - \sin\beta + \cos\beta + \cos(\beta+\theta_1))/16,\\
b_3 &= \sqrt{2}(-\sin(\beta+\theta_1) + \sqrt{2}\sin\theta_1 - \cos\beta - \sin\beta - \cos(\beta+\theta_1))/16,\\
b_4 &= -\sin\theta_1/4, \quad b_5 = (1-\cos\theta_1)/4,\\
c_0 &= \sqrt{2}(\sqrt{2} - \cos(\beta+\theta_1) + \sin(\beta+\theta_1))/8, \quad c_1 = \sqrt{2}(\sin(\beta+\theta_1) + \cos(\beta+\theta_1))/8,\\
c_2 &= \sqrt{2}(\sin(\beta+\theta_1) + \cos(\beta+\theta_1))/8, \quad c_3 = -\sqrt{2}(\cos(\beta+\theta_1) - \sin(\beta+\theta_1) + \sqrt{2})/8,\\
d_0 &= -\sqrt{2}(\cos(\beta+\theta_1) + \cos\beta + \sin\beta + \sin(\beta+\theta_1) + \sqrt{2}\sin\theta_1)/16,\\
d_1 &= \sqrt{2}(\cos\beta - \sin\beta - \sqrt{2} - \sin(\beta+\theta_1) - \sqrt{2}\cos\theta_1 + \cos(\beta+\theta_1))/16,\\
d_2 &= -\sqrt{2}(\sqrt{2}\cos\theta_1 - \sin(\beta+\theta_1) + \sqrt{2} - \sin\beta + \cos\beta + \cos(\beta+\theta_1))/16,\\
d_3 &= \sqrt{2}(\sin(\beta+\theta_1) - \sqrt{2}\sin\theta_1 + \cos\beta + \sin\beta + \cos(\beta+\theta_1))/16,\\
d_4 &= \sqrt{2}(\sin(\beta+\theta_1) + \cos(\beta+\theta_1) - \sin\beta - \cos\beta)/8,\\
d_5 &= \sqrt{2}(\sin(\beta+\theta_1) - \cos(\beta+\theta_1) + \cos\beta - \sin\beta)/8.
\end{aligned}
$$

For the choices of

$$\beta = \pi - \arctan\left(\frac{6+\sqrt{151}}{-3+2\sqrt{151}}\right), \quad \theta_1 = -\pi - \arctan\left(\frac{-19+\sqrt{151}}{19+\sqrt{151}}\right),$$

the corresponding $\Phi, \Psi \in W^{1.53797}(\mathbb{R})$ and Φ has accuracy of order 2 with $\mathbf{y}_0 = (1,1), \mathbf{y}_1 = \frac{1}{4}(7,9)$. Here and in the following we use the smoothness estimate for scaling functions provided in [7]. $\square$

Example 4.3. Let ${}_3\mathbf{H}, {}_3\mathbf{G}$ be the orthogonal filters defined by (3.10) with $\gamma = 3$. Here we choose $\mathbf{w}_0$ to be $-\mathbf{r}_{\frac{3\pi}{4}}\mathbf{D}_0$, $\mathbf{v}_0, \mathbf{u}_1, \mathbf{u}_2$ to be $\mathbf{r}_\beta, -\mathbf{r}_{\theta_1}\mathbf{D}_0, \mathbf{r}_{\theta_2}$ respectively, and $\pm$ in $\mathbf{B}_0$ to be $+$. We refrain from providing the parameter expressions of the matrix coefficients $\mathbf{h}_0, \cdots, \mathbf{h}_7, \mathbf{g}_0, \cdots, \mathbf{g}_7$ of ${}_3\mathbf{H}, {}_3\mathbf{G}$ here. For the choices of

$$\beta = \sqrt{2}(502+3f)/1000, \quad \theta_1 = 217 - 7f/1600, \quad \theta_2 = 217 + 7f/1600,$$

where $f := \sqrt{4111}$, the resulting Φ, Ψ are in $W^{2.09532}(\mathbb{R})$ and Φ has accuracy of order 3 with $\mathbf{y}_0 = (1,1), \mathbf{y}_1 = \frac{1}{4}(11,13), \mathbf{y}_2 = \frac{1}{16}(121,169)$. (See Φ, Ψ in Figure 1.) The corresponding matrix coefficients

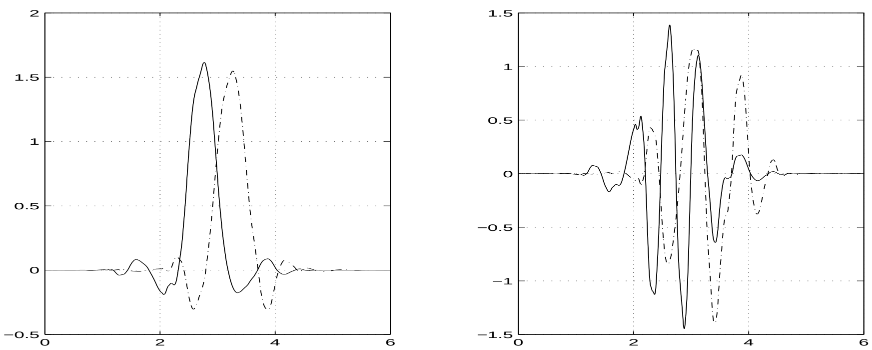

FIGURE 1. Scaling function Φ (on the left) and the balanced multiwavelet Ψ (on the right) with $\Phi, \Psi \in W^{2.09532}(\mathbb{R})$.

are given by (1.9) and (3.4) with

$$
\begin{aligned}
&[a_0, a_1, a_2, a_3, a_4, a_5] = 2^{-4}10^{-3}(-59 - f, 247 + 3f, -87 - 3f, -1061 + f, 1120, 7840),\\
&[c_0, c_1, c_2, c_3, c_4, c_5] = 2^{-9}10^{-3}(f^2 + 59f, -3f^2 - 247f, 3f^2 + 87f, 1061f - f^2,\\
&\qquad\qquad -72478 - 1302f, 217434 - 434f),\\
&[b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7] = 2^{-10}10^{-3}(413 + 7f, -1729 - 21f, 609 + 21f, 7427 - 7f,\\
&\qquad\qquad 6473 + 107f, -81309 - 321f, 82429 + 321f, 497687 - 107f),\\
&[d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7] = 2^{-10}10^{-3}(-413 - 7f, 1729 + 21f, -609 - 21f, 7f - 7427,\\
&\qquad\qquad 45565 + 775f, -136545 - 2325f, 2325f - 5695, 438115 - 775f). \quad \square
\end{aligned}
$$

Example 4.4. Let ${}_4\mathbf{H}, {}_4\mathbf{G}$ be the orthogonal filters defined by (3.10) with $\gamma = 4$. For the choices of $+$ in $\mathbf{B}_0$, $\mathbf{w}_0 = \mathbf{r}_{-\frac{\pi}{4}}$, $\mathbf{v}_0 = \mathbf{r}_\beta, \mathbf{u}_1 = \mathbf{r}_{\theta_1}, \mathbf{u}_2 = -\mathbf{r}_{\theta_2}\mathbf{D}_0$, $\mathbf{u}_3 = -\mathbf{r}_{\theta_3}\mathbf{D}_0$ with

$$\beta = -\pi/5, \theta_1 = .344651483483599, \theta_2 = .811516467802877, \theta_3 = -.115691736305177,$$

the resulting Φ, Ψ are in $W^{2.35834}(\mathbb{R})$ and Φ has accuracy of order 3 with $\mathbf{y}_0 = (1,1), \mathbf{y}_1 = \frac{1}{4}(15, 17), \mathbf{y}_2 = \frac{1}{16}(225, 289)$. (See Φ, Ψ in Figure 2.) We can also construct other balanced multiwavelets with high accuracy and good smoothness if we use other choices of $\pm$ in $\mathbf{B}_0$ or in $\mathbf{v}_0, \mathbf{u}_j$. Here we do not give the details. $\square$

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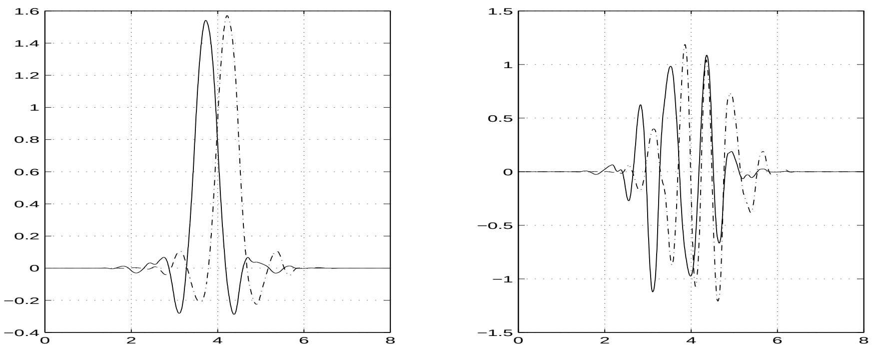

FIGURE 2. Scaling function Φ (on the left) and the balanced multiwavelet Ψ (on the right) with $\Phi, \Psi \in W^{2.35834}(\mathbb{R})$.

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