PARAMETERIZATION OF M–CHANNEL ORTHOGONAL MULTIFILTER BANKS

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Abstract. A complete parameterization for the m–channel FIR orthogonal multifilter banks is provided based on the lattice structure of the paraunitary systems. Two forms of complete factorization of the m–channel FIR orthogonal multifilter banks for symmetric/antisymmetric scaling functions and multiwavelets with the same symmetric center $\frac{1}{2}(1+\gamma+\frac{\gamma}{m-1})$ for some nonnegative integer γ are obtained. For the case of multiplicity 2 and dilation factor $m=2$, the result of the factorization shows that if the scaling function Φ and multiwavelet Ψ are symmetric/antisymmetric about the same symmetric center $\gamma+\frac{1}{2}$ for some nonnegative integer γ, then one of the component of Φ (respectively Ψ) is symmetric and the other is antisymmetric. Two examples of the construction of symmetric/antisymmetric orthogonal multiwavelets of multiplicity 3 with dilation factor 2 and multiplicity 2 with dilation factor 3 are presented to demonstrate the use of these parameterizations of orthogonal multifilter banks.

1. Introduction

For $m \in \mathbb{Z}_+$, $m \geq 2$, Ψ_0 is an r–dimensional column vector function satisfying

$$\Psi_0(x) = m \sum_{k\in\mathbb{Z}} \mathbf{h}_0(k)\Psi_0(mx-k), \tag{1.1}$$

or equivalently

$$\widehat{\Psi}_0(\omega) = \mathbf{H}_0(\omega/m)\widehat{\Psi}_0(\omega/m),$$

where $\mathbf{h}_0(k)$ are $r\times r$ real matrices and $\mathbf{H}_0(\omega)=\sum_k \mathbf{h}_0(k)e^{-ik\omega}$. The vector Ψ_0 is called an $(m,\mathbf{H}_0)$ refinable vector (matrix refinable function). A compactly supported refinable vector $\Psi_0=(\psi_{j,0})_{j=1}^r$ is called an *orthogonal scaling function* if the integer shifts $\psi_{j,0}(\cdot-k), 1\leq j\leq r, k\in\mathbb{Z}$, form an orthonormal basis of their closed linear span in $L^2(\mathbb{R})$. A set of r–dimensional column vector functions $\Psi_\ell=(\psi_{j,\ell})_{j=1}^r$, $1\leq \ell < m$, is called a set of *multiwavelets of dilation factor* m if $\psi_{j,\ell}(m^d x-k), 1\leq j\leq r, 1\leq \ell < m, d,k\in\mathbb{Z}$, constitute an orthonormal basis of $L^2(\mathbb{R})$. If vector-valued functions $\Psi_\ell(x), 1\leq\ell<m$ are defined by

$$\Psi_\ell(x) = m \sum_{k\in\mathbb{Z}} \mathbf{h}_\ell(k)\Psi_0(mx-k), \tag{1.2}$$

or equivalently by

$$\widehat{\Psi}_\ell(\omega) = \mathbf{H}_\ell(\omega/m)\widehat{\Psi}_0(\omega/m),$$

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for some matrix filters $\mathbf{H}_\ell$, where Ψ_0 is $(m, \mathbf{H}_0)$ refinable vector, then a necessary condition for Ψ_0 to be an orthogonal scaling function and $\Psi_\ell, 0 \le \ell < m$ to be a set of orthogonal multiwavelets is that (see e.g., [8], [24] and [7])

$$\sum_{k=0}^{m-1} \mathbf{H}_\ell(\frac{\omega+2k\pi}{m})\mathbf{H}^*_{\ell'}(\frac{\omega+2k\pi}{m}) = \delta_{\ell-\ell'}\mathbf{I}_r, \quad 0 \le \ell, \ell' < m. \tag{1.3}$$

Throughout this paper, $\mathbf{B}^*$ and $\mathbf{B}^T$ denote respectively the Hermitian adjoint and transpose of a matrix $\mathbf{B}$, $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, and δ is the Dirac sequence such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Conversely, if $\mathbf{H}_\ell, 0 \le \ell < m$, are FIR and satisfy (1.3), and the compactly supported $(m, \mathbf{H}_0)$ refinable function Ψ_0 is L^2-stable, then Ψ_0 is an orthogonal scaling function and Ψ_ℓ, $1 \le \ell < m$ defined by (1.2) are a set of multiwavelets (see [5], [14]). In this case, we say that $\mathbf{H}_0$ generates an orthogonal scaling function Ψ_0 and say that $\mathcal{H}_m := \{\mathbf{H}_\ell, 0 \le \ell < m\}$ generates orthogonal multiwavelets Ψ_ℓ, $1 \le \ell < m$. The set $\mathcal{H}_m$ is called an m–channel multiwavelet filter bank (often abbreviated (m–*channel*) *multifilter bank*). A matrix filter $\mathbf{H}$ is called *causal* if $\mathbf{h}(k) = \mathbf{0}, k < 0$, and is a *finite impulse response* (*FIR*) filter if there exists a positive integer N such that $\mathbf{h}(k) = \mathbf{0}, |k| > N$. For a multifilter bank $\mathcal{H}_m$, we say it is *orthogonal* if it satisfies (1.3).

Assume that $\mathbf{H}_0$ is a causal FIR matrix filter, i.e., $\mathbf{h}_0(k) = \mathbf{0}$ if $k < 0$ or $k > N$ for some positive integer N. For positive integers N, $m \ge 2$, let $N(m)$ denote the largest integer smaller than $N/(m-1)$. Let $\mathcal{T}_{\mathbf{H}_0}$ be the matrix defined by

$$\mathcal{T}_{\mathbf{H}_0} := (m\mathcal{A}_{mi-j})_{-N(m)\le i,j \le N(m)}, \tag{1.4}$$

where $\mathcal{A}_j$ is the $r^2 \times r^2$ matrix defined by

$$\mathcal{A}_j := \sum_{\kappa=0}^{N} \mathbf{h}_0(\kappa - j) \otimes \mathbf{h}_0(\kappa),$$

and $\mathbf{h}_0(\kappa - j) \otimes \mathbf{h}_0(\kappa)$ denotes the Kronecker product of $\mathbf{h}_0(\kappa - j), \mathbf{h}_0(\kappa)$. $\mathcal{T}_{\mathbf{H}_0}$ is the representing matrix of the transition operator $\mathbf{T}_{\mathbf{H}_0}$ associated to $\mathbf{H}_0$. It is known that $\mathbf{H}_0$ generates an orthogonal scaling function if and only if $\sum_{k=0}^{m-1} \mathbf{H}_0(\frac{\omega+2k\pi}{m})\mathbf{H}_0^*(\frac{\omega+2k\pi}{m}) = \mathbf{I}_r$, $\mathbf{H}_0(0)$ and $\mathcal{T}_{\mathbf{H}_0}$ satisfy Condition E (see [24], [5], [22], [16]). A matrix $\mathbf{B}$ is said to satisfy *Condition E* if the spectral radius of $\mathbf{B}$ is 1, 1 is the only eigenvalue on the unit circle and 1 is simple. Therefore to construct orthogonal multiwavelets, we need only to find orthogonal multifilter banks $\mathcal{H}_m$ with $\mathbf{H}_0(0)$ and $\mathcal{T}_{\mathbf{H}_0}$ satisfying Condition E. In some cases, in the construction of orthogonal multiwavelets with some desired properties, the parameterization of the multifilter banks is required, see [17], [18]. This paper discusses such parameterizations.

This paper is organized as follows. In §2, we discuss the complete factorization of the m–channel FIR orthogonal multifilter banks and the symmetry of the scaling function and multiwavelets. In §3, we present complete factorizations of the m–channel FIR orthogonal multifilter banks for symmetric/antisymmetric scaling functions and multiwavelets with the same symmetric center for the even m case, while in §4 we provide complete factorizations for the odd m case. The complete factorizations for the case $m = 2, r = 2$ are discussed in more detail in §5. In the last part of this paper, §6, symmetric/antisymmetric multiwavelets of multiplicity 3 with dilation factor 2 supported in [0, 3], and symmetric/antisymmetric multiwavelets of multiplicity 2 with dilation factor 3 supported in [0, 2.5] are constructed in two examples based on the parameterizations of the symmetric orthogonal multifilter banks provided in this paper.

2. Preliminaries

Let $\mathbb{Z}_+$ denote the set of all nonnegative integers. Denote $z := e^{i\omega}$, and for $m \geq 2$, denote $W := e^{-2\pi i/m}$. By the standard abuse of notation, for an FIR matrix filter $\mathbf{H}$, we let $\mathbf{H}(z) = \sum_{k\in\mathbb{Z}} \mathbf{h}(k)z^{-k}$ denote $\mathbf{H}(\omega)$. For an m–channel causal FIR multifilter bank $\mathcal{H}_m = \{\mathbf{H}_\ell, 0 \leq \ell < m\}$, let $\mathbf{P}_m$ denote its *modulation matrix* defined by

$$\mathbf{P}_m(\omega) := \left[\mathbf{H}_\ell(zW^k)\right]_{0\leq \ell,k<m}. \tag{2.1}$$

Then $\mathcal{H}_m$ being orthogonal is equivalent to that $\mathbf{P}_m(\omega)$ is *paraunitary* (or *lossless*), i.e., $\mathbf{P}_m(\omega)$ is unitary for all $\omega \in [-\pi, \pi)$

$$\mathbf{P}_m(\omega)\mathbf{P}_m^*(\omega) = \mathbf{I}_{rm}. \tag{2.2}$$

Write

$$\mathbf{H}_\ell(z) = \sum_{k=0}^{m-1} z^{-k}\mathbf{H}_{(\ell,k)}(z^m), \quad \mathbf{h}_{(\ell,k)}(n) = \mathbf{h}_\ell(mn+k), \quad 0 \leq \ell < m.$$

Then the *polyphase matrix* $\mathbf{E}_p(z)$ of $\mathcal{H}_m$ is defined by

$$\mathbf{E}_p(z) := \left[\mathbf{H}_{(\ell,k)}(z)\right]_{0\leq \ell,k<m}.$$

The relationship of the modulation and polyphase matrices of $\mathcal{H}_m$ is given by

$$\mathbf{P}_m(\omega) = \sqrt{m}\mathbf{E}_p(z^m)\mathbf{U}_m(z), \quad z = e^{i\omega},$$

where $\mathbf{U}_m(z)$ is the rm by rm paraunitary matrix defined by

$$\mathbf{U}_m(z) := \frac{\sqrt{m}}{m}\left[(zW^k)^{-\ell}\mathbf{I}_r\right]_{0\leq \ell,k<m}.$$

The fact that $\mathbf{U}_m(z)\mathbf{U}_m^*(z) = \mathbf{I}_{rm}$ implies that $\mathbf{P}_m(\omega)\mathbf{P}_m^*(\omega) = \mathbf{I}_{rm}$ if and only if

$$(\sqrt{m}\mathbf{E}_p(z))(\sqrt{m}\mathbf{E}_p(z))^* = \mathbf{I}_{rm}.$$

Thus a causal FIR multifilter bank $\mathcal{H}_m$ is orthogonal if and only if $\sqrt{m}\mathbf{E}_p(z)$ is paraunitary. If $\sqrt{m}\mathbf{E}_p(z)$ is paraunitary, then it can be factored as (see e.g., [27], [29] and references therein)

$$\mathbf{E}_p(z) = \frac{\sqrt{m}}{m}\mathbf{V}_\gamma(z)\mathbf{V}_{\gamma-1}(z)\cdots\mathbf{V}_1(z)\mathbf{U}_0, \tag{2.3}$$

where $z = e^{i\omega}$, $\mathbf{U}_0 \in O(mr)$ and

$$\mathbf{V}_k(z) := \mathbf{P}_k + (\mathbf{I}_{rm} - \mathbf{P}_k)z^{-1}, \quad \text{for some projection matrix } \mathbf{P}_k. \tag{2.4}$$

In this paper for a positive integer n, $O(n)$ denotes the set consisting of all $n \times n$ orthogonal matrices. If $\text{rank}(\mathbf{P}_k) = 1$, then γ is the (McMillan) degree of $\sqrt{m}\mathbf{E}_p(z)$. By definition, the orthogonal multifilter bank $\mathcal{H}_m$ is factorized as

$$\begin{bmatrix} \mathbf{H}_0(\omega) \\ \mathbf{H}_1(\omega) \\ \cdots \\ \mathbf{H}_{m-1}(\omega) \end{bmatrix} = \frac{\sqrt{m}}{m}\mathbf{V}_\gamma(z^m)\mathbf{V}_{\gamma-1}(z^m)\cdots\mathbf{V}_1(z^m)\mathbf{U}_0 \begin{bmatrix} \mathbf{I}_r \\ z^{-1}\mathbf{I}_r \\ \cdots \\ z^{1-m}\mathbf{I}_r \end{bmatrix}, \quad z = e^{i\omega}. \tag{2.5}$$

Theorem 2.1. *A causal FIR multifilter bank $\mathcal{H}_m$ is orthogonal if and only if there exists a $\gamma \in \mathbb{Z}_+$, projection matrices $\mathbf{P}_k, 0 \le k \le \gamma$, and an orthogonal matrix $\mathbf{U}_0$ such that $\mathcal{H}_m$ is given by* (2.5).

It was shown in [4] (see also [19], [16]) that if the compactly supported $(m, \mathbf{H}_0)$ refinable vector φ is L^2-stable, then $\mathbf{H}_0(0)$ satisfies Condition E and $\mathbf{H}_0$ satisfies the vanishing moment conditions of order at least 1. In [18], for the case $m = 2$, a simpler parameterization of the orthogonal multifilter banks was provided with these properties of $\mathbf{H}_0$ taken into account. As for the higher dimensional case, one can get some expressions of the orthogonal filter banks similar to (2.5). However the expressions will not be complete.

Based on the parameterization of the multifilter banks, we can construct the scaling functions and multiwavelets with desired properties. In this paper we consider the symmetry of the scaling functions and multiwavelets and discuss the parameterization of the FIR orthogonal multifilter banks which generate symmetric/antisymmetric scaling functions and multiwavelets. To this end, we first have the following two lemmas about the symmetry of the refinable functions and multiwavelets.

Lemma 2.1. *Assume that $\mathbf{P}$ is an FIR matrix filter and $\varphi = (\varphi_1, \cdots, \varphi_r)^T$ is a compactly supported $(m, \mathbf{P})$ refinable vector with $\widehat{\varphi}(0) \ne 0$. If $\mathbf{P}$ satisfies*

$$\mathbf{D_c}(m\omega)\mathbf{P}(-\omega)\mathbf{D_c}(-\omega) = \mathbf{P}(\omega), \tag{2.6}$$

for some $\mathbf{c} = (c_1, \cdots, c_r) \in \mathbb{R}^r$, then φ_j is symmetric/antisymmetric about $\frac{c_j}{2(m-1)}$, i.e.,

$$\varphi_j(\frac{c_j}{m-1} - x) = \pm\varphi_j(x), \quad 1 \le j \le r, \tag{2.7}$$

where for a vector $\mathbf{c} = (c_1, \cdots, c_r) \in \mathbb{R}^r$,

$$\mathbf{D_c}(\omega) := diag(\pm e^{-ic_1\omega/(m-1)}, \pm e^{-ic_2\omega/(m-1)}, \cdots, \pm e^{-ic_r\omega/(m-1)}). \tag{2.8}$$

Conversely, if φ is L^2–stable and $mc_j = c_i mod(m-1)$, $1 \le i, j \le r$, then (2.7) implies (2.6).

Lemma 2.2. *Assume that $\mathbf{P}$ is an FIR matrix filter satisfying (2.6) for some $\mathbf{c} = (c_1, \cdots, c_r) \in \mathbb{R}^r$, and φ is a compactly supported $(m, \mathbf{P})$ refinable vector with $\widehat{\varphi}(0) \ne 0$. Let $\Psi = (\psi_1, \cdots, \psi_r)^T$ be a vector-valued function defined by*

$$\Psi(x) = m\sum_k \mathbf{q}(k)\varphi(mx - k)$$

for some FIR matrix filter $\mathbf{Q}$. If $\mathbf{Q}$ satisfies

$$\mathbf{D_d}(m\omega)\mathbf{Q}(-\omega)\mathbf{D_c}(-\omega) = \mathbf{Q}(\omega) \tag{2.9}$$

for some $\mathbf{d} = (d_1, \cdots, d_r) \in \mathbb{R}^r$, *then*

$$\psi_j(\frac{d_j}{m-1} - x) = \pm\psi_j(x), \quad 1 \le j \le r. \tag{2.10}$$

Conversely, if ϕ *is* L^2*-stable and* $md_j = c_i \bmod(m-1)$, $1 \le i, j \le r$, *then (2.10) implies (2.9).*

In Lemmas 2.1, 2.2, the sign $+$ (or $-$) in $\mathbf{D_c}$ and $\mathbf{D_d}$ coincide with $+$ (or $-$) in (2.7) and (2.10) respectively. For FIR matrix filters $\mathbf{P}, \mathbf{Q}$, (2.6) and (2.9) also imply that $mc_j = c_i \bmod(m-1)$ and $md_j = c_i \bmod(m-1)$ respectively.

The symmetry of the scaling functions and multiwavelets was also considered in [2] and [28]. Comparing with their results, more conditions such as $mc_j = c_i \bmod(m-1)$ are added here. We find in some cases such conditions cannot be dropped. In the following we give the proof of Lemma 2.1. The proof of Lemma 2.2 is similar, and it is omitted here.

Proof of Lemma 2.1. If $\mathbf{P}$ satisfies (2.6), $\mathbf{D_c}(0)\mathbf{P}(0)\mathbf{D_c}(0)\widehat{\varphi}(0) = \mathbf{P}(0)\widehat{\varphi}(0) = \widehat{\varphi}(0)$. Thus $\mathbf{D_c}(0)\widehat{\varphi}(0)$ is also a right 1-eigenvector of $\mathbf{P}(0)$. Hence $\mathbf{D_c}(0)\widehat{\varphi}(0) = c_0\widehat{\varphi}(0)$ for nonzero constant c_0. Thus

$$\widehat{\varphi}(\omega) = \mathbf{D_c}(\omega)\Pi_{j=0}^{\infty}\mathbf{P}(-\omega/m^j)\mathbf{D_c}(0)\widehat{\varphi}(0) = c_0\mathbf{D_c}(\omega)\Pi_{j=0}^{\infty}\mathbf{P}(-\omega/m^j)\widehat{\varphi}(0) = c_0\mathbf{D_c}(\omega)\widehat{\varphi}(-\omega).$$

Since the diagonal elements of $\mathbf{D_c}(0)$ are ± 1, $c_0 = \pm 1$. Thus φ has the symmetric property. Note that if the jth component φ_j of φ is antisymmetric, then $\widehat{\varphi}_j(0) = 0$. Thus $\widehat{\varphi}(0) = c_0\mathbf{D_c}(0)\widehat{\varphi}(0) = c_0\mathbf{I}_r\widehat{\varphi}(0) = c_0\widehat{\varphi}(0)$. That is $c_0 = 1$. Therefore $\widehat{\varphi}(\omega) = \mathbf{D_c}(\omega)\widehat{\varphi}(-\omega)$, or equivalently φ satisfies (2.7).

Conversely, if φ satisfies (2.7), then $\widehat{\varphi}(\omega) = \mathbf{D_c}(\omega)\widehat{\varphi}(-\omega)$. Thus

$$\begin{aligned}\mathbf{P}(\omega)\widehat{\varphi}(\omega) &= \widehat{\varphi}(m\omega) = \mathbf{D_c}(m\omega)\widehat{\varphi}(-m\omega)\\ &= \mathbf{D_c}(m\omega)\mathbf{P}(-\omega)\widehat{\varphi}(-\omega) = \mathbf{D_c}(m\omega)\mathbf{P}(-\omega)\mathbf{D_c}(-\omega)\widehat{\varphi}(\omega).\end{aligned}$$

Denote

$$\mathbf{L}(\omega) := \mathbf{P}(\omega) - \mathbf{D_c}(m\omega)\mathbf{P}(-\omega)\mathbf{D_c}(-\omega).$$

Then $\mathbf{L}(\omega)\widehat{\varphi}(\omega) = \mathbf{0}$. Note that if $mc_j = c_i \bmod(m-1)$, $1 \le i, j, \le r$, then $\mathbf{L}(\omega + 2k\pi) = \mathbf{L}(\omega)$ for any $k \in \mathbb{Z}$. Thus we have

$$\mathbf{L}(\omega)G_\varphi(\omega)\mathbf{L}(\omega)^* = \mathbf{0}, \quad \omega \in [0, 2\pi),$$

where $G_\varphi(\omega) := \sum_{k\in\mathbb{Z}}\widehat{\varphi}(\omega + 2k\pi)\widehat{\varphi}(\omega + 2k\pi)^*$. Since φ is L^2–stable, $G_\varphi(\omega) > 0$. Thus each row of $\mathbf{L}(\omega)$ is the zero vector. Hence $\mathbf{L}(\omega) = \mathbf{0}$. □

In the rest of this section, we discuss the number of the symmetry of components of the symmetric scaling function and multiwavelets. Assume that $\mathcal{H}_m = \{\mathbf{H}_\ell, 0 \le \ell < m\}$ is an orthogonal causal FIR multifilter bank generating scaling functions Ψ_0 and orthogonal multiwavelets $\Psi_\ell, 1 \le \ell < m$. Suppose $\mathbf{h}_\ell(k) = \mathbf{0}$ if $k \notin [0, (\gamma+1)m - 1]$ for some $\gamma \in \mathbb{Z}_+\backslash\{0\}$. Then $\Psi_\ell, 0 \le \ell < m$ are supported in $[0, 1 + \gamma + \frac{\gamma}{m-1}]$ (see [16]). By Lemmas 2.1, 2.2, $\Psi_\ell, 0 \le \ell < m$ are symmetric/antisymmetric about $\frac{1}{2}(1 + \gamma + \frac{\gamma}{m-1})$ if and only if $\mathbf{H}_\ell$ satisfy

$$z^{-((\gamma+1)m-1)}\mathbf{D}_\ell\mathbf{H}_\ell(z^{-1})\mathbf{D}_0 = \mathbf{H}_\ell(z), \quad 0 \le \ell < m, z = e^{i\omega}, \omega \in [-\pi, \pi), \tag{2.11}$$

or equivalently

$$\mathbf{D}_\ell\mathbf{h}_\ell((\gamma+1)m - 1 - k)\mathbf{D}_0 = \mathbf{h}_\ell(k), \quad 0 \le k \le (\gamma+1)m - 1, 0 \le \ell < m, \tag{2.12}$$

where $\mathbf{D}_\ell, 0 \le \ell < m$ are $r \times r$ diagonal matrices with diagonal elements 1 or -1. Let $\mathbf{E}_p(z) = \left[\mathbf{H}_{(\ell,k)}(z)\right]_{0\le \ell,k<m}$ denote the polyphase matrix of $\mathcal{H}_m$. If $\mathbf{H}_\ell$ satisfies (2.11) or equivalently $\mathbf{h}_\ell$ satisfies (2.12), then

$$\begin{aligned}\mathbf{H}_{(\ell,k)}(z) &= \sum_{n=0}^{\gamma} \mathbf{h}_\ell(mn+k)z^{-n} = \mathbf{D}_\ell \sum_{n=0}^{\gamma} \mathbf{h}_\ell((\gamma+1)m-1-(mn+k))z^{-n}\mathbf{D}_0 \\ &= \mathbf{D}_\ell z^{-\gamma} \sum_{n=0}^{\gamma} \mathbf{h}_\ell((\gamma-n)m+m-k-1))z^{\gamma-n}\mathbf{D}_0 = \mathbf{D}_\ell z^{-\gamma}\mathbf{H}_{(\ell,m-k-1)}(z^{-1})\mathbf{D}_0.\end{aligned}$$

Thus $\mathbf{H}_\ell, 0 \le \ell < m$, satisfy (2.11) if and only if

$$\begin{aligned}\mathbf{E}_p(z) &= z^{-\gamma}\left[\mathbf{D}_\ell \mathbf{H}_{(\ell,m-k-1)}(z)\mathbf{D}_0\right]_{0\le\ell,k<m} \\ &= z^{-\gamma}\mathrm{diag}(\mathbf{D}_0, \cdots, \mathbf{D}_{m-1})\mathbf{E}_p(z^{-1})(\mathbf{J}_m \otimes \mathbf{D}_0),\end{aligned} \tag{2.13}$$

where for a positive integer n, $\mathbf{J}_n$ denotes the $n \times n$ antidiagonal (or exchange) matrix

$$\mathbf{J}_n := \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{bmatrix}. \tag{2.14}$$

As in the scalar case (see [26]), we have the following theorem about the number of symmetry of components $\psi_{j,\ell}, 1 \le j \le r, 0 \le \ell < m$ of scaling function $\Psi_0 = (\psi_{j,0})_{j=1}^r$ and multiwavelets $\Psi_\ell = (\psi_{j,\ell})_{j=1}^r, 1 \le \ell < m$.

Theorem 2.2. *Suppose multifilter bank* $\mathbf{H}_\ell(\omega) = \sum_{k=0}^{(\gamma+1)m-1} \mathbf{h}_\ell(k)e^{-ik\omega}$, $0 \le \ell < m$ *for some* $\gamma \in \mathbb{Z}_+$ *generates symmetric/antisymmetric orthogonal scaling function and multiwavelets* $\Psi_\ell = (\psi_{j,\ell})_{j=1}^r, 0 \le \ell < m$, *then* $\mathbf{H}_\ell$ *satisfies* (2.11) *for some diagonal matrices* $\mathbf{D}_\ell$ *with diagonal elements* ± 1, $0 \le \ell < m$, *and that*

(i). if rm *is even, there are* $rm/2$ *symmetric and* $rm/2$ *antisymmetric components* $\psi_{j,\ell}$*;*

(ii). if rm *is odd and the* $(\frac{r+1}{2}, \frac{r+1}{2})-$*entry of* $\mathbf{D}_0$ *is* 1, *there are* $(rm+1)/2$ *symmetric and* $(rm-1)/2$ *antisymmetric components* $\psi_{j,\ell}$, *while if* rm *is odd and the* $(\frac{r+1}{2}, \frac{r+1}{2})-$*entry of* $\mathbf{D}_0$ *is* -1, *there are* $(rm-1)/2$ *symmetric and* $(rm+1)/2$ *antisymmetric components* $\psi_{j,\ell}$.

Proof. If $\{\mathbf{H}_\ell,\ 0 \le \ell < m\}$ generates symmetric/antisymmetric orthogonal scaling function and multiwavelets, then Lemmas 2.1, 2.2 imply that $\mathbf{H}_\ell$ satisfies (2.11). The proof of (i), (ii) can be carried out similarly as in [26]. Denote $\mathbf{D} := \mathrm{diag}(\mathbf{D}_0, \cdots, \mathbf{D}_{m-1})$. Since $\mathbf{H}_\ell, 0 \le \ell < m$ satisfy (2.11), the corresponding polyphase $\mathbf{E}_p(z)$ satisfies (2.13). Let $\mathrm{Tr}(\mathbf{D})$ denote the trance of $\mathbf{D}$. By (2.13) for $z = 1$,

$$\begin{aligned}\mathrm{Tr}(\mathbf{D}) &= \mathrm{Tr}(\mathbf{E}_p(1)(\mathbf{J}_m \otimes \mathbf{D}_0)\mathbf{E}_p(1)^{-1}) \\ &= \mathrm{Tr}(\mathbf{E}_p(1)^{-1}\mathbf{E}_p(1)(\mathbf{J}_m \otimes \mathbf{D}_0)) = \mathrm{Tr}(\mathbf{J}_m \otimes \mathbf{D}_0).\end{aligned}$$

If m is even, then clearly $\mathrm{Tr}(\mathbf{D}) = 0$. If m is odd, then $\mathrm{Tr}(\mathbf{D}) = \mathrm{Tr}(\mathbf{D}_0)$, and $\mathrm{Tr}(\mathbf{D}_0) = 0$ if r is even, $\mathrm{Tr}(\mathbf{D}_0) = (\frac{r+1}{2}, \frac{r+1}{2})-$entry of $\mathbf{D}_0$ if r is odd. These facts lead to (i), (ii). ☐

For a multifilter bank $\mathcal{H}_m$, we say it is symmetric if it satisfies (2.11) for some $\gamma \in \mathbb{Z}_+$. In the following, we discuss the parameterization of the symmetric FIR orthogonal multifilter banks for the case that rm is even. In this case, by Theorem 2.2, half of all components $\psi_{j,\ell}, 1 \le j \le r, 0 \le \ell < m$ of scaling function and multiwavelets are symmetric, and the other half are antisymmetric. Thus half of the diagonal elements of $\operatorname{diag}(\mathbf{D}_0, \cdots, \mathbf{D}_{m-1})$ are 1, and the other half are -1. Therefore there exists an $rm \times rm$ permutation matrix $\mathbf{M}_0$ such that

$$\mathbf{M}_0 \operatorname{diag}(\mathbf{D}_0, \cdots, \mathbf{D}_{m-1}) \mathbf{M}_0^T = \operatorname{diag}(\mathbf{I}_{rm/2}, -\mathbf{I}_{rm/2}). \tag{2.15}$$

The parameterizations of the symmetric FIR orthogonal multifilter banks for the even m and odd m cases are discussed in §3 and §4, respectively. The factorization is studied in more detail in §5 for the case $m = 2, r = 2$.

3. Parameterization of symmetric multifilter banks for even m

In this section, we assume that m is even, i.e., $m = 2m_1$ for some positive integer m_1. Let $\mathbf{J}_{m_1}$ denote the exchange matrix defined by (2.14), and denote $\mathbf{J}_{r,m_1} := \mathbf{J}_{m_1} \otimes \mathbf{I}_r$, the Kronecker product of $\mathbf{J}_{m_1}$ and $\mathbf{I}_r$. Assume that $\mathcal{H}_m = \{\mathbf{H}_\ell, 0 \le \ell < m\}$ is a causal FIR multifilter bank. Let $\mathbf{E}_p(z) = \left[\mathbf{H}_{(\ell,k)}(z)\right]_{0 \le \ell, k < m}$ be its polyphase matrix. Suppose $\mathbf{H}_\ell, 0 \le \ell < m$, satisfy (2.11) for some diagonal matrices $\mathbf{D}_\ell$, and let $\mathbf{M}_0$ be the $rm \times rm$ permutation matrix satisfying (2.15). Denote

$$\mathbf{E}_p^{(1)}(z) := \mathbf{M}_0 \mathbf{E}_p(z) \operatorname{diag}(\mathbf{I}_{rm_1}, \mathbf{I}_{m_1} \otimes \mathbf{D}_0),$$

then $\mathbf{E}_p(z)$ satisfies (2.13) if and only if $\mathbf{E}_p^{(1)}(z)$ satisfies

$$\mathbf{E}_p^{(1)}(z) = z^{-\gamma} \operatorname{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1}) \mathbf{E}_p^{(1)}(z^{-1}) (\mathbf{J}_m \otimes \mathbf{I}_r). \tag{3.1}$$

Let $\mathbf{B}$ be the $mr \times mr$ orthogonal matrix defined by

$$\mathbf{B} := \frac{\sqrt{2}}{2} \begin{bmatrix} \mathbf{I}_{rm_1} & \mathbf{J}_{r,m_1} \\ \mathbf{J}_{r,m_1} & -\mathbf{I}_{rm_1} \end{bmatrix}.$$

Then one has

$$\mathbf{B}(\mathbf{J}_m \otimes \mathbf{I}_r)\mathbf{B} = \operatorname{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1}).$$

Denote

$$\mathcal{E}_\gamma(z) := \mathbf{E}_p^{(1)}(z)\mathbf{B} = \mathbf{M}_0 \mathbf{E}_p(z) \operatorname{diag}(\mathbf{I}_{rm_1}, \mathbf{I}_{m_1} \otimes \mathbf{D}_0)\mathbf{B}, \tag{3.2}$$

or equivalently

$$\mathbf{E}_p(z) = \frac{\sqrt{2}}{2} \mathbf{M}_0^T \mathcal{E}_\gamma(z) \begin{bmatrix} \mathbf{I}_{rm_1} & \mathbf{J}_{m_1} \otimes \mathbf{D}_0 \\ \mathbf{J}_{r,m_1} & -\mathbf{I}_{m_1} \otimes \mathbf{D}_0 \end{bmatrix}, \tag{3.3}$$

here we have applied the mixed-product property of the Kronecker product, i.e., for matrices $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4$ (with appropriate sizes), we have (see [11])

$$(\mathbf{B}_1 \otimes \mathbf{B}_2)(\mathbf{B}_3 \otimes \mathbf{B}_4) = (\mathbf{B}_1\mathbf{B}_3) \otimes (\mathbf{B}_2\mathbf{B}_4).$$

With the relation (3.2), then one can check that $\mathbf{E}_p^{(1)}(z)$ satisfies (3.1) or equivalently $\mathbf{H}_\ell, 0 \le \ell < m$, satisfy (2.11) if and only if $\mathcal{E}_\gamma(z)$ satisfies

$$\mathcal{E}_\gamma(z) = z^{-\gamma}\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1})\mathcal{E}_\gamma(z^{-1})\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1}), \tag{3.4}$$

and $\mathcal{H}_m$ is orthogonal if and only if $\mathcal{E}_\gamma(z)$ is paraunitary. For a causal FIR paraunitary matrix $\mathcal{E}_\gamma(z)$ satisfying (3.4), we have the following factorization theorem.

Theorem 3.1. *A causal FIR matrix $\mathcal{E}_\gamma(z)$ is paraunitary and satisfies* (3.4) *if and only if it can be factorized as*

$$\mathcal{E}_\gamma(z) = \mathbf{V}_\gamma(z)\mathbf{V}_{\gamma-1}(z)\cdots\mathbf{V}_1(z)\begin{bmatrix} \mathbf{w}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_0 \end{bmatrix}, \tag{3.5}$$

where $\mathbf{w}_0, \mathbf{v}_0 \in O(rm_1)$,

$$\mathbf{V}_k(z) = \mathbf{A}_k + (\mathbf{I}_{rm} - \mathbf{A}_k)z^{-1}, \quad \mathbf{A}_k = \frac{1}{2}\begin{bmatrix} \mathbf{I}_{rm_1} & \mathbf{v}_k \\ \mathbf{v}_k^T & \mathbf{I}_{rm_1} \end{bmatrix}, \quad \mathbf{v}_k \in O(rm_1). \tag{3.6}$$

Proof. One can check that for $\mathbf{V}_k(z)$ defined by (3.6) with some $\mathbf{v}_k \in O(rm_1)$, $\mathbf{V}_k(z)\mathbf{V}_k(z^{-1})^T = \mathbf{I}_{rm}$ and $\mathbf{V}_k(z)$ satisfies (3.4). These facts imply that $\mathcal{E}_\gamma(z)$, defined by (3.5) for $\gamma \in \mathbb{Z}_+$, is causal, paraunitary and satisfies (3.4).

For the proof of the converse part, we show the above factorization by induction on the order γ as in [25]. If $\gamma = 0$, the result is evident. For $\mathcal{E}_{\gamma+1}(z)$ of order $\gamma+1$, we reduce the order by 1 to complete the proof, i.e., we show that $\mathcal{E}_{\gamma+1}(z)$ can be written as

$$\mathcal{E}_{\gamma+1}(z) = \mathbf{V}_{\gamma+1}(z)\mathcal{E}_\gamma(z),$$

where $\mathbf{V}_{\gamma+1}(z)$ is given by (3.6) with some $\mathbf{v}_{\gamma+1} \in O(rm_1)$ and $\mathcal{E}_\gamma(z)$ is a causal FIR paraunitary matrix of order γ and satisfies (3.4).

Since $\mathbf{V}_{\gamma+1}(z), \mathcal{E}_{\gamma+1}(z)$ are paraunitary and satisfy (3.4), $\mathcal{E}_\gamma(z)$ given by

$$\mathcal{E}_\gamma(z) = \mathbf{V}_{\gamma+1}(z)^{-1}\mathcal{E}_{\gamma+1}(z) = \mathbf{V}_{\gamma+1}(z^{-1})\mathcal{E}_{\gamma+1}(z), \quad z = e^{i\omega}, \tag{3.7}$$

is also paraunitary and satisfies (3.4) as well. Thus we need only to show that $\mathcal{E}_\gamma(z)$ given by (3.7) is causal. Assume that

$$\mathcal{E}_{\gamma+1}(z) = \mathbf{e}_{\gamma+1}(0) + \mathbf{e}_{\gamma+1}(1)z^{-1} + \cdots + \mathbf{e}_{\gamma+1}(\gamma+1)z^{-(\gamma+1)}, \quad \mathbf{e}_{\gamma+1}(\gamma+1) \neq \mathbf{0}, \tag{3.8}$$

and

$$\mathcal{E}_\gamma(z) = \mathbf{e}_\gamma(0) + \mathbf{e}_\gamma(1)z^{-1} + \cdots + \mathbf{e}_\gamma(\gamma)z^{-\gamma}, \quad \mathbf{e}_\gamma(\gamma) \neq \mathbf{0}.$$

By (3.7),

$$\mathcal{E}_\gamma(z) = \frac{1}{2}\begin{bmatrix} \mathbf{I}_{rm_1} & \mathbf{v}_{\gamma+1} \\ \mathbf{v}_{\gamma+1}^T & \mathbf{I}_{rm_1} \end{bmatrix}\mathcal{E}_{\gamma+1}(z) + \frac{1}{2}\begin{bmatrix} \mathbf{I}_{rm_1} & -\mathbf{v}_{\gamma+1} \\ -\mathbf{v}_{\gamma+1}^T & \mathbf{I}_{rm_1} \end{bmatrix} z\mathcal{E}_{\gamma+1}(z).$$

The second term on the right-hand side is responsible for the noncausality. In particular, the noncausal part of the second term is given by

$$\begin{bmatrix} \mathbf{I}_{rm_1} & -\mathbf{v}_{\gamma+1} \\ -\mathbf{v}_{\gamma+1}^T & \mathbf{I}_{rm_1} \end{bmatrix}\mathbf{e}_{\gamma+1}(0).$$

Thus we shall find $\mathbf{v}_{\gamma+1} \in O(rm_1)$ such that

$$[\mathbf{I}_{rm_1},\ -\mathbf{v}_{\gamma+1}]\mathbf{e}_{\gamma+1}(0) = \mathbf{0}. \tag{3.9}$$

The fact that $\mathcal{E}_{\gamma+1}(z)$ satisfies (3.4) implies

$$\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1})\mathbf{e}_{\gamma+1}(0)\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1}) = \mathbf{e}_{\gamma+1}(\gamma+1), \tag{3.10}$$

and the paraunitariness of $\mathcal{E}_{\gamma+1}(z)$ implies that

$$\mathbf{e}_{\gamma+1}(\gamma+1)^T\mathbf{e}_{\gamma+1}(0) = \mathbf{0}. \tag{3.11}$$

By (3.10) and (3.11), we have

$$\mathbf{e}_{\gamma+1}(0)^T\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1})\mathbf{e}_{\gamma+1}(0) = \mathbf{0}. \tag{3.12}$$

Since $\mathrm{rank}(\mathbf{e}_{\gamma+1}(0)) = \mathrm{rank}(\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1})\mathbf{e}_{\gamma+1}(0))$,

$$\mathrm{rank}(\mathbf{e}_{\gamma+1}(0)) = s \le rm/2 = rm_1.$$

By (3.12), for any matrix $\mathbf{B}$ (with appropriate size),

$$\mathbf{B}^T\mathbf{e}_{\gamma+1}(0)^T\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1})\mathbf{e}_{\gamma+1}(0)\mathbf{B} = \mathbf{0}. \tag{3.13}$$

Let $rm \times s$ matrix $\mathbf{B}$ be so chosen that the s columns of $\mathbf{e}_{\gamma+1}(0)\mathbf{B}$, denoted by $\mathbf{x}_i, 1 \le i \le s$, form an orthonormal basis for the columns of the matrix $\mathbf{e}_{\gamma+1}(0)$, i.e.,

$$\mathbf{e}_{\gamma+1}(0)\mathbf{B} = [\mathbf{x}_1, \cdots, \mathbf{x}_s] =: \mathbf{X}_1$$

with $\mathbf{X}_1^T\mathbf{X}_1 = \mathbf{I}_s$, and each column of $\mathbf{e}_{\gamma+1}(0)$ is a linear combinations of $\mathbf{x}_1, \cdots, \mathbf{x}_s$. Denote

$$\mathbf{X}_1 =: \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Z}_1 \end{bmatrix}, \quad \mathbf{Y}_1, \mathbf{Z}_1 \text{ are } rm_1 \times s \text{ matrices.}$$

Then by (3.13), we have

$$[\mathbf{Y}_1^T,\ \mathbf{Z}_1^T]\mathrm{diag}(\mathbf{I}_{rm_1}, -\mathbf{I}_{rm_1}) \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Z}_1 \end{bmatrix} = \mathbf{0}$$

and hence $\mathbf{Y}_1^T\mathbf{Y}_1 = \mathbf{Z}_1^T\mathbf{Z}_1$. This fact and $\mathbf{X}_1^T\mathbf{X}_1 = \mathbf{I}_s$ imply

$$\mathbf{Y}_1^T\mathbf{Y}_1 = \mathbf{Z}_1^T\mathbf{Z}_1 = \frac{1}{2}\mathbf{I}_s. \tag{3.14}$$

Therefore $\sqrt{2}\mathbf{Y}_1, \sqrt{2}\mathbf{Z}_1$ are $rm_1 \times s$ matrices of orthonormal columns. Let $\mathbf{Y}_2, \mathbf{Z}_2$ be the $rm_1 \times (rm_1 - s)$ matrices such that $\sqrt{2}[\mathbf{Y}_1,\ \mathbf{Y}_2], \sqrt{2}[\mathbf{Z}_1,\ \mathbf{Z}_2]$ are two $rm_1 \times rm_1$ orthogonal matrices. By the orthogonality, we have $[\mathbf{Y}_2^T,\ -\mathbf{Z}_2^T]\mathbf{X}_1 = \mathbf{Y}_2^T\mathbf{Y}_1 - \mathbf{Z}_2^T\mathbf{Z}_1 = \mathbf{0}$; and by (3.14), $[\mathbf{Y}_1^T,\ -\mathbf{Z}_1^T]\mathbf{X}_1 = \mathbf{Y}_1^T\mathbf{Y}_1 - \mathbf{Z}_1^T\mathbf{Z}_1 = \mathbf{0}$. Thus we have

$$\begin{bmatrix} \mathbf{Y}_1^T & -\mathbf{Z}_1^T \\ \mathbf{Y}_2^T & -\mathbf{Z}_2^T \end{bmatrix} \mathbf{x}_j = \mathbf{0}, \quad 1 \le j \le s.$$

Since each column of $\mathbf{e}_{\gamma+1}(0)$ is a linear combinations of $\mathbf{x}_j, 1 \le j \le s$, we have

$$\begin{bmatrix} \mathbf{Y}_1^T & -\mathbf{Z}_1^T \\ \mathbf{Y}_2^T & -\mathbf{Z}_2^T \end{bmatrix} \mathbf{e}_{\gamma+1}(0) = \mathbf{0}.$$

Therefore if we let $\mathbf{O}_{\gamma+1} = \sqrt{2}[\mathbf{Y}_1,\ \mathbf{Y}_2], \mathbf{W}_{\gamma+1} = \sqrt{2}[\mathbf{Z}_1,\ \mathbf{Z}_2]$, then

$$[\mathbf{O}^T_{\gamma+1},\ -\mathbf{W}^T_{\gamma+1}]\mathbf{e}_{\gamma+1}(0) = \sqrt{2}\begin{bmatrix} \mathbf{Y}_1^T & -\mathbf{Z}_1^T \\ \mathbf{Y}_2^T & -\mathbf{Z}_2^T \end{bmatrix}\mathbf{e}_{\gamma+1}(0) = \mathbf{0}.$$

Let $\mathbf{v}_{\gamma+1} = \mathbf{O}_{\gamma+1}\mathbf{W}^T_{\gamma+1}$, then $\mathbf{v}_{\gamma+1}$ is orthogonal and satisfies (3.9). The proof of the theorem is completed. □

In the next theorem we provide another form of the factorization for the causal FIR paraunitary matrix $\mathcal{E}_\gamma(z)$ which satisfies (3.4).

Theorem 3.2. *A causal FIR matrix $\mathcal{E}_\gamma(z)$ is paraunitary and satisfies* (3.4) *if and only if it can be factorized as*

$$\mathcal{E}_\gamma(z) = \mathbf{K}_\gamma\Lambda(z)\mathbf{K}_{\gamma-1}\Lambda(z)\cdots\mathbf{K}_1\Lambda(z)\,diag(\mathbf{W}_0, \mathbf{U}_0), \tag{3.15}$$

where $\mathbf{K}_j = diag(\mathbf{I}_{rm_1}, \mathbf{U}_j)$, $\mathbf{W}_0, \mathbf{U}_0, \mathbf{U}_j \in O(rm_1)$, $1 \le j \le \gamma$, and

$$\Lambda(z) = \frac{1}{2}\begin{bmatrix} \mathbf{I}_{rm/2} & \mathbf{I}_{rm/2} \\ \mathbf{I}_{rm/2} & \mathbf{I}_{rm/2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{I}_{rm/2} & -\mathbf{I}_{rm/2} \\ -\mathbf{I}_{rm/2} & \mathbf{I}_{rm/2} \end{bmatrix} z^{-1}. \tag{3.16}$$

Proof. One can check that $\Lambda(z)\Lambda(z^{-1})^T = \mathbf{I}_{rm}$ and $\Lambda(z)$ satisfies (3.4). These facts imply that if $\mathcal{E}_\gamma(z)$ is defined by (3.15) for $\gamma \in \mathbb{Z}_+$, then $\mathcal{E}_\gamma(z)$ is causal, paraunitary and satisfies (3.4). For the proof of the converse part, as the proof of Theorem 3.1, one needs to show that for any causal FIR order $\gamma + 1$ paraunitary matrix $\mathcal{E}_{\gamma+1}(z)$ which satisfies (3.4), there exists $\mathbf{U}_{\gamma+1} \in O(rm_1)$ such that $\mathcal{E}_\gamma(z)$, defined by

$$\mathcal{E}_\gamma(z) := \Lambda(z)^{-1}\mathrm{diag}(\mathbf{I}_{rm_1}, \mathbf{U}^T_{\gamma+1})\mathcal{E}_{\gamma+1}(z),$$

is causal. Assume that $\mathcal{E}_{\gamma+1}(z)$ is given by (3.8). Since

$$\Lambda(z)^{-1} = \Lambda(z^{-1}) = \frac{1}{2}\begin{bmatrix} \mathbf{I}_{rm_1} & \mathbf{I}_{rm_1} \\ \mathbf{I}_{rm_1} & \mathbf{I}_{rm_1} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{I}_{rm_1} & -\mathbf{I}_{rm_1} \\ -\mathbf{I}_{rm_1} & \mathbf{I}_{rm_1} \end{bmatrix} z, \quad z = e^{i\omega},$$

we need only to find $\mathbf{U}_{\gamma+1} \in O(rm_1)$ such that

$$\begin{bmatrix} \mathbf{I}_{rm_1} & -\mathbf{I}_{rm_1} \\ -\mathbf{I}_{rm_1} & \mathbf{I}_{rm_1} \end{bmatrix}\mathrm{diag}(\mathbf{I}_{rm_1}, \mathbf{U}^T_{\gamma+1})\mathbf{e}_{\gamma+1}(0) = \begin{bmatrix} \mathbf{I}_{rm_1} & -\mathbf{U}^T_{\gamma+1} \\ -\mathbf{I}_{rm_1} & \mathbf{U}^T_{\gamma+1} \end{bmatrix}\mathbf{e}_{\gamma+1}(0) = \mathbf{0},$$

or equivalently

$$[\mathbf{I}_{rm_1},\ -\mathbf{U}^T_{\gamma+1}]\mathbf{e}_{\gamma+1}(0) = \mathbf{0},$$

which has been shown in the proof of Theorem 3.1. □

Remark 1. *A real $n \times n$ orthogonal matrix is determined by $n(n-1)/2$ parameters. Thus by Theorem 3.1 or Theorem 3.2, the degree of freedom for $\mathcal{E}_\gamma(z)$ is $(\gamma+2)rm(rm-2)/8$. A factorization similar to* (3.15) *for $\mathcal{E}_\gamma(z)$ can be derived from the factorization of the linear phase paraunitary systems in* [26], [9]. *However, the number of the parameters in this factorization for $\mathcal{E}_\gamma(z)$ is $(\gamma+1)rm(rm-2)/4$. Therefore, comparing to* (3.15), *the factorization got in such a way has $\gamma rm(rm-2)/8$ redundant parameters.*

By (3.3) and Theorems 3.1, 3.2, we have the following theorem.

Theorem 3.3. *A causal FIR multifilter bank* $\mathbf{H}_\ell(\omega) = \sum_{k=0}^{(\gamma+1)m-1} \mathbf{h}_\ell(k)e^{-ik\omega}$, $0 \le \ell < m$ *for some* $\gamma \in \mathbb{Z}_+$ *and even* $m = 2m_1$ *is orthogonal and satisfies* (2.11) *if and only if it is factorized as*

$$\begin{bmatrix} \mathbf{H}_0(\omega) \\ \vdots \\ \mathbf{H}_{m-1}(\omega) \end{bmatrix} = \frac{\sqrt{2m}}{2m}\mathbf{M}_0^T\mathbf{V}_\gamma(z^m)\cdots\mathbf{V}_1(z^m)\begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_0(\mathbf{J}_{m_1}\otimes\mathbf{D}_0) \\ \mathbf{v}_0\mathbf{J}_{r,m_1} & -\mathbf{v}_0(\mathbf{I}_{m_1}\otimes\mathbf{D}_0) \end{bmatrix}\begin{bmatrix} \mathbf{I}_r \\ \vdots \\ z^{1-m}\mathbf{I}_r \end{bmatrix}, \tag{3.17}$$

or it is factorized in another form as

$$\begin{bmatrix} \mathbf{H}_0(\omega) \\ \vdots \\ \mathbf{H}_{m-1}(\omega) \end{bmatrix} = \frac{\sqrt{2m}}{2m}\mathbf{M}_0^T\mathbf{K}_\gamma\Lambda(z^m)\cdots\mathbf{K}_1\Lambda(z^m)\begin{bmatrix} \mathbf{W}_0 & \mathbf{W}_0(\mathbf{J}_{m_1}\otimes\mathbf{D}_0) \\ \mathbf{U}_0\mathbf{J}_{r,m_1} & -\mathbf{U}_0(\mathbf{I}_{m_1}\otimes\mathbf{D}_0) \end{bmatrix}\begin{bmatrix} \mathbf{I}_r \\ \vdots \\ z^{1-m}\mathbf{I}_r \end{bmatrix}, \tag{3.18}$$

where $z = e^{i\omega}, \mathbf{M}_0$ *is a permutation matrix,* $\mathbf{K}_j = \mathrm{diag}(\mathbf{I}_{rm_1}, \mathbf{U}_j)$, $\mathbf{W}_0, \mathbf{U}_j, \mathbf{w}_0, \mathbf{v}_0 \in O(rm_1)$, *and* $\mathbf{V}_k(z)$ *are defined by* (3.6) *with* $\mathbf{v}_k \in O(rm_1)$, *while* $\Lambda(z)$ *is defined by* (3.16).

4. Parameterization of symmetric multifilter banks for odd m

In this section we consider the case that m is odd. By our assumption, in this case the multiplicity r is even, i.e., $r = 2r_1$ for some positive integer r_1. Let $\mathcal{H}_m = \{\mathbf{H}_\ell, 0 \le \ell < m\}$ be a causal FIR multifilter bank, and $\mathbf{E}_p(z)$ denote its polyphase matrix. If $\mathbf{H}_\ell, 0 \le \ell < m$, satisfy (2.11), then $\mathbf{E}_p(z)$ satisfies (2.13) for some diagonal matrices $\mathbf{D}_\ell, 0 \le \ell < m$. Let $\mathbf{M}_0$ be the permutation matrix satisfying (2.15). Denote

$$\mathbf{E}_p^{(1)}(z) := \mathbf{M}_0\mathbf{E}_p(z)\mathrm{diag}(\mathbf{I}_{r_1(m-1)}, \mathbf{I}_{\frac{m+1}{2}}\otimes\mathbf{D}_0),$$

and denote

$$\mathbf{C}_0 := \begin{bmatrix} & & \mathbf{J}_{r,\frac{m-1}{2}} \\ & \mathbf{D}_0 & \\ \mathbf{J}_{r,\frac{m-1}{2}} & & \end{bmatrix}, \quad \mathbf{J}_{r,\frac{m-1}{2}} := \mathbf{J}_{\frac{m-1}{2}}\otimes\mathbf{I}_r.$$

By the fact

$$\mathrm{diag}(\mathbf{I}_{r_1(m-1)}, \mathbf{I}_{\frac{m+1}{2}}\otimes\mathbf{D}_0)(\mathbf{J}_m\otimes\mathbf{D}_0)\mathrm{diag}(\mathbf{I}_{r_1(m-1)}, \mathbf{I}_{\frac{m+1}{2}}\otimes\mathbf{D}_0) = \mathbf{C}_0,$$

$\mathbf{E}_p(z)$ satisfies (2.13) if and only if $\mathbf{E}_p^{(1)}(z)$ satisfies

$$\mathbf{E}_p^{(1)}(z) = z^{-\gamma}\mathrm{diag}(\mathbf{I}_{r_1m}, -\mathbf{I}_{r_1m})\mathbf{E}_p^{(1)}(z^{-1})\mathbf{C}_0. \tag{4.1}$$

Denote

$$\mathbf{B}_1 := \frac{\sqrt{2}}{2}\begin{bmatrix} \mathbf{I}_{r_1(m-1)} & \mathbf{0} & \mathbf{J}_{r,\frac{m-1}{2}} \\ \mathbf{0} & \sqrt{2}\mathbf{I}_r & \mathbf{0} \\ \mathbf{J}_{r,\frac{m-1}{2}} & \mathbf{0} & -\mathbf{I}_{r_1(m-1)} \end{bmatrix}.$$

Then $\mathbf{B}_1^T = \mathbf{B}_1, \mathbf{B}_1^2 = \mathbf{I}_{rm}$, and

$$\mathbf{B}_1\mathbf{C}_0\mathbf{B}_1 = \mathrm{diag}(\mathbf{I}_{r_1m}, -\mathbf{I}_{r_1m}).$$

Thus if we denote

$$\mathcal{E}_\gamma(z) := \mathbf{E}_p^{(1)}(z)\mathbf{B}_1 = \mathbf{M}_0\mathbf{E}_p(z)\mathrm{diag}(\mathbf{I}_{r_1(m-1)}, \mathbf{I}_{\frac{m+1}{2}}\otimes\mathbf{D}_0)\mathbf{B}_1,$$

or equivalently

$$\mathbf{E}_p(z) = \mathbf{M}_0^T \mathcal{E}_\gamma(z) \mathbf{B}_2 \tag{4.2}$$

where

$$\mathbf{B}_2 := \mathbf{B}_1 \begin{bmatrix} \mathbf{I}_{r_1(m-1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\frac{m+1}{2}} \otimes \mathbf{D}_0 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} \mathbf{I}_{r_1(m-1)} & \mathbf{0} & \mathbf{J}_{\frac{m-1}{2}} \otimes \mathbf{D}_0 \\ \mathbf{0} & \sqrt{2}\mathbf{D}_0 & \mathbf{0} \\ \mathbf{J}_{r,\frac{m-1}{2}} & \mathbf{0} & -\mathbf{I}_{\frac{m-1}{2}} \otimes \mathbf{D}_0 \end{bmatrix}, \tag{4.3}$$

then $\mathbf{E}_p^{(1)}(z)$ satisfies (4.1) or equivalently $\mathbf{H}_\ell, 0 \le \ell < m$ satisfies (2.11) if and only if $\mathcal{E}_\gamma(z)$ satisfies

$$\mathcal{E}_\gamma(z) = z^{-\gamma} \mathrm{diag}(\mathbf{I}_{r_1 m}, -\mathbf{I}_{r_1 m}) \mathcal{E}_\gamma(z^{-1}) \mathrm{diag}(\mathbf{I}_{r_1 m}, -\mathbf{I}_{r_1 m}). \tag{4.4}$$

Furthermore, $\mathcal{H}_m$ is orthogonal if and only if $\mathcal{E}_\gamma(z)$ is paraunitary. By Theorems 3.1, 3.2, for a causal FIR paraunitary matrix $\mathcal{E}_\gamma(z)$ satisfying (4.4), it can be factorized in the form as (3.5) or in another form as (3.15). Thus we have the following.

Theorem 4.1. *A causal FIR multifilter bank* $\mathbf{H}_\ell(\omega) = \sum_{k=0}^{(\gamma+1)m-1} \mathbf{h}_\ell(k) e^{-ik\omega}$, $0 \le \ell < m$ *for some nonnegative even integer* $\gamma = 2\gamma_1$ *and odd* m *is orthogonal and satisfies* (2.11) *if and only if it is factorized as*

$$\begin{bmatrix} \mathbf{H}_0(\omega) \\ \vdots \\ \mathbf{H}_{m-1}(\omega) \end{bmatrix} = \frac{\sqrt{m}}{m} \mathbf{M}_0^T \mathbf{V}_\gamma(z^m) \cdots \mathbf{V}_1(z^m) \begin{bmatrix} \mathbf{w}_0 & \\ & \mathbf{v}_0 \end{bmatrix} \mathbf{B}_2 \begin{bmatrix} \mathbf{I}_r \\ \vdots \\ z^{1-m}\mathbf{I}_r \end{bmatrix}, \tag{4.5}$$

or it is factorized in another form as

$$\begin{bmatrix} \mathbf{H}_0(\omega) \\ \vdots \\ \mathbf{H}_{m-1}(\omega) \end{bmatrix} = \frac{\sqrt{m}}{m} \mathbf{M}_0^T \mathbf{K}_\gamma \Lambda(z^m) \cdots \mathbf{K}_1 \Lambda(z^m) \begin{bmatrix} \mathbf{W}_0 & \\ & \mathbf{U}_0 \end{bmatrix} \mathbf{B}_2 \begin{bmatrix} \mathbf{I}_r \\ \vdots \\ z^{1-m}\mathbf{I}_r \end{bmatrix}, \tag{4.6}$$

where $z = e^{i\omega}, \mathbf{M}_0$ *is a permutation matrix,* $\mathbf{K}_j = diag(\mathbf{I}_{r_1 m}, \mathbf{U}_j)$, $\mathbf{W}_0, \mathbf{U}_0, \mathbf{w}_0, \mathbf{v}_0, \mathbf{U}_j \in O(r_1 m)$, $\mathbf{B}_2$ *is defined by* (4.3), *and* $\mathbf{V}_k(z)$ *are defined by* (3.6) *with* $\mathbf{v}_k \in O(r_1 m)$, *while* $\Lambda(z)$ *is defined by* (3.16).

5. Parameterization of symmetric multifilter banks for $m = 2, r = 2$

In this section, we consider the case that $m = 2, r = 2$ in more detail. In this case, we use Φ, Ψ to denote the scaling function and multiwavelet, and let $\mathbf{P}, \mathbf{Q}$ denote the corresponding multifilter bank. By Theorem 3.3, for a causal FIR multifilter bank $\{\mathbf{P}, \mathbf{Q}\}$, it generates symmetric/antisymmetric orthogonal scaling function and multiwavelet if and only if it is given by (3.17) (also by (3.18)) with $m = 2$ and the orthogonal matrices $\mathbf{W}_0, \mathbf{U}_k, \mathbf{w}_0, \mathbf{v}_k$ given by $\mathbf{R}_\theta \mathbf{I}_\pm$, where

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{I}_\pm = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

Based on such parameterizations, we have the following proposition.

Proposition 5.1. *If a causal FIR multifilter bank* $\mathbf{P}_\gamma(\omega) = \sum_{k=0}^{2(\gamma+1)-1} \mathbf{p}_\gamma(k)e^{-ik\omega}$, $\mathbf{Q}_\gamma(\omega) = \sum_{k=0}^{2(\gamma+1)-1} \mathbf{q}_\gamma(k)e^{-ik\omega}$ *for some* $\gamma \in \mathbb{Z}_+$ *generates symmetric/antisymmetric orthogonal scaling function* Φ *and multiwavelet* Ψ *about symmetric center* $\gamma + \frac{1}{2}$, *then one component of each of* Φ, Ψ *is symmetric and the other component of each of them is antisymmetric.*

Proof. Suppose causal FIR multifilter bank $\{\mathbf{P}_\gamma, \mathbf{Q}_\gamma\}$ generates symmetric/antisymmetric scaling function and multiwavelet $\Phi = (\phi_1, \phi_2)^T, \Psi = (\psi_1, \psi_2)^T$. Then it is given by

$$\begin{bmatrix} \mathbf{P}_\gamma(\omega) \\ \mathbf{Q}_\gamma(\omega) \end{bmatrix} = \frac{1}{2}\mathbf{M}_0^T \mathbf{V}_\gamma(z^2) \cdots \mathbf{V}_1(z^2) \begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_0\mathbf{D}_0 \\ \mathbf{v}_0 & -\mathbf{v}_0\mathbf{D}_0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 \\ z^{-1}\mathbf{I}_2 \end{bmatrix}, \quad z = e^{i\omega},$$

for some permutation matrix $\mathbf{M}_0$, and $\mathbf{w}_0, \mathbf{v}_0 \in O(2)$,

$$\mathbf{V}_k(z) = \frac{1}{2}\begin{bmatrix} \mathbf{I}_2 & \mathbf{v}_k \\ \mathbf{v}_k^T & \mathbf{I}_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mathbf{I}_2 & -\mathbf{v}_k \\ -\mathbf{v}_k^T & \mathbf{I}_2 \end{bmatrix} z^{-1}, \quad \mathbf{v}_k \in O(2). \tag{5.1}$$

If ϕ_1, ϕ_2 are symmetric, then ψ_1, ψ_2 are antisymmetric, and in this case $\mathbf{D}_0 = \mathbf{I}_2, \mathbf{D}_1 = -\mathbf{I}_2, \mathbf{M}_0 = \mathbf{I}_4$. Thus

$$\begin{bmatrix} \mathbf{P}_\gamma(0) \\ \mathbf{Q}_\gamma(0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_0 \\ \mathbf{v}_0 & -\mathbf{v}_0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_0 \\ \mathbf{0} \end{bmatrix},$$

i.e., $\mathbf{P}_\gamma(0) = \mathbf{w}_0$. For orthogonal matrix $\mathbf{w}_0$, the modulus of each eigenvalue of $\mathbf{w}_0$ is 1, which is a contradiction to that $\mathbf{P}_\gamma(0)$ satisfies Condition E. If ϕ_1, ϕ_2 are antisymmetric, then $\widehat{\Phi}(0) = \mathbf{0}$, and hence $\Phi = \mathbf{0}$, which leads to the contradiction. □

In fact, as shown in the proof of Proposition 5.1, there is no two–channel causal FIR multifilter bank which generates orthogonal scaling function Φ and multiwavelet Ψ with all components of Φ or Ψ symmetric or antisymmetric about the same symmetric center.

By Proposition 5.1, if we construct symmetric/antisymmetric Φ, Ψ with the same symmetric center, then one component of each of them is symmetric and another component is antisymmetric. The 2×2 orthogonal matrices $\mathbf{w}_0$ and $\mathbf{W}_0$ can be determined if the necessary condition that $\mathbf{H}(0)$ satisfies condition E is assured. In particular, if we want to construct Φ, Ψ with the first components of them are symmetric and the second components are antisymmetric, then one can find $\mathbf{W}_0 = \mathbf{w}_0 = \mathbf{I}_\pm$. In this case,

$$\mathbf{D}_1 = \mathbf{D}_0 = \mathrm{diag}(1, -1) =: \mathcal{D}_0,$$

and the permutation matrix $\mathcal{M}_0$ is given by

$$\mathcal{M}_0 = \mathrm{diag}(1, J_2, 1),$$

where J_2 is the exchange matrix defined by (2.14). Denote

$$\begin{bmatrix} \mathbf{P}_0(\omega) \\ \mathbf{Q}_0(\omega) \end{bmatrix} = \frac{1}{2}\mathcal{M}_0 \begin{bmatrix} \mathbf{I}_\pm & \mathbf{I}_\pm\mathcal{D}_0 \\ \mathbf{v}_0 & -\mathbf{v}_0\mathcal{D}_0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 \\ z^{-1}\mathbf{I}_2 \end{bmatrix}. \tag{5.2}$$

Theorem 5.1. *A causal FIR multifilter bank* $\mathbf{P}_\gamma(\omega) = \sum_{k=0}^{2(\gamma+1)-1} \mathbf{p}_\gamma(k)e^{-ik\omega}$, $\mathbf{Q}_\gamma(\omega) = \sum_{k=0}^{2(\gamma+1)-1} \mathbf{q}_\gamma(k)e^{-ik\omega}$ *for some* $\gamma \in \mathbb{Z}_+$ *generates symmetric/antisymmetric orthogonal scaling function* $\Phi = (\phi_1, \phi_2)^T$ *and multiwavelet* $\Psi = (\psi_1, \psi_2)^T$ *with* ϕ_1, ψ_1 *symmetric and* ϕ_2, ψ_2 *antisymmetric about* $\gamma + \frac{1}{2}$ *if and only if the* *matrix* $\mathcal{T}_{\mathbf{P}_\gamma}$ *associated to* $\mathbf{P}_\gamma$ *satisfies Condition E, and* $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ *can be factorized as*

$$\begin{bmatrix} \mathbf{P}_\gamma(\omega) \\ \mathbf{Q}_\gamma(\omega) \end{bmatrix} = \mathcal{M}_0 \mathbf{V}_\gamma(z^2) \cdots \mathbf{V}_1(z^2) \mathcal{M}_0 \begin{bmatrix} \mathbf{P}_0(\omega) \\ \mathbf{Q}_0(\omega) \end{bmatrix}, \tag{5.3}$$

where $z = e^{i\omega}, \mathbf{P}_0, \mathbf{Q}_0$ *are defined by* (5.2) *for some* $\mathbf{v}_0 \in O(2)$*, and* $\mathbf{V}_k(z)$ *are defined by* (5.1) *for* $\mathbf{v}_k \in O(2)$*; or* $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ *can be factorized in another form as*

$$\begin{bmatrix} \mathbf{P}_\gamma(\omega) \\ \mathbf{Q}_\gamma(\omega) \end{bmatrix} = \mathcal{M}_0 \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_\gamma \end{bmatrix} \Lambda(z^2) \cdots \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \Lambda(z^2) \mathcal{M}_0 \begin{bmatrix} \mathbf{P}_0(\omega) \\ \mathbf{Q}_0(\omega) \end{bmatrix}, \tag{5.4}$$

where $z = e^{i\omega}, \mathbf{U}_k \in O(2), 1 \le k \le \gamma$*,* $\mathbf{P}_0, \mathbf{Q}_0$ *are defined by* (5.2) *for some* $\mathbf{v}_0 \in O(2)$*, and*

$$\Lambda(z) = \frac{1}{2} \begin{bmatrix} \mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{I}_2 & -\mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix} z^{-1}. \tag{5.5}$$

Remark 2. *The parametric expression* (5.3) *for the orthogonal multifilter banks was also given in* [18]*. Here it is shown that such expression is complete. Scaling functions and multiwavelets constructed based on* (5.3) *or* (5.4) *are symmetric/antisymmetric about* $\gamma + \frac{1}{2}$*. In* [18] *parametric expression of the orthogonal multifilter banks for symmetric/antisymmetric scaling functions and multiwavelets with symmetric center* $\gamma + 1, \gamma \in \mathbb{Z}_+$*, was also provided.*

If $\mathbf{v}_k$ in (5.1) and $\mathbf{U}_k$ in (5.4) are chosen to be $\mathbf{v}_k = \mathbf{R}_{\theta_k}$, $\mathbf{U}_k = \mathbf{R}_{\alpha_k}, \theta_k, \gamma_k \in [-\pi, \pi)$ for all $1 \le k \le \gamma$, then $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ defined by (5.3) or by (5.4) have the relation as shown in the following proposition.

Proposition 5.2. *Assume that* $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ *are defined by* (5.3) *or by* (5.4) *with* $\mathbf{V}_k(z)$ *defined by* (5.1) *and* $\mathbf{v}_k = \mathbf{R}_{\theta_k}$*,* $\mathbf{U}_k = \mathbf{R}_{\alpha_k}, 1 \le k \le \gamma$*. Then the matrix coefficients for* $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ *satisfy one of the following relations:*

$$(A).\ \mathbf{q}_\gamma(k) = (-1)^{k+1} \mathbf{p}_\gamma(k) \mathbf{J}_2 \mathcal{D}_0, \quad (B).\ \mathbf{q}_\gamma(k) = (-1)^{k} \mathbf{p}_\gamma(k) \mathbf{J}_2 \mathcal{D}_0,$$
$$(C).\ \mathbf{q}_\gamma(k) = (-1)^{k+1} \mathcal{D}_0 \mathbf{p}_\gamma(k) \mathbf{J}_2 \mathcal{D}_0, \quad (D).\ \mathbf{q}_\gamma(k) = (-1)^{k} \mathcal{D}_0 \mathbf{p}_\gamma(k) \mathbf{J}_2 \mathcal{D}_0,$$

depending on the choices of $\mathbf{P}_0(\omega), \mathbf{Q}_0(\omega)$*:* $\left[\mathbf{P}_0(\omega)^T, \mathbf{Q}_0(\omega)^T\right]^T$ *is respectively*

$$(a).\ \frac{1}{2}\mathcal{M}_0 \begin{bmatrix} \mathbf{I}_2 + \mathcal{D}_0 z^{-1} \\ \mathbf{R}_{\theta_0}(\mathbf{I}_2 - \mathcal{D}_0 z^{-1}) \end{bmatrix}, \quad (b).\ \frac{1}{2}\mathcal{M}_0 \begin{bmatrix} \mathcal{D}_0 + \mathbf{I}_2 z^{-1} \\ \mathbf{R}_{\theta_0}(\mathcal{D}_0 - \mathbf{I}_2 z^{-1}) \end{bmatrix},$$
$$(c).\ \frac{1}{2}\mathcal{M}_0 \begin{bmatrix} \mathbf{I}_2 + \mathcal{D}_0 z^{-1} \\ \mathbf{R}_{\theta_0}(\mathcal{D}_0 - \mathbf{I}_2 z^{-1}) \end{bmatrix}, \quad (d).\ \frac{1}{2}\mathcal{M}_0 \begin{bmatrix} \mathcal{D}_0 + \mathbf{I}_2 z^{-1} \\ \mathbf{R}_{\theta_0}(\mathbf{I}_2 - \mathcal{D}_0 z^{-1}) \end{bmatrix}.$$

Proof. Relation (A) for $\mathbf{p}_\gamma, \mathbf{q}_\gamma$ is equivalent to $\mathbf{Q}_\gamma(z) = -\mathbf{P}_\gamma(-z)\mathbf{J}_2\mathcal{D}_0$, i.e.,

$$\begin{bmatrix} \mathbf{P}_\gamma(z) \\ \mathbf{Q}_\gamma(z) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P}_\gamma(-z) \\ \mathbf{Q}_\gamma(-z) \end{bmatrix} \mathbf{J}_2 \mathcal{D}_0.$$

Notice that $\mathcal{M}_0 \begin{bmatrix} \mathbf{0} & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0} \end{bmatrix} \mathcal{M}_0 = -\mathrm{diag}(\mathbf{J}_2\mathcal{D}_0, \mathbf{J}_2\mathcal{D}_0)$. Thus (A) is equivalent to

$$\mathcal{M}_0 \begin{bmatrix} \mathbf{P}_\gamma(z) \\ \mathbf{Q}_\gamma(z) \end{bmatrix} = - \begin{bmatrix} \mathbf{J}_2\mathcal{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2\mathcal{D}_0 \end{bmatrix} \mathcal{M}_0 \begin{bmatrix} \mathbf{P}_\gamma(-z) \\ \mathbf{Q}_\gamma(-z) \end{bmatrix} \mathbf{J}_2 \mathcal{D}_0. \tag{5.6}$$

Assume that $\mathbf{P}_0(\omega), \mathbf{Q}_0(\omega)$ are defined by (a). Then one checks that $\mathbf{P}_0, \mathbf{Q}_0$ satisfy (5.6) by the facts $\mathbf{J}_2\mathcal{D}_0\mathbf{R}_{\theta_0}\mathbf{J}_2\mathcal{D}_0 = -\mathbf{R}_{\theta_0}$ and $\mathbf{J}_2\mathcal{D}_0\mathbf{R}_{\theta_0}\mathcal{D}_0\mathbf{J}_2\mathcal{D}_0 = \mathbf{R}_{\theta_0}\mathcal{D}_0$. To show that $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ defined by (5.3) with $\mathbf{v}_k = \mathbf{R}_{\theta_k}$ and $\mathbf{P}_0, \mathbf{Q}_0$ given by (a) satisfy (5.6), we need only to prove that if $\mathbf{P}_{\gamma-1}, \mathbf{Q}_{\gamma-1}$ defined by (5.3) with $\mathbf{v}_k = \mathbf{R}_{\theta_k}$ and $\mathbf{P}_0, \mathbf{Q}_0$ given by (a) satisfy (5.6), then $\mathbf{P}_\gamma(z), \mathbf{Q}_\gamma(z)$ defined by

$$\begin{bmatrix} \mathbf{P}_\gamma(z) \\ \mathbf{Q}_\gamma(z) \end{bmatrix} = \mathcal{M}_0\mathbf{V}_\gamma(z^2)\mathcal{M}_0 \begin{bmatrix} \mathbf{P}_{\gamma-1}(z) \\ \mathbf{Q}_{\gamma-1}(z) \end{bmatrix}$$

also satisfy (5.6), where $\mathbf{V}_\gamma(z)$ defined by (5.1) with $\mathbf{v}_\gamma = \mathbf{R}_{\theta_\gamma}, \theta_\gamma \in [-\pi, \pi)$. In fact we have

$$\begin{aligned}
&\begin{bmatrix} \mathbf{J}_2\mathcal{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2\mathcal{D}_0 \end{bmatrix} \mathcal{M}_0 \begin{bmatrix} \mathbf{P}_\gamma(-z) \\ \mathbf{Q}_\gamma(-z) \end{bmatrix} \mathbf{J}_2\mathcal{D}_0 \\
&= \begin{bmatrix} \mathbf{J}_2\mathcal{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2\mathcal{D}_0 \end{bmatrix} \mathbf{V}_\gamma(z^2)\mathcal{M}_0 \begin{bmatrix} P_{\gamma-1}(-z) \\ \mathbf{Q}_{\gamma-1}(-z) \end{bmatrix} \mathbf{J}_2\mathcal{D}_0 \\
&= \begin{bmatrix} \mathbf{J}_2\mathcal{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2\mathcal{D}_0 \end{bmatrix} \mathbf{V}_\gamma(z^2) \begin{bmatrix} \mathbf{J}_2\mathcal{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2\mathcal{D}_0 \end{bmatrix} \mathcal{M}_0 \begin{bmatrix} {}_{\gamma} - 1H(z) \\ \mathbf{Q}_{\gamma-1}(z) \end{bmatrix} \\
&= -\mathbf{V}_\gamma(z^2)\mathcal{M}_0 \begin{bmatrix} {}_{2\gamma-1}\mathbf{H}(z) \\ {}_{2\gamma-1}\mathbf{Q}(z) \end{bmatrix} = -\mathcal{M}_0 \begin{bmatrix} \mathbf{P}_\gamma(z) \\ \mathbf{Q}_\gamma(z) \end{bmatrix},
\end{aligned}$$

where the second last equation follows from the fact that $\mathbf{J}_2\mathcal{D}_0\mathbf{R}_{\theta_\gamma}\mathbf{J}_2\mathcal{D}_0 = -\mathbf{R}_{\theta_\gamma}$ and hence $\mathrm{diag}(\mathbf{J}_2\mathcal{D}_0, \mathbf{J}_2\mathcal{D}_0)\mathbf{V}_\gamma(z^2)\mathrm{dia}$ $-\mathbf{V}_\gamma(z^2)$. If $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ are defined by (5.4) with $\mathbf{U}_k = \mathbf{R}_{\alpha_k}$ and $\mathbf{P}_0, \mathbf{Q}_0$ given by (a), then one can show similarly that they satisfy (5.6). If $\mathbf{P}_0, \mathbf{Q}_0$ are defined by (b), or (c), or (d) respectively, and $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ are defined by (5.4) or by (5.3) with $\mathbf{v}_k = \mathbf{R}_{\theta_k}, \mathbf{U}_k = \mathbf{R}_{\alpha_k}$, then $\mathbf{P}_\gamma, \mathbf{Q}_\gamma$ satisfy (B), or (C), or (D). The proof is similar and details are omitted here. ☐

6. Examples

Based on the parameterizations of the orthogonal multifilter banks, we can construct multiwavelets with various properties, e.g., multiwavelets with good smoothness and multiwavelets with optimum time-frequency resolution, see [17], [18]. In this section, we construct in two examples the multiwavelets of multiplicity 3 with dilation factor $m = 2$ and multiwavelets of multiplicity 2 with dilation factor $m = 3$ to demonstrate further the use of these parameterizations provided in the above sections. We hope that the scaling functions have some approximation property and are as smooth as possible. For an $(m, \mathbf{P})$ refinable function Φ, under condition that Φ is L^2-stable, the approximation order of Φ is equivalent to the order of the vanishing moment conditions for $\mathbf{P}$ (see e.g., [10], [21], [12], [6], [1], [16]). $\mathbf{P}(\omega)$ is said to satisfy the *vanishing moment conditions (sum rules)* of order $K \in \mathbb{Z}_+$ if there exist $1 \times r$ vectors $\mathbf{l}_0^k$ with $\mathbf{l}_0^0 \neq 0$, $k \in \mathbb{Z}_+, k < K$, such that

$$\sum_{0 \le s \le k} \begin{pmatrix} k \\ s \end{pmatrix} (mi)^{s-k}\mathbf{l}_0^s(D^{k-s}\mathbf{P})(2\pi j/m) = \delta_j m^{-k}\mathbf{l}_0^k, \quad 0 \le j \le m-1.$$

The regularity estimate of the refinable vectors Φ was studied in [3], [24], [20], [15], [16], [13] and [23]. Here we use the regularity estimate provided in [16]. It was shown in [16] that each component of the $(m, \mathbf{P})$ refinable vector Φ is the Sobolev space $W^s(\mathbb{R})$ for any $s < s_0 := -\log_{2m}\rho(\mathbf{T_P}|_{V^0})$, where $\mathbf{T_P}$ is the transition operator related to $\mathbf{P}$, V^0 is an invariant subspace under $\mathbf{T_P}$, and $\rho(\mathbf{T_P}|_{V^0})$ is the spectral radius of the restricted operator $\mathbf{T_P}|_{V^0}$. Based on the parameterization for $\mathbf{P}$, we can construct smooth scaling function by minimizing $\rho(\mathbf{T_P}|_{V^0})$.

Example 4.1. In this example, we construct scaling function Φ and multiwavelet Ψ of multiplicity $r = 3$ with dilation factor $m = 2$, $\text{supp}(\Phi)$, $\text{supp}(\Psi) \subset [0,\ 3]$, and ϕ_1, ψ_2, ψ_3 are symmetric, while ϕ_2, ϕ_3, ψ_1 are antisymmetric. In this case,

$$\mathbf{D}_0 = \text{diag}(1, -1, -1), \quad \mathbf{D}_1 = -\mathbf{D}_0.$$

By (3.17), the multifilter bank $\mathbf{P}, \mathbf{Q}$ is given by

$$\begin{bmatrix} \mathbf{P}(\omega) \\ \mathbf{Q}(\omega) \end{bmatrix} = \frac{1}{2}\mathbf{M}_0^T \mathbf{V}_1(z^2) \begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_0\mathbf{D}_0 \\ \mathbf{v}_0 & -\mathbf{v}_0\mathbf{D}_0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 \\ z^{-1}\mathbf{I}_3 \end{bmatrix}$$
$$= \frac{1}{4}\mathbf{M}_0^T \Big(\begin{bmatrix} \mathbf{E} & \mathbf{EF} \\ \mathbf{GF}^T & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{E} & -\mathbf{EF} \\ -\mathbf{GF}^T & \mathbf{G} \end{bmatrix} z^{-2} \Big) \Big(\begin{bmatrix} \mathbf{I}_3 \\ \mathbf{I}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{I}_3 \\ -\mathbf{I}_3 \end{bmatrix} \mathbf{D}_0 z^{-1} \Big),$$

where $z = e^{i\omega}, \mathbf{E} = \mathbf{w}_0, \mathbf{F} = \mathbf{w}_0^T \mathbf{v}_1 \mathbf{v}_0, \mathbf{G}_0 = \mathbf{v}_0 \in O(3)$. Orthogonal matrices can be given by Euler angles or Givens rotations, see e.g. [29], [30]. A 3×3 orthogonal matrix can be expressed as $\mathbf{U}(\theta_1, \theta_2, \theta_3)\text{diag}(1, \pm 1, \pm 1)$, where

$$\mathbf{U}(\theta_1, \theta_2, \theta_3) := \tag{6.1}$$
$$\begin{bmatrix} \cos\theta_2 \cos\theta_3 & \cos\theta_2 \sin\theta_3 & \sin\theta_2 \\ -\cos\theta_1 \sin\theta_3 - \sin\theta_1 \sin\theta_2 \cos\theta_3 & \cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3 & \sin\theta_1 \cos\theta_2 \\ \sin\theta_1 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3 & -\sin\theta_1 \cos\theta_3 - \cos\theta_1 \sin\theta_2 \sin\theta_3 & \cos\theta_1 \cos\theta_2 \end{bmatrix}.$$

Here we choose

$$\mathbf{E} = \mathbf{U}(\theta_1, \theta_2, \theta_3), \quad \mathbf{F} = \mathbf{U}(\alpha_1, \alpha_2, \alpha_3), \quad \mathbf{G} = \mathbf{U}(\beta_1, \beta_2, \beta_3). \tag{6.2}$$

By the symmetry of Φ, $\mathbf{l}_0^0 = \widehat{\Phi}(0)^T = (1, 0, 0)$ is a left 1-eigenvector of $\mathbf{H}(0)$. By $\mathbf{l}_0^0\mathbf{H}(0) = \mathbf{l}_0^0$, $\mathbf{l}_0^0\mathbf{H}(\pi) = \mathbf{0}$, we get $\theta_2 = \theta_3 = 0$. $\mathbf{H}$ is independent of θ_1. Here we choose $\theta_1 = 0$. Thus $\mathbf{E} = \mathbf{I}_3$. There are six free parameters for $\mathbf{H}$. For the choices of

$$\alpha_1 = 2.63203213185982,\ \alpha_2 = -.27741087230381,\ \alpha_3 = 3.09917642215873,$$
$$\beta_1 = 2.96875069023522,\ \beta_2 = 1.13820946588261,\ \beta_3 = -1.32902204611443,$$

the corresponding Φ provides approximation order 2, and $\Phi, \Psi \in W^{1.9077}(\mathbb{R})$, or $\Phi, \Psi \in C^{1.4077}(\mathbb{R})$. Φ, Ψ are shown in Fig. 1.

In [7], scaling function Φ and multiwavelet Ψ of multiplicity 3 were constructed by fractal interpolation. The supports of Φ, Ψ are in $[0,\ 2]$, $\Phi, \Psi \in C^{1-\epsilon}(\mathbb{R})$ for any $\epsilon > 0$, but they do not possess symmetry.

Example 4.2. In this example, we construct scaling function Ψ_0 and multiwavelets Ψ_1, Ψ_2 of multiplicity $r = 2$ with dilation factor $m = 3$, the supports of Ψ_0, Ψ_1, Ψ_2 are on $[0,\ 2.5]$, and the first components of them are symmetric, while the second components are antisymmetric. In this case

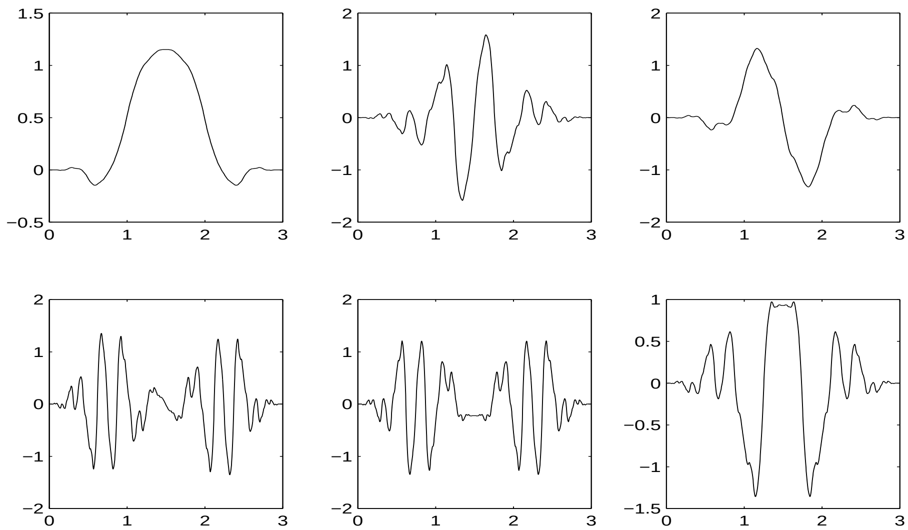

FIGURE 1. Scaling function Φ (the first row) and multiwavelet Ψ (the second row) with $\Phi, \Psi \in C^{1.4077}(\mathbb{R})$.

$\mathbf{D}_0 = \mathbf{D}_1 = \mathbf{D}_2 = \mathcal{D}_0$, recalling $\mathcal{D}_0 = \text{diag}(1, -1)$. By (3.17), the multifilter bank is given by

$$\begin{bmatrix} \mathbf{H}_0(\omega) \\ \mathbf{H}_1(\omega) \\ \mathbf{H}_2(\omega) \end{bmatrix} = \frac{\sqrt{6}}{12}\mathbf{M}_0^T\Big(\begin{bmatrix} \mathbf{E} & \mathbf{EF} \\ \mathbf{GF}^T & \mathbf{G} \end{bmatrix} + \begin{bmatrix} \mathbf{E} & -\mathbf{EF} \\ -\mathbf{GF}^T & \mathbf{G} \end{bmatrix} z^{-3}\Big)$$

$$\cdot([\mathbf{I}_2, \mathbf{0}, \mathbf{I}_2]^T + \left[\mathbf{0}, \sqrt{2}\mathcal{D}_0, \mathbf{0}\right]^T z^{-1} + [\mathbf{I}_2, \mathbf{0}, \mathbf{I}_2]^T \mathcal{D}_0 z^{-2}), \quad z = e^{i\omega},$$

where $\mathbf{E}, \mathbf{F}, \mathbf{G} \in O(3)$. Here again $\mathbf{E}, \mathbf{F}, \mathbf{G}$ are chosen as in (6.2). To assure that $\mathbf{H}_0(0)$ and $\mathbf{H}_0$ satisfy Condition E and the vanishing moment conditions of order one respectively, $\mathbf{E}$ satisfies

$$\frac{\sqrt{6}}{6}\mathbf{E}[2,\ 0,\ \sqrt{2}]^T = [1,\ 0,\ 0]^T.$$

Thus $\mathbf{E} = \mathbf{U}(\theta_1, \arcsin\frac{\sqrt{3}}{3}, 0)$. $\mathbf{H}_0$ is independent of θ_1 and β_1, and we choose $\theta_1 = 0, \beta_1 = 0$. Therefore there are five free parameters for $\mathbf{H}_0$. For the choices of

$$\alpha_1 = -1.30360937891366,\ \ \alpha_2 = -.10783832558925,\ \ \alpha_3 = -1.78266196529869,$$

$$\beta_2 = .88677581545706,\ \ \beta_3 = -.43432057665091,$$

Φ_0 provides approximation order 2, and $\Psi_\ell \in W^{1.7890}(\mathbb{R})$, or $\Psi_\ell \in C^{1.2890}(\mathbb{R})$, $0 \le \ell \le 2$. Ψ_ℓ are shown in Fig. 2. By choosing another group of α_j, β_j, we can construct Ψ_ℓ with $\Psi_\ell \in C^{1.1151}(\mathbb{R})$, $0 \le \ell \le 2$, and Ψ_0 providing approximation order 3.

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REFERENCES

[1] C. Cabrelli, C. Heil and U. Molter, Accuracy of lattice translates of several multidimensional refinable functions, J. Approx. Theory 95 (1998), 5-52.

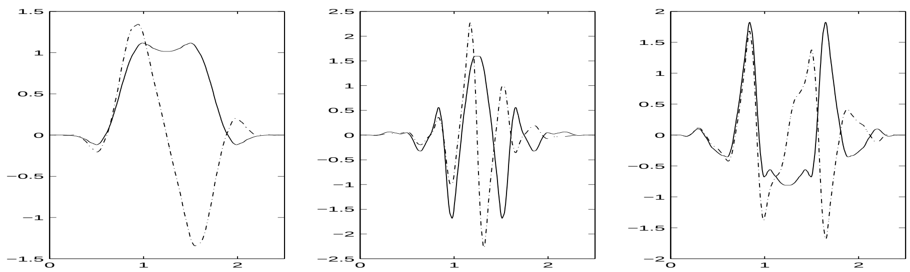

FIGURE 2. Scaling function Ψ_0 (the left) and multiwavelet Ψ_1 (the central), Ψ_2 (the right) with $\Psi_0, \Psi_1, \Psi_2 \in C^{1.2890}(\mathbb{R})$.

[2] C. K. Chui and J. Lian, A study on orthonormal multi-wavelets, J. Appl. Numer. Math. 20 (1996), 273-298.

[3] A. Cohen, I. Daubechies and G. Plonka, Regularity of refinable function vectors, J. Fourier Anal. and Appl. 3 (1997), 295-324.

[4] W. Dahman and C. Micchelli, Biorthogonal wavelet expansions, Constr. Approx. 13 (1997), 293-328.

[5] C. de Boor, R. DeVore and A. Ron, The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994), 37-78.

[6] —, Approximation orders of FSI spaces in $L_2(\mathbb{R})$, Constr. Approx. 14 (1998), 411-427.

[7] G. Donovan, J. S. Geronimo and D. P. Hardin, Intertwining multiresolution analysis and the construction of piecewise polynomial wavelets, SIAM J. Math. Anal. 27 (1996), 1791-1815.

[8] T. N. Goodman, S. L. Lee and W. S. Tang, Wavelets in wandering subspaces, Trans. Amer. Math. Soc. 338 (1993), 639-654.

[9] R. A. Gopinath and C. S. Burrus, Unitary FIR filter banks and symmetry, Rice Univ., Tech Rep. CML TR92-12, 1992.

[10] C. Heil, G. Strang and V. Strela, Approximation by translates of refinable functions, Numer. Math. 73 (1996), 75-94.

[11] R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1991.

[12] R. Q. Jia, S. Riemenschneider and D. X. Zhou, Approximation by multiple refinable functions, Canad. J. Math. 49 (1997), 944-962.

[13] —, Smoothness of multiple refinable functions and multiple wavelets, Comp. Math. 67 (1998), 1533-1563.

[14] R. Q. Jia and Z. W. Shen, Multiresolution and wavelets, Proc. Edinburgh Math. Soc. 7 (1994), 271-300.

[15] Q. T. Jiang, On the regularity of matrix refinable functions, SIAM J. Math. Anal. 29 (1998), 1157-1176.

[16] —, Multivariate matrix refinable functions with arbitrary matrix dilation, Trans. Amer. Math. Soc. 351 (1999), 2407-2438.

[17] —, Orthogonal multiwavelets with optimum time-frequency resolution, IEEE Trans. on Signal Proc. 46 (1998), 830-844.

[18] —, On the design of multifilter banks and orthogonal multiwavelet bases, IEEE Trans. on Signal Proc. 46 (1998), 3292-3303.

[19] Q. T. Jiang and Z. W. Shen, On the existence and weak stability of matrix refinable functions, Constr. Approx. 15 (1999), 337-353.

[20] C. Micchelli and T. Sauer, Regularity of multiwavelets, Advances in Comp. Math. 7 (1997), 455-545.

[21] G. Plonka, Approximation order provided by refinable function vectors, Constr. Approx. 13 (1997), 221-244.

[22] —, On stability of scaling vectors, in *Surface Fitting and Multiresolution Methods*, eds. A. Le Mehaute, C. Rabut and L. L. Schumaker, Vanderbilt University Press, Nashville 1997, pp. 293-300.

[23] A. Ron and Z. W. Shen, The Sobolev regularity of refinable functions, Preprint, National University of Singapore, 1997.

[24] Z. W. Shen, Refinable function vectors, SIAM J. Math. Anal. 29 (1998), 235-250.

[25] A. K. Soman and P. P. Vaidyanathan, A complete factorization with pairwise mirror-image symmetry in the frequency domain, IEEE Trans. on Signal Proc. 43 (1995), 1002-1004.

[26] A. K. Soman, P. P. Vaidyanathan and T. Nguyen, Linear phase paraunitary filter banks: theory, factorizations and design, IEEE Trans. on Signal Proc. 41 (1993), 3480-3496.

[27] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, 1996.

[28] V. Strela, Multiwavelets: regularity, orthogonality and symmetry via two-scale similarity transform, Stud. Appl. Math. 98 (1997), 335-354.

[29] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Englewood Cliffs, NJ: Prentice Hall, 1993.

[30] M. Vilenkin, *Special Functions and Theory of Group Representations*, AMS, Providence, 1968.

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