Orthogonal multiwavelets with optimum time-frequency resolution

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Abstract— **A procedure to design orthogonal multiwavelets with good time-frequency resolution is introduced. Formulas to compute the time-durations and the frequency-bandwidths of scaling functions and multiwavelets are obtained. Parameter expressions for the matrix coefficients of the multifilter banks that generate symmetric/antisymmetric scaling functions and multiwavelets supported in $[0, N]$ are presented for $N = 2, \cdots, 6$. Orthogonal multiwavelets with optimum time-frequency resolution are constructed and some optimal multifilter banks are provided.**

Keywords— **Multifilter bank, Scaling function, Orthogonal multiwavelet, Optimum time-frequency resolution.**

I. Introduction

RECENTLY , the construction of multiwavelets, wavelets generated by a finite set of scaling functions, has been studied by many authors (see e.g., [11], [10], [9], [30], [1], [6], [5], [21], and [22]).

A set of functions $\psi_1, \cdots, \psi_r \in L^2(R)$ are called **orthogonal multiwavelets of multiplicity** r if $\psi_1(2^j x - k), \cdots, \psi_r(2^j x - k), j, k \in Z$, form an orthonormal basis of $L^2(R)$. Wavelet construction is associated with multiresolution analysis (MRA) developed by Mallat and Meyer (see [23] and [3]). As in the scalar case, multiwavelet construction is associated with MRA of multiplicity r which was first studied by Goodman, Lee and Tang ([11], [10]). More precisely, an **MRA of multiplicity** r ([11]) is a nested sequence of closed subspaces V_j in $L^2(R)$ satisfying the following conditions:

- (1°) $V_j \subset V_{j+1}, j \in Z$;
- (2°) $\bigcap_{j\in Z} V_j = \{0\}$;
- (3°) $\bigcup_{j\in Z} V_j$ is dense in $L^2(R)$;
- (4°) $f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$;
- (5°) There exist r functions $\phi_1, \cdots, \phi_r$ such that the collection of integer translates $\{\phi_j(\cdot - k) : 1 \le j \le r, k \in Z\}$ is a Riesz basis of V_0.

Such functions $\phi_1, \cdots, \phi_r$ are called **scaling functions**, and they are said to generate the MRA (V_j). If there is a set of compactly supported scaling functions whose integer translates form an orthonormal basis of V_0, then (V_j) is called an orthogonal MRA. For an orthogonal MRA (V_j), let $W_j := V_{j+1} \ominus V_j$, the orthogonal complement of V_j in V_{j+1}, then Conditions (1°), (2°) and (3°) imply that $W_j \perp W_k$ if $j \neq k$ and $\sum_{j\in Z} \oplus W_j = L^2(R)$. If the integer translates of a set of functions $\psi_1, \cdots, \psi_r$ form an orthonormal basis of W_0, then $\psi_1, \cdots, \psi_r$ are a set of orthogonal multiwavelets.

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For a set of functions $f_1, \cdots, f_r$, write $\mathbf{F} = (f_1, \cdots, f_r)^T$, where $\mathbf{M}^T$ denotes the transpose of matrix $\mathbf{M}$. We shall call $\mathbf{F}$ is stable (orthogonal) if the integer translates of $f_1, \cdots, f_r$ form a Riesz basis (an orthonormal basis) of their closed linear span in $L^2(R)$, and call $\mathbf{F}$ a scaling function (a multiwavelet) if $f_1, \cdots, f_r$ are a set of scaling functions (a set of orthogonal multiwavelets).

Assume that $\mathbf{P}$ is a matrix filter with matrix coefficients $\mathbf{P}_k$ satisfying $\mathbf{P}_k = \mathbf{0}, |k| > N$, for some positive integer N, and $\Phi = (\phi_1, \cdots, \phi_r)^T$ is a compactly supported $(2, \mathbf{P})$ refinable vector, i.e., Φ is a vector-valued function satisfying

$$\Phi(x) = 2 \sum_{k\in Z} \mathbf{P}_k \Phi(2x - k), \tag{1}$$

or equivalently satisfying

$$\widehat{\Phi}(\omega) = \mathbf{P}(\omega/2)\widehat{\Phi}(\omega/2). \tag{2}$$

Assume that $\Phi \in L^2(R)$, i.e., $\phi_j \in L^2(R), j = 1, \cdots, r$, and define the closed subspaces $V_j(\Phi)$ of $L^2(R)$ by

$$V_0(\Phi) := \overline{\text{span}}\{\phi_i(x - k), 1 \le i \le r, k \in Z\},$$

and $V_j(\Phi) := \{f : f(2^{-j}x) \in V_0(\Phi)\}, j \in Z$. By the refinability of Φ and the definition of $(V_j(\Phi))$, it is clear that $(V_j(\Phi))$ satisfies (1°) and (4°). Condition (5°) is equivalent to the stability of Φ. For compactly supported $\Phi \in L^2(R)$, it was shown in [15] (see also [4] and [29]) that Condition (2°) and Condition (3°) follow from the other conditions (1°), (4°) and (5°). Therefore, to check whether $(V_j(\Phi))$ is an MRA or not we need only to check the stability of Φ. If the compactly supported $(2, \mathbf{P})$ refinable vector Φ is stable (or equivalently Φ generates an MRA), we say $\mathbf{P}$ generates the scaling function Φ. In this case, $\Phi(\omega) = \lim_{n\to\infty} \Phi_n(\omega)$, where Φ_n is defined by

$$\widehat{\Phi}_n(\omega) := \mathbf{P}(\omega/2) \cdots \mathbf{P}(\omega/2^n)\mathbf{v}_0 \frac{\sin(\omega/2^{n+1})}{\omega/2^{n+1}} e^{-i\omega/2^{n+1}}, \tag{3}$$

and $\mathbf{v}_0$ is the normalized right 1-eigenvector of $\mathbf{P}(0)$ (see e.g., [2] and [18]).

Let $G_\Phi(\omega)$ denote the Gram matrix of the $(2, \mathbf{P})$ refinable vector $\Phi \in L^2(R)$ defined by

$$G_\Phi(\omega) := \sum_{k\in Z} \widehat{\Phi}(\omega + 2\pi k)\widehat{\Phi}^*(\omega + 2\pi k).$$

Throughout this paper, $\mathbf{M}^*$ denotes the Hermitian adjoint of matrix $\mathbf{M}$. Then Φ is stable (or equivalently Φ is a scaling function) if and only if $G_\Phi(\omega)$ is positive definite for all $\omega \in T$, and Φ is orthogonal if and only if $G_\Phi(\omega) = \mathbf{I}_r$ and $\mathbf{P}$ is a matrix **Conjugate Quadrature Filter (CQF)**, i.e., (see e.g., [11] and [9])

$$\mathbf{P}(\omega)\mathbf{P}^*(\omega) + \mathbf{P}(\omega+\pi)\mathbf{P}^*(\omega+\pi) = \mathbf{I}_r, \quad \omega \in T. \tag{4}$$

where $\mathbf{I}_r$ denotes the $r \times r$ identity matrix. We use $\mathbf{0}_r$ for the $r \times r$ zero matrix.

Suppose the scaling function Φ generated by $\mathbf{P}$ is orthogonal. Let $\{\mathbf{Q}_k\}$ be another finitely supported $r \times r$ matrix sequence, and $\Psi = (\psi_1, \cdots, \psi_r)^T$ be the vector-valued function defined by

$$\Psi(x) := 2\sum_{k\in Z} \mathbf{Q}_k \Phi(2x-k), \tag{5}$$

or equivalently by

$$\widehat{\Psi}(\omega) = \mathbf{Q}(\omega/2)\widehat{\Phi}(\omega/2), \tag{6}$$

where $\mathbf{Q}(\omega) := \sum_{k\in Z} \mathbf{Q}_k e^{-ik\omega}$. If Ψ is a compactly supported orthogonal multiwavelet, we say that $\mathbf{P}, \mathbf{Q}$ generate the multiwavelet Ψ (or $\mathbf{P}, \mathbf{Q}$ generate an orthogonal multiwavelet basis). The pair $\{\mathbf{P}, \mathbf{Q}\}$ is called a multiwavelet filter bank (often abbreviated **multifilter bank**), and $\mathbf{P}$ ($\mathbf{Q}$, respectively) is called a **matrix lowpass filter** (**matrix highpass filter**, respectively). For a multifilter bank, the matrix filters $\mathbf{P}, \mathbf{Q}$ are called **finite impulse responses (FIR)** if there exists an integer N such that $\mathbf{P}_k = \mathbf{0}, \mathbf{Q}_k = \mathbf{0}, |k| > N$. The orthogonal conditions for Ψ defined by (5) are equivalent to

$$\mathbf{P}(\omega)\mathbf{Q}^*(\omega) + \mathbf{P}(\omega+\pi)\mathbf{Q}^*(\omega+\pi) = \mathbf{0}_r, \quad \omega \in T, \tag{7}$$

and

$$\mathbf{Q}(\omega)\mathbf{Q}^*(\omega) + \mathbf{Q}(\omega+\pi)\mathbf{Q}^*(\omega+\pi) = \mathbf{I}_r, \quad \omega \in T. \tag{8}$$

Spline multiwavelets were first constructed in [10]. The first example of orthogonal multiwavelet was constructed in [9] and [6] (usually called the GHM-multiwavelet), and more examples were provided in [30], [1] and [5]. However as in the scalar case, in the constructions of multiwavelets, the main emphasis is on the approximation and regularity properties of scaling functions and multiwavelets. The approximation order and regularity of scaling functions and wavelets are very important for some applications, but in digital signal processing applications, the value of approximation order and regularity is still unknown. On the other hand, in many still-image and video processing applications, the time-frequency localization property of the decomposition technique is an important consideration and the time-frequency localization of scaling functions and wavelets is much more important than the approximation order and regularity properties (see [12] and [25]).

For the scalar case, the design of optimum time-frequency resolution (**OPTFR**) wavelets was first considered in [7]. In [32] and [25], further studies were carried out and more optimal filters were designed. In this paper we shall discuss the construction of OPTFR-multiwavelets.

This paper is organized as follows. In Section II, we review some results on the approximation order and regularity of scaling functions and multiwavelets which will be used to design OPTFR-scaling functions and multiwavelets with some approximation and regularity properties. In Section III, we derive formulas to compute the areas of the resolution cells for scaling functions and multiwavelets. In Section IV, for $2 \le N \le 6$, parameter expressions for the matrix coefficients of the multifilter banks which generate symmetric/antisymmetric scaling functions and multiwavelets supported in $[0, N]$ are provided, and the OPTFR-multiwavelets are constructed. The conclusions are given in Section V. The proofs of the propositions in Section III are presented in the Appendix, where some optimal multifilter banks are also provided.

Results in Section III have straightforward generalizations to the case where the dilation factor 2 in (1) is replaced by any integer $M \ge 2$, and even to the multivariate case where the dilation factor is a matrix. In this paper, scaling functions, multiwavelets and the matrix coefficients of the multifilter banks discussed are real.

II. Theory

Suppose $\mathbf{P}$ is an FIR matrix filter, and Φ is a compactly supported $(2, \mathbf{P})$ refinable vector. It is useful to transform the characterizations of the stability and orthonormality of Φ in terms of $\mathbf{P}$. Assume $\text{supp}\{\mathbf{P}_k\} \subset [0, N]$ for a positive integer N, i.e., $\mathbf{P}_k = \mathbf{0}$ if $k < 0$ or $k > N$. Let V_N denote the space of all $r\times r$ matrices with trigonometric polynomial entries whose Fourier coefficients are real and supported in $[1-N, N-1]$. The **transition operator $\mathbf{T_P}$** corresponding to $\mathbf{P}$ is defined on V_N by

$$\mathbf{T_P}\mathbf{H}(\omega) := \mathbf{P}(\frac{\omega}{2})\mathbf{H}(\frac{\omega}{2})\mathbf{P}^*(\frac{\omega}{2}) + \mathbf{P}(\frac{\omega}{2}+\pi)\mathbf{H}(\frac{\omega}{2}+\pi)\mathbf{P}^*(\frac{\omega}{2}+\pi), \mathbf{H} \in V_N. \tag{9}$$

It was shown in [29] that Φ is stable if and only if $\mathbf{T_P}$ satisfies Condition E and the 1-eigenvector of $\mathbf{T_P}$ is positive (or negative) definite everywhere; Φ is orthogonal if and only if $\mathbf{P}$ is a CQF and $\mathbf{T_P}$ satisfies Condition E (for the case $r = 1$, similar characterization was provided by Lawton and it is called Lawton's condition, see [20] and [3]). For a matrix or an operator $\mathbf{A}$, we say $\mathbf{A}$ satisfies **Condition E** if the spectral radius of $\mathbf{A}$ is 1, 1 is the unique eigenvalue of $\mathbf{A}$ on the unit circle and 1 is simple.

Since eigenvalues of a finite matrix can be computed directly, it is useful in practice to represent the operator $\mathbf{T_P}$ as a finite matrix, and such representation was provided in [16].

Let $\mathbf{M}$ be an $r \times r$ matrix, $\mathbf{M}(j)$ be the jth column of $\mathbf{M}$, i.e., $\mathbf{M} = (\mathbf{M}(1), \cdots, \mathbf{M}(r))$, and define $r^2 \times 1$ vector $\text{vec}(\mathbf{M})$ by

$$\text{vec}(\mathbf{M}) := (\mathbf{M}(1)^T, \cdots, \mathbf{M}(r)^T)^T. \tag{10}$$

For $\mathbf{H} = \sum_{j=1-N}^{N-1} \mathbf{H}_j e^{-i\omega j} \in V_N$, let $\text{vec}(\mathbf{H})$ be the $((2N-1)r^2) \times 1$ vector defined by

$$\text{vec}(\mathbf{H}) := \left((\text{vec}(\mathbf{H}_{1-N}))^T, \cdots, (\text{vec}(\mathbf{H}_{N-1}))^T\right)^T.$$

For two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, let $\mathbf{A} \otimes \mathbf{B} := (a_{ij}\mathbf{B})$ denote the Kronecker product of $\mathbf{A}, \mathbf{B}$. Then for any matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (with appropriate sizes), we have (see [14])

$$\mathrm{vec}(\mathbf{ABC}^T) = (\mathbf{C} \otimes \mathbf{A})\mathrm{vec}(\mathbf{B}). \tag{11}$$

It was shown in [16] that the representation matrix of the operator $\mathbf{T_P}$ is

$$\mathcal{T}_{\mathbf{P}} := (2\mathcal{A}_{2i-j})_{1-N\le i,j\le N-1}, \tag{12}$$

and that

$$\mathrm{vec}(\mathbf{T_P H}) = \mathcal{T}_{\mathbf{P}}\mathrm{vec}(\mathbf{H}), \quad \mathbf{H} \in V_N,$$

where $\mathcal{A}_j$ is the $r^2 \times r^2$ matrix defined by

$$\mathcal{A}_j := \sum_{\kappa=0}^{N} \mathbf{P}_{\kappa-j} \otimes \mathbf{P}_\kappa.$$

Suppose that the compactly supported $(2, \mathbf{P})$ refinable vector Φ is stable. Then the property that Φ has approximation of order m is equivalent to that $\mathbf{P}$ satisfies the **vanishing moment conditions** of order m (see e.g., [13] and [27]). We say that $\mathbf{P}$ satisfies the vanishing moment conditions of order m if there exist real $1 \times r$ row vectors $\mathbf{l}_0^\beta$ with $\mathbf{l}_0^0 \neq 0$, $0 \le \beta < m$, such that

$$\begin{cases} \sum_{0\le\alpha\le\beta} \binom{\beta}{\alpha}(2i)^{\alpha-\beta}\mathbf{l}_0^\alpha D^{\beta-\alpha}\mathbf{P}(0) = 2^{-\beta}\mathbf{l}_0^\beta, \\ \sum_{0\le\alpha\le\beta} \binom{\beta}{\alpha}(2i)^{\alpha-\beta}\mathbf{l}_0^\alpha D^{\beta-\alpha}\mathbf{P}(\pi) = \mathbf{0}, \end{cases} \tag{13}$$

where $D^{\beta-\alpha}\mathbf{P}(\omega)$ denotes the matrix formed by the $(\beta-\alpha)$th derivatives of the entries of $\mathbf{P}(\omega)$.

The regularity estimates of a refinable vector Φ were given in [2], [29], [24] and [16]. Here we shall use the estimate provided in [16].

For $s \ge 0$, we say that a function f is in the Sobolev space $W^s(R)$ if $(1+|\omega|^2)^{\frac{s}{2}}\widehat{f}(\omega) \in L^2(R)$, where $\widehat{f}$ denotes the Fourier transform of f. Let $C^\gamma(R)$ denote the space defined as follows: if $\gamma = n + \gamma_1$ with $n \in Z_+$ and $0 \le \gamma_1 < 1$, then $f \in C^\gamma(R)$ if and only if $f \in C^{(n)}(R)$ and $f^{(n)}$ is uniformly Hölder continuous with exponent γ_1, i.e.

$$|f^{(n)}(x+y) - f^{(n)}(x)| \le c|y|^{\gamma_1},$$

for some constant c independent of $x, y \in R$. We have the well-known inclusion

$$W^s(R) \subset C^\gamma(R) \quad \text{for } s > \gamma + \frac{1}{2}.$$

Assume that the FIR matrix filter $\mathbf{P}$ satisfies the vanishing moment conditions of order m, i.e., $\mathbf{P}$ satisfies (13) for some $1 \times r$ vectors $\mathbf{l}_0^\beta$, $0 \le \beta < m$ with $\mathbf{l}_0^0 \neq 0$. Let $m_0 \le m$ be the largest nonnegative integer such that there exist $1 \times r$ vectors $\mathbf{l}_0^\beta$, $m \le \beta \le m + m_0 - 1$ satisfying

$$\sum_{0\le\alpha\le\beta} \binom{\beta}{\alpha}(i2)^{\beta-\alpha}\mathbf{l}_0^\alpha D^{\beta-\alpha}\mathbf{P}(0) = 2^{-\beta}\mathbf{l}_0^\beta. \tag{14}$$

If each of the numbers of $2^{-\beta}, m \le \beta \le m + m_0 - 1$ is not an eigenvalue of $\mathbf{P}(0)$ for some $m_0 \in Z_+$, then the vectors $\mathbf{l}_0^\beta$ can be chosen iteratively by

$$\mathbf{l}_0^\beta\left(2^{-\beta}\mathbf{I}_r - \mathbf{P}(0)\right) = \sum_{0\le\alpha<\beta} \binom{\beta}{\alpha}(i2)^{\alpha-\beta}\mathbf{l}_0^\alpha D^{\beta-\alpha}\mathbf{P}(0).$$

For $\kappa \in Z$, define row vectors $\mathbf{l}_\kappa^\beta$ by

$$\mathbf{l}_\kappa^\beta := \sum_{0\le\alpha\le\beta} \binom{\beta}{\alpha}(-\kappa)^{\beta-\alpha}\mathbf{l}_0^\alpha, \quad 0 \le \beta < m + m_0, \tag{15}$$

and then define the $1 \times ((2N-1)r^2)$ vectors $\mathbf{L}_N^\beta$ by

$$\mathbf{L}_N^\beta := (\mathbf{l}^\beta(1-N), \cdots, \mathbf{l}^\beta(N-1)), \tag{16}$$

with

$$\mathbf{l}^\beta(\kappa) := \sum_{0\le\alpha\le\beta} (-1)^\alpha \binom{\beta}{\alpha}\mathbf{l}_{-\kappa}^\alpha \otimes \mathbf{l}_0^{\beta-\alpha}, \quad \kappa \in Z.$$

Then as shown in [16], $\mathbf{L}_N^\beta \mathcal{T}_{\mathbf{P}} = 2^{-\beta}\mathbf{L}_N^\beta$. Therefore if $\mathbf{L}_N^\beta \neq \mathbf{0}$, $2^{-\beta}$ is an eigenvalue of $\mathcal{T}_{\mathbf{P}}$ with a corresponding left eigenvector $\mathbf{L}_N^\beta$. These vectors play an important role in the regularity estimate of refinable vectors.

For $1 \le j \le r$, denote $\mathbf{e}_j := (\delta_j(k))_{k=1}^r$, the standard unit vectors in R^r, and let ${}_j\mathbf{l}_N^\alpha$, ${}_j\mathbf{r}_N^\alpha$ $(\alpha < m)$ denote the $1 \times ((2N-1)r^2)$ vectors defined respectively by

$$\begin{aligned} {}_j\mathbf{l}_N^\alpha &:= ({}_j\mathbf{l}^\alpha(1-N), \cdots, {}_j\mathbf{l}^\alpha(N-1)), \\ {}_j\mathbf{r}_N^\alpha &:= ({}_j\mathbf{r}^\alpha(1-N), \cdots, {}_j\mathbf{r}^\alpha(N-1)), \end{aligned}$$

where

$${}_j\mathbf{l}^\alpha(\kappa) := \mathbf{e}_j^T \otimes \mathbf{l}_\kappa^\alpha, \quad {}_j\mathbf{r}^\alpha(\kappa) := \mathbf{l}_{-\kappa}^\alpha \otimes \mathbf{e}_j^T, \quad \kappa \in Z.$$

Let $\mathcal{L}_N$ be the $r^2(2N-1)$ by $m + m_0$ matrix defined by

$$\mathcal{L}_N := ((\mathbf{L}_N^0)^T, \cdots, (\mathbf{L}_N^{m+m_0-1})^T),$$

where $\mathbf{L}_N^\beta$ are the vectors defined by (16). For $1 \le j \le r$, let L_j and R_j be the $r^2(2N-1)$ by m matrices defined respectively by

$$\begin{aligned} L_j &:= (({}_j\mathbf{l}_N^0)^T, \cdots, ({}_j\mathbf{l}_N^{m-1})^T), \\ R_j &:= (({}_j\mathbf{r}_N^0)^T, \cdots, ({}_j\mathbf{r}_N^{m-1})^T). \end{aligned}$$

Finally define the $r^2(2N-1)$ by $m + m_0 + 2rm$ matrix $\mathbf{M}_N$:

$$\mathbf{M}_N := (\mathcal{L}_N, L_1, \cdots, L_r, R_1, \cdots, R_r).$$

Let V_N^0 denote the subspace of V_N defined by

$$V_N^0 := \{\mathbf{H} \in V_N : \quad (\mathbf{M}_N)^T\mathrm{vec}(\mathbf{H}) = 0\}.$$

Then V_N^0 is invariant under $\mathbf{T_P}$. Let $\mathbf{T_P}|_{V_N^0}$ denote the restriction of $\mathbf{T_P}$ to V_N^0. It was obtained in [16] that Φ is in the Sobolev space $W^{s_0-\epsilon}(R)$ for any $\epsilon > 0$, where $s_0 := -\log_4(\rho(\mathbf{T_P}|_{V_N^0}))$, and $\rho(\mathbf{T_P}|_{V_N^0})$ denotes the spectral radius of $\mathbf{T_P}|_{V_N^0}$. Hence $\Phi \in C^{s_0-\frac{1}{2}-\epsilon}(R)$ for any $\epsilon > 0$.

It was shown in [17] that if λ_0 is a nonzero eigenvalue of $\mathcal{T}_{\mathbf{P}}$, then λ_0 is an eigenvalue of $\mathbf{T}_{\mathbf{P}}|_{V_N^0}$ if and only if $\operatorname{rank}((\mathbf{M}_N)^T(\mathbf{u}_1,\cdots,\mathbf{u}_l)) < l$, where $\mathbf{u}_1,\cdots,\mathbf{u}_l$ are a basis of the λ_0-eigenspace of $\mathcal{T}_{\mathbf{P}}$. Therefore $\rho(\mathbf{T}_{\mathbf{P}}|_{V_N^0})$ is the maximum of the moduli of all the eigenvalues λ_0 of $\mathcal{T}_{\mathbf{P}}$ satisfying $\operatorname{rank}((\mathbf{M}_N)^T(\mathbf{u}_1,\cdots,\mathbf{u}_l)) < l$.

Assume that $\{\mathbf{P},\mathbf{Q}\}$ is an FIR multifilter bank generating the orthogonal scaling function Φ and multiwavelet Ψ. It was shown in [18] that $\mathbf{P}(0)$ satisfies Condition E and $\mathbf{P}$ satisfies the vanishing moment condition of order at least one, i.e., there exists a nonzero $1\times r$ vector $\mathbf{l}_0^0$ such that

$$\mathbf{l}_0^0\mathbf{P}(0)=\mathbf{l}_0^0,\quad \mathbf{l}_0^0\mathbf{P}(\pi)=\mathbf{0}. \tag{17}$$

By (4), $\mathbf{l}_0^0\mathbf{P}(0)^T=\mathbf{l}_0^0$. Thus $\mathbf{P}(0)(\mathbf{l}_0^0)^T=(\mathbf{l}_0^0)^T$, i.e., $(\mathbf{l}_0^0)^T$ is a right 1-eigenvector of $\mathbf{P}(0)$. By (2), $\widehat{\Phi}(0)$ is a right 1-eigenvector of $\mathbf{P}(0)$. Therefore up to a constant, $(\mathbf{l}_0^0)^T=\widehat{\Phi}(0)$ since 1 is a simple eigenvalue of $\mathbf{P}(0)$. Assume Ψ is the corresponding multiwavelet with matrix highpass filter $\mathbf{Q}$. By (7), $\mathbf{l}_0^0\mathbf{P}(0)\mathbf{Q}(0)^T+\mathbf{l}_0^0\mathbf{P}(\pi)\mathbf{Q}(\pi)^T=0$. Thus $\mathbf{l}_0^0\mathbf{Q}(0)^T=0$ and hence $\mathbf{Q}(0)\widehat{\Phi}(0)=0$. By (6), we have $\widehat{\Psi}(0)=0$.

Proposition 1: Assume that $\{\mathbf{P},\mathbf{Q}\}$ is an FIR multifilter bank generating the multiwavelet Ψ. Then each component of Ψ is a bandpass function, i.e., $\widehat{\Psi}(0)=0$.

For a window function f (with some smoothness and decay at infinity), the **time-duration** Δ_f of f is defined by

$$\Delta_f^2 := \int_{-\infty}^{+\infty}(t-\bar{t})^2|f(t)|^2dt/E,$$

where $\bar{t}$ is the center in the time domain defined as

$$\bar{t} := \int_{-\infty}^{+\infty} t|f(t)|^2dt/E,\quad E:=\int_{-\infty}^{+\infty}|f(t)|^2dt.$$

The **frequency-bandwidth** of f denoted by $\Delta_{\widehat{f}}$ is defined in the same way with f replaced by $\widehat{f}$. Then

$$\Delta_f\Delta_{\widehat{f}} \geq 1/2. \tag{18}$$

Equation (18) is called the uncertainty principle. Equality in (18) holds if and only if f is the Gaussian $exp(-ct^2)$, for some constant $c\in R$ ([19], [26]). The product of time-duration and frequency-bandwidth $\Delta_f\Delta_{\widehat{f}}$ is called the **resolution cell** (called **information cell** in [8]).

By Proposition 1, each component ψ_j of the orthogonal multiwavelet Ψ is a bandpass function. Therefore, as in the scalar case, we shall also discuss the frequency-bandwidth $\widetilde{\Delta}_{\widehat{\psi}_j}$ of ψ_j defined by ([28], [12], [7])

$$\widetilde{\Delta}_{\widehat{\psi}_j}^2 := \frac{\int_0^{+\infty}(\omega-\overline{\overline{\omega}})^2|\widehat{\psi}_j(\omega)|^2d\omega}{\int_0^{+\infty}|\widehat{\psi}_j(\omega)|^2d\omega}, \tag{19}$$

where

$$\overline{\overline{\omega}} := \frac{\int_0^{+\infty}\omega|\widehat{\psi}_j(\omega)|^2d\omega}{\int_0^{+\infty}|\widehat{\psi}_j(\omega)|^2d\omega}.$$

One can check that for real ψ_j, $\widetilde{\Delta}_{\widehat{\psi}_j}^2=\Delta_{\widehat{\psi}_j}^2-(\overline{\overline{\omega}})^2$. If $\widehat{\psi}_j(0)=0$, then $\Delta_{\psi_j}\widetilde{\Delta}_{\widehat{\psi}_j}>1/2$ holds (see [7] and [12]).

In the final part of this section, we give the procedure to construct the OPTFR-multiwavelets. At first let us recall the general procedure to construct multiwavelets. We first construct FIR matrix lowpass filter $\mathbf{P}=\sum_k\mathbf{P}_ke^{-ik\omega}$ such that $\mathbf{P}$ is a matrix CQF and the corresponding compactly supported refinable vector Φ generates an orthogonal MRA (we need only to check that matrix $\mathcal{T}_{\mathbf{P}}$ defined by (12) satisfies Condition E). Then we construct the matrix highpass filter $\mathbf{Q}=\sum_k\mathbf{Q}_ke^{-ik\omega}$ such that $\mathbf{Q}$ satisfies (7) and (8). The compactly supported vector-valued function Ψ defined by (5) is the orthogonal multiwavelet. The CQF condition for $\mathbf{P}$, and the orthogonal conditions (7), (8) for $\mathbf{Q}$ are equivalent to

$$\sum_{k\in Z}\mathbf{P}_k\mathbf{P}_{k+2j}^T=\frac{1}{2}\delta_j\mathbf{I}_r,\quad j\in Z, \tag{20}$$

and

$$\sum_{k\in Z}\mathbf{P}_k\mathbf{Q}_{k+2j}^T=0,\quad j\in Z, \tag{21}$$

$$\sum_{k\in Z}\mathbf{Q}_k\mathbf{Q}_{k+2j}^T=\frac{1}{2}\delta_j\mathbf{I}_r,\quad j\in Z. \tag{22}$$

In the construction of the OPTFR-multiwavelets, the parameter expressions for the matrix coefficients $\mathbf{P}_k,\mathbf{Q}_k$ of the multifilter bank are required. Therefore the procedure to construct the OPTFR-multiwavelets is as follows:

1) to find parameter expressions for the matrix coefficients $\mathbf{P}_k,\mathbf{Q}_k$ with $\mathbf{P}_k,\mathbf{Q}_k$ satisfying (20), (21) and (22);

2) to fix the values of the parameters for $\mathbf{P}_k,\mathbf{Q}_k$ by minimizing the areas of the resolution cells $\Delta_{\phi_j}\Delta_{\widehat{\phi}_j}$ and $\Delta_{\psi_j}\Delta_{\widehat{\psi}_j}$ (or $\Delta_{\phi_j}\Delta_{\widehat{\phi}_j}$ and $\Delta_{\psi_j}\widetilde{\Delta}_{\widehat{\psi}_j}$), where $\Phi=(\phi_1,\cdots,\phi_r)^T$ is a $(2,\mathbf{P})$ refinable vector and $\Psi=(\psi_1,\cdots,\psi_r)^T$ is the vector-valued function defined by (5);

3) to check that the matrix $\mathcal{T}_{\mathbf{P}}$ corresponding to the optimal matrix lowpass filter $\mathbf{P}$ satisfies Condition E.

In this paper two cases in (2) will be considered. In the next section, formulas to compute $\Delta_{\phi_j},\Delta_{\widehat{\phi}_j},\Delta_{\psi_j}$ and $\Delta_{\widehat{\psi}_j}$ are drawn.

III. Energy moments in the time-frequency plane

Assume that $\mathbf{P}(\omega)=\sum_{k=0}^N\mathbf{P}_ke^{-ik\omega}$ is an FIR matrix filter and $\Phi=(\phi_1,\cdots,\phi_r)^T\in L^2(R)$ is a compactly supported $(2,\mathbf{P})$ refinable vector. Let $\Psi=(\psi_1,\cdots,\psi_r)^T\in L^2(R)$ be the vector-valued function defined by

$$\Psi(x)=2\sum_{k=0}^N\mathbf{Q}_k\Phi(2x-k), \tag{23}$$

for some $r\times r$ matrices $\mathbf{Q}_k$. In this section, we provide formulas to represent the energy moments of Φ and Ψ in the time-frequency plane in terms of $\mathbf{P}_k$ and $\mathbf{Q}_k$, from which the areas of the resolution cells $\Delta_{\phi_j}\Delta_{\widehat{\phi}_j}$, $\Delta_{\psi_j}\Delta_{\widehat{\psi}_j}$ can be computed. In this section, $\mathbf{P},\mathbf{Q}$ need not have to satisfy

(4), (7) or (8). The formulas for energy moments in the time domain and frequency domain will be carried out in Section III.A and Section III.B, respectively.

A. Energy Moments in the Time Domain

For a real $r\times 1$ vector function $\mathbf{F} = (f_1, \cdots, f_r)^T \in L^2(R)$ supported in $[0, N]$, define the energy moments of $\mathbf{F}$ in the time domain by

$$\mathbf{I}_F^\beta(y) := \int_{-\infty}^{+\infty} x^\beta \mathbf{F}(x)\mathbf{F}^T(x-y)dx, \quad \beta \in Z_+,$$

and define the $(2N-1)r^2 \times 1$ vector

$$\text{vec}(\mathbf{I}_\mathbf{F}^\beta) := (\text{vec}(\mathbf{I}_\mathbf{F}^\beta(j)))_{j=N-1}^{1-N}, \tag{24}$$

where for an $r \times r$ matrix $\mathbf{B}$, $\text{vec}(\mathbf{B})$ is the $r^2 \times 1$ vector defined by (10).

Suppose $\mathbf{P}(\omega) = \sum_{k=0}^N \mathbf{P}_k e^{-ik\omega}$ is an FIR matrix filter. In the following we assume that $\mathbf{P}$ satisfies the vanishing moment conditions of order at least one. Let $\mathbf{l}_0^0$ be the left 1-eigenvector of $\mathbf{P}(0)$ with $\mathbf{l}_0^0\mathbf{P}(\pi) = 0$. Let $\mathbf{L}_N^0$ denote the $1 \times (2N-1)r^2$ vector defined by (16), i.e., the vector

$$\mathbf{L}_N^0 = (\mathbf{l}_0^0 \otimes \mathbf{l}_0^0, \cdots, \mathbf{l}_0^0 \otimes \mathbf{l}_0^0).$$

For $\beta \in Z_+$, denote

$$\mathcal{T}_\mathbf{P}^\beta := (2\mathcal{A}_{2i-j}^\beta)_{1-N\le i,j \le N-1}, \tag{25}$$

where $\mathcal{A}_j^\beta := \sum_{k=0}^N k^\beta \mathbf{P}_{k-j} \otimes \mathbf{P}_k$. We know that $\mathcal{T}_\mathbf{P}^0$ is the matrix $\mathcal{T}_\mathbf{P}$ defined by (12). Then we have

Proposition 2: Suppose $\mathbf{P} = \sum_{k=0}^N \mathbf{P}_k e^{-ik\omega}$ is an FIR matrix filter and $\Phi \in L^2(R)$ is a compactly supported $(2, \mathbf{P})$ refinable vector. Let $\text{vec}(\mathbf{I}_\Phi^\beta)$ and $\mathcal{T}_\mathbf{P}^\beta$ be the vectors and matrices defined by (24) and (25), respectively, then

$$\text{vec}(\mathbf{I}_\Phi^\beta) = 2^{-\beta} \sum_{0\le\alpha\le\beta} \binom{\beta}{\alpha} \mathcal{T}_\mathbf{P}^\alpha \text{vec}(\mathbf{I}_\Phi^{\beta-\alpha}), \quad \beta \in Z_+. \tag{26}$$

Furthermore, if Φ is stable, then $\text{vec}(\mathbf{I}_\Phi^\beta)$ are uniquely determined by (26) and the requirement

$$\mathbf{L}_N^0 \text{vec}(\mathbf{I}_\Phi^0) = |\mathbf{l}_0^0 \widehat{\Phi}(0)|^2. \tag{27}$$

The proof of Proposition 2 is provided in the Appendix.

Let Ψ be the vector-valued function defined by (23). For $\beta \in Z_+$, define

$$\mathcal{T}_\mathbf{Q}^\beta := (2\mathcal{B}_{2i-j}^\beta)_{1-N\le i,j\le N-1}, \tag{28}$$

where $\mathcal{B}_j^\beta := \sum_{k=0}^N k^\beta \mathbf{Q}_{k-j} \otimes \mathbf{Q}_k$. Then by a similar derivation as in the proof of Proposition 2, we have

Proposition 3: Let $\mathcal{T}_\mathbf{Q}^\beta$ be the matrix defined by (28), and $\text{vec}(\mathbf{I}_\Phi^\beta)$, $\text{vec}(\mathbf{I}_\Psi^\beta)$ be the vectors defined by (24). Then

$$\text{vec}(\mathbf{I}_\Psi^\beta) = 2^{-\beta} \sum_{0\le\alpha\le\beta} \binom{\beta}{\alpha} \mathcal{T}_\mathbf{Q}^\alpha \text{vec}(\mathbf{I}_\Phi^{\beta-\alpha}), \quad \beta \in Z_+. \tag{29}$$

We note that if every component of $\Phi = (\phi_1, \cdots, \phi_r)^T$ is normalized , i.e., $\|\phi_j\|_2 = 1$, then the time-duration Δ_{ϕ_j} of ϕ_j is given by

$$\Delta_{\phi_j}^2 = (\mathbf{e}_j \otimes \mathbf{e}_j)^T \text{vec}(\mathbf{I}_\Phi^2(0)) - ((\mathbf{e}_j \otimes \mathbf{e}_j)^T \text{vec}(\mathbf{I}_\Phi^1(0)))^2.$$

Similarly the time-duration of each component of Ψ is given in terms of $\text{vec}(\mathbf{I}_\Psi^1(0))$ and $\text{vec}(\mathbf{I}_\Psi^2(0))$. If Φ is stable, by Proposition 2, $\text{vec}(\mathbf{I}_\Phi^0)$ is the right 1-eigenvector of $\mathcal{T}_\mathbf{P}$ satisfying (27). By (26),

$$(2^\beta \mathbf{I} - \mathcal{T}_\mathbf{P})\text{vec}(\mathbf{I}_\Phi^\beta) = \sum_{0<\alpha\le\beta} \binom{\beta}{\alpha} \mathcal{T}_\mathbf{P}^\alpha \text{vec}(\mathbf{I}_\Phi^{\beta-\alpha}).$$

Since the spectral radius of $\mathcal{T}_\mathbf{P}$ is not greater than 1, $\text{vec}(\mathbf{I}_\Phi^\beta)$ for $\beta = 1, 2$ are uniquely determined by $\text{vec}(\mathbf{I}_\Phi^\alpha)$, $0 \le \alpha < \beta$, and $\text{vec}(\mathbf{I}_\Psi^\beta), 0 \le \beta \le 2$, are uniquely determined by (29). Therefore Propositions 2, 3 provide the formulas for the computation of the time-durations of Φ and Ψ.

For the GHM-multiwavelet, by Proposition 2 and Proposition 3, the time-durations Δ_{ϕ_1}, Δ_{ϕ_2} for the scaling functions and Δ_{ψ_1}, Δ_{ψ_2} for the multiwavelets are respectively

.1324, .1974, .2497, .3026.

B. Energy Moments in the Frequency Domain

To compute the frequency-bandwidths of scaling functions and multiwavelets, we define the energy moments of a vector-valued function in the frequency domain. For a real $r \times 1$ vector function $\mathbf{F} = (f_1, \cdots, f_r)^T \in L^2(R)$ supported in $[0, N]$, if $\mathbf{F}$ is in Sobolev space $W^s(R)$ for some $s \ge 0$, define for β, $0 \le \beta \le 2s$, the energy moments of $\mathbf{F}$ in the frequency domain by

$$\mathbf{D}_\mathbf{F}^\beta(\xi) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^\beta \widehat{\mathbf{F}}(\omega)\widehat{\mathbf{F}}^*(\omega) e^{i\omega\xi} d\omega,$$

and define the $(2N-1)r^2 \times 1$ vector

$$\text{vec}(\mathbf{D}_\mathbf{F}^\beta) := (\text{vec}(\mathbf{D}_\mathbf{F}^\beta(j)))_{j=N-1}^{1-N}. \tag{30}$$

We have

Proposition 4: Suppose $\mathbf{P} = \sum_{k=0}^N \mathbf{P}_k e^{-ik\omega}$ is an FIR matrix filter, and Φ is a compactly supported $(2, \mathbf{P})$ refinable vector with $\Phi \in W^s(R)$ for some $s \ge 0$. For any $\beta \in Z_+, \beta \le 2s$, let $\text{vec}(\mathbf{D}_\Phi^\beta)$ be the vectors defined by (30). Then

$$\mathcal{T}_\mathbf{P} \text{vec}(\mathbf{D}_\Phi^\beta) = 2^{-\beta} \text{vec}(\mathbf{D}_\Phi^\beta). \tag{31}$$

Furthermore, if $\mathbf{P}$ satisfies the vanishing moment conditions of order m for some vector $\mathbf{l}_0^\beta$, and satisfies (14) for some $m_0 \in Z_+$, then for any $\beta \in Z_+$, $\beta \le \min(m + m_0 - 1, 2s)$, and for any $\alpha \in Z_+$, $\alpha \le \beta$,

$$\mathbf{L}_N^\alpha \text{vec}(\mathbf{D}_\Phi^\beta) = \begin{cases} 0, & 0 \le \alpha < \beta, \\ i^\beta \beta! |\mathbf{l}_0^0 \widehat{\Phi}(0)|^2, & \alpha = \beta, \end{cases} \tag{32}$$

where $\mathbf{L}_N^\alpha$ are the vectors defined by (16).

The proof of Proposition 4 is presented in the Appendix.

For Ψ defined by (23), define $\text{vec}(\mathbf{D}^\beta_\Psi)$ by (30). A similar proof as that of Proposition 4 gives

Proposition 5: Suppose $\mathbf{P} = \sum_{k=0}^N \mathbf{P}_k e^{-ik\omega}$ is an FIR matrix filter, and Φ is a compactly supported $(2, \mathbf{P})$ refinable vector with $\Phi \in W^s(R)$ for some $s \geq 0$. Let Ψ be the vector-valued function defined by (23), and for any $\beta \in Z_+$, $\beta \leq 2s$, let $\text{vec}(\mathbf{D}^\beta_\Phi)$ and $\text{vec}(\mathbf{D}^\beta_\Psi)$ be the vectors defined by (30). Then

$$\text{vec}(\mathbf{D}^\beta_\Psi) = 2^\beta \mathcal{T}_\mathbf{Q} \text{vec}(\mathbf{D}^\beta_\Phi). \tag{33}$$

In the scalar case, $r = 1$, the above results were obtained in [31].

For real $\Phi = (\phi_1, \cdots, \phi_r)^T$ and $\Psi = (\psi_1, \cdots, \psi_r)^T$ with $\|\phi_j\| = 1, \|\psi_j\| = 1$, the frequency-bandwidths $\Delta_{\widehat{\phi_j}}$, $\Delta_{\widehat{\psi_j}}$ of ϕ_j, ψ_j are respectively given by

$$\Delta^2_{\widehat{\phi_j}} = (\mathbf{e}_j \otimes \mathbf{e}_j)^T \text{vec}(\mathbf{D}^2_\Phi(0)),$$
$$\Delta^2_{\widehat{\psi_j}} = (\mathbf{e}_j \otimes \mathbf{e}_j)^T \text{vec}(\mathbf{D}^2_\Psi(0)).$$

By Proposition 5, the vectors $\text{vec}(\mathbf{D}^\beta_\Psi)$ are determined by $\text{vec}(\mathbf{D}^\beta_\Phi)$. Therefore in order to decide on the frequency-bandwidths of orthogonal scaling functions and multiwavelets, we need to decide on $\text{vec}(\mathbf{D}^2_\Phi)$. By Proposition 4, $\text{vec}(\mathbf{D}^2_\Phi)$ is a right 1/4-eigenvector of $\mathcal{T}_\mathbf{P}$ satisfying (32). If

i) $\mathbf{P}$ satisfies the vanishing moment conditions of order at least two;

ii) there exists a vector $\mathbf{l}^2_0$ satisfying (14) for $m = 2, m_0 = 1$;

iii) $\Phi \in W^1(R)$;

iv) 1/4 is a simple eigenvalue of $\mathcal{T}_\mathbf{P}$;

then $\text{vec}(D^2_\Phi)$ is uniquely determined by (31) and the requirement $\mathbf{L}^2_N \text{vec}(\mathbf{D}^2_\Phi) = -2|\mathbf{l}^0_0 \widehat{\Phi}(0)|^2$. Usually the multiwavelets constructed have some smoothness, and we note that if 1/4 is not an eigenvalue of $\mathbf{P}(0)$ (this condition can be easily met in practice), then $\mathbf{P}$ satisfies (ii). Thus in the design of the OPTFR-multiwavelets, if we use Propositions 4, 5 to compute the frequency-bandwidths of scaling functions and multiwavelets, we mainly consider conditions (i) and (iv). When we consider the area of the resolution cell $\Delta_{\psi_j} \widetilde{\Delta}_{\widehat{\psi_j}}$, by $\widetilde{\Delta}^2_{\widehat{\psi_j}} = \Delta^2_{\widehat{\psi_j}} - (\overline{\widetilde{\omega}})^2$, what we need to compute is the center $\overline{\widetilde{\omega}}$ of ψ_j in the frequency domain since Δ_{ψ_j} and $\Delta_{\widehat{\psi_j}}$ can be obtained by the above formulas. In this case, we will use the cascade algorithm to approximate multiwavelets, i.e., we will compute the centers $\overline{\widetilde{\omega}}$ of the components of Ψ_n, where $\Psi_n(x) = 2\sum_k \mathbf{Q}_k \Phi_n(2x-k)$ and Φ_n is defined by (3) for some n (e.g. $n = 8$). In the case that 1/4 is an eigenvalue of $\mathcal{T}_\mathbf{P}$ but it is not simple, or that we do not want to construct the scaling functions which provide approximation order 2, we also use the cascade algorithm to compute approximately the frequency-bandwidths of scaling functions and multiwavelets.

For the GHM-multiwavelet, the matrix lowpass filter $\mathbf{P}$ satisfies the vanishing moment conditions of order 2, the corresponding transition operator $\mathcal{T}_\mathbf{P}$ satisfies Condition E and 1/4 is a simple eigenvalue of $\mathcal{T}_\mathbf{P}$. The scaling function Φ and multiwavelet Ψ are in $W^{1.5-\epsilon}(R)$ for any $\epsilon > 0$. Thus we can use Proposition 4 and Proposition 5 to compute the frequency-bandwidths of Φ and Ψ. The frequency-bandwidths of ϕ_1, ϕ_2, ψ_1 and ψ_2 are respectively

$$4.2762, \quad 4.9281, \quad 13.6172, \quad 10.6100.$$

Thus the areas of the resolution cells for ϕ_1, ϕ_2, ψ_1 and ψ_2 are respectively

$$.5662, \quad .9727, \quad 3.3999, \quad 3.2104.$$

IV. Constructions of OPTFR-multiwavelets

In this section, we will construct the OPTFR-wavelets of multiplicity 2 with support lengths from 2 to 6. Here the symmetry property of scaling functions and multiwavelets is considered, and we shall construct scaling functions and multiwavelets with the first component being symmetric and the second one antisymmetric. For $2 \leq N \leq 6$, let ${}_N\Phi$, ${}_N\Psi$ denote the symmetric/antisymmetric scaling functions and multiwavelets supported in $[0, N]$ with corresponding multifilter banks ${}_N\mathbf{P}(\omega) = \sum_{k=0}^N \mathbf{P}_k e^{-ik\omega}$, ${}_N\mathbf{Q}(\omega) = \sum_{k=0}^N \mathbf{Q}_k e^{-ik\omega}$. We will construct ${}_N\mathbf{P}, {}_N\mathbf{Q}$ such that $\mathbf{P}_k$, $\mathbf{Q}_k$ satisfy (see e.g., [1])

$$\mathbf{P}_k = U\mathbf{P}_{N-k}U, \quad \mathbf{Q}_k = U\mathbf{Q}_{N-k}U, \quad k = 0, \cdots, N,$$

where U is the 2×2 unitary matrix defined by $U := \text{diag}(1, -1)$.

In the following, we first find the parameter expressions for the matrix coefficients $\mathbf{P}_k$ and $\mathbf{Q}_k$. With these explicit expressions, we construct the OPTFR-multiwavelets by minimizing the sum

$${}_NS := \sum_{j=1}^2 \Delta_{{}_N\phi_j} \Delta_{\widehat{{}_N\phi_j}} + \Delta_{{}_N\psi_j} \Delta_{\widehat{{}_N\psi_j}} \quad \text{or}$$
$${}_N\widetilde{S} := \sum_{j=1}^2 \Delta_{{}_N\phi_j} \Delta_{\widehat{{}_N\phi_j}} + \Delta_{{}_N\psi_j} \widetilde{\Delta}_{\widehat{{}_N\psi_j}}.$$

Let ${}_N\Phi^o$ and ${}_N\Psi^o$ (${}_N\widetilde{\Phi}^o$ and ${}_N\widetilde{\Psi}^o$, respectively) denote the OPTFR-scaling functions and multiwavelets by minimizing ${}_NS$ (${}_N\widetilde{S}$, respectively). The optimal problem for the sums of the areas of resolution cells ${}_NS, {}_N\widetilde{S}$ is non-linear. We will use the simplex search algorithm to minimize ${}_NS, {}_N\widetilde{S}$. The minimizations are performed with different random initial values and the best results are chosen. In the following, we also construct the smoothest scaling functions ${}_N\Phi^s$ and multiwavelets ${}_N\Psi^s$ based on the smoothness estimates provided above and the parameter expressions for the matrix coefficients $\mathbf{P}_k$ and $\mathbf{Q}_k$. ${}_N\Phi^s$ and ${}_N\Psi^s$ can be used as good starting values for the constructions of the OPTFR-multiwavelets.

Example 1: In this example, we first give the explicit expressions for the matrix coefficients $\mathbf{P}_k$ and $\mathbf{Q}_k$ for symmetric/anti symmetric scaling function ${}_2\Phi$ and multiwavelet ${}_2\Psi$ supported in $[0, 2]$, and then construct the OPTFR-multiwavelet. Assume $\mathbf{P}_k = \mathbf{0}, \mathbf{Q}_k = \mathbf{0}, k <$

$0, k > 2$. By (20) for $j = 1$, $\mathbf{P}_0\mathbf{P}_2^T = 0$. This and the fact that both $\mathbf{P}_0, \mathbf{P}_2$ are not zero matrix imply that rank($\mathbf{P}_0$) =rank($\mathbf{P}_2$) = 1. Thus we have

$$\mathbf{P}_0 = \frac{1}{2}\begin{pmatrix} a_1 & c_0 a_1 \\ a_2 & c_0 a_2 \end{pmatrix}, \quad \mathbf{P}_1 = \frac{1}{2}\begin{pmatrix} a_3 & 0 \\ 0 & a_4 \end{pmatrix},$$
$$\mathbf{P}_2 = \frac{1}{2}\begin{pmatrix} a_1 & -c_0 a_1 \\ -a_2 & c_0 a_2 \end{pmatrix}$$

for some $c_0 \in R$. By $\mathbf{P}_0\mathbf{P}_2^T = 0$ again, $c_0^2 = 1$. We choose $c_0 = 1$ (the choice of $c_0 = -1$ only reduces the sign changes of the second component of ${}_2\Phi$). Therefore

$$\mathbf{P}(0) = \begin{pmatrix} a_1 + \frac{1}{2}a_3 & 0 \\ 0 & a_2 + \frac{1}{2}a_4 \end{pmatrix},$$
$$\mathbf{P}(\pi) = \begin{pmatrix} a_1 - \frac{1}{2}a_3 & 0 \\ 0 & a_2 - \frac{1}{2}a_4 \end{pmatrix}.$$

By the symmetry of ${}_2\Phi$, ${}_2\widehat{\Phi}(0) = (1, 0)^T$ is the unique normalized 1-eigenvector of $\mathbf{P}(0)$. By (17), $a_1 = 1/2, a_3 = 1$.

One can check that the CQF condition (20) for $j = 0$ is equivalent to

$$4a_1^2 + a_4^2 = 2.$$

Thus

$$a_2 = \frac{\sqrt{2}}{2}\sin\theta, \quad a_4 = \sqrt{2}\cos\theta, \quad \theta \in [-\pi, \pi),$$

and

$$\mathbf{P}_0 = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ \sqrt{2}\sin\theta & \sqrt{2}\sin\theta \end{pmatrix},$$
$$\mathbf{P}_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2}\cos\theta \end{pmatrix}, \quad \theta \in [-\pi, \pi),$$

with $\mathbf{P}_2 = U\mathbf{P}_0U$.

Assume $\mathbf{Q}_k, 0 \le k \le 2$, are given by

$$\mathbf{Q}_0 = \frac{1}{2}\begin{pmatrix} b_1 & b_1 \\ b_2 & b_2 \end{pmatrix}, \quad \mathbf{Q}_1 = \frac{1}{2}\begin{pmatrix} b_3 & 0 \\ 0 & b_4 \end{pmatrix},$$

and $\mathbf{Q}_2 = U\mathbf{Q}_0U$. Then (21) is equivalent to

$$4b_1^2 + b_3^2 = 2, \quad 4b_2^2 + b_4^2 = 2,$$

and (22) is equivalent to

$$2b_1 + b_3 = 0, \quad 4ab_2 + b_4\sqrt{2 - 4a^2} = 0.$$

Thus we have

$$b_1 = \pm 1/2, \quad b_3 = \mp 1, \quad b_2 = \pm\frac{\sqrt{2}}{2}\cos\theta, \quad b_4 = \mp\sqrt{2}\sin\theta.$$

The choice of positive or negative sign for b_j only reduces the sign changes of ${}_2\Psi$. We choose

$$\mathbf{Q}_0 = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ \sqrt{2}\cos\theta & \sqrt{2}\cos\theta \end{pmatrix},$$
$$\mathbf{Q}_1 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -\sqrt{2}\sin\theta \end{pmatrix},$$
$$\mathbf{Q}_2 = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -\sqrt{2}\cos\theta & \sqrt{2}\cos\theta \end{pmatrix}.$$

For the case $N = 2$, the explicit expressions for $\mathbf{P}_k, \mathbf{Q}_k$ are provided above. For

$$\theta = -1.20942920288819(-\arccos(\sqrt{2}/4)),$$

the corresponding transition operator $T_{\mathbf{P}}$ (i.e., matrix $\mathcal{T}_{\mathbf{P}}$) satisfies condition E, ${}_2\Phi$ provides approximation order 2 with ${}_2\Phi \in W^{1.0545}(R)$. For such a choice of θ, the multiwavelet constructed above is the one constructed in [1], and in this case the areas of the resolution cells for ϕ_1, ϕ_2, ψ_1 and ψ_2 are respectively

$$1.1486, \quad 2.6412, \quad 3.8171, \quad 3.6006.$$

For other choices of θ, $\mathbf{P}$ does not satisfy the vanishing moment conditions of order 2, and we cannot use Propositions 4, 5 to compute the frequency-bandwidths of ${}_2\Phi$ and ${}_2\Psi$. Instead, we use the cascade algorithm to compute approximately the frequency-bandwidths of ${}_2\Phi$ and ${}_2\Psi$.

The minimum of the sum ${}_2S$ is attained at $\theta = -1.10157463780242$. For such choice of θ, the corresponding OPTFR-scaling function ${}_2\Phi^o$ and multiwavelets ${}_2\Psi^o$ are in $W^{.9482}(R)$ and the areas of the resolution cells for ${}_2\phi_1^o$, ${}_2\phi_2^o$, ${}_2\psi_1^o$, ${}_2\psi_2^o$ are respectively

$$1.0597, \quad 2.1828, \quad 3.3948, \quad 3.3671.$$

For sum ${}_2\widetilde{S}$, its minimum is attained at $\theta = -1.09579052470259$, and the areas of the resolution cells for the corresponding OPTFR-scaling function and multiwavelet ${}_2\widetilde{\phi}_1^o$, ${}_2\widetilde{\phi}_2^o$, and ${}_2\widetilde{\psi}_1^o$, ${}_2\widetilde{\psi}_2^o$ are respectively

$$1.0788, \quad 2.2189, \quad 2.1916, \quad 1.8993.$$

In Example 1 and the following examples, the areas of the resolution cells for the OPTFR-multiwavelet ${}_N\widetilde{\psi}_j^o$ constructed by minimizing the sum ${}_N\widetilde{S}$ should be understood as $\Delta_{{}_N\widetilde{\psi}_j^o}\widetilde{\Delta}_{\widehat{{}_N\widetilde{\psi}_j^o}}$.

Example 2: In this example, we shall present the explicit expressions for the matrix coefficients of the lowpass and highpass filters ${}_3\mathbf{P}, {}_3\mathbf{Q}$ for the scaling functions ${}_3\Phi$ and multiwavelets ${}_3\Psi$ supported in $[0, 3]$, and then construct the OPTFR-multiwavelets.

Suppose the matrix coefficients $\mathbf{P}_k$ of the lowpass filter ${}_3\mathbf{P}$ for scaling function ${}_3\Phi$ are given by

$$\mathbf{P}_0 = \frac{1}{2}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad \mathbf{P}_1 = \frac{1}{2}\begin{pmatrix} a_5 & a_6 \\ a_7 & a_8 \end{pmatrix},$$

and $\mathbf{P}_2 = U\mathbf{P}_1U, \mathbf{P}_3 = U\mathbf{P}_0U, \mathbf{P}_k = \mathbf{0}, k < 0, k > 3$. Then

$$\mathbf{P}(0) = \begin{pmatrix} a_1 + a_5 & 0 \\ 0 & a_4 + a_8 \end{pmatrix}$$

and

$$\mathbf{P}(\pi) = \begin{pmatrix} 0 & a_2 - a_6 \\ a_3 - a_7 & 0 \end{pmatrix}.$$

By (17) with $\mathbf{l}_0^0 = (1, 0)$, one obtains

$$a_5 = 1 - a_1, \quad a_6 = a_2.$$

And one can check that (20) is equivalent to

$$\begin{cases} a_2^2 = a_1(1-a_1) \\ a_3^2 + a_4^2 + a_7^2 + a_8^2 = 1 \\ a_4 a_8 = a_3 a_7 \\ a_2(a_4+a_8) = a_1(a_7-a_3) + a_3. \end{cases} \tag{34}$$

By the first equation in (34), we have

$$a_1 = \frac{1}{2}(1+\sin\xi), \quad a_2 = \frac{1}{2}\cos\xi, \quad \xi \in [-\pi,\pi),$$

and by the second and third equation in (34),

$$(a_3 - a_7)^2 + (a_4 + a_8)^2 = 1.$$

Thus

$$\begin{cases} a_3 - a_7 = & \sin\eta \\ a_4 + a_8 = & \cos\eta, \quad \eta \in [-\pi,\pi). \end{cases} \tag{35}$$

Equation (35) and the last equation in (34) lead to

$$\begin{aligned} a_3 &= \frac{1}{2}\sin\eta + \frac{1}{2}(\cos\xi\cos\eta + \sin\xi\sin\eta) \\ &= \cos(\frac{\pi}{4} - \frac{\xi}{2})\sin(\frac{\pi}{4} - \frac{\xi}{2} + \eta), \end{aligned}$$

and

$$\begin{aligned} a_7 &= -\frac{1}{2}\sin\eta + \frac{1}{2}\cos(\xi - \eta) \\ &= \sin(\frac{\pi}{4} - \frac{\xi}{2})\cos(\frac{\pi}{4} - \frac{\xi}{2} + \eta). \end{aligned}$$

By (35) and the third equation in (34), we have

$$(a_8 - \frac{\cos\eta}{2})^2 = \frac{\sin^2(\eta - \xi)}{4}.$$

Thus

$$a_8 = \frac{\cos\eta}{2} \pm \frac{\sin(\eta-\xi)}{2}.$$

If we hope that there exist highpass filters such that (7) and (8) hold, we shall choose the plus sign in a_8. Hence

$$a_8 = \sin(\frac{\pi}{4} - \frac{\xi}{2})\sin(\frac{\pi}{4} - \frac{\xi}{2} + \eta)$$

and

$$a_4 = \cos(\frac{\pi}{4} - \frac{\xi}{2})\sin(\frac{\pi}{4} - \frac{\xi}{2} + \eta).$$

After making the changes of variables:

$$\theta = \frac{\pi}{2} - \xi, \quad \zeta = \frac{\pi}{4} - \frac{\xi}{2} + \eta,$$

we have

$$\begin{aligned} \mathbf{P}_0 &= \frac{1}{2}\cos\frac{\theta}{2}\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\zeta & \cos\zeta \end{pmatrix}, \\ \mathbf{P}_1 &= \frac{1}{2}\sin\frac{\theta}{2}\begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \\ -\cos\zeta & -\sin\zeta \end{pmatrix}, \theta,\zeta \in [-\pi,\pi) \end{aligned}$$

and $\mathbf{P}_2 = U\mathbf{P}_1 U$, $\mathbf{P}_3 = U\mathbf{P}_0 U$.

Using the above derivation, one can obtain that the matrix coefficients $\mathbf{Q}_k$ for the corresponding highpass filter ${}_3\mathbf{Q}$ are given by (omit the details here)

$$\begin{aligned} \mathbf{Q}_0 &= \frac{1}{2}\cos\frac{\theta}{2}\begin{pmatrix} \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ -\cos\zeta & \sin\zeta \end{pmatrix}, \\ \mathbf{Q}_1 &= \frac{1}{2}\sin\frac{\theta}{2}\begin{pmatrix} -\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ -\sin\zeta & \cos\zeta \end{pmatrix}, \end{aligned}$$

and $\mathbf{Q}_2 = U\mathbf{Q}_1 U$, $\mathbf{Q}_3 = U\mathbf{Q}_0 U, \mathbf{Q}_k = \mathbf{0}, k < 0, k > 3$.

In this case, there are two parameters θ, ζ to choose to obtain the scaling functions ${}_3\Phi$ and multiwavelets ${}_3\Psi$. By minimizing $\rho(\mathbf{T}_{{}_2\mathbf{P}}|_{V_3^o})$ we obtain the choice

$$(\theta,\zeta) = (3.41911388444093, -.04773984485434).$$

The corresponding scaling functions ${}_3\Phi^s$ (providing approximation order 2) and multiwavelets ${}_3\Psi^s$ are the smoothest and they are in $C^{1.2668}(R)$.

The property that the scaling function ${}_3\Phi$ has approximation order 2 is equivalent to the following condition under the assumption that ${}_3\Phi$ is stable:

$$\cos\theta - \cos(\zeta + \frac{\theta}{2}) - 2\cos(\zeta - \frac{\theta}{2}) + \frac{1}{2} = 0. \tag{36}$$

We then minimize ${}_3S$ under the constrained condition (36). The sum ${}_3S$ attains the minimum at

$$(\theta,\zeta) = (-2.87441379981790, 3.07512747587073).$$

In this case, the corresponding OPTFR-scaling function and multiwavelet ${}_3\Phi^o$, ${}_3\Psi^o$ are in $C^{1.2470}(R)$, and the areas of the resolution cells for ${}_3\phi_1^o$, ${}_3\phi_2^o$, ${}_3\psi_1^o$, ${}_3\psi_2^o$ are respectively

$$.6447, \quad 1.6951, \quad 3.1872, \quad 3.7523.$$

For the sum ${}_3\widetilde{S}$, its minimum is attained at (with ${}_3\widetilde{\Phi}^o$ providing approximation order 1)

$$(\theta,\zeta) = (2.84273685649241 - 3.0690176412534),$$

and the areas of the resolution cells for the corresponding OPTFR-scaling function and multiwavelet ${}_3\widetilde{\phi}_1^o$, ${}_3\widetilde{\phi}_2^o$, ${}_3\widetilde{\phi}_1^o$, ${}_3\widetilde{\psi}_2^o$ are respectively

$$.6388, \quad 1.6777, \quad 1.2135, \quad 1.07654.$$

In the following three examples, the explicit expressions for the matrix coefficients of multifilter banks $\{{}_N\mathbf{P}, {}_N\mathbf{Q}\}$ can be obtained similarly as in Examples 1 and 2, and the details are omitted here.

Example 3: In this example, we shall construct the OPTFR-multiwavelets supported in $[0, 4]$. The explicit expressions for the matrix coefficients of the matrix lowpass filter ${}_4\mathbf{P}$ are given by

$$\begin{aligned} \mathbf{P}_0 &= \tfrac{1}{8}\begin{pmatrix} 1-\sqrt{2}\sin\theta & 1-\sqrt{2}\sin\theta \\ \sqrt{2}(\sin\xi - \sin(\theta+\xi-\frac{\pi}{4})) & \sqrt{2}(\sin\xi - \sin(\theta+\xi-\frac{\pi}{4})) \end{pmatrix}, \\ \mathbf{P}_1 &= \tfrac{1}{4}\begin{pmatrix} 1 & \sqrt{2}\cos\theta \\ \sqrt{2}\cos(\theta+\xi-\frac{\pi}{4}) & \sqrt{2}\cos\xi \end{pmatrix} \\ \mathbf{P}_2 &= \tfrac{1}{4}\begin{pmatrix} 1+\sqrt{2}\sin\theta & 0 \\ 0 & \sqrt{2}(\sin\xi + \sin(\theta+\xi-\frac{\pi}{4})) \end{pmatrix}, \end{aligned}$$

and $\mathbf{P}_3 = U\mathbf{P}_1U, \mathbf{P}_4 = U\mathbf{P}_0U, \mathbf{P}_k = \mathbf{0}, k < 0, k > 4$, where $\theta, \xi \in [-\pi, \pi)$. The matrix coefficients for the corresponding matrix highpass filter ${}_4\mathbf{Q}$ are given by

$$\mathbf{Q}_0 = -\frac{1}{8}\begin{pmatrix} 1-\sqrt{2}\cos\theta & 1-\sqrt{2}\cos\theta \\ \sqrt{2}(\cos\xi - \cos(\theta+\xi-\frac{\pi}{4})) & \sqrt{2}(\cos\xi - \cos(\theta+\xi-\frac{\pi}{4})) \end{pmatrix},$$
$$\mathbf{Q}_1 = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2}\sin\theta \\ \sqrt{2}\sin(\theta+\xi-\frac{\pi}{4}) & \sqrt{2}\sin\xi \end{pmatrix},$$
$$\mathbf{Q}_2 = -\frac{1}{4}\begin{pmatrix} 1+\sqrt{2}\cos\theta & 0 \\ 0 & \sqrt{2}(\cos\xi + \cos(\theta+\xi-\frac{\pi}{4})) \end{pmatrix}$$

and $\mathbf{Q}_3 = U\mathbf{Q}_1U, \mathbf{Q}_4 = U\mathbf{Q}_0U, \mathbf{Q}_k = \mathbf{0}, k < 0, k > 4$.

There are also two variables θ and ξ to choose to obtain the scaling functions and multiwavelets. For

$$(\theta, \xi) = (1.01075860019276, 2.43693014166954),$$

we obtain the smoothest scaling function ${}_4\Phi^s$ and multiwavelet ${}_4\Psi^s$ with ${}_4\Phi^s$ providing approximation order 2, and ${}_4\Phi^s$, ${}_4\Psi^s \in C^{1.3161}(R)$. Here again we shall construct OPTFR-scaling function ${}_4\Phi^o$ and multiwavelet ${}_4\Phi^o$ with ${}_4\Phi^o$ providing approximation order 2. In this case, under the condition that ${}_4\Phi$ is stable, the property that the scaling function ${}_4\Phi$ has approximation order 2 is equivalent to

$$1 + 4\sin(\theta+\xi) - 2\sqrt{2}\cos\xi - 2\sin(\theta + \frac{\pi}{4}) = 0. \quad (37)$$

Thus we minimize the sum ${}_4S$ under (37). ${}_4S$ attains its minimum at

$$(\theta, \xi) = (.96630781393588, 2.51067760935378).$$

The corresponding ${}_4\Phi^o, {}_4\Psi^o$ are in $C^{1.2639}(R)$, and the areas of the resolution cells for ${}_4\phi_1^o$, ${}_4\phi_2^o$, ${}_4\psi_1^o$, ${}_4\psi_2^o$ are respectively

.7234, 1.7606, 2.9741, 3.1591.

The sum ${}_4\widetilde{S}$ attains its minimum at (with ${}_4\widetilde{\Phi}^o$ providing approximation order 1)

$$(\theta, \xi) = (1.03184046099418, 2.45611030970033),$$

and the areas of the resolution cells for ${}_4\widetilde{\phi}_1^o$, ${}_4\widetilde{\phi}_2^o$, ${}_4\widetilde{\psi}_1^o$, ${}_4\widetilde{\psi}_2^o$ are respectively

.6728, 1.7273, 1.1111, .80642.

Example 4: In this example, we construct the OPTFR-multiwavelets supported in $[0, 5]$. In this case, the explicit expression for the matrix coefficients of the multifilter bank for scaling functions ${}_5\Phi$ and multiwavelets ${}_5\Psi$ are given by

$$\mathbf{P}_0 = \frac{1}{2}\sin(\theta+\eta)\sin(\theta+\xi+\eta)\begin{pmatrix} \cos\xi & \sin\xi \\ \cos\eta & \sin\eta \end{pmatrix},$$
$$\mathbf{P}_1 = \frac{1}{2}\cos(\theta+\eta)\sin(\theta+\xi+\eta)\begin{pmatrix} \sin\xi & \cos\xi \\ \sin\eta & \cos\eta \end{pmatrix},$$
$$\mathbf{P}_2 = \frac{1}{2}\cos(\theta+\xi+\eta)\cdot \begin{pmatrix} \cos(\theta+\xi+\eta) & \sin(\theta+\xi+\eta) \\ \cos\theta & -\sin\theta \end{pmatrix},$$
$$\mathbf{P}_j = U\mathbf{P}_{5-j}U, 3 \le j \le 5, \quad \mathbf{P}_k = \mathbf{0}, k < 0, k > 5,$$

and

$$\mathbf{Q}_0 = \frac{1}{2}\sin(\theta+\eta)\sin(\theta+\xi+\eta)\begin{pmatrix} -\sin\xi & \cos\xi \\ -\sin\eta & \cos\eta \end{pmatrix},$$
$$\mathbf{Q}_1 = \frac{1}{2}\cos(\theta+\eta)\sin(\theta+\xi+\eta)\begin{pmatrix} \cos\xi & -\sin\xi \\ \cos\eta & -\sin\eta \end{pmatrix},$$
$$\mathbf{Q}_2 = \frac{1}{2}\cos(\theta+\xi+\eta)\cdot \begin{pmatrix} -\sin(\theta+\xi+\eta) & \cos(\theta+\xi+\eta) \\ \sin\theta & \cos\theta \end{pmatrix},$$
$$\mathbf{Q}_j = U\mathbf{Q}_{5-j}U, 3 \le j \le 5, \quad \mathbf{Q}_k = \mathbf{0}, k < 0, k > 5,$$

where $\theta, \xi, \eta \in [-\pi, \pi)$. There are now three variables θ, ξ and η to choose to obtain the scaling functions and multiwavelets. For

$$(\theta, \xi, \eta) = -(2.97243117364381, 3.02526503395165, .07926225995205),$$

the corresponding scaling function ${}_5\Phi$ provides approximation order 4 with ${}_5\Phi \in C^{1.6556}(R)$; and for

$$(\theta, \xi, \eta) = (.18237114886620, -3.02211022204529, 3.05559127520425),$$

we find the smoothest scaling function ${}_5\Phi^s$ with ${}_5\Phi^s$ providing approximation order 3, and ${}_5\Phi^s$ is in $C^{1.6777}(R)$.

Under the assumption that the scaling function ${}_5\Phi$ is stable, the property that ${}_5\Phi$ has approximation order 2 is equivalent to

$$\sin(\theta+\xi+\eta)(\cos(\theta+2\eta) + \frac{1}{2}\sin(\theta+\xi-\eta)) = \frac{1}{4}\sin(\xi+\eta) - \frac{1}{8} \quad (38)$$

By minimizing ${}_5S$ under the constrained condition (38), we obtain the optimal multifilter banks with

$$(\theta, \xi, \eta) = (.48385785530695, 2.99910363068828, -.45541559556097).$$

The corresponding OPTFR-scaling function and multiwavelet ${}_5\Phi^o$, ${}_5\Psi^o$ are in $C^{1.2018}(R)$, and the areas of the resolution cells for ${}_5\phi_1^o$, ${}_5\phi_2^o$, ${}_5\psi_1^o$, ${}_5\psi_2^o$ are respectively

.5960, 1.7747, 3.1169, 3.4689.

The sum ${}_5\widetilde{S}$ attains the minimum at (with ${}_5\widetilde{\Phi}^o$ providing approximation order 1)

$$(\theta, \xi, \eta) = (.35152175378550, -.09720137580057, -.37822377697579),$$

and the areas of the resolution cells for ${}_5\widetilde{\phi}_1^o$, ${}_5\widetilde{\phi}_2^o$ ${}_5\widetilde{\psi}_1^o$, ${}_5\widetilde{\psi}_2^o$ are respectively

.6037, 1.7209, 1.0884, .9190.

Example 5: In the last example of this paper, we construct the OPTFR-multiwavelets supported in $[0, 6]$. The explicit expression for the matrix coefficients $\mathbf{P}_k, \mathbf{Q}_k$ of the multifilter bank $\{{}_6\mathbf{P}, {}_6\mathbf{Q}\}$ are given by

$$\mathbf{P}_0 = \frac{\sqrt{2}}{4}\sin\theta\cos\xi \cdot \begin{pmatrix} -\cos(\theta+\xi+\frac{\pi}{4}) & -\cos(\theta+\xi+\frac{\pi}{4}) \\ \cos\eta & \cos\eta \end{pmatrix},$$
$$\mathbf{P}_1 = \frac{\sqrt{2}}{4}\sin\theta \begin{pmatrix} \sin(\theta+\frac{\pi}{4}) & -\sin(\theta+2\xi+\frac{\pi}{4}) \\ \sin(\xi-\eta) & \sin(\xi+\eta) \end{pmatrix},$$
$$\mathbf{P}_2 = \frac{\sqrt{2}}{4} \begin{pmatrix} \frac{\sqrt{2}}{2}+\cos\xi\sin\theta\cos(\theta+\xi+\frac{\pi}{4}) & \sin(2\theta+\frac{\pi}{4})-\sin\theta\cos\xi\cos(\theta+\xi+\frac{\pi}{4}) \\ \cos\theta\sin(\xi+\eta)+\sin\theta\sin\xi\sin\eta & \cos\theta\sin(\xi+\eta)-\sin\theta\sin\xi\sin\eta \end{pmatrix},$$
$$\mathbf{P}_3 = \frac{\sqrt{2}}{2}\cos\theta \begin{pmatrix} \cos(\theta+\frac{\pi}{4}) & 0 \\ 0 & -\cos(\xi+\eta) \end{pmatrix},$$
$$\mathbf{P}_j = U\mathbf{P}_{6-j}U, 4 \le j \le 6, \quad \mathbf{P}_k = \mathbf{0}, k<0, k>6,$$

and

$$\mathbf{Q}_0 = \tfrac{\sqrt{2}}{4}\sin\theta\cos\xi \cdot \begin{pmatrix} \sin(\theta+\xi+\frac{\pi}{4}) & \sin(\theta+\xi+\frac{\pi}{4}) \\ -\sin\eta & -\sin\eta \end{pmatrix},$$
$$\mathbf{Q}_1 = \tfrac{\sqrt{2}}{4}\sin\theta \begin{pmatrix} \cos(\theta+\frac{\pi}{4}) & \cos(\theta+2\xi+\frac{\pi}{4}) \\ -\cos(\xi-\eta) & \cos(\xi+\eta) \end{pmatrix},$$
$$\mathbf{Q}_2 = \tfrac{\sqrt{2}}{4} \begin{pmatrix} \frac{\sqrt{2}}{2}-\cos\xi\sin\theta\sin(\theta+\xi+\frac{\pi}{4}) & \cos(2\theta+\frac{\pi}{4})+\sin\theta\cos\xi\sin(\theta+\xi+\frac{\pi}{4}) \\ \cos\theta\cos(\xi+\eta)+\sin\theta\sin\xi\cos\eta & \cos\theta\cos(\xi+\eta)-\sin\theta\sin\xi\cos\eta \end{pmatrix},$$
$$\mathbf{Q}_3 = \tfrac{\sqrt{2}}{2}\cos\theta \begin{pmatrix} -\sin(\theta+\frac{\pi}{4}) & 0 \\ 0 & \sin(\xi+\eta) \end{pmatrix},$$
$$\mathbf{Q}_j = U\mathbf{Q}_{6-j}U, 4 \le j \le 6, \quad \mathbf{Q}_k = \mathbf{0}, k<0, k>6,$$

where $\theta, \xi, \eta \in [-\pi, \pi)$.

For

$$(\theta, \xi, \eta) = (-.09478663741893, 1.97467895132030, 2.11772714811323),$$

the corresponding scaling function ${}_6\Phi$ provides approximation order 4 with ${}_6\Phi \in C^{1.6520}(R)$, and for

$$(\theta, \xi, \eta) = (-.08786114688669, -1.11213061730812, -1.05315350967819),$$

the corresponding scaling function ${}_6\Phi^s$ is smoothest with ${}_6\Phi^s$ providing approximation order 3. The smoothest scaling function ${}_6\Phi^s$ and the corresponding multiwavelet ${}_6\Psi^s$ are in $C^{1.8634}(R)$. One can check that if the scaling function ${}_6\Phi$ is stable, then the property that ${}_6\Phi$ has approximation order 2 is equivalent to

$$\frac{\sqrt{2}}{2}\cos(\theta+\xi+\eta)+\sin\theta(\sin(\theta+2\xi)-2\sin(\eta-\xi+\frac{\pi}{4}))+\frac{1}{4}=0. \tag{39}$$

Again by minimizing ${}_6S$ under (39), we obtain the optimal multifilter bank with

$$(\theta, \xi, \eta) = (-.10137179232227, -1.63191646567871, -.46550166911210).$$

In this case the corresponding OPTFR-scaling function and multiwavelet ${}_6\Phi^o, {}_6\Psi^o \in C^{1.3542}(R)$. The areas of the resolution cells for ${}_6\phi_1^o$, ${}_6\phi_2^o$, ${}_6\psi_1^o$, ${}_6\psi_2^o$ are respectively

$$.7033, \quad 1.7666, \quad 2.9726, \quad 3.1532.$$

The sum ${}_6\widetilde{S}$ attains its minimum at (with ${}_6\widetilde{\Phi}^o$ providing approximation order 1)

$$(\theta, \xi, \eta) = (-.09644181482851, -1.17135241959765, -1.00341129573443).$$

The areas of the resolution cells for ${}_6\widetilde{\phi}_1^o$, ${}_6\widetilde{\phi}_2^o$, ${}_6\widetilde{\psi}_1^o$, ${}_6\widetilde{\psi}_2^o$ are respectively

$$.7117, \quad 1.6884, \quad 1.0805, \quad .7969.$$

From the values of θ, ξ, η provided in the above examples and the explicit expressions for $\mathbf{P}_k$, $\mathbf{Q}_k$, we have the corresponding optimal multifilter banks. Here we would like to provide the optimal multifilter banks for ${}_N\Phi^o$, ${}_N\Psi^o$ and ${}_N\widetilde{\Phi}^o$, ${}_N\widetilde{\Psi}^o$ with $N = 4, 6$ in the Appendix. The functions ${}_4\Phi^o$, ${}_4\Psi^o$ and ${}_6\Phi^o$, ${}_6\Psi^o$ are shown in Figure 1 and Figure 2, respectively; and ${}_3\Phi^s$, ${}_3\Psi^s$ and ${}_5\Phi^s$, ${}_5\Psi^s$ are shown in Figure 3 and Figure 4, respectively.

For the scaling functions ${}_N\Phi$ constructed in the above examples, ${}_N\widehat{\Phi}(0) = (1,0)^T$ by the symmetric/antisymmetric property of ${}_N\Phi$. However in image processing applications, the balanced multiwavelets are required (see [22]). As in [22], a multiwavelet Ψ is called **balanced** if its corresponding scaling function Φ satisfies $\widehat{\Phi}(0) = (1,1)^T/\sqrt{2}$, and in this case we also call Φ balanced. By a rotation of angle $\pi/4$, we can get the balanced orthogonal scaling functions and multiwavelets, denoted by ${}_N\Phi^b$ and ${}_N\Psi^b$, respectively, from the symmetric/antisymmetric ${}_N\Phi$ and ${}_N\Psi$:

$${}_N\Phi^b = \mathbf{R}_0\,{}_N\Phi, \quad {}_N\Psi^b = \mathbf{R}_0\,{}_N\Psi, \tag{40}$$

where $\mathbf{R}_0$ is the rotation by angle $\pi/4$ in the (x_2, x_1)-plane, i.e.

$$\mathbf{R}_0 := \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

In this case the multifilter bank corresponding to ${}_N\Phi^b, {}_N\Psi^b$ is

$${}_N\mathbf{P}^b(\omega) = \mathbf{R}_0\,{}_N\mathbf{P}(\omega)\mathbf{R}_0^T, {}_N\mathbf{Q}^b(\omega) = \mathbf{R}_0\,{}_N\mathbf{Q}(\omega)\mathbf{R}_0^T. \tag{41}$$

For ${}_N\Phi^b = ({}_N\phi_1^b, {}_N\phi_2^b)^T$ and ${}_N\Psi^b = ({}_N\psi_1^b, {}_N\psi_2^b)^T$, they lost the symmetry, but they possess another property: the first components are the reflections of the second components about their center point $N/2$, i.e.

$${}_N\phi_2^b(N-x) = {}_N\phi_1^b(x), \quad {}_N\psi_2^b(N-x) = {}_N\psi_1^b(x).$$

By (40), for the OPTFR-scaling functions ${}_N\Phi^o$ and multiwavelets ${}_N\Psi^o$ (${}_N\widetilde{\Phi}^o$ and ${}_N\widetilde{\Psi}^o$, respectively) constructed above, we have the corresponding balanced scaling functions and multiwavelets, denoted by ${}_N\Phi^{bo}$ and ${}_N\Psi^{bo}$ (by ${}_N\widetilde{\Phi}^{bo}$ and ${}_N\widetilde{\Psi}^{bo}$, respectively). In Table 1, we list the areas of the resolution cells for such balanced scaling functions and multiwavelets. Figure 5 and Figure 6 show the graphs of ${}_4\Phi^{bo}$, ${}_4\Phi^{bo}$ and ${}_6\Phi^{bo}$, ${}_6\Phi^{bo}$, respectively.

V. Conclusions

A procedure to design orthogonal multiwavelets with good time-frequency resolution has been introduced. The formulas to compute the time-durations and the frequency-bandwidths of scaling functions and multiwavelets are derived. For $2 \le N \le 6$, parameter expressions for the matrix coefficients of the multifilter banks which generate symmetric/antisymmetric scaling functions and multiwavelets supported in $[0, N]$ are presented. Orthogonal multiwavelets with optimum time-frequency resolution are constructed and some optimal multifilter banks are provided. Future research problems are: (i) to design more optimal multifilter banks; (ii) to use the optimal multifilter banks in image processing applications.

Appendix

A. Proof of Proposition 2

Proof: By definition, we have

$$\begin{aligned}
&\mathbf{I}_\Phi^\beta(y)\\
&= 4\sum_{k,n}\mathbf{P}_k\int_{-\infty}^{+\infty} x^\beta\Phi(2x-k)\Phi^*(2x-2y-n)dx\mathbf{P}_n^T\\
&= \frac{1}{2^{\beta-1}}\sum_{k,n}\mathbf{P}_k\int_{-\infty}^{+\infty}(x+k)^\beta\cdot\\
&\qquad \Phi(x)\Phi^*(x+k-2y-n)dx\mathbf{P}_n^T\\
&= \frac{1}{2^{\beta-1}}\sum_{k,n}\mathbf{P}_k\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}k^\alpha\int_{-\infty}^{+\infty}x^{\beta-\alpha}\cdot\\
&\qquad \Phi(x)\Phi^*(x-(2y+n-k))dx\mathbf{P}_n^T\\
&= \frac{1}{2^{\beta-1}}\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}\sum_{k,n}k^\alpha\mathbf{P}_k\mathbf{I}_\Phi^{\beta-\alpha}(2y+n-k)\mathbf{P}_n^T.
\end{aligned}$$

Thus by (11)

$$\operatorname{vec}(\mathbf{I}_\Phi^\beta(y)) = \frac{1}{2^{\beta-1}}\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}\sum_{k,n}k^\alpha(\mathbf{P}_n\otimes\mathbf{P}_k)\cdot \operatorname{vec}(\mathbf{I}_\Phi^{\beta-\alpha}(2y+n-k)).$$

For $j\in[1-N, N-1]$, we have

$$\begin{aligned}
&\operatorname{vec}(\mathbf{I}_\Phi^\beta(j))\\
&= \frac{1}{2^{\beta-1}}\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}\sum_{k,n}k^\alpha(\mathbf{P}_n\otimes\mathbf{P}_k)\cdot\\
&\qquad \operatorname{vec}(\mathbf{I}_\Phi^{\beta-\alpha}(2j+n-k))\\
&= \frac{1}{2^{\beta-1}}\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}\sum_n\sum_k k^\alpha(\mathbf{P}_{k-(2j-n)}\otimes\mathbf{P}_k)\cdot\\
&\qquad \operatorname{vec}(\mathbf{I}_\Phi^{\beta-\alpha}(n))\\
&= \frac{1}{2^\beta}\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}\sum_{n=1-N}^{N-1}2\sum_{k=0}^{N}k^\alpha(\mathbf{P}_{k-(2j-n)}\otimes\mathbf{P}_k)\cdot\\
&\qquad \operatorname{vec}(\mathbf{I}_\Phi^{\beta-\alpha}(n))\\
&= \frac{1}{2^\beta}\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}(\mathcal{T}_\mathbf{P}^\alpha\operatorname{vec}(\mathbf{I}_\Phi^{\beta-\alpha}))(j).
\end{aligned}$$

Therefore, we have

$$\operatorname{vec}(\mathbf{I}_\Phi^\beta) = \frac{1}{2^\beta}\sum_{0\le\alpha\le\beta}\binom{\beta}{\alpha}\mathcal{T}_\mathbf{P}^\alpha\operatorname{vec}(\mathbf{I}_\Phi^{\beta-\alpha}),$$

which is (26).

We now show that $\operatorname{vec}(\mathbf{I}_\Phi^0)$ satisfies (27). By the refinability of Φ, $\mathbf{l}_0^0\widehat{\Phi}(2k\pi) = \mathbf{0}$, $k\in Z\backslash\{0\}$ (see [18]). Thus

$$\begin{aligned}
\mathbf{L}_N^0\operatorname{vec}(\mathbf{I}_\Phi^0) &= \sum_{j=1-N}^{N-1}(\mathbf{l}_0^0\otimes\mathbf{l}_0^0)\operatorname{vec}(\mathbf{I}_\Phi^0(j))\\
&= \sum_{j=1-N}^{N-1}\mathbf{l}_0^0\operatorname{vec}(\mathbf{I}_\Phi^0(j))(\mathbf{l}_0^0)^T\\
&= \mathbf{l}_0^0\sum_{j=1-N}^{N-1}\int\Phi(x)\Phi(x-j)^T dx(\mathbf{l}_0^0)^T\\
&= \mathbf{l}_0^0\sum_k\widehat{\Phi}(2k\pi)\widehat{\Phi}(2k\pi)^*(\mathbf{l}_0^0)^T = |\mathbf{l}_0^0\widehat{\Phi}(0)|^2.
\end{aligned}$$

By (26) for $\beta = 0$, $\operatorname{vec}(\mathbf{I}_\Phi^0) = \mathcal{T}_\mathbf{P}\operatorname{vec}(\mathbf{I}_\Phi^0)$. Thus if Φ is stable, then $\mathcal{T}_\mathbf{P}$ satisfies Condition E and $\operatorname{vec}(\mathbf{I}_\Phi^0)$ is the unique right 1-eigenvector of $\mathcal{T}_\mathbf{P}$ with the requirement $\mathbf{L}_N^0\operatorname{vec}(\mathbf{I}_\Phi^0) = |\mathbf{l}_0^0\widehat{\Phi}(0)|^2 \ne 0$. To find $\operatorname{vec}(\mathbf{I}_\Phi^\beta)$ for $\beta\ge 1$, (26) can rewritten as

$$(2^\beta\mathbf{I}-\mathcal{T}_\mathbf{P})\operatorname{vec}(\mathbf{I}_\Phi^\beta) = \sum_{0<\alpha\le\beta}\binom{\beta}{\alpha}\mathcal{T}_\mathbf{P}^\alpha\operatorname{vec}(\mathbf{I}_\Phi^{\beta-\alpha}).$$

Since the spectral radius of $\mathcal{T}_\mathbf{P}$ is not greater than 1, $\operatorname{vec}(\mathbf{I}_\Phi^\beta)$ is uniquely determined by $\operatorname{vec}(\mathbf{I}_\Phi^\alpha)$, $\alpha<\beta$. ■

B. Proof of Proposition 3

Proof: By definition, we have

$$\begin{aligned}
&\mathbf{D}_\Phi^\beta(\xi)\\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\omega^\beta\mathbf{P}(\frac{\omega}{2})\widehat{\Phi}(\frac{\omega}{2})\widehat{\Phi}^*(\frac{\omega}{2})\mathbf{P}^*(\frac{\omega}{2})e^{i\omega\xi}d\omega\\
&= \frac{2^\beta}{\pi}\int_{-\infty}^{+\infty}\omega^\beta\mathbf{P}(\omega)\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)\mathbf{P}^*(\omega)e^{i2\omega\xi}d\omega\\
&= \frac{2^\beta}{\pi}\sum_n\sum_k\mathbf{P}_k\int_{-\infty}^{+\infty}\omega^\beta\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)e^{i\omega(2\xi-k+n)}d\omega\mathbf{P}_n^T\\
&= \frac{2^\beta}{\pi}\sum_n\sum_k\mathbf{P}_k\int_{-\infty}^{+\infty}\omega^\beta\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)e^{i\omega(2\xi-n)}d\omega\mathbf{P}_{k-n}^T\\
&= 2^{\beta+1}\sum_n\sum_k\mathbf{P}_k\mathbf{D}_\Phi^\beta(2\xi-n)\mathbf{P}_{k-n}^T.
\end{aligned}$$

By (11),

$$\operatorname{vec}(\mathbf{D}_\Phi^\beta(\xi)) = 2^{\beta+1}\sum_{n=-N}^{N}\sum_{k=0}^{N}(\mathbf{P}_{k-n}\otimes\mathbf{P}_k)\operatorname{vec}(\mathbf{D}_\Phi^\beta(2\xi-n)).$$

For $j\in[1-N, N-1]$,

$$\operatorname{vec}(\mathbf{D}_\Phi^\beta(j))$$

$$= 2^{\beta+1} \sum_{n=-N}^{N} \sum_{k=0}^{N} (\mathbf{P}_{k-n} \otimes \mathbf{P}_k) \mathrm{vec}(\mathbf{D}_\Phi^\beta(2j-n))$$
$$= 2^{\beta} \sum_{n=1-N}^{N-1} 2 \sum_{k=0}^{N} (\mathbf{P}_{k-(2j-n)} \otimes \mathbf{P}_k) \mathrm{vec}(\mathbf{D}_\Phi^\beta(n))$$
$$= 2^{\beta} (\mathcal{T}_\mathbf{P} \mathrm{vec}(\mathbf{D}_\Phi^\beta))(j).$$

Thus we have proved (31).

The proof of (32) is the same as in [16]. We omit the details here. ∎

C. Multifilter banks in Example 3 and Example 5

${}_4\mathbf{P}, {}_4\mathbf{Q}$ for ${}_4\Phi^o$ and ${}_4\Psi^o$:

$$\mathbf{P}_0 = \begin{pmatrix} -.02045061057401 & -.02045061057401 \\ .02738509680987 & .02738509680987 \end{pmatrix}$$
$$\mathbf{P}_1 = \begin{pmatrix} .25 & .20093899455952 \\ -.31835529662014 & -.28549014734181 \end{pmatrix}$$
$$\mathbf{P}_2 = \begin{pmatrix} .54090122114802 & 0 \\ 0 & .36234070660108 \end{pmatrix}$$
$$\mathbf{Q}_0 = \begin{pmatrix} -.02453050272024 & -.02453050272024 \\ -.01643257463917 & -.01643257463917 \end{pmatrix}$$
$$\mathbf{Q}_1 = \begin{pmatrix} .25 & .29090122114802 \\ .15378525649067 & .20855545011040 \end{pmatrix}$$
$$\mathbf{Q}_2 = \begin{pmatrix} -.45093899455952 & 0 \\ 0 & .60384544396195 \end{pmatrix}$$

and $\mathbf{P}_j = U\mathbf{P}_{4-j}U, \mathbf{Q}_j = U\mathbf{Q}_{4-j}U$, $j = 3, 4$.

${}_4\mathbf{P}, {}_4\mathbf{Q}$ for ${}_4\widetilde{\Phi}^o$ and ${}_4\widetilde{\Psi}^o$:

$$\mathbf{P}_0 = \begin{pmatrix} -.02671772204867 & -.02671772204867 \\ .03676526136168 & .03676526136168 \end{pmatrix}$$
$$\mathbf{P}_1 = \begin{pmatrix} .25 & .18145779472221 \\ -.32002243244560 & -.27369055553292 \end{pmatrix}$$
$$\mathbf{P}_2 = \begin{pmatrix} .55343544409734 & 0 \\ 0 & .37410093198947 \end{pmatrix}$$
$$\mathbf{Q}_0 = \begin{pmatrix} -.03427110263889 & -.03427110263889 \\ -.02316593845634 & -.02316593845634 \end{pmatrix}$$
$$\mathbf{Q}_1 = \begin{pmatrix} .25 & .30343544409734 \\ .15028520463305 & .22381572735642 \end{pmatrix}$$
$$\mathbf{Q}_2 = \begin{pmatrix} -.43145779472221 & 0 \\ 0 & .59371298797852 \end{pmatrix}$$

and $\mathbf{P}_j = U\mathbf{P}_{4-j}U, \mathbf{Q}_j = U\mathbf{Q}_{4-j}U$, $j = 3, 4$.

${}_6\mathbf{P}, {}_6\mathbf{Q}$ for ${}_6\Phi^o$ and ${}_6\Psi^o$:

$$\mathbf{P}_0 = \begin{pmatrix} -.00127499039395 & -.00127499039395 \\ .00195291533336 & .00195291533336 \end{pmatrix}$$
$$\mathbf{P}_1 = \begin{pmatrix} -.02260941238924 & -.01905941799280 \\ .03289326667537 & .03093129087626 \end{pmatrix}$$
$$\mathbf{P}_2 = \begin{pmatrix} .25127499039395 & .19326509672541 \\ -.32011146096034 & -.28805112655501 \end{pmatrix}$$
$$\mathbf{P}_3 = \begin{pmatrix} .54521882477849 & 0 \\ 0 & .35357845394700 \end{pmatrix}$$
$$\mathbf{Q}_0 = \begin{pmatrix} -.00177499719822 & -.00177499719822 \\ .00098098789956 & .00098098789956 \end{pmatrix}$$
$$\mathbf{Q}_1 = \begin{pmatrix} -.02772995644032 & -.03027993722822 \\ .01407725186930 & .01798308253603 \end{pmatrix}$$
$$\mathbf{Q}_2 = \begin{pmatrix} .25177499719822 & .29344382758027 \\ -.14487694819769 & -.20870150574932 \end{pmatrix}$$
$$\mathbf{Q}_3 = \begin{pmatrix} -.44454008711936 & 0 \\ 0 & -.60816258751535 \end{pmatrix}$$

and $\mathbf{P}_j = U\mathbf{P}_{6-j}U, \mathbf{Q}_j = U\mathbf{Q}_{6-j}U$, $j = 4, 5, 6$.

${}_6\mathbf{P}, {}_6\mathbf{Q}$ for ${}_6\widetilde{\Phi}^o$ and ${}_6\widetilde{\Psi}^o$:

$$\mathbf{P}_0 = \begin{pmatrix} .01172922944199 & .01172922944199 \\ -.00711561604177 & -.00711561604177 \end{pmatrix}$$
$$\mathbf{P}_1 = \begin{pmatrix} -.02164317453772 & -.03392743452984 \\ .00569063308134 & .02802165119852 \end{pmatrix}$$
$$\mathbf{P}_2 = \begin{pmatrix} .23827077055801 & .20917065732884 \\ -.31610352864132 & -.26320370755727 \end{pmatrix}$$
$$\mathbf{P}_3 = \begin{pmatrix} .54328634907545 & 0 \\ 0 & .39970866211417 \end{pmatrix}$$
$$\mathbf{Q}_0 = \begin{pmatrix} .00614212999606 & .00614212999606 \\ -.01116550905859 & -.01116550905859 \end{pmatrix}$$
$$\mathbf{Q}_1 = \begin{pmatrix} -.02627928605658 & -.00282082717259 \\ .03356552658380 & .01933429450025 \end{pmatrix}$$
$$\mathbf{Q}_2 = \begin{pmatrix} .24385787000394 & .29942847907150 \\ -.18299818891716 & -.21671047319701 \end{pmatrix}$$
$$\mathbf{Q}_3 = \begin{pmatrix} -.44744142788684 & 0 \\ 0 & -.57930723619859 \end{pmatrix}$$

and $\mathbf{P}_j = U\mathbf{P}_{6-j}U, \mathbf{Q}_j = U\mathbf{Q}_{6-j}U$, $j = 4, 5, 6$.

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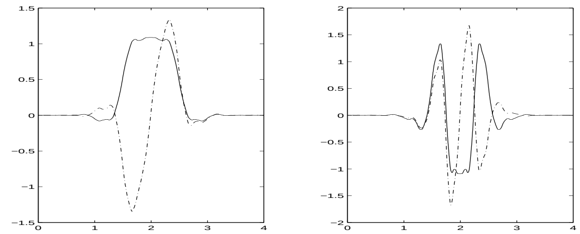

Fig. 1. OPTFR-scaling function ${}_4\Phi^o$ (the left) and multiwavelet ${}_4\Psi^o$ (the right).

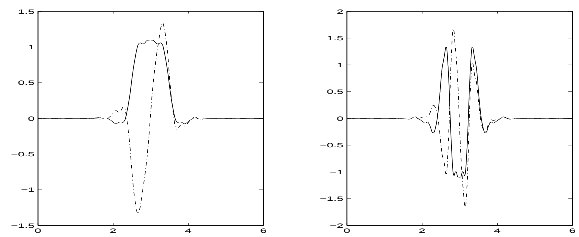

Fig. 2. OPTFR-scaling function ${}_6\Phi^o$ (the left) and multiwavelet ${}_6\Psi^o$ (the right).

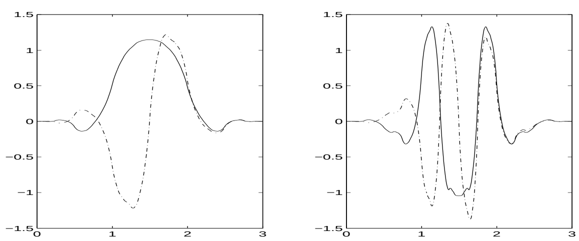

Fig. 3. Symmetric/antisymmetric scaling function ${}_3\Phi^s$ (the left) and multiwavelet ${}_3\Psi^s$ (the right) with ${}_3\Phi^s, {}_3\Psi^s \in C^{1.2668}(R)$.

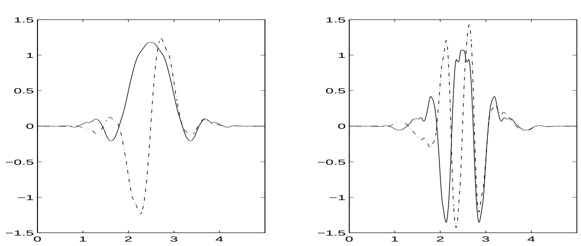

Fig. 4. Symmetric/antisymmetric scaling function ${}_5\Phi^s$ (the left) and multiwavelet ${}_5\Psi^s$ (the right) with ${}_5\Phi^s, {}_5\Psi^s \in C^{1.6777}(R)$.

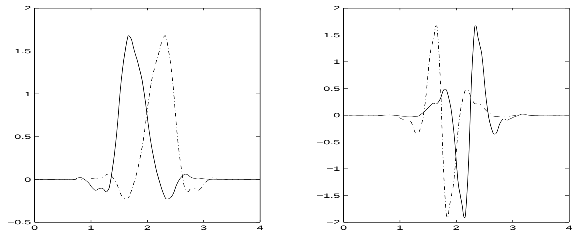

Fig. 5. Balanced scaling function ${}_4\Phi^{bo}$ (the left) and multiwavelet ${}_4\Psi^{bo}$ (the right).

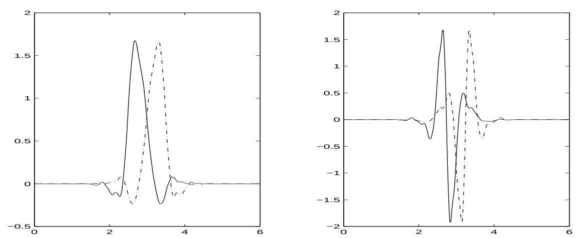

Fig. 6. Balanced scaling function ${}_6\Phi^{bo}$ (the left) and multiwavelet ${}_6\Psi^{bo}$ (the right).

TABLE 1. The areas of the resolution cells for the balanced scaling functions and multiwavelets.

N	$\Delta_{{}_N\phi_1^{bo}}\Delta_{\widehat{{}_N\phi_1^{bo}}}$	$\Delta_{{}_N\psi_1^{bo}}\Delta_{\widehat{{}_N\psi_1^{bo}}}$	$\Delta_{{}_N\widetilde{\phi}_1^{bo}}\Delta_{\widehat{{}_N\widetilde{\phi}_1^{bo}}}$	$\Delta_{{}_N\widetilde{\psi}_1^{bo}}\widetilde{\Delta}_{\widehat{{}_N\widetilde{\psi}_1^{bo}}}$
2	.7950	2.1968	.7931	1.2793
3	.6520	2.0160	.6628	.7283
4	.6533	1.9802	.6658	.7156
5	.7127	2.2472	.6884	.7130
6	.6628	2.0065	.6840	.6860