ROTATION INVARIANT AMBIGUITY FUNCTIONS

QINGTANG JIANG

ABSTRACT. Let $W(\psi; x, y)$ be the wideband ambiguity function. It is obtained in this note that $y^{-\frac{\alpha+2}{2}} W(\psi; x, y)(\alpha > -1)$ is $SO(2)$-invariant if and only if the Fourier transform of ψ is a Laguerre function.

1. INTRODUCTION

For $g \in L^2(\mathbb{R})$, the continuous Gabor transform (or windowed Fourier transform) of $f \in L^2(\mathbb{R})$ with analyzing function g is defined by

$$\Psi_g f(x, y) := \int_{-\infty}^{+\infty} f(t) e^{-2\pi i y t} \overline{g(t-x)} dt. \tag{1.1}$$

It was introduced by Gabor for study of communication theory ([4]). In (1.1), x is the time variable and y is the frequency variable. The transform $\Psi_g f(x, y)$ of f is formed by shifting the window function g so that it is centered at x, then taking the Fourier transform. In this way, $\Psi_g f(x, y)$ displays the frequency content of f near time x. For $f \in L^2(\mathbb{R})$, it can be reconstructed from $\Psi_g f(x, y)$:

$$f(t) = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^2} \Psi_g f(x, y) g(t-x) e^{2\pi i y t} dx dy.$$

For $f, g \in L^2(\mathbb{R})$, the radar cross-ambiguity function of f, g is defined to be

$$H(f, g; x, y) := \int_{-\infty}^{+\infty} f(t + \frac{1}{2}x) \overline{g(t - \frac{1}{2}x)} e^{-2\pi i y t} dt. \tag{1.2}$$

From (1.1), (1.2), one knows $\Psi_g f(x, y)$ is exactly $H(f, g; x, y)$ except for a phase factor. Both $\Psi_g f(x, y)$ and $H(f, g; x, y)$ are related to the representation of the Weyl-Heisenberg group ([15], [12]). For $f \in L^2(\mathbb{R})$, denote $H(f; x, y) := H(f, f; x, y)$.

1991 *Mathematics Subject Classification.* 42C05, 42C99.

Key words and phrases. Ambiguity function, rotation invariant, Laguerre function.

Function $H(f; , x, y)$ is called the radar auto-ambiguity function or narrowband ambiguity function with respect to signal f. Ambiguity functions play an important role in radar analysis and design since they were introduced by Woodward (see [20], [19], [15]). Properties of $H(f, f; x, y)$, $H(f; x, y)$ and their applications can be found in many literatures, e.g. [19], [15]. One of these properties is that $H(f; x, y)$, as a function on $\mathbb{R}^2$, is $SO(2)$-invariant (or rotation-invariant) if and only if $f(x)$ is a Hermite function, i.e. there exists a nonnegative integer m such that $f(x) = ch_m(x)e^{-x^2}$, here $h_m(x)$ is the Hermite polynomial of degree m, see [19], [15].

Originally proposed as an alternative to windowed Fourier transform, wavelet transform has its applications in many fields (see [6], [2]). Let $H^2(\mathbb{R})$ denote the Hardy space, the subspace of $L^2(\mathbb{R})$ consisting of functions ψ with supp$\widehat{\psi} \subset [0, +\infty)$. The continuous wavelet transform of $f \in H^2(\mathbb{R})$ with analyzing function $\psi \in H^2(\mathbb{R})$, denoted by W_ψ, is defined by

$$W_\psi f(x, y) := \frac{1}{\sqrt{y}} \int_{-\infty}^{+\infty} f(t)\overline{\psi(\frac{t-x}{y})}dt. \tag{1.3}$$

Continuous wavelet transform is associated to the square integrable representation of the affine group "$ax + b$" (see [5]). When ψ satisfies the following condition

$$C_\psi := 2\pi \int_0^{+\infty} |\widehat{\psi}(\omega)|^2 \frac{d\omega}{\omega} < +\infty, \tag{1.4}$$

then $f(x)$ can be reconstructed from $W_\psi f(x, y)$ as from $\Psi_g f(x, y)$. In this case

$$f(x) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{\mathbb{R}} W_\psi f(b, a) \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}) \frac{dadb}{a^2}. \tag{1.5}$$

Equation (1.5) holds at least "in the weak sense", i.e. taking inner product of both sides of (1.5) with any $g \in H^2(\mathbb{R})$ and commuting the inner product with the integral over a, b in the right hand side, leads to a true formula, which in fact is the Moyal formula. The convergence of the integral in (1.5) also holds in the following "strong sense" (see [2]):

$$\lim_{\delta\to 0, A, B\to+\infty} \|f(x) - C_\psi^{-1} \int_{\delta<a<A} \int_{|b|<B} W_\psi f(b, a) \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}) \frac{dadb}{a^2}\|_2 = 0.$$

For $\psi, f \in H^2(\mathbb{R})$, let $W(\psi, f; x, y) := W_\psi f(x, y)$ be the wideband cross-ambiguity function of ψ, f and $W(\psi; x, y) := W(\psi, \psi; x, y)$ the wideband ambiguity function.

Such ambiguity functions were studied by Swick in [16](1967), [17](1969). The renewed interest in the wideband functions ([1] , [12], [10], [14], [18], [21], [8]) seems to have been inspired by the development of wavelet analysis. In this note, we will consider the $SO(2)$-invariant properties of $W(\psi; x, y)$. The rotation invariance of wideband ambiguity functions would be of interest for applications in radar/sonar analysis or design. In the following, when considering the $SO(2)$-invariant properties of ambiguity functions, we will assume that $\psi \in H^2(\mathbb{R})$, $\widehat{\psi}$ is real and $\psi(x)$ having some smooth and decaying properties at infinity which insure that $\widehat{\psi}''(\omega)$ exists on $\mathbb{R}_+^* := (0, +\infty)$ and $\widehat{\psi}(\omega)\widehat{\psi}'(\omega), \omega\widehat{\psi}(\omega)\widehat{\psi}''(\omega) \in L^1(\mathbb{R}_+^*)$. Let $\mathcal{A}$ denote the set of all such functions.

2. MAIN RESULTS

For $\psi \in \mathcal{A}$, let $W(\psi; x, y)$ be the wideband ambiguity function of ψ defined as above. As a function on the upper half plane U, if $W(\psi; x, y)$ is called $SO(2)$-invariant when it satisfies

$$W(\psi; x_\theta, y_\theta) = W(\psi; x, y), \text{ with } x_\theta + iy_\theta = \frac{(x+iy)\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{(x+iy)\sin\frac{\theta}{2} + \cos\frac{\theta}{2}},$$

then we can get as below that there is no $SO(2)$-invariant $\psi \in \mathcal{A}$. In the following we will consider the ambiguity functions with a different dilation factor.

For $\alpha > -1$, let $L^{\alpha 2}(U)$ denote the function space consisting of functions on the upper half plane U square integrable with measure $y^\alpha dxdy$. For $\psi \in H^2(\mathbb{R})$, define the wavelet transform of $f \in H^2$ with a different dilation by

$$W^\alpha(\psi, f; x, y) = W_\psi^\alpha f(x, y) := \frac{1}{y^\nu} W_\psi f(x, y),$$

where

$$\nu := \alpha + 2$$

and $h = \frac{1}{\nu}$ is the Planck constant in the terminology of quantum mechanics. If ψ satisfies (1.4), W_ψ^α is an isometry (up to a constant) from $H^2(\mathbb{R})$ into $L^{\alpha 2}(U)$. We will consider the rotation invariant properties of the ambiguity function

$$W^\alpha(\psi; x, y) := W^\alpha(\psi, \psi; x, y) = \frac{1}{y^{\nu+\frac{1}{2}}} \int_{-\infty}^{+\infty} \psi(t)\overline{\psi(\frac{t-x}{y})}dt. \tag{2.1}$$

Let $SL(2,\mathbb{R})$ denote the special linear group. For $g \in SL(2,\mathbb{R})$, it acts on U via the transformations

$$g : z \to gz := g(z) = \frac{az+b}{cz+d}, \quad \text{with} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and it induces the action on $L^{\alpha 2}(U)$ via

$$U_g^\nu : F(z) \to F(gz)\{g'(z)\}^{\frac{\nu}{2}} = F(gz)(cz+d)^{-\nu}. \tag{2.2}$$

Let $SO(2)$ be the special rotation group, the maximal compact subgroup of $SL(2,\mathbb{R})$, then elements $g \in SO(2)$ are given by

$$g = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \text{ with } g^{-1} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \ 0 \le \theta < 2\pi.$$

Let R_θ be the restriction of U_g^ν to $SO(2)$ given by

$$R_\theta F(z) := \frac{(i\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^\nu}{(z\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^\nu} F\left(\frac{z\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{z\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}\right),$$

where $c_\theta := (i\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^\nu$. Adding the constant c_θ in the definition of R_θ is to assure that $R_\theta F(i) = F(i)$. In fact the point i is the rotation center. If $R_\theta F(z) = F(z)$, then $F(z)$ is called $SO(2)$-invariant.

Let $\mathbb{Z}_+$ denote the set of all nonnegative integers and in this note we would consider the problem in the case $\alpha \in \mathbb{Z}_+$. We have

Theorem 2.1. *For $\alpha \in \mathbb{Z}_+$ and $\psi \in \mathcal{A}$, let $W^\alpha(\psi; x, y)$ be the ambiguity function of ψ defined by (2.1), then $W^\alpha(\psi; x, y)$ is $SO(2)$-invariant if and only if there exists $k \in \mathbb{Z}_+$, $k < \frac{\alpha+1}{2}$, such that*

$$\widehat{\psi}(\omega) = \begin{cases} c(2\omega)^{\frac{\alpha+1}{2}-k} L_k^{(\alpha+1-2k)}(2\omega)e^{-\omega}, & for\ \omega \ge 0, \\ 0, & for\ \omega < 0, \end{cases} \tag{2.3}$$

where c is a nonzero constant and $L_k^{(\alpha)}(\omega)$ is the Laguerre polynomial of degree k.

Proof. "$\Longrightarrow$" If there exists $\psi \in \mathcal{A}$ such that $W^\alpha(\psi; x, y)$ is $SO(2)$-invariant, then $R_\theta W^\alpha(\psi; x, y) = W^\alpha(\psi; x, y)$ for all θ and hence

$$\left.\frac{d(R_\theta W^\alpha(\psi; x, y))}{d\theta}\right|_{\theta=0} = 0. \tag{2.4}$$

For appropriate functions $F(z)$ on U, it follows by a direct calculation

$$\left.\frac{dR_\theta F(z)}{d\theta}\right|_{\theta=0} = \frac{i\nu}{2}F(z) - \frac{\nu}{2}zF(z) + \frac{y^2-x^2-1}{2}\frac{\partial F(x,y)}{\partial x} - xy\frac{\partial F(x,y)}{\partial y}.$$

From the definition of $W_\psi^\alpha f(x,y)$,

$$W_\psi^\alpha f(x,y) = \frac{y^{-\frac{\nu-1}{2}}}{2\pi}\int_0^{+\infty}\widehat{\psi}(y\omega)\widehat{f}(\omega)e^{i\omega x}d\omega.$$

Therefore we would consider functions with the form of

$$F(z) = y^{-\frac{\nu-1}{2}}\int_0^{+\infty}h(y\omega)g(\omega)e^{i\omega x}d\omega,$$

where $z = x+iy$, $h(\omega)$ and $g(\omega)$ are real functions on $\mathbb{R}_+^*$ with $h''(\omega), g''(\omega)$ existing and having some decay properties at infinity. For such kind function $F(z)$,

$$\begin{aligned}(2.5)\qquad & -2i\left.\frac{dR_\theta F(z)}{d\theta}\right|_{\theta=0} \\ &= \nu(1-y)F(z) + ixF(z) + (y^2-x^2-1)y^{-\frac{\nu-1}{2}}\int_0^{+\infty}h(y\omega)\omega g(\omega)e^{i\omega x}d\omega \\ &\quad +2ixyy^{-\frac{\nu-1}{2}}\int_0^{+\infty}h'(y\omega)\omega g(\omega)e^{i\omega x}d\omega.\end{aligned}$$

Let D_ν denote the differential operator of functions on $\mathbb{R}_+^*$ defined by

$$D_\nu := -\omega^2\frac{d^2}{d\omega^2} - \omega\frac{d}{d\omega} + \omega^2 - \nu\omega + \frac{(\nu-1)^2}{4}. \tag{2.6}$$

Then by a direct calculation and (2.5), one has

$$\begin{aligned}&\int_0^{+\infty}h(y\omega)\frac{1}{\omega}D_\nu g(\omega)e^{i\omega x}d\omega - \int_0^{+\infty}g(\omega)\frac{1}{\omega}D_\nu h(y\omega)e^{i\omega x}d\omega \\ (2.7)\qquad &= -2iy^{\frac{\nu-1}{2}}\left.\frac{dR_\theta F(z)}{d\theta}\right|_{\theta=0}.\end{aligned}$$

Let $F(z) = W^\alpha(\psi; x, y)$, then (2.4) and (2.7) lead to

$$\int_0^\infty \widehat{\psi}(y\omega)\frac{1}{\omega}D_\nu\widehat{\psi}(\omega)e^{i\omega x}d\omega = \int_0^\infty \widehat{\psi}(\omega)\frac{1}{\omega}D_\nu\widehat{\psi}(y\omega)e^{i\omega x}d\omega,$$

and hence $\widehat{\psi}(y\omega)D_\nu\widehat{\psi}(\omega) = \widehat{\psi}(\omega)D_\nu\widehat{\psi}(y\omega)$ for all $y,\omega \in R_+^*$. Therefore one can get that $\widehat{\psi}(\omega)$ is an eigenfunction of D_ν. The differential operator D_ν has spectra (see [3], [13]):

$$\sigma(D_\nu) = \left\{(\frac{\nu-1}{2})^2 - (\frac{\nu-1}{2}-k)^2, k\in\mathbb{Z}_+, k < \frac{\nu-1}{2}\right\} \cup \left\{[(\frac{\nu-1}{2})^2, +\infty)\right\}.$$

For $k < \frac{\nu-1}{2}$, denote $\lambda_k := (\frac{\nu-1}{2})^2 - (\frac{\nu-1}{2} - k)^2$, and let $\widehat{\psi}_k(\omega)$ be the eigenfunction of D_ν corresponding to λ_k, i.e.

$$D_\nu \widehat{\psi}_k(\omega) = \lambda_k \widehat{\psi}_k(\omega). \tag{2.8}$$

And Let φ be the function defined by $\widehat{\psi}_k(\omega) = (2\omega)^{-\frac{1}{2}}\varphi(2\omega)$. Then by (2.8),

$$\varphi''(t) + (-\frac{1}{4} + \frac{\nu}{2t} + \frac{1 + 4\lambda_k - (\nu-1)^2}{4t^2})\varphi(t) = 0. \tag{2.9}$$

Equation (2.9) is just the " Whittaker's differential equation" (see [11]) and it has solution $M_{\mathcal{N},\mu_k}$, the Whittaker's function, given by

$$M_{\mathcal{N},\mu_k}(t) = e^{-\frac{t}{2}} t^{\mu_k + \frac{1}{2}} {}_1F_1(\mu_k + \frac{1}{2} - \mathcal{N}; 1 + 2\mu_k; t),$$

where $\mathcal{N} = \frac{\nu}{2}$, $\mu_k = \frac{\nu-1}{2} - k$. Thus

$$\begin{aligned} \widehat{\psi}_k(\omega) &= (2\omega)^{-\frac{1}{2}} M_{\mathcal{N},\mu_k}(2\omega) = (2\omega)^{\frac{\nu-1-2k}{2}} e^{-\omega} {}_1F_1(-k; \nu - 1 - 2k; 2\omega) \\ &= (2\omega)^{\frac{\alpha+1-2k}{2}} e^{-\omega} L_k^{(\alpha+1-2k)}(2\omega). \end{aligned}$$

For the continuous spectra λ of D_ν, let $\widehat{\psi}_\lambda$ be the corresponding eigenfunction, then one can get as above that

$$\widehat{\psi}_\lambda(\omega) = (2\omega)^{\mu_\lambda} e^{-\omega} {}_1F_1(\mu_\lambda + \frac{1}{2} - \frac{\nu}{2}; 1 + 2\mu_\lambda; 2\omega),$$

where $\mu_\lambda = \pm i\sqrt{\lambda - (\frac{\nu-1}{2})^2}$. Such ψ_λ is not in $\mathcal{A}$ since $\widehat{\psi}_\lambda(\omega)$ is not a real function.

"$\Longleftarrow$" For any $k \in \mathbb{Z}_+, k < \frac{\alpha+1}{2}$, let $\psi_k \in \mathcal{A}$ given by

$$\widehat{\psi}_k(\omega) = \begin{cases} (2\omega)^{\frac{\alpha+1}{2} - k} L_k^{(\alpha+1-2k)}(2\omega) e^{-\omega}, & for\ \omega \geq 0, \\ 0, & for\ \omega < 0. \end{cases}$$

Then

$$\begin{aligned} W^\alpha(\psi_k; x, y) &= \frac{1}{2\pi y^{\frac{\alpha+1}{2}}} \int_0^{+\infty} \widehat{\psi}_k(y\omega)\widehat{\psi}_k(\omega) e^{i\omega x} d\omega \\ &= \frac{2^{\alpha-2k}}{\pi y^k} \int_0^{+\infty} \omega^{\alpha+1-2k} L_k^{(\alpha+1-2k)}(2\omega) L_k^{(\alpha+1-2k)}(2y\omega) e^{-\omega(y+1-ix)} d\omega. \end{aligned}$$

Denote $p := \frac{y+1-ix}{2} = \frac{1-iz}{2}$ with $z = x + iy$, one can get

$$\begin{aligned}
W^\alpha(\psi_k; x, y) &= \frac{1}{4\pi} y^{-k} \int_0^\infty \omega^{\alpha+1-2k} L_k^{(\alpha+1-2k)}(\omega) L_k^{(\alpha+1-2k)}(y\omega) e^{-\omega p} d\omega \\
&= y^{-k} \frac{\Gamma(\alpha+2)}{4\pi (k!)^2} \frac{(p-1)^k (p-y)^k}{p^{\alpha+2}} {}_2F_1(-k, -k; -\alpha-1; \frac{p(p-1-y)}{(p-1)(p-y)}) \\
&= y^{-k} \frac{\Gamma(\alpha+2)}{4\pi (k!)^2} \frac{(p-1)^k (p-y)^k}{p^{\alpha+2}} (\frac{y}{(p-1)(p-y)})^k \cdot \\
&\quad \cdot {}_2F_1(-k, k-\alpha-1; -\alpha-1; \frac{p(p-1-y)}{-y}) \\
&= \frac{2^\alpha \Gamma(\alpha+2)}{\pi (k!)^2} \frac{1}{(1-iz)^{\alpha+2}} {}_2F_1(-k, k-\alpha-1; -\alpha-1; \frac{|1-iz|^2}{4y}),
\end{aligned}$$

where ${}_2F_1(a, b; c; t) := \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} t^n$ is the hypergeometric function with $(a)_0 := 1, (a)_n := a(a+1)\cdots(a+n-1)$.

For $z \in U, g^{-1} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in SO(2)$, denote

$$\omega := u + iv = g(z) = \frac{z\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{z\sin\frac{\theta}{2} + \cos\frac{\theta}{2}},$$

then $\frac{|1-i\omega|^2}{4v} = \frac{|1-iz|^2}{4y}$ and

$$\begin{aligned}
&R_\theta W^\alpha(\psi_k; x, y) \\
&= \frac{(i\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^{\alpha+2}}{(z\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^{\alpha+2}} \frac{2^\alpha \Gamma(\alpha+2)}{\pi (k!)^2 (1-i\omega)^{\alpha+2}} {}_2F_1(-k, k-\alpha-1; -\alpha-1; \frac{|1-i\omega|^2}{4v}) \\
&= \frac{2^\alpha \Gamma(\alpha+2)}{\pi (k!)^2 (1-iz)^{\alpha+2}} {}_2F_1(-k, k-\alpha-1; -\alpha-1; \frac{|1-iz|^2}{4y}) \\
&= W^\alpha(\psi_k; x, y).
\end{aligned}$$

That is $W^\alpha(\psi_k; x, y)$ is $SO(2)$-invariant. The proof of Theorem 1 is completed. □

Remark 1. *The differential operator D_ν given by* (2.6) *is equivalent to the Casimir operator of the representation U^ν of $SL(2, R)$ given by* (2.2). *In fact, the Casimir operator $\Box_\nu$ is given by*

$$\Box_\nu := -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + i\nu y(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}),$$

and we have (see [7], [3])

$$\Box_\nu W^\alpha(\psi, f; x, y) = \frac{1}{2\pi y^{\frac{\alpha+1}{2}}} \int_0^\infty (D_\nu \widehat{\psi})(y\omega) \widehat{f}(\omega) e^{i\omega x} d\omega.$$

We shall also note here that function ψ *given by* (2.3) *satisfies* (1.4) *since* $\frac{\alpha+1}{2} - k > 0$.

Remark 2. *If* α *is not an integer and* ψ *is the function defined by* (2.3), *then for* $\theta \in [0, 2\pi]$, $R_\theta W^\alpha(\psi; x, y)$ *equals to* $W^\alpha(\psi; x, y)$ (*up to a constant on the unit circle*).

Acknowledgements. The author would like to express his thanks to the anonymous referee for many helpful suggestions to this paper.

References

1. L. Auslander and I. Gertner, *Wideband ambiguity functions and the* $a \cdot x + b$ *group, in "Signal Processing: Part I - Signal Processing Theory",* L. Auslander et al eds, Springer-Verlag, New York, 1990, 1-12.
2. I. Daubechies, *"Ten Lectures on Wavelets",* SIAM, Philadelphia, 1992.
3. I. Daubechies, J. Klauder and T. Paul , *Wiener measures for path integrals with affine kinematic variables,* J. Math. Phys., 28(1987), 85-102.
4. D. Gabor, *Theory of communication,* J. Inst. Electr. Eng., 93(1964), 429-457.
5. A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape,* SIAM. J. Math. Anal., 15 (1984) , 723-736.
6. C. Heil and D. Walnut, *Continuous and discrete wavelet transform,* SIAM. Rev., 31(1989), 628-666.
7. Q. Jiang and L. Peng, *Casimir operator and wavelet transform, in "Harmonic Analysis in China",* Hongkong, Kluwer Academic Publishers, 1995.
8. G. Kaiser, *"A Friendly Guide to Wavelet"*, Birkhäuser, 1994.
9. E. Kalnins and W. Miller, *A note on group contractions and radar ambiguity functions, in " Radar and Sonar, Part II",* F. Alberto Grünbaum et al eds, Springer-Verlag, New York, 1992, 71-82.
10. P. Maas, *Wideband approximation and wavelet transform, in " Radar and Sonar, Part II",* F. Alberto Grünbaum et al eds, Springer-Verlag, New York, 1992, 83-88.
11. W. Magnus, F. Oberhettinger and R. Soni, *"Formulas and Theorems for the Special Functions of Mathematical Physics"*, Springer-Verlag Berlin, Heidelberg, 1966.
12. W. Miller, *Topics in harmonic analysis with applications to radar and sonar, in " Radar and Sonar, Part I",* R. Bluhat et al eds, Springer-Verlag, New York, 1991, 66-168.

13. P. Morse, *Diatomic molecules according to the wave mechanics. II. Vibrational levels,* Physical Review, 34(1929), 57-64.

14. H. Naparst, *Dense target signal processing,* IEEE Trans. Inform. Theory, 37(1991), 317-327.

15. W. Schempp, *"Harmonic Analysis on the Heisenberg Nilpotent Lie Group, with Applications to Signal Theory",* Longman, 1986.

16. D. Swick, *An ambiguity function independent of assumption about bandwidth and carrier frequency,* NRL Report 6471, Washington, DC, 1966.

17. D. Swick, *A review of wide-band ambiguity function,* NRL Report 6994, Washington, DC, 1969.

18. L. Weiss, *Wavelets and wideband correlation processing*, IEEE Signal Proc. Magazine, Jan. 1994, 13-32.

19. C. Wilcox, *The synthesis problem for radar ambiguity functions, in " Radar and Sonar, Part I",* R. Bluhat et al eds, Springer-Verlag, New York, 1991, 229-260.

20. P. Woodward, *"Probability and Information Theory with Applications to Radar"*, 2nd ed, Pergamon Press, New York, 1964.

21. R. Young , *"Wavelet Theory and Its Applications"*, Kluwer Academic Publishers, Boston, 1993.

Department of Mathematics, Peking University, Beijing 100871, P. R. China.

Current address: Department of Mathematics, National University of Singapore, Lower Kent Ridge Road, Singapore 119260.

E-mail: qjiang@haar.math.nus.sg.