$\sqrt{3}$-subdivision schemes: maximal sum rule orders

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Abstract

Subdivision with finitely supported masks is an efficient method to create discrete multiscale representations of smooth surfaces for CAGD applications. Recently a new subdivision scheme for triangular meshes, called $\sqrt{3}$-*subdivision*, has been studied. In comparison to dyadic subdivision which is based on the dilation matrix $2I$, $\sqrt{3}$-subdivision is based on a dilation M with $\det M = 3$. This has certain advantages, for example a slower growth for the number of control points.

This paper concerns the problem of achieving maximal sum rule orders for stationary $\sqrt{3}$-subdivision schemes with given mask support which is important because the sum rule order characterizes the order of the polynomial reproduction, and provides an upper bound on the Sobolev smoothness of the surface. We study both interpolating and approximating schemes for a natural family of symmetric mask support sets related to squares of sidelength $2n$ in $\mathbb{Z}^2$, and obtain exact formulas for the maximal sum rule order for arbitrary n. For approximating schemes, the solution is simple, and schemes with maximal sum rule order are realized by an explicit family of schemes based on repeated averaging [15].

In the interpolating case, we use properties of multivariate Lagrange polynomial interpolation to prove the existence of interpolating schemes with maximal sum rule orders. Those can be found by solving a linear system which can be reduced in size by using symmetries. From this, we construct some new examples of smooth (C^2, C^3) interpolating $\sqrt{3}$-subdivision schemes with maximal sum rule order and symmetric masks. The construction of associated dual schemes is also discussed.

1 Introduction

Stationary subdivision is an efficient method to build smooth surfaces in CAGD in a multiscale fashion. A subdivision scheme starts with a set of points with connectivity relations between them, the so called *coarse control mesh*. By repeatedly applying a set of refinement rules, we get a sequence of finer and finer control meshes which converges to a limit surface. The refinement rules consist of a topological rule (or so-called *split operation*) which determines the connectivity of the finer meshes, and local averaging rules (usually given by *coefficient stencils*) which give the exact location of the control points in the finer mesh by taking linear combinations of nearby control points in the coarser mesh. If the positions of the coarse control points remain unchanged during subdivision, the subdivision scheme is called *interpolating*, otherwise it is *approximating*.

The topological rules for subdivision with triangular meshes are usually based on the 1-to-4 split (dyadic split) operation which inserts a new vertex for every edge of the given mesh, and then connects the new vertices with neighboring old and new vertices appropriately. Recently a new subdivision scheme for triangular meshes, called $\sqrt{3}$-*subdivision*, has been studied in [12], [13], [15]. The new scheme first inserts a new vertex for every face of the given mesh, then discards the original edges, and finally connects new and old vertices. Two steps of this topological rule correspond to trisection of the original triangular mesh. Figure 1 illustrates this refinement process. Compared to dyadic subdivision, $\sqrt{3}$-subdivision has certain advantages (see [12], [13]), among them a slower growth of the number of control points during subdivision.

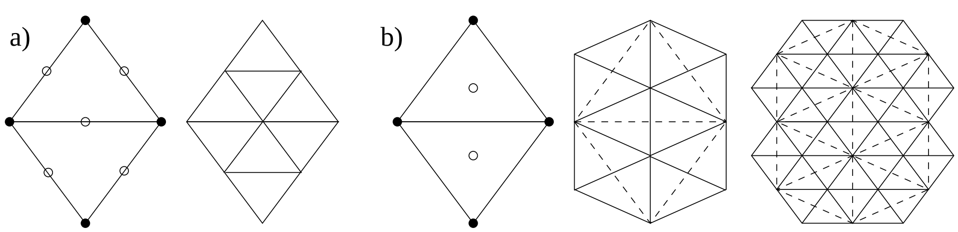

Figure 1: Topological rules for a) dyadic and b) $\sqrt{3}$-subdivision

In this paper, we will only deal with the so-called *regular* case, i.e., when all vertices of the triangular mesh have valence 6. In this case, the structure of the limit surface of a subdivision scheme is determined by the study of an associated refinement equation. We say that ϕ is a *refinable function* provided that ϕ satisfies the *refinement equation*

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^2} P_\alpha \phi(Mx - \alpha) , \qquad x \in \mathbb{R}^2 , \tag{1}$$

where $P := \{P_\alpha : \alpha \in \mathbb{Z}^2\}$, the *mask* of the refinement equation, is a finitely supported sequence of real numbers, and M is the *dilation matrix*. The mask P contains the information about the stencils, the dilation matrix M is solely defined by the topological rule of the subdivision scheme. For the dyadic split, $M = 2I$, where I is the 2×2 identity matrix, while for the $\sqrt{3}$-split, the dilation matrix is

$$M = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} . \tag{2}$$

The importance of the refinable function ϕ is based on the obvious fact that a parametric representation of the limit surface of the corresponding subdivision scheme can be obtained by taking linear combinations of the integer translates of ϕ. In particular, the smoothness of the surface is related to the smoothness of ϕ.

There are many aspects one has to deal with when designing subdivision schemes. If we restrict our attention to the regular case, then one of the basic issues is to obtain highly smooth limit surfaces with interpolating and approximating schemes that have small coefficient stencils. In terms of refinable functions, this is the question about mask support versus achievable Hölder (or Sobolev) smoothness of ϕ. Since smoothness properties depend nonlinearly on the mask, this problem is hard to attack directly. Therefore, it is often replaced by finding the *maximal sum rule order* for masks P with support in a given set $\Omega \subset \mathbb{Z}^2$. Under mild conditions on ϕ, the sum rule order characterizes the approximation properties of the subdivision scheme and represents an upper bound for the Sobolev smoothness of the associated ϕ. Since sum rule orders are characterized by a linear condition, this weaker problem can be treated more easily. Its dual formulation, i.e., the construction of masks with smallest support for a given sum rule order, is also of interest.

Let us mention some research on this problem. Smooth interpolating dyadic subdivision schemes were constructed in [16], for this case the problem of maximal sum rule orders was studied in [4]. An interesting approach to the dual problem is implicitly contained in [17], where an algorithmic description of a basis for the ideal of all symbols with given sum rule order is given, from which masks with small support can be deduced.

Analogous questions for $\sqrt{2}$-subdivision schemes which are based on quadrilateral control meshes and related to the quincunx dilation matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} ,$$

were discussed in [5]. A method for the construction of interpolating schemes with arbitrarily large sum rule orders for general dilation was proposed in [2]. In [3], the projection method was introduced to construct interpolating schemes. However, the sum rule orders of the masks constructed by these two methods are not maximal, in the sense that the same sum rule order could theoretically be achieved with much smaller mask support.

In this paper, we study the problem of achieving maximal sum rule orders for both interpolating and approximating $\sqrt{3}$-subdivision schemes for a natural family of support sets. While in the approximating case the solution is straightforward, and optimal schemes are known from [15, 11], proving the existence of interpolating schemes with optimal sum rule orders required some technical effort. In Section 2 we give the necessary definitions and preliminary material. Section 3 contains the main results and their proofs. In the concluding section we provide some evidence on the actual smoothness properties of the schemes with maximal sum rule orders. We also demonstrate by examples how to construct dual schemes which allows the construction of biorthogonal multiresolution analyses and wavelets related to the $\sqrt{3}$-split.

2 Preliminary material

Consider the refinement equation (1) with the dilation matrix M given by (2). For the representers of the coset space $\mathbb{Z}^2/M\mathbb{Z}^2$, we will choose

$$\gamma_0 := (0,0), \quad \gamma_1 := (1,0), \quad \gamma_2 := (0,1). \tag{3}$$

We also define

$$\tilde{\omega}^0 := (0,0)\ , \quad \tilde{\omega}^1 := (\frac{2\pi}{3}, \frac{4\pi}{3})\ , \quad \tilde{\omega}^2 := (\frac{4\pi}{3}, \frac{2\pi}{3})\ . \tag{4}$$

For a mask $P = \{P_\alpha\}$, we call

$$P(\omega) := \frac{1}{3} \sum_{\alpha \in \mathbf{Z}^2} P_\alpha e^{-i\alpha\omega}.$$

its *symbol* resp. the symbol of (1). We assume that $P(0) = 1$. This is the necessary and sufficient condition for the refinement equation to have a (distributional) solution ϕ with $\hat{\phi}(0) \neq 0$.

We say that P has the *sum rule order* k if $P(0) = 1$ and

$$D^\mu P(\tilde{\omega}^j) = 0, \quad j = 1, 2, \quad |\mu| < k, \tag{5}$$

where $\mu = (\mu_1, \mu_2) \in \mathbb{Z}_+^2$ are multi-indices, $|\mu| = \mu_1 + \mu_2$, and $D^\mu = D_1^{\mu_1} D_2^{\mu_2}$. This gives formally $k(k+1)+1$ linear equations for the coefficients of the mask. The solvability of this system (and, thus, the existence of a mask with sum rule order k) clearly depends on the mask support and possibly further constraints on its coefficients such as symmetries to be observed. As was already mentioned, the sum rule order k equals the approximation order of the refinable function ϕ under a mild condition on ϕ, see [8].

A $\sqrt{3}$-subdivision scheme is interpolating if and only if the associated mask $P = (P_\alpha)$ satisfies

$$P_{M\alpha} = \delta_\alpha, \quad \alpha \in \mathbb{Z}^2. \tag{6}$$

In this case, we call the mask P *interpolatory.*

Let us look at the mask P given by the coefficient array

$$P \hat{=} \begin{bmatrix} 0 & c & c & 0 & 0 & 0 & 0 \\ c & b & 0 & b & c & 0 & 0 \\ c & 0 & a & a & 0 & c & 0 \\ 0 & b & a & \mathbf{1} & a & b & 0 \\ 0 & c & 0 & a & a & 0 & c \\ 0 & 0 & c & b & 0 & b & c \\ 0 & 0 & 0 & 0 & c & c & 0 \end{bmatrix}, \tag{7}$$

corresponding to the centered index box $[-3,3]^2$, i.e., $P_\alpha = 0$ if $\alpha \notin [-3,3]^2$ and for $\alpha \in [-3,3]^2$, P_α is the $(4-\alpha_2, \alpha_1+4)$-th entry of the matrix on the right side of (7). This mask was discussed in [11], and is obviously interpolatory. Its coefficient stencil for newly inserted control points is shown in Figure 2 a). That the mask (7) contains only 3 free parameters a, b, c is dictated by the geometric symmetries of the stencil support. As was shown in [11] this P has sum rule of order 4 if and only if

$$\begin{cases} 3a + 3b + 6c = 1, \\ a + 4b + 14c = 0, \\ a - 8b + 20c = 0. \end{cases} \tag{8}$$

Note that many of the conditions in (5) become linearly dependent if the mask and its support satisfy additional symmetries. The linear system (8) has a unique solution

$$a = 32/81\,,\ b = -1/81\,,\ c = -2/81.$$

The corresponding $\sqrt{3}$-subdivision scheme was first introduced by Labsik and Greiner in [13]. It can be checked that it cannot have sum rule order 5. Thus, the maximal sum rule order for interpolatory schemes with (symmetric) stencils as shown in Figure 2 a) is 4, and achieved by a unique scheme.

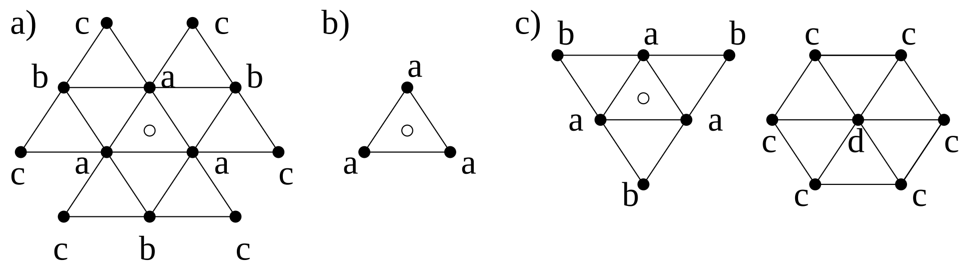

Figure 2: Stencils for simple interpolating and approximating schemes

Suppose $b = c = 0$, i.e., the corresponding stencil is much smaller, see Figure 2 b). Then $P(0) = 1$ if and only if $a = \frac{1}{3}$, but P has still sum rules of order 2 (but not 3). This scheme, which corresponds to linear interpolation in the $\sqrt{3}$-setting, was introduced in [15] as $VFV(1)$ scheme. We use $P_1(\omega)$ to denote its symbol given by

$$P_1(\omega) = \frac{1}{9}(3 + 2(\cos\omega_1 + \cos\omega_2 + \cos(\omega_1 - \omega_2)). \tag{9}$$

Thus, the maximal sum rule order of interpolating (and approximating) schemes with this smaller support is 2.

Let us look at some approximating schemes with coefficient stencils as shown in Figure 2 c). The corresponding mask P is given by the coefficient array

$$P \hat{=} \begin{bmatrix} b & c & b & 0 & 0 \\ c & a & a & c & 0 \\ b & a & \mathbf{d} & a & b \\ 0 & c & a & a & c \\ 0 & 0 & b & c & b \end{bmatrix}. \tag{10}$$

It has sum rules of order 4 if and only if

$$\left\{\begin{array}{rcrcrcrl} 6a & + & 6b & + & 6c & + & d & = 3, \\ -3a & - & 3b & + & 6c & + & d & = 0, \\ a & + & 4b & - & 6c & & & = 0, \\ a & - & 8b & & & & & = 0. \end{array}\right. \tag{11}$$

For details, see [11]. The linear system (11) has again a unique solution

$$a = 8/27\,,\ b = 1/27\,,\ c = 2/27\,,\ d = 5/9.$$

The resulting mask is $P_2(\omega) = P_1(\omega)^2$, does not have sum rule order 5, and the corresponding subdivision scheme is called $VFV(2)$ scheme in [15]. Thus, the $VFV(2)$ scheme achieves the maximal sum rule order 4 for approximating schemes with the above support.

Suppose $b = 0$ (i.e., the stencil for newly inserted control points becomes smaller). Then P cannot have sum rule order 4. However, P has sum rule order 3 if and only if

$$\left\{\begin{array}{rcrcrl} 6a & + & 6c & + & d & = 3, \\ -3a & + & 6c & + & d & = 0, \\ a & - & 6c & & & = 0. \end{array}\right. \tag{12}$$

The linear system (12) has a unique solution

$$a = 1/3\,,\ c = 1/18\,,\ d = 2/3.$$

The corresponding scheme was introduced by Kobbelt in [12].

From the above examples, we find that in order to construct $\sqrt{3}$-subdivision schemes with maximal sum rule order for given mask support (and symmetry constraints on the coefficients), one just solves a ladder of linear systems corresponding to increasing sum rule orders. In all considered cases, the maximal sum rule order was achieved when the corresponding system had a unique solution. It is natural to ask to which extent these empirical findings generalize, and whether one can determine the maximal sum rule order beforehand, i.e., without examining the whole ladder. Such results will be given in the next section.

The symmetry properties of support sets and masks occuring in $\sqrt{3}$-subdivision are dictated by the symmetries of a uniform hexagonal triangular mesh, and the requirement that a sound triangular subdivision scheme should not have any directional preferences. The hexagonal triangulation, a portion of which is shown in Figure 3 a), best reflects the geometric properties of a generic regular triangulation of $\mathbb{R}^2$. After transferring to the three-directional mesh with vertex set $\mathbb{Z}^2$ as shown in Figure 3 b) which is used for the analysis based on the refinement equation (1) with dilation (2), we arrive at the following natural definition of symmetry: A set $\Omega \subset \mathbb{Z}^2$ is called *symmetric* if it is invariant under linear transformations given by the matrices

$$V = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

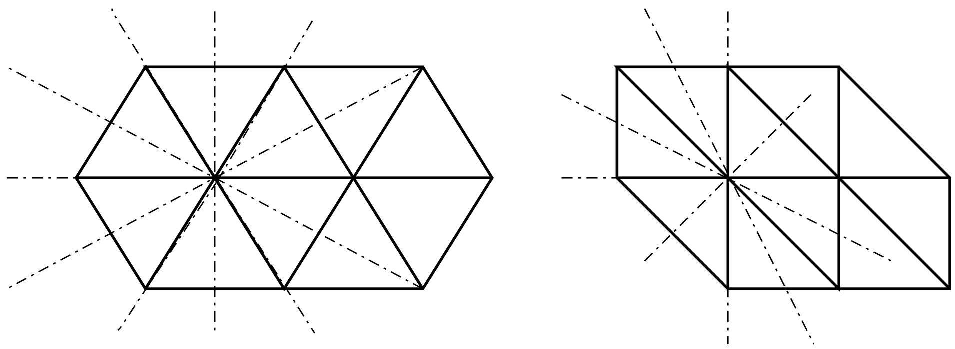

Figure 3: Symmetries of uniform hexagonal and three-directional triangulations

and their superpositions. Here, and in the following I denotes the 2×2 identity matrix. Analogously, a mask is called symmetric if

$$P_{V\alpha} = P_\alpha, \quad P_{W\alpha} = P_\alpha, \quad P_{-\alpha} = P_\alpha, \qquad \alpha \in \mathbb{Z}^2 . \tag{14}$$

Clearly, the support set of a symmetric mask is automatically symmetric. Note that the masks of the above examples are all symmetric. The two matrices V, W from (13) generate a multiplicative group $\mathcal{G}$ containing 12 different matrices. For future reference, we set $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, where

$$\begin{aligned} \mathcal{G}_1 &= \{V_1, \ldots, V_6\}, \quad V_1 = -I,\ V_2 = -V,\ V_3 = -V^2,\ V_4 = W,\ V_5 = WV,\ V_6 = WV^2, \\ \mathcal{G}_2 &= \{V_7, \ldots, V_{12}\}, \quad V_{s+6} = -V_s,\ s = 1, \ldots, 6. \end{aligned}$$

Mask symmetries automatically translate into symmetry properties of the refinable function ϕ. It is easy to prove that for finitely supported masks satisfying $P(0) = 1$ the (distributional) solution ϕ of the refinement equation satisfies

$$\phi(x_1, x_2) = \phi(x_2, -x_1 - x_2) = \phi(x_2, x_1) = \phi(-x_1, -x_2), \tag{15}$$

if and only if P is symmetric in the above sense. Moreover, in this case we have $\phi(x) = \phi(V_s x)$ for any of the above V_s.

We conclude this preliminary section with a helpful observation which allows us to separate symmetry issues from the investigation of sum rule orders.

Lemma 1 *Let $\Omega \in \mathbb{Z}^2$ be finite and symmetric. If there exists a mask P with support in Ω and sum rules of order k, then the formula*

$$\tilde{P}_\alpha := \frac{1}{12} \sum_{s=1}^{12} P_{V_s\alpha} , \qquad \alpha \in \mathbb{Z}^2 ,$$

delivers a symmetric mask $\tilde{P}$ with the same properties.

Proof. To see this (and also to prepare for our further considerations), recall that if P has support in Ω and satisfies sum rules of order k then

$$\sum_{\alpha\in\Omega} P_\alpha = 3\,, \tag{16}$$

$$\sum_{\alpha\in\Omega}(U\alpha)^\mu P_\alpha \tilde{z}^{2\alpha_1+\alpha_2} = \sum_{\alpha\in\Omega}(U\alpha)^\mu P_\alpha \tilde{z}^{\alpha_1+2\alpha_2} = 0 \tag{17}$$

for all $0 \le |\mu| < k$ and *any* (complex) invertible matrix U. Here,

$$\tilde{z} := (-1+i\sqrt{3})/2. \tag{18}$$

This immediately follows from the definition (5). Vice versa, if P has support in Ω and satisfies (16-17) for *some* invertible U then P satisfies sum rules of order k.

Set

$$\Omega^l = \Omega \cap (\gamma_l + M\mathbb{Z}^2)\,, \qquad l = 0,1,2\,, \tag{19}$$

and

$$S^l_{U,\mu}(P) = \sum_{\alpha\in\Omega^l}(U\alpha)^\mu P_\alpha\,, \quad l = 0,1,2\,. \tag{20}$$

Using the fact that

$$\frac{1}{2}(\tilde{z}^{2\alpha_1+\alpha_2} + \tilde{z}^{\alpha_1+2\alpha_2}) = \begin{cases} 1\,, & \alpha\in\gamma_0+\mathbb{Z}^2/M\mathbb{Z}^2\,, \\ -1/2\,, & \alpha\in\gamma_l+\mathbb{Z}^2/M\mathbb{Z}^2,\ l=1,2\,, \end{cases}$$

$$\frac{1}{2}(\tilde{z}^{2\alpha_1+\alpha_2} - \tilde{z}^{\alpha_1+2\alpha_2}) = \begin{cases} 0\,, & \alpha\in\gamma_0+\mathbb{Z}^2/M\mathbb{Z}^2\,, \\ (-1)^l i\sqrt{3}/2\,, & \alpha\in\gamma_l+\mathbb{Z}^2/M\mathbb{Z}^2,\ l=1,2\,, \end{cases}$$

we obtain the following linear system equivalent to (16-17):

$$S^0_{U,0}(P) + S^1_{U,0}(P) + S^2_{U,0}(P) = 3\,, \tag{21}$$

$$2S^0_{U,\mu}(P) - S^1_{U,\mu}(P) - S^2_{U,\mu}(P) = S^1_{U,\mu}(P) - S^2_{U,\mu}(P) = 0\,, \quad 0\le|\mu|<k\,, \tag{22}$$

again for all U.

Observe that if we consider any of the matrices $U = V_s$, $s = 1,\dots,6$, from $\mathcal{G}_1$ we have

$$U\alpha\in\Omega^0 \Longleftrightarrow \alpha\in\Omega^0\,, \quad U\alpha\in\Omega^1 \Longleftrightarrow \alpha\in\Omega^2\,, \quad U\alpha\in\Omega^2 \Longleftrightarrow \alpha\in\Omega^1\,,$$

while for $U = V_s$, $s = 7,\dots,12$, from $\mathcal{G}_2$ we have

$$U\alpha\in\Omega^l \Longleftrightarrow \alpha\in\Omega^l\,, \quad l=0,1,2\,.$$

Thus, if P has support in Ω then for $V_s\in\mathcal{G}$ the rotated mask $P_{V_s} := \{P_{V_s\alpha}\}$ satisfies

$$S^0_{V_s,\mu}(P_{V_s}) = \sum_{\alpha\in\Omega^0}(V_s\alpha)^\mu P_{V_s\alpha} = \sum_{V_s\alpha\in\Omega^0}(V_s\alpha)^\mu P_{V_s\alpha} = S^0_{I,\mu}(P)\,,$$

and

$$S^1_{V_s,\mu}(P_{V_s}) = S^2_{I,\mu}(P) \ , \quad S^2_{V_s,\mu}(P_{V_s}) = S^1_{I,\mu}(P) \ ,$$

for $s = 1, \dots, 6$, while

$$S^l_{V_s,\mu}(P_{V_s}) = S^l_{I,\mu}(P) \ , \qquad s = 7, \dots, 12 \, , \quad l = 0, 1, 2 \ .$$

In any case, this shows that if P satisfies (21-22) then P_{V_s}, $V_s \in \mathcal{G}$, also satisfies (21-22). This proves Lemma 1. ♣

The proof of Lemma 1 could be simplified by establishing $D^\mu \tilde{P}(\tilde{\omega}^j) = 0$, $|\mu| < k, j = 1, 2$, or the sum rule conditions in [8], directly. The reason for pointing out the equivalence of (21-22) and (5) is to prepare for our further considerations. See [8] for the derivation of conditions similar to (21-22) and equivalent to the sum rule order conditions for the case of a general dilation matrix.

3 Maximal sum rule orders

In this section we will first derive a simple necessary and sufficient condition for the existence of a real-valued masks $P = \{P_\alpha\}$ with given finite symmetric support Ω satisfying sum rules of order k for the dilation matrix M given by (2), and then apply this condition to find the maximal sum rule order for a certain family of support sets. As follows from Lemma 1, we do not have to bother about any symmetry conditions for the masks, they follow automatically once existence is proved. Without loss of generality, we will assume that $0 \in \Omega$.

Recall that P with support in Ω has sum rule order k if the conditions (21-22) are satisfied for some (and then all) invertible U. In this section, we will assume that U is real. We will look at (21-22) as some linear system $AP = B$ for the unknown vector $P = \{P_\alpha, \ \alpha \in \Omega\}$ with a real-valued rectangular matrix A of dimension $(1+k(k+1)) \times |\Omega|$, and a right-hand side B that contains exactly one non-zero element (for the first equation). Its solvability is governed by the rank condition

$$\text{rank}(A) = \text{rank}([A \ B]) \ , \tag{23}$$

which is equivalent to requiring that $vA = 0$ implies $vB = 0$ for any $1 \times (1 + k(k+1))$ vector v (see [6, p.224]). Thus, taking into account the special form of B, we see that (23) is satisfied if and only if the following statement is true: Whenever

$$a/2 + \sum_{0 \le |\beta| < k} b_\beta (U\alpha)^\beta = 0 \ , \qquad \alpha \in \Omega^0 \ ,$$

$$a - \sum_{0 \le |\beta| < k} b_\beta (U\alpha)^\beta - \sum_{0 \le |\beta| < k} c_\beta (U\alpha)^\beta = 0 \ , \qquad \alpha \in \Omega^1 \ ,$$

$$a - \sum_{0 \le |\beta| < k} b_\beta (U\alpha)^\beta + \sum_{0 \le |\beta| < k} c_\beta (U\alpha)^\beta = 0 \ , \qquad \alpha \in \Omega^2 \ ,$$

holds for some real a (the multiplier for the first equation) and real vectors $b = \{b_\beta\}$, $c = \{c_\beta\}$, we should have $a = 0$.

Let us introduce the polynomials

$$b(x) = \sum_{0 \le |\beta| < k} b_\beta x^\beta \ , \qquad c(x) = \sum_{0 \le |\beta| < k} c_\beta x^\beta$$

of total degree $< k$, i.e., $b(x), c(x) \in \Pi_k$. Here and below Π_k denotes the set of all (two variable) polynomials with total degree $< k$. Then the above necessary and sufficient condition for the solvability of (21-22) means that two $b(x), c(x) \in \Pi_k$ cannot satisfy

$$\begin{aligned} b(U\alpha) &= -a/2 \ , \qquad \alpha \in \Omega^0 \ , \\ b(U\alpha) + c(U\alpha) &= a \ , \qquad \alpha \in \Omega^1 \ , \\ b(U\alpha) - c(U\alpha) &= a \ , \qquad \alpha \in \Omega^2 \ , \end{aligned}$$

with some constant a other than $a = 0$. Set

$$p_1(x) := b(x) + c(x) - a \ , \qquad p_2(x) := b(x) - c(x) - a \ .$$

Then the previous statement says that there are no two polynomials $p_1(x), p_2(x) \in \Pi_k$ such that $p_l(U\alpha) = 0$ for $\alpha \in \Omega^l$, $l = 1, 2$, and

$$p_1(U\alpha) + p_2(U\alpha) = -3a \ , \qquad \alpha \in \Omega^0 \ ,$$

with some constant $a \neq 0$. Without loss of generality, we can assume that $p_2(x) = p_1(-x)$ (if $p_1(x), p_2(x)$ are polynomials with the above properties then so are

$$\tilde{p}_1(x) = \frac{1}{2}(p_1(x) + p_2(-x)) \ , \quad \tilde{p}_2(x) = \frac{1}{2}(p_1(-x) + p_2(x)) = \tilde{p}_1(-x) \ .)$$

Thus, we have arrived at the following characterization:

Theorem 1 *Let $\Omega \in \mathbb{Z}^2$ be finite and symmetric. Then there exists a mask P with support in Ω and satisfying sum rules of order k if and only if there is no invertible U and no polynomial $p(x) \in \Pi_k$ such that*

$$p(U\alpha) = 0 \ , \quad \alpha \in \Omega^1 \ , \qquad p(U\alpha) + p(-U\alpha) = 1 \ , \quad \alpha \in \Omega^0 \ . \tag{24}$$

The invertible linear map $U : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is kept for convenience (the formulation of the theorem remains true with any fixed U, e.g., one can set $U = I$).

Corollary 1 *If there are no nontrivial polynomials of total degree $< k$ vanishing on Ω^1 then there exists a symmetric mask P with support in Ω with sum rule order k. Moreover, this mask can be chosen interpolatory (i.e., $P_0 = 1$ and $P_\alpha = 0$ for $0 \neq \alpha \in \Omega^0$). The result is invariant with respect to invertible linear transformations U.*

Note that the condition used in Corollary 1 is only sufficient. The last statement in the corollary follows by applying the previous result to the modified set $\tilde{\Omega} := \{0\} \cup \Omega^1 \cup \Omega^2$ (a mask with support in $\tilde{\Omega}$ must be interpolatory!).

Theorem 1 and Corollary 1 reduce the existence of masks with support in a given finite symmetric set Ω and with specified order of sum rules to properties of polynomials from Π_k with zeros in Ω^1. Since Ω^1 is a finite subset of the lattice $\mathbb{Z}^2$ but otherwise may have a complicated structure (depending on the specification of Ω), the verification of the above conditions, though not hopeless, remains a problem. As far as we know, there are no general statements about the class of lattice subsets which cannot be in the zero set of a nontrivial polynomial from Π_k.

In the remainder of this section we will apply the above characterizations to find the maximal sum rule orders for approximating and interpolating $\sqrt{3}$-subdivision schemes with mask support in the sets

$$\Omega_n = \{\alpha \in \mathbb{Z}^2 \,:\, \max(|\alpha_1 + \alpha_2|, |\alpha_1|, |\alpha_2|) \le n\}, \qquad n = 1, 2, \ldots . \tag{25}$$

Obviously, the examples given in Section 2 have support sets from this family for some specific $n \le 3$. Although there might be other situations of interest, we find this family very natural for $\sqrt{3}$-subdivision in the following way. Families of $\sqrt{3}$-subdivision schemes can be characterized by their stencil supports for new and old control points (for interpolating schemes, only the stencils for new control points matter). The larger the stencils, the more flexible and complicated the schemes! Figure 4 shows a natural hierarchy of triangular complexes whose vertex sets serve as support sets for stencils of level $n = 1, 2, \ldots$ for new and old vertices in $\sqrt{3}$-schemes. The rule of enlarging the triangular complex when going from level n to level $n+1$ is based on adding the triangle-neighbors of all triangles in triangular complex. The triangle-neighbors of a triangle are the 3 flaps attached through its edges (only for the degenerate triangular complex at level $n = 1$ all triangles attached to the old vertex are added). See [15] for more background information on why adding flaps is very natural and convenient in this context. The reader may easily verify that approximating schemes with mask support in Ω_n are exactly those with stencils of level n, while interpolating schemes with mask support in Ω_n have stencils of level n for the new vertices (and of level 1 for the old vertices). Clearly, there are more combinations of possible interest (such as taking n-th level stencils for new and $(n+1)$-th level for old vertices, as this would cover Kobbelt's scheme with $n = 1$) but we will not pursue them.

We start with deriving upper bounds for the possible sum rule order of masks with support in Ω_n. To do this, we change the two dimensional masks into one dimensional ones as in [4]. At this point, it is convenient to use

$$U_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

because $U_0 \Omega_n^1$ possesses then a simple geometric structure which makes the upper bounds for k easier to explain. Figure 5 shows the sets $U_0 \Omega_n^1$ for $n \le 7$. In Figure 5 and in the

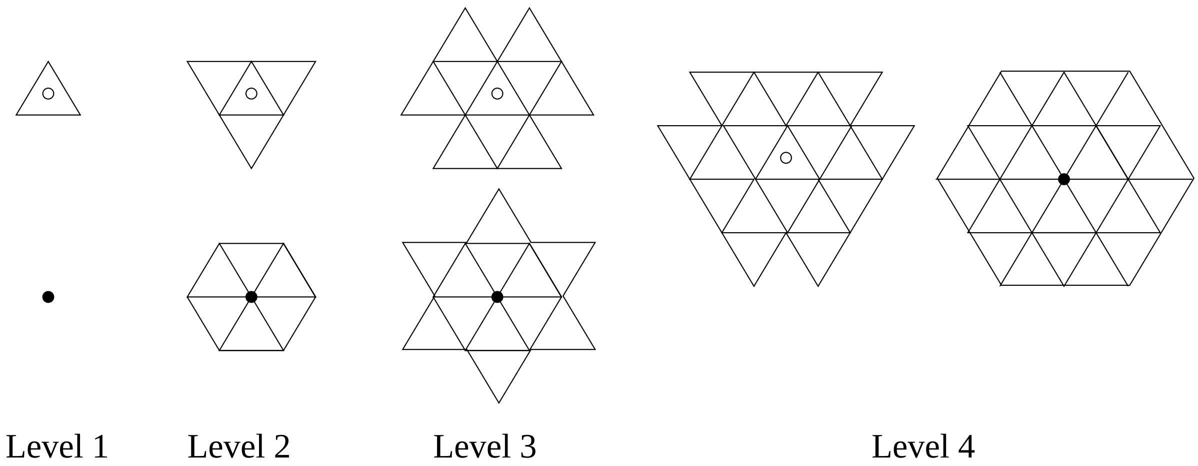

Figure 4: Stencil hierarchy for $\sqrt{3}$-subdivision schemes

following, for $n \in \mathbb{Z}_+$, $k(n)$ denotes the number given by

$$k(n) := n + [(n+2)/3]. \tag{26}$$

Lemma 2 *a) With the above U_0, the set $U_0\Omega_n^1$ is contained in the union of exactly $k(n)$ vertical lines $x_1 = 3l+1$, $l = -[2(n+1)/3], \ldots, [2(n-1)/3]$. By symmetry, $U_0\Omega_n^2$ is contained in the union of the $k(n)$ vertical lines $x_1 = -(3l+1)$, $l = -[2(n+1)/3], \ldots, [2(n-1)/3]$. Finally, $U_0\Omega_n^0$ is contained in the $4n+1-2k(n)$ vertical lines $x_1 = 3l$, $l = -2n+k(n), \ldots, 2n-k(n)$.*
b) The maximal sum rule order for interpolatory masks with support in Ω_n is $\le k(n)$.
c) The maximal sum rule order for arbitrary masks with support in Ω_n is $\le 2n$.

Proof. Part a) follows since $(x_1, x_2) \in U_0\Omega_n^1$ implies

$$x_1 = 1 \bmod 3 , \qquad -2n \le x_1 \le 2n ,$$

by the definitions of U_0 and Ω_n^1. Thus, $x_1 = 3l+1$ for some $l = -l_1, \ldots, l_2$, where $l_1 := [2(n+1)/3]$ and $l_2 := [2(n-1)/3]$. This shows that $U_0\Omega_n^1$ is contained in the line system specified above (note that indeed $k(n) = l_2 + l_1 + 1$). The reasoning for the other statements in Part a) is analogous.

For Part b), consider the polynomial

$$p(x_1, x_2) = \frac{1}{2} \prod_{l=-l_1}^{l_2} (1 - x_1/(3l+1)) .$$

Obviously, this polynomial belongs to $\Pi_{k(n)+1}$, its zero set contains $U_0\Omega_n^1$ by Part a), and $p(0) = 1/2$. If we choose $\tilde{\Omega}_n = \{0\} \cup \Omega_n^1 \cup \Omega_n^2$ in Theorem 1, we have $\tilde{\Omega}_n^1 = \Omega_n^1$ and

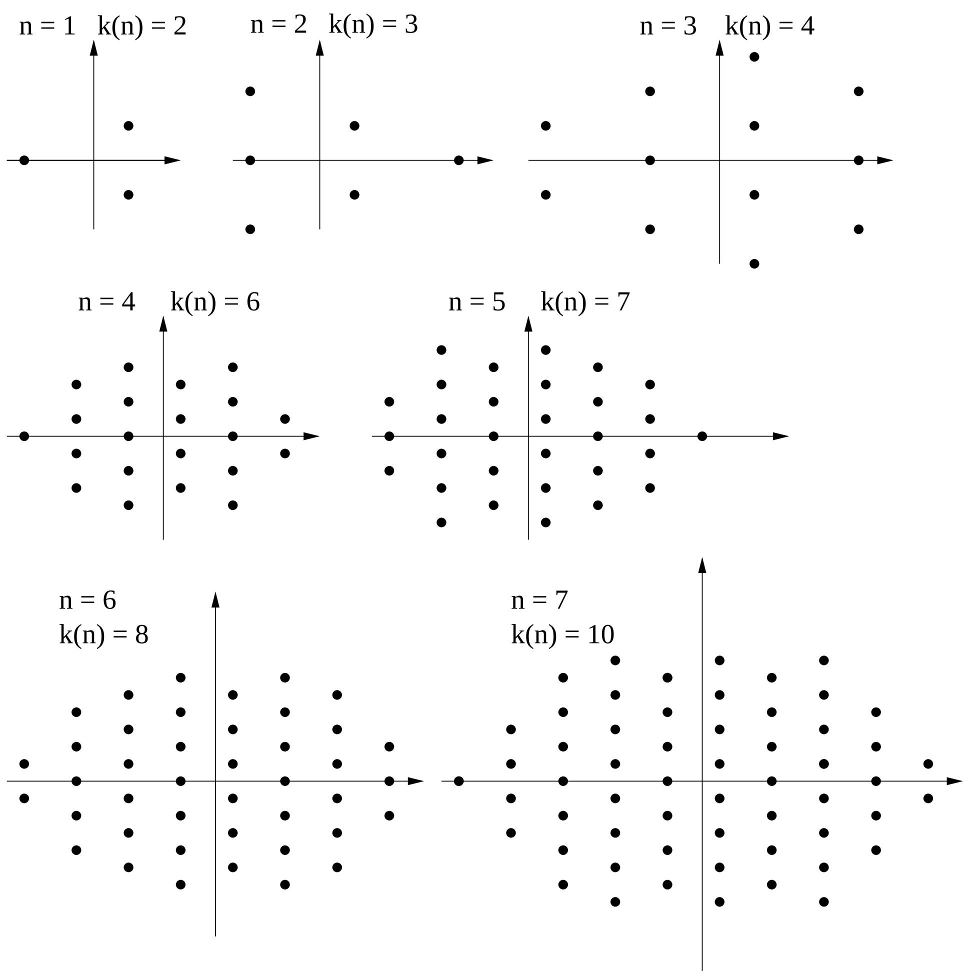

Figure 5: The sets $U_0\Omega_n^1$

$\tilde{\Omega}_n^0 = \{0\}$, and we see that a mask with support in $\tilde{\Omega}_n$ and sum rules of order $k(n)+1$ cannot exist. Consequently, there is no interpolatory mask with support in Ω_n and sum rules of order $k(n)+1$.

Finally, for Part c), in view of Theorem 1 and Part a), it is sufficient to construct a univariate polynomial $p(t)$ of degree $2n$ that satisfies

$$p(3l+1) = 0\,, \quad l = -l_1, \ldots, l_2\,, \qquad p(-3l) + p(3l) = 1\,, \quad l = 0, \ldots, 2n - k(n)\,. \tag{27}$$

These are exactly $2n+1$ linear conditions for the $2n+1$ unknown coefficients of $p(t)$. Since all our attempts to directly verify the non-singularity of this linear system failed we resort on the following indirect argument. By repeating the proof of Theorem 1 we can easily show that (27) has *no* solution if and only if there exists a univariate trigonometric polynomial

$$h(\omega) = \sum_{l=-2n}^{2n} h_l e^{-il\omega}$$

such that

$$h(0) = 1\,, \quad h^{(r)}(\frac{2\pi}{3}) = h^{(r)}(\frac{4\pi}{3}) = 0\,, \quad r = 0, \ldots, 2n\,.$$

The latter condition is equivalent to the univariate mask $h = \{h_l\,,\ l = -2n, \ldots, 2n\}$ having sum rule order $2n+1$ for dilation factor 3. By writing $h(\omega) = z^{-2n} q(z)$, where $z = e^{-i\omega}$ and $q(z) = z^{2n} \sum_{l=-2n}^{2n} h_l z^l$ is an algebraic polynomial of degree $\leq 4n$, it follows that the existence of an $h(\omega)$ with the above properties implies

$$q^{(r)}(\tilde{z}) = q^{(r)}(\tilde{z}^2) = 0\,, \quad r = 0, \ldots, 2n\,,$$

where $\tilde{z}$ is given by (18). Since the degree of $q(z)$ is $\leq 4n$ we obtain $q(z) \equiv 0$ which is in contradiction with $h(0) = 1$. Thus, (27) has a solution, and Lemma 2 is established. ♣

Let P_1 be the mask with the associated symbol given by (9). It is clear that the mask P_n associated with the symbol

$$P_n(\omega) = P_1(\omega)^n \tag{28}$$

has support in Ω_n and possesses sum rules of order $2n$ since the mask P_1 has support in Ω_1 and satisfies sum rules of order 2. Subdivision schemes with these masks have been introduced in [15], see also [11]. Thus, the bound in Part c) is sharp and we have

Theorem 2 *The maximal sum rule order for masks P with support in Ω_n is $2n$. It is achieved by the symmetric masks with symbol $P_n(\omega)$ defined in (28).*

From (9), we know the critical points of P_1 are characterized by either $\omega_2 = -\omega_1 \bmod(2\pi)$ or $\omega_1 - \omega_2 = \pi \bmod(2\pi)$, which implies

$$\min_{\omega \in \mathbf{R}^2} P_1(\omega) = P_1(2\pi/3, 4\pi/3) = 0\,.$$

Thus $P_n \geq 0$. This property is useful for computing exact Hölder exponents for the associated refinable function ϕ_n.

Either by the numerical calculations in [11], or by the results in Section 4.3 below that the transition operator T_P associated with P satisfies Condition E (i.e., 1 is a simple eigenvalue of T_P and other eigenvalues lie inside the unit disk) and that $P_1(\omega)$ has a biorthogonal dual $\tilde{P}(\omega)$ such that $T_{\tilde{P}}$ associated with $\tilde{P}$ also satisfies Condition E, we know ϕ_1 is L_2-stable. One can also obtain the L_∞-stability of ϕ_1 from the facts that P_1 is interpolating, $P_1(\omega) \geq 0$, and the subdivision operator S_{P_1} associated with P_1 satisfies Condition E (see [9]). Thus, for any $\omega \in \mathbb{R}^2$, there is a $k \in \mathbb{Z}^2$ such that $\hat{\phi}_1(\omega + k)$ is not zero (this is a necessary and sufficient condition for L_p-stability, see [10]). Since $\hat{\phi}_n(\omega) = \hat{\phi}_1(\omega)^n$, for any $\omega \in \mathbb{R}^2$, there is a $k \in \mathbb{Z}^2$ such that $\hat{\phi}_n(\omega + k)$ is not zero. Therefore ϕ_n is also L_p-stable (for any $1 \leq p \leq \infty$).

It is conjectured from the computations reported in [11] (see also Section 4.2) that the Hölder and Sobolev smoothness exponents of ϕ_n approach their trivial upper bound $2n$ given by their sum rule orders, i.e.,

$$\lim_{n\to\infty}(2n - s_\infty(\phi_n)) = \lim_{n\to\infty}(2n - s_2(\phi_n)) = 0\ .$$

This is in contrast to the analogous family $\{\varphi_n\}$of odd degree box splines for dyadic refinement for which

$$s_\infty(\varphi_n) = 2n - 1\ , \qquad s_2(\varphi_n) = 2n - 1/2\ , \qquad n \geq 1\ .$$

Either delete the following paragraph or indicate where the source is On the other hand, the above ϕ_n are not the *smoothest* ϕ related to a symmetric refinement mask with support in Ω_n. E.g., there is a box spline example (with unstable shifts!) with mask support in Ω_2 and only sum rule order 3 which has Sobolev exponent 4.5 and Hölder exponent 4. These numbers exceed the corresponding values for ϕ_2 (compare Table 2 below).

Theorem 3 *The maximal sum rule order for interpolatory masks P with support in Ω_n is $k(n)$, where $k(n)$ is given by (26).*

We will use Corollary 1 and the following known result (see [14, Section 11.4]).

Lemma 3 *Suppose $Z = \{x_{u,v} : u = 1, \ldots, v,\ v = 1, \ldots, r\}$ is such a set of $r(r+1)/2$ different points in $\mathbb{R}^2$ that there exist r different lines ℓ_v in $\mathbb{R}^2$ satisfying*

$$x_{u,v} \in \ell_v \setminus \cup_{w>v}\, \ell_w\ , \quad u = 1, \ldots, v\ ,$$

for $v = 1, \ldots, r$. Then the Lagrange interpolation problem

$$p(x_{u,v}) = f_{u,v}\ , \qquad u = 1, \ldots, v\ ,\ v = 1, \ldots, r\ ,$$

has a unique solution $p(x) \in \Pi_r$. In particular, if $p(x) \in \Pi_r$ vanishes on Z then $p(x) \equiv 0$.

It is convenient for us to prove Theorem 3 by transforming Ω_n into the set of lattice points within a uniform hexagon. Denote

$$U := \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Then $U\Omega_n$ is the set of lattice points from $U\mathbb{Z}^2$ within the hexagon with vertices $(\pm n, 0), (\pm \frac{n}{2}, \pm \frac{\sqrt{3}}{2} n)$. One can check that $(x_1, x_2) \in U\Omega_n^1$ if and only if $(x_1, x_2) \in U\Omega_n$ and

$$x_1 - \sqrt{3} x_2 = 1 \bmod 3. \tag{29}$$

Since Ω_n^1 is invariant under V (defined in (13)), $U\Omega_n^1$ is invariant under

$$U_1 := UVU^{-1} = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}.$$

Denote

$$v := (0, -\sqrt{3}), \quad w := U_1 v = (\frac{3}{2}, \frac{\sqrt{3}}{2}). \tag{30}$$

By Corollary 1, to prove Theorem 3, it is enough to show that there is no polynomial p of total degree smaller than k ($k := k(n)$) such that

$$p(\alpha) = 0, \quad \alpha \in U\Omega_n^1.$$

Without loss of generality, we can assume $\deg(p) = k - 1$ (otherwise, multiply by an appropriate monomial). On the contrary, assume that such a polynomial exists. Suppose for a moment that we were able to identify a set Z_n such that it satisfies the conditions in Lemma 3 with $r = k - [\frac{k}{2}]$ and

$$Z_n - sv \in U\Omega_n^1, \quad \forall\, 0 \le s \le [\frac{k}{2}]. \tag{31}$$

The existence of such a Z_n will be formally established in Lemma 4 below. From (31) we find by induction that

$$\nabla_v^{[\frac{k}{2}]} p(\alpha) = 0 \quad \forall\, \alpha \in Z_n,$$

where for $v \in \mathbb{R}^2$, ∇_v is the difference operator defined by

$$\nabla_v p := p - p(\cdot - v), \quad v \in \mathbb{R}^2,$$

and $\nabla_v^j := \nabla_v(\nabla_v^{j-1})$, $j \ge 2$. Since $\nabla_v^{[\frac{k}{2}]} p$ is a polynomial of total degree $< r = k - [\frac{k}{2}]$, and Z_n satisfies the assumptions of Lemma 3 with this r, we conclude that $\nabla_v^{[\frac{k}{2}]} p$ is identically zero.

Since $U\Omega_n^1$ is invariant under the rotation U_1, we have

$$U_1 Z_n - sw = U_1(Z_n - sv) \in U\Omega_n^1, \quad \forall\, 0 \le s \le [\frac{k}{2}].$$

Thus, using $Z_n' := U_1 Z_n$ resp. w instead of Z_n resp. v, we conclude in complete analogy that $\nabla_w^{[\frac{k}{2}]} p$ also vanishes. Since $\{v, w\}$ is a basis of $\mathbb{R}^2$, this implies

$$p(x_1, x_2) = \sum_{i<[\frac{k}{2}], j<[\frac{k}{2}]} c_{ij} x_1^i x_2^j.$$

Therefore $\deg(p) \le 2[\frac{k}{2}] - 2 < k - 1$, a contradiction to $\deg(p) = k - 1$. (See [7] for a discussion of the solutions to difference equations.)

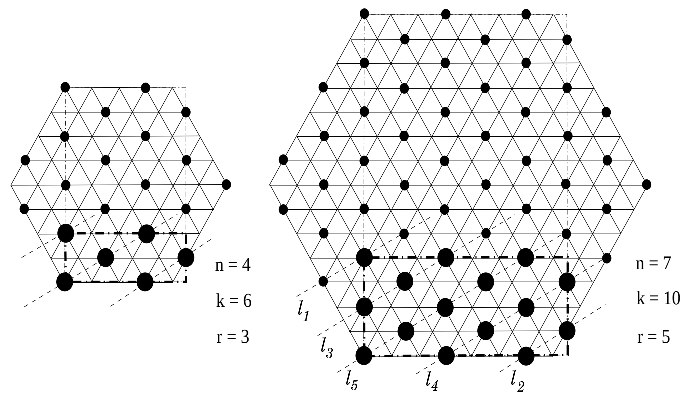

Figure 6: The sets $U\Omega_n^1$ and Z_n for $n = 3m + 1$ and $m = 1, 2$

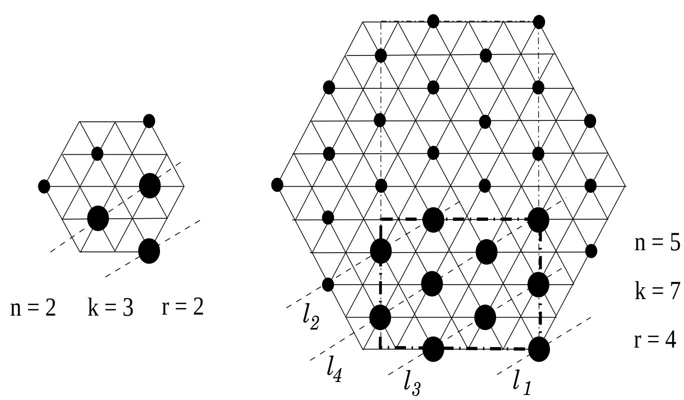

Figure 7: The sets $U\Omega_n^1$ and Z_n for $n = 3m + 2$ and $m = 0, 1$

To finish with the proof of Theorem 3, it remains to construct sets Z_n with the properties mentioned above. In Figures 6-8, the used notation is visualized for some small n, the lattice points in $U\Omega_n^1$ are highlighted by dots, the elements of $Z_n \subset U\Omega_n^1$ are indicated by bigger dots. In the figures, the points $Z_n = Z^o \cup Z^e$ will lie on two sets of lines in the direction of the vector w (see (30), Z^o on the odd labeled lines and Z^e on the even labeled lines. Because of slight technical differences, we distinguish the three

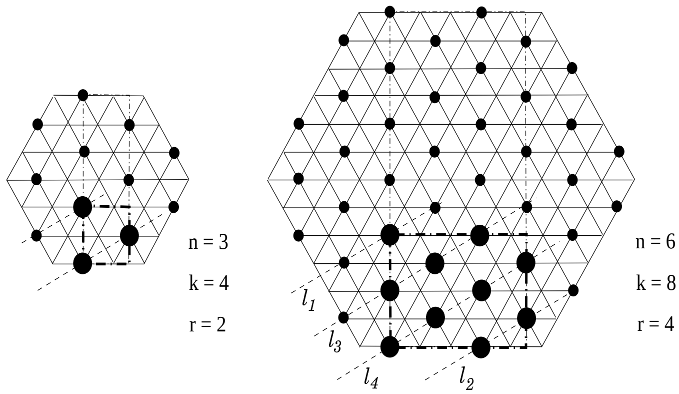

Figure 8: The sets $U\Omega_n^1$ and Z_n for $n = 3m + 3$ and $m = 0, 1$

cases $n = 3m + 1$, $n = 3m + 2$, and $n = 3m + 3$. The formal definition runs as follows. We define $Z_n = Z^o \cup Z^e$ for $n = 3m + 1$ by

$$Z^o := \{(-\frac{3m+1}{2}, (\frac{1-m}{2} - j)\sqrt{3}) + sw,\ 0 \le s \le 2j-2, 1 \le j \le m+1\},$$
$$Z^e := \{(\frac{3m+5}{2} - 3j, -\frac{3m+1}{2}\sqrt{3}) + sw,\ 0 \le s \le 2j-1, 1 \le j \le m\};$$

for $n = 3m + 2$ by

$$Z^o := \{(\frac{3m+8}{2} - 3j, -\frac{3m+2}{2}\sqrt{3}) + sw,\ 0 \le s \le 2j-2, 1 \le j \le m+1\},$$
$$Z^e := \{(-\frac{3m+1}{2}, (\frac{1-m}{2} - j)\sqrt{3}) + sw,\ 0 \le s \le 2j-1, 1 \le j \le m+1\};$$

and for $n = 3m + 3$ by

$$Z^o := \{(-\frac{3m+1}{2}, \sqrt{3}(\frac{1-m}{2} - j)) + sw,\ 0 \le s \le 2j-2, 1 \le j \le m+1\},$$
$$Z^e := \{(\frac{3m+5}{2} - 3j, -\frac{3m+3}{2}\sqrt{3}) + sw,\ 0 \le s \le 2j-1, 1 \le j \le m+1\}.$$

We also introduce the rectangular container sets for the points Z_n

$$R_{3m+1} := \{(x_1, x_2) \in U\mathbb{Z}^2 : -\frac{3m+1}{2} \le x_1 \le \frac{3m+2}{2}, -\frac{3m+1}{2}\sqrt{3} \le x_2 \le -\frac{m+1}{2}\sqrt{3}\},$$
$$R_{3m+2} := \{(x_1, x_2) \in U\mathbb{Z}^2 : -\frac{3m+1}{2} \le x_1 \le \frac{3m+2}{2}, -\frac{3m+2}{2}\sqrt{3} \le x_2 \le -\frac{m}{2}\sqrt{3}\},$$
$$R_{3m+3} := \{(x_1, x_2) \in U\mathbb{Z}^2 : -\frac{3m+1}{2} \le x_1 \le \frac{3m+2}{2}, -\frac{3m+3}{2}\sqrt{3} \le x_2 \le -\frac{m+1}{2}\sqrt{3}\}.$$

The regions R_n are depicted by bold dot-dashed lines in Figures 6-8. Clearly, $Z_n \subset R_n \subset U\Omega_n$, and since the points in Z_n satisfy (29), we also have $Z_n \in U\Omega_n^1$.

Lemma 4 *For all n, with $k = k(n)$ defined by (26), the above defined set Z_n satisfies (31), and the assumptions of Lemma 3 for $r = k - [\frac{k}{2}]$.*

Proof. We give the details for the case $n = 3m + 1$, where we have $k = 4m + 2$ and $r = 2m+1$. First of all, the assumptions for Lemma 3 immediately follow from the definition of Z_n since the set Z^o is the union of $m+1$ parallel lines (ordered by the index j), each containing exactly $2j - 1$ points (indexed by s). These correspond to the lines $\ell_1, \ldots, \ell_{2m+1}$. Similarly, Z^e is the union of m parallel lines $\ell_2, \ldots, \ell_{2m}$ each containing exactly $2j$ points ($j = 1, \ldots, m$).

To establish (31) for $n = 3m + 1$, it is enough to show that

$$-iv + Z_n \subset U\Omega_n \ , \qquad i = 0, \ldots, 2m+1 \ , \tag{32}$$

since shifts of points from Z_n by multiples of v again satisfy (29). But for any such i

$$\begin{aligned} &-iv + Z_{3m+1} \subset -iv + R_{3m+1} \\ &\quad \subset R'_{3m+1} := \{(x_1, x_2) \in U\mathbb{Z}^2 : \ -\frac{3m+1}{2} \leq x_1 \leq \frac{3m+2}{2}, \ |x_2| \leq \frac{3m+1}{2}\sqrt{3}\}. \end{aligned}$$

The rectangular region containing R'_{3m+1} is indicated by a dot-dashed line in Figure 6. Since the set $U\mathbb{Z}^2 \cap [-\frac{n}{2}, \frac{n}{2}] \times [-\frac{\sqrt{3}}{2}n, \frac{\sqrt{3}}{2}n]$ is obviously contained in Ω_n, we only have to check those lattice points from R'_{3m+1} which are on the vertical line $x_1 = \frac{3m+2}{2}$. Note that $(\frac{3m+2}{2}, \pm\frac{3m+1}{2}\sqrt{3}) \notin U\mathbb{Z}^2$, and if $-\frac{3m}{2}\sqrt{3} \leq x_2 \leq \frac{3m}{2}\sqrt{3}$ and $(\frac{3m+2}{2}, x_2) \in U\mathbb{Z}^2$, then $(\frac{3m+2}{2}, x_2) \in U\Omega_{3m+1}$. This establishes (32), and completes the proof of Lemma 4 for $n = 3m + 1$.

The proof for the cases $n = 3m + 2$ and $n = 3m + 3$ is completely analogous, and the details are omitted here. Lemma 4 and Theorem 3 are proved. ♣

Remark 1. The argument for the proof of Theorem 3 also gives clues on how to select a subset $\Omega' \subset \Omega_n^1$ of size $|\Omega'| = k(k+1)/2$ such that the reduced system

$$\sum_{\alpha \in \Omega'} \alpha^\beta P_\alpha = \delta_\beta \ , \qquad 0 \leq |\beta| < k \ , \tag{33}$$

has a unique solution. Such sets will be called *defining*. Note that if Ω' is defining in the above sense then (33) is equivalent to (21-22) with $U = I$ and $\Omega = \tilde{\Omega}_n$, i.e., it is easy to produce symmetric interpolatory masks of the maximal sum rule order $k = k(n)$, if we solve (33). Without going into details, we mention that it follows from the proof that a defining set is given by

$$\Omega' = \bigcup_{i=0}^{[\frac{k}{2}]} (-iv + Z_n) \cup (-iw + U_1 Z_n) \ .$$

But this choice is by no means the only defining set.

Remark 2. The above method can also be used to find out whether the support of a symmetric interpolatory mask P could be made slightly smaller than all of $\tilde{\Omega}_n$. Indeed, if Ω_n^* is a proper symmetric subset of $\tilde{\Omega}_n$ then an interpolatory mask with support in Ω_n^* and sum rule order $k = k(n)$ exists if we are able to find a replacement Z_n^* for the set Z_n such that it satisfies the conclusions of Lemma 4 with $U\Omega_n^1$ replaced by $U(\Omega_n^*)^1$. The earliest this might happen is for $n = 6$. Set

$$\Omega_6^* = \Omega_6 \backslash \{V_s\alpha \ : \ s = 1, \ldots 12, \ \alpha = (2,4)\} \ .$$

Figure 9 shows a possible choice for Z_6^* for which the conclusions of Lemma 4 are obviously satisfied. The empty circles indicate the images of $\alpha = (2,4)$ under the action of $\mathcal{G}$ that were deleted from Ω_6^1. We conjecture that a similar reduction of mask size can be achieved for all $n = 3m + 3$, $m \geq 1$, but also for some other n.

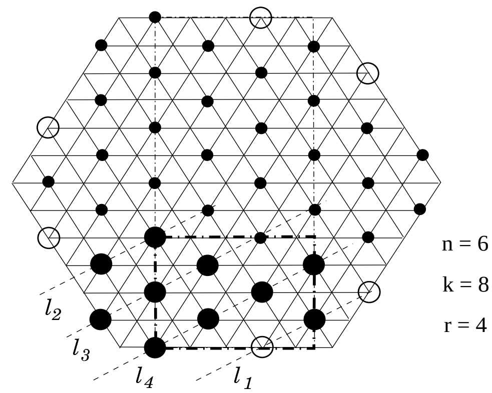

Figure 9: The sets $U(\Omega_6^*)^1$ and Z_6^*

4 Examples and further remarks

4.1 Reduction of linear systems

Although the results of the previous section tell us exactly how to find a linear system whose solutions give all (non-symmetric) masks of maximal sum rule order with support in any of the Ω_n and, subsequently, find the corresponding symmetric mask(s) via Lemma 1, this is cumbersome, and one wishes to come up with a smaller system by directly using the symmetry conditions. Since such reduced systems could be useful for small-sized Ω, we give a few details on their derivation for the case of interpolatory masks.

Let V and W be the matrices defined in (13). Then one has $V^3 = I_2$ and

$$V(\gamma_j + M\mathbb{Z}^2) = \gamma_j + M\mathbb{Z}^2, \quad j = 0, 1, 2.$$

The eigenvalues of the matrix V are

$$\lambda_1 := \tilde{z} = \frac{-1 + \sqrt{3}i}{2}, \quad \lambda_2 := \tilde{z}^2 = \frac{-1 - \sqrt{3}i}{2}.$$

Denote

$$\Lambda := \begin{bmatrix} 1 & -\lambda_2 \\ 1 & -\lambda_1 \end{bmatrix}.$$

It can be easily verified that

$$\Lambda V \Lambda^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \Lambda W \Lambda^{-1} = -\begin{bmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{bmatrix}.$$

Thus, for all $\alpha \in \mathbb{Z}^2$

$$(\Lambda V \Lambda^{-1}\alpha)^\mu = \lambda_1^{\mu_1}\lambda_2^{\mu_2}\alpha^\mu, \qquad (\Lambda W \Lambda^{-1}\alpha)^\mu = (-1)^{|\mu|}\lambda_1^{\mu_2}\lambda_2^{\mu_1}\alpha^{\mu'},$$

where $\mu' = (\mu_2, \mu_1)$.

Suppose P is interpolatory and symmetric, with support in Ω. Setting $U = \Lambda$, and using the notation (20) together with (21-22) we see that (5) is equivalent to

$$\delta_\mu = S^1_{\Lambda,\mu}(P) = S^2_{\Lambda,\mu}(P), \qquad 0 \le |\mu| < k.$$

The mask symmetry implies that $P_{-\beta} = P_\beta$ and since $-I \in \mathcal{G}_1$ we have $-\Omega^1 = \Omega^2$, so we get

$$S^1_{\Lambda,\mu}(P) = (-1)^{|\mu|}\sum_{\beta\in\Omega^1}(-\Lambda\beta)^\mu P_{-\beta} = (-1)^{|\mu|}\sum_{\beta\in\Omega^2}(\Lambda\beta)^\mu P_\beta = (-1)^{|\mu|}S^2_{\Lambda,\mu}(P).$$

Therefore, an interpolatory and symmetric mask P has sum rules of order k if and only if

$$S^1_{\Lambda,\mu} = \delta_\mu, \quad 0 \le |\mu| < k. \tag{34}$$

A further reduction of the number of equations can be achieved by using the other symmetry properties. E.g., since $P_{V\beta} = P_\beta$ and $V \in \mathcal{G}_2$, we have $V\Omega^1 = \Omega^1$ and

$$\begin{aligned} S^1_{\Lambda,\mu}(P) &= \sum_{V\beta\in\Omega^1}(\Lambda V\beta)^\mu P_{V\beta} = \sum_{\beta\in\Omega^1}((\Lambda V\Lambda^{-1})(\Lambda\beta))^\mu P_\beta \\ &= \lambda_1^{\mu_1}\lambda_2^{\mu_2}\sum_{\beta\in\Omega^1}(\Lambda\beta)^\mu P_\beta = \tilde{z}^{\mu_1-\mu_2}S^1_{\Lambda,\mu}(P). \end{aligned}$$

Thus, for any multi-index μ such that $\mu_2 - \mu_1 \notin 3\mathbb{Z}$ we automatically have $S^1_{\Lambda,\mu}(P) = 0$, and we can drop these equations from (34). Moreover, applying the same arguments with $W \in \mathcal{G}_1$, we get

$$\begin{aligned} S^1_{\Lambda,\mu}(P) &= \sum_{W\beta\in\Omega^1}(\Lambda W\beta)^\mu P_{W\beta} = \sum_{\beta\in\Omega^2}((\Lambda W\Lambda^{-1})(\Lambda\beta))^\mu P_\beta \\ &= (-1)^{|\mu|}\lambda_1^{\mu_2}\lambda_2^{\mu_1}\sum_{\beta\in\Omega^2}(\Lambda\beta)^{\mu'} P_\beta = \tilde{z}^{\mu_2-\mu_1}S^1_{\Lambda,\mu'}(P). \end{aligned}$$

Thus, only μ with $\mu_1 \ge \mu_2$ need to be considered.

These findings and (34) imply that a symmetric interpolatory mask has sum rule order k if and only if

$$\sum_{\beta\in\Omega^1} (\Lambda\beta)^{(m+3\ell,m)} P_\beta = \begin{cases} 1, & \text{if } (m,\ell)=(0,0), \\ 0, & \text{otherwise,} \end{cases} \qquad \forall\, m,\ell \in \mathbb{Z}_+,\ 2m+3\ell<k. \tag{35}$$

One can avoid complex coefficients by observing that (35) is equivalent to

$$\sum_{\beta\in\Omega^1} \underbrace{\frac{1}{6}(\sum_{s=7}^{12}(\Lambda V_s\beta)^{(m+3\ell,m)})}_{=A_\beta^{(m,\ell)}} P_\beta = \begin{cases} 1, & \text{if } (m,\ell)=(0,0), \\ 0, & \text{otherwise,} \end{cases} \qquad \forall\, m,\ell \in \mathbb{Z}_+, 2m+3\ell<k, \tag{36}$$

where we again have used the symmetry of P and $V_s\Omega^1 = \Omega^1$, $s = 7,\dots,12$. It turns out that for any $\beta \in \mathbb{Z}^l$ and all $s,l \in \mathbb{Z}_+$, the expressions for averaged coefficients $A_\beta^{(m,\ell)}$ are real-valued and can be computed as

$$A_\beta^{(m,\ell)} = (\beta_1^2+\beta_1\beta_2+\beta_2^2)^m \left[-\frac{1}{2}(\beta_1-\beta_2)^{3\ell} + \frac{3}{2}\sum_{r=0}^{\ell} \begin{pmatrix} 3\ell \\ 3r \end{pmatrix} \beta_1^{3(\ell-r)}(-\beta_2)^{3r} \right] \tag{37}$$

We leave this as an exercise for the reader. Clearly, $A_{V_s\beta}^{(m,\ell)} = A_\beta^{(m,\ell)}$ for all $s = 7,\dots,12$. Similar results can be obtained for approximating schemes.

For small support sets Ω, the use of (36-37) represents an alternative way of finding the linear systems for interpolatory masks with maximal sum rule orders. E.g., taking Ω_n with $n \le 3$, we recover the known results from [11]. However, for large n, it is easier to explore the results of Section 3 directly.

4.2 More Examples

For symmetric interpolatory P, we get interpolatory masks with maximal sum rule order and support in Ω_n by solving equation (5). Here we give some more examples for the values $4 \le n \le 7$. For $n \le 3$ the corresponding schemes can be found in [11] and Section

2. The symbol P is formally given by the coefficient array

$$P \hat{=} \begin{bmatrix} g_1 & g_2 & 0 & g_3 & g_3 & 0 & g_2 & g_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_2 & 0 & f_1 & f_2 & 0 & f_2 & f_1 & 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1 & e_1 & 0 & e_2 & e_2 & 0 & e_1 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_3 & f_2 & 0 & d_1 & d_2 & 0 & d_2 & d_1 & 0 & f_2 & g_3 & 0 & 0 & 0 & 0 \\ g_3 & 0 & e_2 & d_2 & 0 & c & c & 0 & d_2 & e_2 & 0 & g_3 & 0 & 0 & 0 \\ 0 & f_2 & e_2 & 0 & c & b & 0 & b & c & 0 & e_2 & f_2 & 0 & 0 & 0 \\ g_2 & f_1 & 0 & d_2 & c & 0 & a & a & 0 & c & d_2 & 0 & f_1 & g_2 & 0 \\ g_1 & 0 & e_1 & d_1 & 0 & b & a & \mathbf{1} & a & b & 0 & d_1 & e_1 & 0 & g_1 \\ 0 & g_2 & f_1 & 0 & d_2 & c & 0 & a & a & 0 & c & d_2 & 0 & f_1 & g_2 \\ 0 & 0 & 0 & f_2 & e_2 & 0 & c & b & 0 & b & c & 0 & e_2 & f_2 & 0 \\ 0 & 0 & 0 & g_3 & 0 & e_2 & d_2 & 0 & c & c & 0 & d_2 & e_2 & 0 & g_3 \\ 0 & 0 & 0 & 0 & g_3 & f_2 & 0 & d_1 & d_2 & 0 & d_2 & d_1 & 0 & f_2 & g_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & f_1 & e_1 & 0 & e_2 & e_2 & 0 & e_1 & f_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_2 & 0 & f_1 & f_2 & 0 & f_2 & f_1 & 0 & g_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_2 & 0 & g_3 & g_3 & 0 & g_2 & g_1 \end{bmatrix}. \tag{38}$$

The interpolating schemes with maximal sum rule order k are shown in Table 1. The regularity exponents s_2 and s_∞ of the resulting scaling functions ϕ are also provided there, where s_2 is the Sobolev smoothness exponent and s_∞ is (an upper bound for) the Hölder smoothness exponent. For calculating smoothness estimates of scaling functions, we refer to [11].

We note that for $n = 6$ the solution is not unique. In fact, as follows from Remark 2, we can delete $\alpha_0 = (2, 4)$ and all its images under $\mathcal{G}$ from $\tilde{\Omega}_6$, and still have the same sum rule order. In other words, the parameter f_2 which corresponds to the coefficients $P_{V_s\alpha}$, $s = 1, \ldots, 12$, is a free parameter for the symmetric interpolating schemes supported in Ω_6 with sum rule order 8. The number $295/4/3^9$ for f_2 was determined by requiring $D_1^7 D_2 P(\tilde{\omega}^1) = 0$ in addition to the conditions for the sum rule order 8. This resulted in slightly better smoothness properties than when taking $f_2 = 0$. It can be verified that the symbols $P(\omega)$ associated with the masks in Table 1 for $n = 4$, $n = 6$ with $f_2 = 0$, and $n = 7$ are nonnegative. So, for these cases the value s_∞ in Table 1 equals the exact Hölder smoothness exponents for the limiting surfaces (and not just an upper bound!). See Figure 10 for the graphs of the associated refinable function ϕ for $n = 4, 5$.

For symmetric approximating P and mask support in Ω_5, the symbol P is given by

n	4	5	6	7
a	$104/3^5$	$3080/3^8$	$2920/3^8+4f_2$	$80500/3^{11}$
b	$-20/3^6$	$-620/3^8$	$-740/3^9-6f_2$	$-65800/3^{13}$
c	$-32/3^6$	$-10/3^5$	$-1090/3^9-f_2$	$-102235/3^{13}$
d_1	$7/3^6$	$55/3^8$	$325/3^9+4f_2$	$34685/3^{13}$
d_2	$4/3^6$	$130/3^8$	$70/3^8+3f_2$	$21875/3^{13}$
e_1	–	$-28/3^8$	$-44/3^9-2f_2$	$-4109/3^{13}$
e_2	–	$-10/3^8$	$10/3^8-3f_2$	$7805/3^{13}$
f_1	–	–	$-20/3^9$	$-4655/3^{13}$
f_2	–	–	f_2	$-3010/3^{13}$
g_1	–	–	–	$715/3^{13}$
g_2	–	–	–	$385/3^{13}$
g_3	–	–	–	$560/3^{13}$
k	6	7	8	10
s_2	3.2804	3.3208	$4.0873(f_2=295/4/3^9)$ $3.9107(f_2=0)$	4.4667
s_∞	2.3465	2.8840	$3.4840(f_2=295/4/3^9)$ $3.0213(f_2=0)$	3.5921

Table 1: Sum rule order, Sobolev and Hölder smoothness for some interpolatory schemes with symbol (38)

the coefficient array

$$P \hat{=} \begin{bmatrix} e_1 & e_3 & e_2 & e_2 & e_3 & e_1 & 0 & 0 & 0 & 0 & 0 \\ e_3 & d_1 & d_2 & d_3 & d_2 & d_1 & e_3 & 0 & 0 & 0 & 0 \\ e_2 & d_2 & c_2 & c_1 & c_1 & c_2 & d_2 & e_2 & 0 & 0 & 0 \\ e_2 & d_3 & c_1 & b_1 & b_2 & b_1 & c_1 & d_3 & e_2 & 0 & 0 \\ e_3 & d_2 & c_1 & b_2 & a & a & b_2 & c_1 & d_2 & e_3 & 0 \\ e_1 & d_1 & c_2 & b_1 & a & \mathbf{a_0} & a & b_1 & c_2 & d_1 & e_1 \\ 0 & e_3 & d_2 & c_1 & b_2 & a & a & b_2 & c_1 & d_2 & e_3 \\ 0 & 0 & e_2 & d_3 & c_1 & b_1 & b_2 & b_1 & c_1 & d_3 & e_2 \\ 0 & 0 & 0 & e_2 & d_2 & c_2 & c_1 & c_1 & c_2 & d_2 & e_2 \\ 0 & 0 & 0 & 0 & e_3 & d_1 & d_2 & d_3 & d_2 & d_1 & e_3 \\ 0 & 0 & 0 & 0 & 0 & e_1 & e_3 & e_2 & e_2 & e_3 & e_1 \end{bmatrix}. \tag{39}$$

Table 2 shows some approximating schemes with maximal sum rule order, and mask support in Ω_n, $2 \le n \le 5$, as well as the regularity of the resulting scaling functions. We note that for these cases the solutions of the system (5) are unique (we also checked this for $n = 6, 7$). Thus, the above symbols coincide with $P_1(\omega)^n$, where $P_1(\omega)$ was defined in (9). Whether this holds for all n remains open.

n	2	3	4	5
a_0	$5/9$	$31/81$	$71/3^5$	$517/3^7$
a	$8/27$	$20/81$	$152/3^6$	$3535/3^9$
b_1	$1/27$	$5/81$	$160/3^7$	$1520/3^9$
b_2	$2/27$	$8/81$	$76/3^6$	$2020/3^9$
c_1	–	$1/81$	$52/3^7$	$70/3^7$
c_2	–	$1/243$	$8/3^6$	$115/3^8$
d_1	–	–	$1/3^7$	$35/3^9$
d_2	–	–	$4/3^7$	$95/3^9$
d_3	–	–	$6/3^7$	$130/3^9$
e_1	–	–	–	$1/3^9$
e_2	–	–	–	$10/3^9$
e_3	–	–	–	$5/3^9$
k	4	6	8	10
s_2	3.9518	5.9961	7.99967	9.9999
s_∞	3.3143	5.7073	7.9036	9.9721

Table 2: Sum rule order, Sobolev and Hölder smoothness for some approximating schemes with symbol (39)

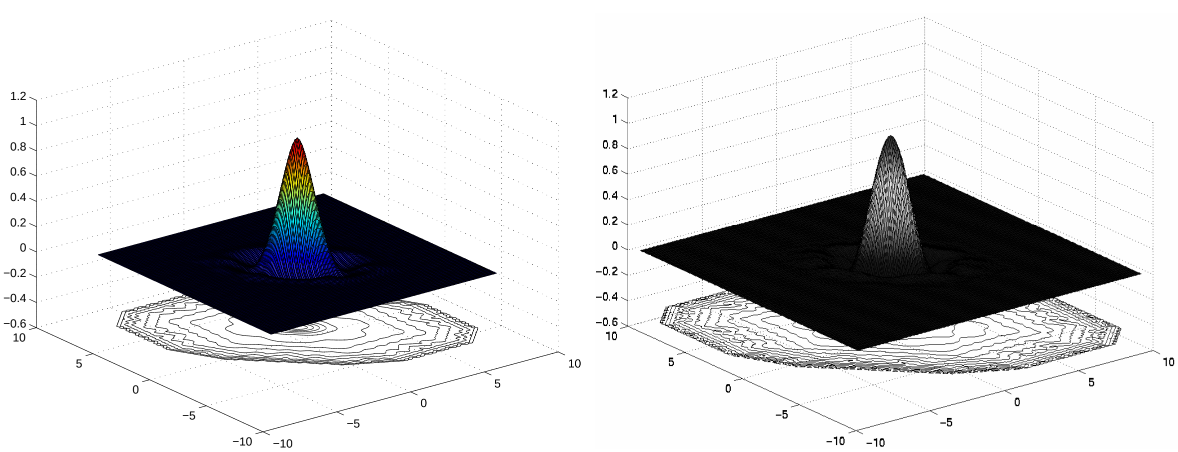

Figure 10: Graphs of ϕ: Interpolating schemes in Table 1 with $n = 4$ and 5

4.3 Dual scaling functions

Masks with maximal sum rule order are appealing for use in subdivision schemes since the schemes are compact, the approximation order of the schemes is high, and surfaces are usually quite smooth. However, they might not be the best candidates for primal scaling functions in the construction of biorthogonal wavelet systems. To explain this point, in the following we construct several dual scaling functions. After primal and dual scaling functions with decent smoothness properties have been found, the construction of the biorthogonal wavelets is standard, details are omitted here.

We say that a mask $\tilde{P}$ is *dual* to P if its symbol $\widetilde{P}(\omega)$ satisfies

$$P(\omega)\overline{\widetilde{P}(\omega)} + P(\omega+\tilde{\omega}_1)\overline{\widetilde{P}(\omega+\tilde{\omega}_1)} + P(\omega+\tilde{\omega}_2)\overline{\widetilde{P}(\omega+\tilde{\omega}_2)} = 1,$$

or, equivalently, if

$$\sum_{\beta\in\mathbf{Z}^2} P_{M\alpha+\beta}\tilde{P}_\beta = 3\delta_\alpha, \quad \alpha\in\mathbb{Z}^2. \tag{40}$$

Let $\tilde{\phi}$ be the solution of the refinement equation with symbol $\tilde{P}$. If the subdivision schemes associated with both P and $\tilde{P}$ are convergent in L^2 norm, then $\tilde{\phi}$ is a dual ϕ, i.e.,

$$\int_{\mathbf{R}^2} \phi(x)\overline{\tilde{\phi}(x-\beta)}dx = \delta(\beta), \quad \beta\in\mathbb{Z}^2.$$

Even when the subdivision scheme associated with P is convergent in L^2 norm and $\tilde{\phi}$ is in L^2, (40) cannot guarantee $\tilde{\phi}$ being a dual of ϕ.

In practice, to find a dual mask $\tilde{P}$ for a given primal mask P with sum rule order k, we start with a large enough symmetric support set for $\tilde{P}$, and solve the equations (40), together with symmetry constraints and conditions for achieving sum rule order $\tilde{k}$ for $\tilde{P}$. Clearly, if we take any of the examples of the previous subsection, to be able to satisfy all these conditions, we should either take the support for $\tilde{P}$ much larger than the support of P or live with $\tilde{k} << k$ which most likely implies poor smoothness properties for the dual subdivision schemes. For this brief discussion, we restrict ourselves to the case of symmetric dual masks $\tilde{P}$ with support in Ω_5, i.e., the dual symbol is given by the coefficient array

$$\tilde{P} \doteq \begin{bmatrix}
\tilde{e}_1 & \tilde{e}_3 & \tilde{e}_2 & \tilde{e}_2 & \tilde{e}_3 & \tilde{e}_1 & 0 & 0 & 0 & 0 & 0 \\
\tilde{e}_3 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & \tilde{d}_2 & \tilde{d}_1 & \tilde{e}_3 & 0 & 0 & 0 & 0 \\
\tilde{e}_2 & \tilde{d}_2 & \tilde{c}_2 & \tilde{c}_1 & \tilde{c}_1 & \tilde{c}_2 & \tilde{d}_2 & \tilde{e}_2 & 0 & 0 & 0 \\
\tilde{e}_2 & \tilde{d}_3 & \tilde{c}_1 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_1 & \tilde{c}_1 & \tilde{d}_3 & \tilde{e}_2 & 0 & 0 \\
\tilde{e}_3 & \tilde{d}_2 & \tilde{c}_1 & \tilde{b}_2 & \tilde{a} & \tilde{a} & \tilde{b}_2 & \tilde{c}_1 & \tilde{d}_2 & \tilde{e}_3 & 0 \\
\tilde{e}_1 & \tilde{d}_1 & \tilde{c}_2 & \tilde{b}_1 & \tilde{a} & \tilde{\mathbf{a}}_{\mathbf{0}} & \tilde{a} & \tilde{b}_1 & \tilde{c}_2 & \tilde{d}_1 & \tilde{e}_1 \\
0 & \tilde{e}_3 & \tilde{d}_2 & \tilde{c}_1 & \tilde{b}_2 & \tilde{a} & \tilde{a} & \tilde{b}_2 & \tilde{c}_1 & \tilde{d}_2 & \tilde{e}_3 \\
0 & 0 & \tilde{e}_2 & \tilde{d}_3 & \tilde{c}_1 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_1 & \tilde{c}_1 & \tilde{d}_3 & \tilde{e}_2 \\
0 & 0 & 0 & \tilde{e}_2 & \tilde{d}_2 & \tilde{c}_2 & \tilde{c}_1 & \tilde{c}_1 & \tilde{c}_2 & \tilde{d}_2 & \tilde{e}_2 \\
0 & 0 & 0 & 0 & \tilde{e}_3 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & \tilde{d}_2 & \tilde{d}_1 & \tilde{e}_3 \\
0 & 0 & 0 & 0 & 0 & \tilde{e}_1 & \tilde{e}_3 & \tilde{e}_2 & \tilde{e}_2 & \tilde{e}_3 & \tilde{e}_1
\end{bmatrix}. \tag{41}$$

With P being the mask of $VFV(1)$, the choice of $\tilde{P}$ with support in Ω_2

$$\tilde{a}_0 = 7/3,\ \tilde{a} = 1/3,\ \tilde{b}_1 = 0,\ \tilde{b}_2 = -2/9,\ \tilde{c}_j = \tilde{d}_j = \tilde{e}_j = 0,$$

turns out to be dual, with $\tilde{k} = 2$. However, the corresponding dual subdivision scheme is not convergent. Therefore, we have to enlarge the support to Ω_3. For the parameters

$$\tilde{a}_0 = 53/27,\ \tilde{a} = 14/27,\ \tilde{b}_1 = -5/27,\ \tilde{b}_2 = -2/9,\ \tilde{c}_2 = 5/81,\ \tilde{c}_1 = \tilde{d}_j = \tilde{e}_j = 0,$$

$\tilde{P}$ has sum rule of order 3, and the dual subdivision scheme is convergent. Thus, the resulting $\tilde{\phi}$ is a dual of the scaling function ϕ. Our calculations show that $\tilde{\phi}$ is in the Sobolev space $W^{0.5715}(\mathbb{R}^2)$.

If we take Kobbelt's scheme as primal mask, the choice of

$$\tilde{a}_0 = \frac{926762}{353199}, \tilde{a} = \frac{2229691}{3178791}, \tilde{b}_1 = -\frac{5747765}{12715164}, \tilde{b}_2 = -\frac{497}{1089}, \tilde{c}_1 = \frac{7}{278},$$
$$\tilde{c}_2 = \frac{603767}{2119194}, \tilde{d}_1 = \frac{-173465}{1816452}, \tilde{d}_2 = \frac{13}{231}, \tilde{d}_3 = \frac{-12}{121}, \tilde{e}_2 = \frac{1}{121}, \tilde{e}_1 = \tilde{e}_3 = 0,$$

corresponds to convergent dual scheme, and the resulting $\tilde{\phi}$ belongs to the Sobolev space $W^{0.0222}(\mathbb{R}^2)$. If one wants to construct a smoother dual scaling function, the support of the symbol must be larger than Ω_5. The same is true if one attempts to find a convergent dual scheme to the $VFV(2)$ scheme, but we will not give the details. Recall that the masks for both Kobbelt's scheme and $VFV(2)$ have support in Ω_2.

By giving up the optimality of the primal scheme, we could try to achieve a better balance between the properties of primal and dual schemes. Let P, $\tilde{P}$ be a pair of primal and dual symmetric masks supported in Ω_2, Ω_4, and parameterized as in (10), (41), respectively. If both of $P, \tilde{P}$ are required to have sum rule of order 3, then the mask for P contains only one free parameter which we choose to be b. It turns out that the duality conditions together with the conditions for achieving $\tilde{k} = 3$ can be satisfied for any value of b. For the choice $b = -0.1751872$ (a root of $13122t^4 + 2673t^3 + 279t^2 + 19t - 29/9$), $\tilde{P}$ even has sum rule order 4. The resulting ϕ and $\tilde{\phi}$ are in $W^{1.4211}(\mathbb{R}^2)$ and $W^{0.7067}(\mathbb{R}^2)$, respectively. For the choice of $b = -1/9$, we have computed that $\phi \in W^{1.8957}(\mathbb{R}^2)$ and $\tilde{\phi} \in W^{0.7853}(\mathbb{R}^2)$.

Altogether, the above discussion suggests that, in order to construct biorthogonal multiresolution analyses with good properties (high smoothness, high sum rule orders, small symmetric masks, etc.), it might not be the best idea to start with an optimized primal scaling function. Instead, it is better to construct primal and dual schemes and optimize their properties simultaneously, see [2] for an alternative approach to this problem.

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