CONVERGENCE OF CASCADE ALGORITHMS IN SOBOLEV SPACES AND INTEGRALS OF WAVELETS

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ABSTRACT. The cascade algorithm with mask a and dilation M generates a sequence ϕ_n, $n = 1, 2, \ldots$, by the iterative process

$$\phi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi_{n-1}(Mx - \alpha) \quad x \in \mathbb{R}^s,$$

from a starting function ϕ_0, where M is a dilation matrix. A complete characterization is given for the strong convergence of cascade algorithms in Sobolev spaces for the case in which M is isotropic. The results on the convergence of cascade algorithms are used to deduce simple conditions for the computation of integrals of products of derivatives of refinable functions and wavelets.

1. INTRODUCTION

For $1 \le p \le \infty$, let $L_p(\mathbb{R}^s)$ denote the Banach space of all complex-valued measurable functions f on $\mathbb{R}^s$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p \, dx \right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of f on $\mathbb{R}^s$. The Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi} \, dx, \qquad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$. For a vector $x = (x_1, \ldots, x_s)$ in $\mathbb{R}^s$, its norm is defined to be

$$\|x\| := \sqrt{x_1^2 + \cdots + x_s^2}.$$

We shall denote the set of all non-negative integers by $\mathbb{N}_0$ and the set of all natural numbers by $\mathbb{N}$. A multi-index is an s-tuple $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$. The length of μ is $|\mu| := \mu_1 + \cdots + \mu_s$, and the factorial of μ is $\mu! := \mu_1! \cdots \mu_s!$. For two multi-indices $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_s)$, we write $\nu \le \mu$ if $\nu_j \le \mu_j$ for $j = 1, \ldots, s$. If $\nu \le \mu$, we define

$$\binom{\mu}{\nu} := \frac{\mu!}{\nu!(\mu - \nu)!}.$$

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The partial derivative of a differentiable function f with respect to the jth coordinate is denoted by $D_j f$, $j = 1, \ldots, s$, and for $\mu = (\mu_1, \ldots, \mu_s)$, D^μ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$.

For a non-negative integer $k \in \mathbb{N}_0$, let Π_k be the linear space of all polynomials in s variables of total degree at most k. We shall write $\Pi_{-1} = \{0\}$. Let $\Pi := \cup_{k=0}^\infty \Pi_k$ be the linear space of all polynomials. The degree of a polynomial q is denoted by $\deg q$. If $q = \sum_\mu c_\mu x^\mu$ is a polynomial, we shall use $q(D)$ to denote the differential operator $\sum_\mu c_\mu D^\mu$. A compactly supported function $f \in L_1(\mathbb{R}^s)$ is said to satisfy the *moment conditions* of order k if $\hat{f}(0) = 1$, and

$$D^\mu \hat{f}(2\pi\beta) = 0 \qquad \forall\, |\mu| < k \ \text{ and } \ \beta \in \mathbb{Z}^s \setminus \{0\}.$$

It is known that f satisfies the moment conditions of order k if and only if for each polynomial $q \in \Pi_{k-1}$,

$$\sum_{\alpha \in \mathbb{Z}^s} q(\alpha) f(\cdot - \alpha) - q$$

is a polynomial of degree less than $\deg q$ (see [19]).

We use $W_p^k(\mathbb{R}^s)$ to denote the Sobolev space that consists of all distributions f such that $D^\mu f \in L_p(\mathbb{R}^s)$ for all multi-indices μ with $|\mu| \le k$, equipped with the norm defined by

$$\|f\|_{W_p^k(\mathbb{R}^s)} := \sum_{|\mu| \le k} \|D^\mu f\|_p.$$

The linear space of all sequences and the linear space of all finitely supported sequences on $\mathbb{Z}^s$ are denoted by $\ell(\mathbb{Z}^s)$ and $\ell_0(\mathbb{Z}^s)$ respectively. For $\alpha \in \mathbb{Z}^s$, we denote by δ_α the element in $\ell_0(\mathbb{Z}^s)$ given by $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \setminus \{\alpha\}$. In particular, we write δ for δ_0. For $j = 1, \ldots, s$, let e_j be the jth coordinate unit vector. The difference operator ∇_j on $\ell(\mathbb{Z}^s)$ is defined by $\nabla_j a := a - a(\cdot - e_j)$, $a \in \ell(\mathbb{Z}^s)$. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, ∇^μ denotes the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$. If $q(x) = \sum_\mu c_\mu x^\mu$ is a polynomial we shall write $q(\nabla)$ to mean the difference operator $\sum_\mu c_\mu \nabla^\mu$.

We are concerned with functional equations of the form

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(M \cdot - \alpha), \tag{1.1}$$

where ϕ is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, a is a finitely supported sequence on $\mathbb{Z}^s$, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The equation (1.1) is called a *refinement equation,* M is called a *dilation matrix* and the sequence a is called a *refinement mask.* Any function satisfying a refinement equation is called a *refinable function.* If a satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m := |\det M|, \tag{1.2}$$

then it is known that there exists a unique compactly supported distribution ϕ satisfying the refinement equation (1.1) normalized so that $\hat{\phi}(0) = 1$. This distribution is called the *normalized solution* of the refinement equation with mask a. Refinement equations and refinable functions play an important role in wavelet analysis and the theory of uniform subdivision. Associated with the refinement equation (1.1) is the *refinement operator* Q_a defined on $L_p(\mathbb{R}^s)$ by

$$Q_a f := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) f(M\cdot-\alpha), \qquad f\in L_p(\mathbb{R}^s). \tag{1.3}$$

Let ϕ_0 be an initial function in $L_p(\mathbb{R}^s)$ with compact support. For $n=1,2,\ldots$, define

$$\phi_n := Q_a\phi_{n-1}. \tag{1.4}$$

Clearly $\phi_n = Q_a^n\phi_0$, $n=0,1,\ldots$. The algorithm (1.4) is called the *cascade algorithm* with mask a and dilation M. Convergence of cascade algorithms has been studied in connection with the solution of refinement equations and the description of curves and surfaces in computer aided geometric design (see [2], [9], [18], [20]).

In one dimension the convergence of a cascade algorithm with dilation 2 depends on the two matrices $A_0 := (a(2i-j-1))_{i,j=1}^N$ and $A_1 := (a(2i-j))_{i,j=1}^N$ associated with the mask $(a(j))_{j=0}^N$. A characterization of convergence in $L_p(\mathbb{R})$ in terms of the joint spectral radius of A_0 and A_1 was given in [9]. In $L_2(\mathbb{R})$ this characterization takes a simpler form which can be described in terms of the *transition matrix* $(a\odot a(2i-j))_{i,j=-N}^N$ associated with the autocorrelation $a\odot a$ of a defined by

$$a\odot a(k) := \sum_{j\in\mathbb{Z}} a(j)\overline{a(j-k)}, \quad k\in\mathbb{Z}.$$

This characterization of strong convergence of cascade algorithms in $L_2(\mathbb{R})$ can be deduced from the results in [9] as well from [7] and [2], and was also obtained independently in [18]. Similar results were established in [15] for strong as well as weak convergence of cascade algorithms in $L_2(\mathbb{R}^s)$ with a general dilation matrix M. These results were further extended to matrix cascade algorithms with dilation matrix $M=2I$ [17] and also to nonstationary cascade algorithms in $L_2(\mathbb{R}^s)$ with an arbitrary dilation matrix M [5]. Weak convergence of derivatives of cascade algorithms was also considered in [5]. In this paper we are interested in the strong convergence in the Sobolev space $W_p^k(\mathbb{R}^s)$ of the cascade sequence $(Q_a^n\phi_0)_{n\in\mathbb{N}}$, i.e.,

$$\lim_{n\to\infty}\left\|Q_a^n\phi_0-\phi\right\|_{W_p^k(\mathbb{R}^s)} = 0. \tag{1.5}$$

In one dimension ($s=1$) with dilation 2 and $p=2$, characterizations of weak and strong convergence of cascade algorithms in $W_2^k(\mathbb{R})$ in terms of the spectral properties of the transition matrix $(a\odot a(2i-j))_{i,j=-N}^N$ for the mask $a=(a(j))_{j=0}^N$ has been studied in [6]. The multivariate problem with a general dilation matrix is more complicated and the techniques in [20] and [6] are not easily extended to higher dimension. Our main objective is to give a characterization of strong convergence of the cascade algorithm (1.4) in the Sobolev space $W_2^k(\mathbb{R}^s)$ for the case in which the dilation matrix M is isotropic, i.e., M is similar to a diagonal matrix diag $(\sigma_1,\ldots,\sigma_s)$ with $|\sigma_1|=\cdots=|\sigma_s|$. We shall show in Section 2 that if (1.5) holds then the initial function ϕ_0 must satisfy the moment conditions of order $k+1$. This motivates us to give the following definition. We say that the cascade algorithm with mask a converges (strongly) in the Sobolev space $W_p^k(\mathbb{R}^s)$ if (1.5) is valid for any compactly supported function $\phi_0\in W_p^k(\mathbb{R}^s)$ satisfying the moment conditions of order $k+1$.

For an $s \times s$ dilation matrix M, let Γ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$, and let Ω be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M^T\mathbb{Z}^s$, where M^T denotes the transpose of M.. Evidently, $\#\Gamma = \#\Omega = |\det M|$. Without loss of any generality, we may assume that $0 \in \Gamma$ and $0 \in \Omega$. We say that a sequence $a \in \ell_0(\mathbb{Z}^s)$ satisfies the sum rules of order k, if for all $p \in \Pi_{k-1}$,

$$\sum_{\beta\in\mathbb{Z}^s} a(M\beta)\, p(M\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\, p(M\beta+\gamma) \qquad \forall\, \gamma \in \Gamma. \tag{1.6}$$

We shall also show in Section 2 that if the cascade algorithm (1.4) converges in the Sobolev space $W_p^k(\mathbb{R}^s)$, then the mask a must satisfy the sum rules of order $k+1$. Our main theorem, Theorem 3.2, is stated and proved in Section 3. Our proof in one direction depends on a result on the convergence of cascade algorithms in $C^k(\mathbb{R}^s)$, the subspace of $W_\infty^k(\mathbb{R}^s)$ that consists of all k times continuously differentiable functions on $\mathbb{R}^s$. The proof here is different from that in [6]. In Section 4 we apply the results of Section 3 on the convergence of cascade algorithms to deduce results on the computation of integrals of products of derivatives of refinable functions and wavelets. These integrals are useful in the construction of wavelets on a finite interval and for the multiscale solution of partial differential equations. Computation of such an integral has been considered earlier by Beylkin [1] and by Dahmen and Micchelli [4]. The approach in [4] is to express the integral as the unique eigenvector of a certain transition matrix, under a linear constraint. Their results are obtained under the assumption that the refinable functions are stable and belong to $C^k(\mathbb{R}^s)$. Our assumption based on the convergence of cascade algorithms is weaker and easier to verify in practice. Furthermore, unlike stability, convergence of cascade algorithm is preserved under convolution.

2. Cascade Algorithms in Sobolev Spaces

Suppose a is a finitely supported sequence on $\mathbb{Z}^s$ satisfying (1.2). Let ϕ be the normalized solution of the refinement equation (1.1). Taking the Fourier transform of both sides of (1.1), we obtain

$$\hat{\phi}(\xi) = H((M^T)^{-1}\xi)\, \hat{\phi}((M^T)^{-1}\xi), \qquad \xi \in \mathbb{R}^s, \tag{2.1}$$

where

$$H(\xi) := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)e^{-i\alpha\cdot\xi}/m, \qquad \xi \in \mathbb{R}^s. \tag{2.2}$$

Clearly, H is a 2π-periodic function and $H(0) = 1$. Taking the Fourier transform of both sides of (1.3), we obtain

$$\widehat{Q_a f}(\xi) = H\big((M^T)^{-1}\xi\big)\, \hat{f}\big((M^T)^{-1}\xi\big), \qquad \xi \in \mathbb{R}^s. \tag{2.3}$$

Let M be an $s \times s$ isotropic matrix with entries in $\mathbb{C}$. Then M is similar to a diagonal matrix $\mathrm{diag}\,(\sigma_1, \ldots, \sigma_s)$ with $|\sigma_1| = \cdots = |\sigma_s|$. In this case, there exists an invertible matrix $\Lambda = (\lambda_{jl})_{1\le j,l\le s}$ such that

$$\Lambda M \Lambda^{-1} = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_s). \tag{2.4}$$

Let f be a differentiable function on $\mathbb{R}^s$. By using the chain rule we obtain

$$\begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} \big(f \circ M^T\big)(x) = M \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} f(M^T x), \qquad x \in \mathbb{R}^s.$$

It follows that

$$\Lambda \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} \big(f \circ M^T\big)(x) = \Lambda M \Lambda^{-1} \Lambda \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} f(M^T x), \qquad x \in \mathbb{R}^s. \tag{2.5}$$

For $j = 1, \ldots, s$, let q_j be the linear polynomial given by

$$q_j(x) := \sum_{l=1}^{s} \lambda_{jl} x_l, \qquad x = (x_1, \ldots, x_s) \in \mathbb{R}^s. \tag{2.6}$$

By (2.4) and (2.5) we deduce that

$$q_j(D)\big(f \circ M^T\big)(x) = \sigma_j q_j(D) f(M^T x), \qquad x \in \mathbb{R}^s.$$

For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, define $q_\mu := q_1^{\mu_1} \cdots q_s^{\mu_s}$. It follows from the above equation that

$$q_\mu(D)\big(f \circ M^T\big)(x) = \sigma^\mu q_\mu(D) f(M^T x), \qquad x \in \mathbb{R}^s,$$

where $\sigma^\mu := \sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s}$. Consequently, for $n = 1, 2, \ldots$,

$$q_\mu(D)\big(f \circ (M^T)^n\big)(x) = \sigma^{\mu n} q_\mu(D) f((M^T)^n x), \qquad x \in \mathbb{R}^s. \tag{2.7}$$

Proposition 2.1. *Suppose M is an $s \times s$ isotropic dilation matrix, and a is an element in $\ell_0(\mathbb{Z}^s)$ satisfying* (1.2). *Let $\phi \in W_p^k(\mathbb{R}^s)$ be the normalized solution of the refinement equation* (1.1) *with mask a, and let Q_a be the linear operator defined in* (1.3). *If ϕ_0 is a compactly supported function in $W_p^k(\mathbb{R}^s)$, $1 \le p \le \infty$, such that*

$$\lim_{n\to\infty} \big\| Q_a^n \phi_0 - \phi \big\|_{W_p^k(\mathbb{R}^s)} = 0,$$

then ϕ_0 satisfies the moment conditions of order $k+1$.

Proof. Since the supports of ϕ_n are uniformly bounded, it suffices to deal with the case $p = 1$. Let $\phi_n := Q_a^n \phi_0$ and $g_n(\xi) := \hat{\phi}_n((M^T)^n \xi)$, $\xi \in \mathbb{R}^s$, $n = 1, 2, \ldots$. A repeated use of (2.3) with $f = \hat{\phi}_0$ yields

$$g_n(\xi) = h_n(\xi) \hat{\phi}_0(\xi), \qquad \xi \in \mathbb{R}^s,$$

where

$$h_n(\xi) := \prod_{j=1}^{n} H\big((M^T)^{j-1}\xi\big), \qquad \xi \in \mathbb{R}^s.$$

For $\beta \in \mathbb{Z}^s$, we have $h_n(2\pi\beta) = h_n(0) = 1$. It follows that

$$\hat{\phi}_0(2\pi\beta) = g_n(2\pi\beta) = \hat{\phi}_n\big((M^T)^n 2\pi\beta\big).$$

In particular, $\hat{\phi}_0(0) = g_n(0) = \hat{\phi}_n(0)$. But $|\hat{\phi}_n(\xi) - \hat{\phi}(\xi)| \le \|\phi_n - \phi\|_{L_1(\mathbb{R}^s)} \to 0$ uniformly in ξ as $n \to \infty$. Hence, $|\hat{\phi}_n(0) - \hat{\phi}(0)| \to 0$ as $n \to \infty$. Therefore, $\hat{\phi}_0(0) = \hat{\phi}(0) = 1$. Similarly, for $\beta \in \mathbb{Z}^s \setminus \{0\}$,

$$\hat{\phi}_0(2\pi\beta) = \lim_{n\to\infty} g_n(2\pi\beta) = \lim_{n\to\infty} \hat{\phi}_n\big((M^T)^n 2\pi\beta\big) = \lim_{n\to\infty} \hat{\phi}\big((M^T)^n 2\pi\beta\big) = 0,$$

where the Riemann-Lebesgue lemma has been used to derive the last equality.

We claim that $D^\mu \hat{\phi}_0(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s \setminus \{0\}$ and all multi-indices μ with $|\mu| \le k$. This has been established for $|\mu| = 0$. Let $0 < r \le k$ and suppose our claim has been verified for all $|\mu| < r$. Let μ be a multi-index with $|\mu| = r$. By using the Leibniz formula for differentiation, we obtain

$$D^\mu g_n(\xi) = \sum_{\nu \le \mu} \binom{\mu}{\nu} D^\nu \hat{\phi}_0(\xi) D^{\mu-\nu} h_n(\xi).$$

For $\nu \le \mu$ and $\nu \ne \mu$, by the induction hypothesis, we have $D^\nu \hat{\phi}_0(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s \setminus \{0\}$. Consequently,

$$D^\mu g_n(2\pi\beta) = D^\mu \hat{\phi}_0(2\pi\beta) \qquad \forall\, \beta \in \mathbb{Z}^s \setminus \{0\} \text{ and } |\mu| = r. \tag{2.8}$$

Since the matrix M is isotropic, it is similar to a diagonal matrix $\operatorname{diag}(\sigma_1, \dots, \sigma_s)$ with $|\sigma_1| = \cdots = |\sigma_s|$. Suppose $\Lambda = (\lambda_{jl})_{1\le j,l\le s}$ is a matrix such that (2.4) is valid. With $\Theta := (\Lambda^T)^{-1}$ it follows from (2.4) that

$$\Theta M^T \Theta^{-1} = \operatorname{diag}(\sigma_1, \dots, \sigma_s).$$

Let $q_\mu := q_1^{\mu_1} \cdots q_s^{\mu_s}$, where q_j, $j = 1, \dots, s$, is the linear polynomial given in (2.6). Since $g_n(\xi) = \hat{\phi}_n((M^T)^n \xi)$, $\xi \in \mathbb{R}^s$, (2.7) shows that

$$q_\mu(D) g_n(\xi) = \sigma^{\mu n} \big(q_\mu(D)\hat{\phi}_n\big)\big((M^T)^n \xi\big), \qquad \xi \in \mathbb{R}^s. \tag{2.9}$$

On the other hand, it follows from (2.8) that

$$q_\mu(D) g_n(2\pi\beta) = q_\mu(D)\hat{\phi}_0(2\pi\beta) \qquad \forall\, \beta \in \mathbb{Z}^s \setminus \{0\} \text{ and } |\mu| = r. \tag{2.10}$$

Let

$$\phi_{\mu,n}(x) := q_\mu(-ix)\phi_n(x) \quad \text{and} \quad \phi_\mu(x) := q_\mu(-ix)\phi(x), \qquad x \in \mathbb{R}^s.$$

Since $(\phi_n)_{n\in\mathbb{N}}$ converges to ϕ in the Sobolev space $W_p^k(\mathbb{R}^s)$, for any multi-index ν with $|\nu| = r$ we have

$$\lim_{n\to\infty} \big\| D^\nu \phi_{\mu,n} - D^\nu \phi_\mu \big\|_{L_1(\mathbb{R}^s)} = 0.$$

Note that the Fourier transforms of $D^\nu \phi_{\mu,n}$ and $D^\nu \phi_\mu$ are

$$(i\xi)^\nu q_\mu(D)\hat{\phi}_n(\xi) \quad \text{and} \quad (i\xi)^\nu q_\mu(D)\hat{\phi}(\xi),$$

respectively. Hence,

$$\lim_{n\to\infty} \|\xi\|^r q_\mu(D)\hat{\phi}_n(\xi) = \lim_{n\to\infty} \|\xi\|^r q_\mu(D)\hat{\phi}(\xi), \qquad \xi \in \mathbb{R}^s, \tag{2.11}$$

where the convergence is uniform in ξ. Since $D^\nu \phi_\mu \in L_1(\mathbb{R}^s)$ for any multi-index ν with $|\nu| = r$, by the Riemann-Lebesgue lemma, $\xi^\nu q_\mu(D)\hat{\phi}(\xi) \to 0$ as $\|\xi\| \to \infty$. Hence, $\|\xi\|^r q_\mu(D)\hat{\phi}(\xi) \to 0$ as $\|\xi\| \to \infty$. Combining this and (2.11), we obtain

$$\lim_{n\to\infty} \|(M^T)^n 2\pi\beta\|^r q_\mu(D)\hat{\phi}_n((M^T)^n 2\pi\beta) = 0, \qquad \beta \in \mathbb{Z}^s \setminus \{0\}. \tag{2.12}$$

Note that $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s}$, where $m = |\det M|$. In light of (2.4), we see that there exists a positive constant C independent of n such that

$$\|(M^T)^n\beta\| = \|\Theta^{-1}\mathrm{diag}\,(\sigma_1^n, \ldots, \sigma_s^n)\Theta\beta\| \geq C m^{n/s}\|\beta\|.$$

This together with (2.9) and (2.12) gives

$$\lim_{n\to\infty} q_\mu(D) g_n(2\pi\beta) = \lim_{n\to\infty} \sigma^{\mu n} q_\mu(D)\hat{\phi}_n((M^T)^n 2\pi\beta) = 0, \qquad \beta \in \mathbb{Z}^s \setminus \{0\}.$$

Finally, this in connection with (2.8) yields

$$q_\mu(D)\hat{\phi}_0(2\pi\beta) = 0 \qquad \forall\, \beta \in \mathbb{Z}^s \setminus \{0\} \text{ and } |\mu| = r.$$

This completes the induction procedure and also the proof of Proposition 2.1. □

As a corollary of Proposition 2.1, we have the following result which was obtained earlier in [10].

Corollary 2.1. *Suppose M is an $s \times s$ isotropic dilation matrix, and a is an element in $\ell_0(\mathbb{Z}^s)$ satisfying* (1.2). *Let $\phi \in W_p^k(\mathbb{R}^s)$ be the normalized solution of the refinement equation* (1.1) *with mask a. Then ϕ satisfies the moment conditions of order $k+1$.*

Proof. Choose the initial function $\phi_0 = \phi$. Then the cascade sequence $(Q_a^n\phi)_{n\in\mathbb{N}}$ satisfies the conditions of Proposition 2.1. It follows that ϕ satisfies the moment conditions of order $k+1$. □

Theorem 2.1. *If the cascade algorithm with mask a converges in the Sobolev space $W_p^k(\mathbb{R}^s)$, $1 \leq p \leq \infty$, then the mask a satisfies the sum rules of order $k+1$.*

Proof. Let f be a stable and compactly supported function in $W_p^k(\mathbb{R}^s)$, and suppose that $(Q_a^n f)_{n\in\mathbb{N}}$ converges in $W_p^k(\mathbb{R}^s)$ to the normalized solution ϕ of the refinement equation (1.1). By Proposition 2.1, f satisfies the moment conditions of order $k+1$. Since $(Q_a^n f)_{n\in\mathbb{N}}$ converges in $W_p^k(\mathbb{R}^s)$ to ϕ, it follows that $(Q_a^n(Q_a f))_{n\in\mathbb{N}}$ also converges to ϕ in $W_p^k(\mathbb{R}^s)$. By Proposition 2.1, $g := Q_a f$ also satisfies the moment conditions of order $k+1$. By (2.3)

$$\hat{g}(\xi) = H\big((M^T)^{-1}\xi\big)\,\hat{f}\big((M^T)^{-1}\xi\big), \qquad \xi \in \mathbb{R}^s. \tag{2.13}$$

In order to prove that a satisfies the sum rules of order $k+1$, it suffices to show that

$$D^\mu H\big((M^T)^{-1}2\pi\omega\big) = 0 \qquad \forall\, \omega \in \Omega \setminus \{0\} \text{ and } |\mu| \leq k. \tag{2.14}$$

This will be done by induction on $|\mu|$, the length of μ.

Let ω be an element of $\Omega \setminus \{0\}$. Since f is stable, there exists some $\gamma \in \mathbb{Z}^s$ such that $\hat{f}((M^T)^{-1}2\pi\omega + 2\pi\gamma) \neq 0$ (see [12], [13]). Let $\beta := \omega + M^T\gamma$. Then $\beta \in \mathbb{Z}^s \setminus \{0\}$, and so $\hat{g}(2\pi\beta) = 0$. This together with (2.13) gives

$$0 = H\big((M^T)^{-1}2\pi\omega\big)\hat{f}\big((M^T)^{-1}2\pi\beta\big).$$

But $\hat{f}((M^T)^{-1}2\pi\beta) \neq 0$. Therefore, $H((M^T)^{-1}2\pi\omega) = 0$. This verifies (2.14) for $|\mu| = 0$.

Let $0 < r \leq k$. Suppose our claim has been verified for $|\mu| < r$. Let μ be a multi-index with $|\mu| = r$. Let $\beta = \omega + M^T\gamma$ be given as above. Applying the Leibniz formula for differentiation to (2.13), we obtain

$$D^\mu \hat{g}(2\pi\beta) = \sum_{\nu \leq \mu} \binom{\mu}{\nu} D^\nu \big(H \circ (M^T)^{-1}\big)(2\pi\beta) D^{\mu-\nu}\big(\hat{f} \circ (M^T)^{-1}\big)(2\pi\beta).$$

In the above series, if $\nu \leq \mu$ and $\nu \neq \mu$, then by the induction hypothesis we have $D^\nu(H \circ (M^T)^{-1})(2\pi\beta) = D^\nu(H \circ (M^T)^{-1})(2\pi\omega) = 0$. Hence,

$$D^\mu \hat{g}(2\pi\beta) = D^\mu \big(H \circ (M^T)^{-1}\big)(2\pi\omega)\, \hat{f}\big((M^T)^{-1}2\pi\beta\big), \qquad |\mu| = r.$$

It follows that

$$q_\mu(D)\hat{g}(2\pi\beta) = q_\mu(D)\big(H \circ (M^T)^{-1}\big)(2\pi\omega)\, \hat{f}\big((M^T)^{-1}2\pi\beta\big), \qquad |\mu| = r.$$

But $q_\mu(D)\hat{g}(2\pi\beta) = 0$, and by (2.7) we have

$$q_\mu(D)\big(H \circ (M^T)^{-1}\big)(2\pi\omega) = \sigma^{-\mu} q_\mu(D) H\big((M^T)^{-1}2\pi\omega\big).$$

Hence,

$$0 = \sigma^{-\mu} q_\mu(D) H\big((M^T)^{-1}2\pi\omega\big)\, \hat{f}\big((M^T)^{-1}2\pi\beta\big).$$

But $\hat{f}((M^T)^{-1}2\pi\beta) \neq 0$. Therefore,

$$q_\mu(D) H\big((M^T)^{-1}2\pi\omega\big) = 0 \qquad \forall\, |\mu| = r \text{ and } \omega \in \Omega \setminus \{0\}.$$

This verifies (2.14) for $|\mu| = r$, thereby completing the induction procedure. The proof of Theorem 2.1 is complete. □

Remark 2.1. *From the proof of Theorem 2.1 we see that the following stronger result holds.*

If the cascade sequence $(Q_a^n \phi_0)_{n \in \mathbb{N}}$ converges in the Sobolev space $W_p^k(\mathbb{R}^s)$ to the normalized solution of the refinement equation (1.1) *for some stable initial function ϕ_0, then the mask a satisfies the sum rules of order $k+1$.*

As an example, let ϕ be the normalized solution of the refinement equation

$$\phi(x) = \frac{1}{2}\,\phi(2x) + \frac{1}{2}\,\phi(2x-1) + \frac{1}{2}\,\phi(2x-2) + \frac{1}{2}\,\phi(2x-3), \qquad x \in \mathbb{R}.$$

Thus, the refinement mask a is given by $a(j) = 1/2$ for $j = 0, 1, 2, 3$, and $a(j) = 0$ for $j \in \mathbb{Z} \setminus \{0,1,2,3\}$. One can check easily $\phi = (h(\cdot) + h(\cdot - 1))/2$, where h is the hat function supported on $[0, 2]$. *Thus $\phi \in W_2^1(\mathbb{R})$. It is easily verified that the cascade algorithm associated with mask a converges in $L_2(\mathbb{R})$. However, the mask a does not satisfy the sum rules of order* 2. *Therefore, the cascade algorithm associated with mask a does not converge in the Sobolev space $W_2^1(\mathbb{R})$. This answers the question raised at the end of the revised version of the paper* [16].

3. Characterization of Convergence

In this section we give a characterization for the convergence of a cascade algorithm in the Sobolev space $W_2^k(\mathbb{R}^s)$ in terms of the corresponding mask. The results are based on the transition operator and subdivision operator associated with a given mask. Let us review their basic properties.

Let a be an element in $\ell_0(\mathbb{Z}^s)$ and let M be a dilation matrix. The transition operator T_a is the linear operator on $\ell_0(\mathbb{Z}^s)$ defined by

$$T_a v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(M\alpha-\beta)v(\beta), \qquad \alpha\in\mathbb{Z}^s, \quad v\in\ell_0(\mathbb{Z}^s). \tag{3.1}$$

Let $G = \operatorname{supp} a := \{\alpha\in\mathbb{Z}^s : a(\alpha)\neq 0\}$, and let

$$K := \sum_{n=1}^{\infty} M^{-n}G := \left\{\sum_{n=1}^{\infty} M^{-n}y_n : y_n\in G \quad \forall n\in\mathbb{N}\right\}.$$

We shall denote the linear space of all sequences supported on K by $\ell(K)$. Then $\ell(K)$ is a finite dimensional invariant subspace of T_a. Moreover, any eigenvector of T_a corresponding to a nonzero eigenvalue is supported in K. See ([8], Theorem 4.2) and ([11], Lemma 3.1) for these facts. For an invariant subspace V of T_a we define the spectral radius of $T_a|_V$ by

$$\rho(T_a|_V) := \rho(T_a|_{\ell(K)\cap V}).$$

In particular, $\rho(T_a) := \rho(T_a|_{\ell(K)})$.

The subdivision operator S_a is the linear operator on $\ell(\mathbb{Z}^s)$ defined by

$$S_a u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha-M\beta)u(\beta), \qquad \alpha\in\mathbb{Z}^s, \quad u\in\ell(\mathbb{Z}^s). \tag{3.2}$$

Let Q_a be the linear operator given in (1.3), and let $\phi_n := Q_a^n\phi_0$, $n=0,1,\ldots$. Then

$$\phi_n = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\phi_0(M^n\cdot-\alpha), \tag{3.3}$$

where the sequences a_n, $n=1,2,\ldots$, are obtained iteratively by the relation

$$a_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{n-1}(\beta)a(\alpha-M\beta), \qquad \alpha\in\mathbb{Z}^s,$$

with $a_1 = a$. Consequently, $a_n = S_a(a_{n-1}) = \cdots = S_a^n\delta$.

When $p=\infty$, it is more appropriate to discuss convergence in the space $C^k(\mathbb{R}^s)$ with norm

$$\|f\|_{C^k(\mathbb{R}^s)} := \sum_{|\mu|\le k} \|D^\mu f\|_\infty, \qquad f\in C^k(\mathbb{R}^s).$$

We say that the cascade algorithm with mask a converges strongly in $C^k(\mathbb{R}^s)$ if

$$\lim_{n\to\infty} \left\|Q_a^n\phi_0-\phi\right\|_{C^k(\mathbb{R}^s)} = 0$$

holds for any compactly supported function $\phi_0\in C^k(\mathbb{R}^s)$ satisfying the moment conditions of order $k+1$.

For a vector $v \in \mathbb{R}^s \setminus \{0\}$, we let ∇_v denote the difference operator defined by $\nabla_v f := f - f(\cdot - v)$ for any f, and we use D_v to denote the directional derivative, i.e.,

$$D_v f(x) := \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}, \qquad x \in \mathbb{R}^s,$$

if the limit exists. It is easily seen that, for a function $f \in C^1(\mathbb{R}^s)$,

$$\nabla_v f(x) = \int_{-1}^{0} D_v f(x+tv)\, dt, \qquad x \in \mathbb{R}^s.$$

Suppose $f \in C^k(\mathbb{R}^s)$ and $v_1, \dots, v_k \in \mathbb{R}^s \setminus \{0\}$. Then

$$\nabla_{v_1} \cdots \nabla_{v_k} f(x) = \int_{-1}^{0} \cdots \int_{-1}^{0} D_{v_1} \cdots D_{v_k} f(x + t_1 v_1 + \cdots + t_k v_k)\, dt_1 \cdots dt_k.$$

It follows that

$$\big\|\nabla_{v_1} \cdots \nabla_{v_k} f\big\|_\infty \le \big\|D_{v_1} \cdots D_{v_k} f\big\|_\infty. \tag{3.4}$$

Let

$$V_k := \Big\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha\in\mathbb{Z}^s} p(\alpha) v(\alpha) = 0 \;\forall p \in \Pi_k \Big\}.$$

It is known that V_k is invariant under T_a if and only if a satisfies the sum rules of order $k+1$ (see [10], Theorem 5.2).

The following theorem gives a necessary condition for convergence of cascade algorithms in $C^k(\mathbb{R}^s)$.

Theorem 3.1. *Suppose M is an $s \times s$ isotropic dilation matrix, and $m := |\det M|$. Let a be a finitely supported mask satisfying* (1.2). *For a stable compactly supported $\phi_0 \in C^k(\mathbb{R}^s)$, if the cascade sequence $(Q_a^n \phi_0)_{n\in\mathbb{N}}$ converges in $C^k(\mathbb{R}^s)$ to the normalized solution of the refinement equation* (1.1), *then V_k is invariant under the transition operator T_a and*

$$\rho(T_a|_{V_k}) < m^{-k/s}.$$

Proof. Let ϕ_0 be a stable compactly supported function in $C^k(\mathbb{R}^s)$. Since the cascade sequence $\big(Q_a^n\phi_0\big)_{n\in\mathbb{N}}$ converges in $C^k(\mathbb{R}^s)$, ϕ_0 satisfies the moment conditions of order $k+1$ by Proposition 2.1. By Theorem 2.1, the mask a satisfies the sum rules of order $k+1$. Thus V_k is invariant under the transition operator T_a.

Let $\phi_n := Q_a^n \phi_0$, $n = 1, 2, \dots$, and let $v_j := M^{-n} e_j$, for $j = 1, \dots, s$. It follows from (3.3) that

$$\nabla_{v_j} \phi_n = \sum_{\alpha\in\mathbb{Z}^s} \nabla_j a_n(\alpha) \phi_0(M^n \cdot - \alpha).$$

Moreover, for a multi-index $\mu = (\mu_1, \dots, \mu_s)$, we have

$$\nabla_{v_j} \nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s} \phi_n = \sum_{\alpha\in\mathbb{Z}^s} \nabla_j \nabla^\mu a_n(\alpha) \phi_0(M^n \cdot - \alpha).$$

Since ϕ_0 is stable in $L_\infty(\mathbb{R}^s)$, there exists a constant C_1 independent of n such that

$$\big\|\nabla_j \nabla^\mu a_n\big\|_\infty \le C_1 \big\|\nabla_{v_j} \nabla_{v_1}^{\mu_1} \cdots \nabla_{v_s}^{\mu_s} \phi_n\big\|_\infty.$$

This in connection with (3.4) yields

$$\big\|\nabla_j \nabla^\mu a_n\big\|_\infty \le C_1 \big\|\nabla_{v_j} D_{v_1}^{\mu_1} \cdots D_{v_s}^{\mu_s} \phi_n\big\|_\infty. \tag{3.5}$$

Since M is isotropic, there exists an invertible matrix Λ such that (2.4) holds with $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s}$. Consequently, $M^{-n} = \Lambda^{-1} \operatorname{diag}(\sigma_1^{-n}, \ldots, \sigma_s^{-n})\Lambda$. Suppose $M^{-n} = (c_{jl})_{1\le j,l\le s}$. Then $D_{v_l} = D_{M^{-n}e_l} = c_{1l}D_1 + \cdots + c_{sl}D_s$. The above discussion shows that $|c_{jl}| \le C_2 m^{-n/s}$ for some constant C_2 independent of n. Therefore, there exists a constant C_3 independent of n such that

$$\big\|\nabla_{v_j} D_{v_1}^{\mu_1} \cdots D_{v_s}^{\mu_s} \phi_n\big\|_\infty \le C_3 m^{-kn/s} \max_{|\mu|=k} \big\|D^\mu \nabla_{M^{-n}e_j} \phi_n\big\|_\infty. \tag{3.6}$$

Combining (3.5) and (3.6), we see that there exists a constant C independent of n such that

$$\max_{|\mu|=k} \big\|m^{kn/s} \nabla_j \nabla^\mu a_n\big\|_\infty \le C \max_{|\mu|=k} \big\|D^\mu \nabla_{M^{-n}e_j} \phi_n\big\|_\infty. \tag{3.7}$$

Let us estimate the right-hand side of (3.7). By the triangle inequality we have

$$\big\|D^\mu \nabla_{M^{-n}e_j} \phi_n\big\|_\infty \le \big\|D^\mu \nabla_{M^{-n}e_j} (\phi_n - \phi)\big\|_\infty + \big\|D^\mu \nabla_{M^{-n}e_j} \phi\big\|_\infty.$$

Since $(\phi_n)_{n\in\mathbb{N}}$ converges to ϕ in $C^k(\mathbb{R}^s)$, we have

$$\lim_{n\to\infty} \big\|D^\mu \nabla_{M^{-n}e_j} (\phi_n - \phi)\big\|_\infty = 0, \qquad |\mu| = k.$$

Furthermore, for $|\mu| = k$, $D^\mu\phi$ is a compactly supported continuous function; hence

$$\lim_{n\to\infty} \big\|D^\mu \nabla_{M^{-n}e_j} \phi\big\|_\infty = \lim_{n\to\infty} \big\|D^\mu\phi - D^\mu\phi(\cdot - M^{-n}e_j)\big\|_\infty = 0.$$

Therefore,

$$\lim_{n\to\infty} \big\|D^\mu \nabla_{M^{-n}e_j} \phi_n\big\|_\infty = 0.$$

This together with (3.7) gives

$$\lim_{n\to\infty} \big\|m^{kn/s} \nabla^\mu a_n\big\|_\infty = 0 \qquad \forall\, |\mu| = k+1. \tag{3.8}$$

Note that $a_n = S_a^n \delta$, where S_a is the subdivision operator given by (3.2). By ([11], Lemma 3.2) we have

$$T_a^n(\nabla^\mu \delta_\beta)(\alpha) = \nabla^\mu S_a^n \delta(M^n\alpha - \beta) \qquad \forall\, \alpha, \beta \in \mathbb{Z}^s.$$

Thus, it follows from (3.8) that

$$\lim_{n\to\infty} \big\|m^{kn/s} T_a^n(\nabla^\mu \delta_\beta)\big\|_\infty = 0 \qquad \forall\, |\mu| = k+1 \text{ and } \beta \in \mathbb{Z}^s.$$

But V_k is spanned by the vectors $\nabla^\mu\delta_\beta$, $|\mu| = k+1$, $\beta \in \mathbb{Z}^s$ (see [11], Theorem 4.3). Therefore, we have

$$\lim_{n\to\infty} \big\|m^{kn/s} T_a^n v\big\|_\infty = 0 \qquad \forall\, v \in V_k.$$

Consequently, $\rho(T_a|_{V_k}) < m^{-k/s}$, as desired. □

For two functions f and g in $L_2(\mathbb{R}^s)$, their correlation $f \odot g$ is defined by

$$f \odot g(x) := \int_{\mathbb{R}^s} f(x+y)\overline{g(y)}\,dy, \qquad x \in \mathbb{R}^s.$$

In other words, $f \odot g$ is the convolution of f with the function $y \mapsto \overline{g(-y)}$, $y \in \mathbb{R}^s$. It is well-known that $f \odot g$ lies in $C_0(\mathbb{R}^s)$, the space of continuous functions on $\mathbb{R}^s$ that vanish at ∞. Clearly,

$$\|f \odot g\|_\infty \le \|f\|_2 \|g\|_2.$$

Similarly, for any two sequences $u, v \in \ell_0(\mathbb{Z}^s)$, their correlation $u \odot v$ is defined by

$$u \odot v(\alpha) := \sum_{\alpha \in \mathbb{Z}^s} u(\alpha+\beta)\overline{v(\beta)}, \qquad \alpha \in \mathbb{Z}^s.$$

The following is the main theorem which gives a characterization for the convergence of cascade algorithms in the Sobolev space $W_2^k(\mathbb{R}^s)$.

Theorem 3.2. *Suppose M is an $s \times s$ isotropic dilation matrix and $m := |\det M|$. Let a be a finitely supported mask satisfying* (1.2), *and let $b := a \odot a/m$. Then the cascade algorithm with mask a converges in $W_2^k(\mathbb{R}^s)$ if and only if V_{2k} is invariant under the transition operator T_b and*

$$\rho(T_b|_{V_{2k}}) < m^{-2k/s}.$$

Proof. Let ϕ_0 be a stable compactly supported function in $W_2^k(\mathbb{R}^s)$ satisfying the moment conditions of order $k+1$. For $n = 0, 1, \ldots$, let $\phi_n := Q_a^n \phi_0$, where Q_a is the refinement operator in (1.3), and let $g_n := \phi_n \odot \phi_n$. We have

$$g_n := \phi_n \odot \phi_n = Q_b^n(\phi_0 \odot \phi_0) = Q_b^n g_0.$$

If $(\phi_n)_{n\in\mathbb{N}}$ converges to ϕ in the space $W_2^k(\mathbb{R}^s)$, then $(g_n)_{n\in\mathbb{N}}$ converges to $g := \phi \odot \phi$ in the space $C^{2k}(\mathbb{R}^s)$. Moreover, ϕ_0 is stable implies that $g_0 = \phi_0 \odot \phi_0$ is stable. By Theorem 3.1, we conclude that V_{2k} is invariant under T_b and $\rho(T_b|_{V_{2k}}) < m^{-2k/s}$.

Conversely, suppose that V_{2k} is invariant under T_b and $\rho(T_b|_{V_{2k}}) < m^{-2k/s}$. Since M is isotropic, there exists an invertible matrix $\Theta = (\theta_{jl})_{1\le j,l\le s}$ such that $\Theta M^T \Theta^{-1}$ is equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_s)$, where $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s}$. Let $p_\mu := p_1^{\mu_1} \cdots p_s^{\mu_s}$, where p_j is the linear polynomial defined by $p_j(x) := \sum_{l=1}^s \theta_{jl} x_l$, $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$. We have the following formula which is similar to (2.7):

$$p_\mu(D)(\phi_0 \circ M^n)(x) = \sigma^{\mu n} p_\mu(D)\phi_0(M^n x), \qquad x \in \mathbb{R}^s,$$

where $\sigma^\mu = \sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s}$. Applying the differential operator $p_\mu(D)$ to both sides of (3.3), we obtain

$$p_\mu(D)\phi_n = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\sigma^{\mu n} p_\mu(D)\phi_0(M^n \cdot - \alpha). \tag{3.9}$$

We want to show that $(p_\mu(D)\phi_n)_{n\in\mathbb{N}}$ converges in the space $L_2(\mathbb{R}^s)$ for all $|\mu| \le k$. Let us consider $\phi_{n+1} - \phi_n$. We have

$$\phi_{n+1} - \phi_n = Q_a^{n+1}\phi_0 - Q_a^n \phi_0 = Q_a^n \varphi_0,$$

where $\varphi_0 := Q_a\phi_0 - \phi_0$. Let $\varphi_n := Q_a^n\varphi_0$ and $f_n := p_\mu(D)\varphi_n = p_\mu(D)(\phi_{n+1} - \phi_n)$ for $n = 0, 1, \ldots$. We deduce from (3.9) that

$$f_n \odot f_n(x) = \sum_{\alpha\in\mathbb{Z}^s} b_n(\alpha) m^{2kn/s} f_0 \odot f_0(M^n x - \alpha), \qquad x \in \mathbb{R}^s, \tag{3.10}$$

where $b_n := S_b^n\delta$. The Fourier transform of the function $f_0 \odot f_0$ is

$$\widehat{f_0 \odot f_0}(\xi) = \left|p_\mu(i\xi)[\hat{\phi}_1(\xi) - \hat{\phi}_0(\xi)]\right|^2, \qquad \xi \in \mathbb{R}^s.$$

Suppose $|\mu| = k$. Then p_μ is a homogeneous polynomial of degree k. It follows that $D^\nu(\widehat{f_0 \odot f_0})(0) = 0$ for all $|\nu| \le 2k$. Since both ϕ_0 and ϕ_1 satisfy the moment conditions of order $k+1$, we have

$$D^\nu(\widehat{f_0 \odot f_0})(2\pi\beta) = 0 \qquad \forall\, |\nu| \le 2k \text{ and } \beta \in \mathbb{Z}^s \setminus \{0\}.$$

Let

$$v(\alpha) := f_0 \odot f_0(\alpha), \qquad \alpha \in \mathbb{Z}^s.$$

By the Poisson summation formula, we obtain for any $p \in \Pi_{2k}$,

$$\sum_{\alpha\in\mathbb{Z}^s} p(\alpha)v(\alpha) = \sum_{\alpha\in\mathbb{Z}^s} p(\alpha)(f_0 \odot f_0)(\alpha) = \sum_{\beta\in\mathbb{Z}^s} (p(iD)\widehat{f_0 \odot f_0})(2\pi\beta) = 0.$$

In other words, $v \in V_{2k}$.

It follows from (3.10) that

$$\|f_n\|_2^2 = f_n \odot f_n(0) = \sum_{\alpha\in\mathbb{Z}^s} b_n(\alpha) m^{2kn/s} v(-\alpha).$$

Furthermore, by ([11], Lemma 3.2) we have

$$T_b^n v(0) = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha) S_b^n\delta(-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} b_n(\alpha) v(-\alpha) = m^{-2kn/s}\|f_n\|_2^2. \tag{3.11}$$

Since V_{2k} is invariant under T_b and $\rho(T_b|_{V_{2k}}) < m^{-2k/s}$, we can find some η with $0 < \eta < 1$ and a constant C independent of n such that

$$\left\|T_b^n v\right\|_\infty \le C(\eta m^{-2k/s})^n \qquad \forall\, n \in \mathbb{N}. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\|f_n\|_2^2 \le C\eta^n \qquad \forall\, n \in \mathbb{N}.$$

But $f_n = p_\mu(D)(\phi_{n+1} - \phi_n)$. Thus, $(p_\mu(D)\phi_n)_{n\in\mathbb{N}}$ converges in $L_2(\mathbb{R}^s)$. Therefore, for all μ with $|\mu| = k$, $(D^\mu\phi_n)_{n\in\mathbb{N}}$ converges to some function in $L_2(\mathbb{R}^s)$.

Now suppose $|\mu| < k$. By the results of [14] we have

$$\operatorname{spec}(T_b|_{\ell(K)\cap V_{2j}}) = \operatorname{spec}(T_b|_{\ell(K)\cap V_{2k}}) \cup \{\sigma^{-\nu} : 2j+1 \le |\nu| \le 2k\},$$

where $\sigma^{-\nu} = \sigma_1^{-\nu_1} \cdots \sigma_s^{-\nu_s}$ and $\operatorname{spec}(T)$ denotes the spectrum of the operator T. Hence, $\rho(T_b|_{\ell(K)\cap V_{2j}}) < m^{-2j/s}$ for $j = 0, 1, \ldots, k$. By what has been proved before, we conclude that, for all $|\mu| \le k$, $(D^\mu\phi_n)_{n\in\mathbb{N}}$ converges to some function in $L_2(\mathbb{R}^s)$. In other words, the cascade algorithm associated with the mask a converges in the Sobolev space $W_2^k(\mathbb{R}^s)$. The proof of the theorem is complete. □

From the proof of Theorem 3.2 we have actually established the following stronger result which gives a necessary condition for the convergence of cascade algorithms in $W_2^k(\mathbb{R}^s)$.

Proposition 3.1. *Let ϕ_0 be a stable and compactly supported function in $W_2^k(\mathbb{R}^s)$. If $(Q_a^n\phi_0)_{n\in\mathbb{N}}$ converges in $W_2^k(\mathbb{R}^s)$ to the normalized solution of the refinement equation* (1.1)*, then V_{2k} is invariant under the transition operator T_b and*

$$\rho(T_b|_{V_{2k}}) < m^{-2k/s}.$$

Proposition 3.1 and Theorem 3.2 give the following corollary.

Corollary 3.1. *Suppose M is an $s\times s$ isotropic dilation matrix. If the normalized solution ϕ of the refinement equation* (1.1) *belongs to $W_2^k(\mathbb{R}^s)$ and is stable, then the cascade algorithm with mask a and dilation M converges in the Sobolev space $W_2^k(\mathbb{R}^s)$.*

Proof. Clearly if the normalized solution ϕ of the refinement equation is in $W_2^k(\mathbb{R}^s)$, then $(Q_a^n\phi)_{n\in\mathbb{N}}$ converges in $W_2^k(\mathbb{R}^s)$. If ϕ is stable, then by Proposition 3.1, V_{2k} is invariant under the transition operator T_b and $\rho(T_b|_{V_{2k}}) < m^{-2k/s}$, where $b = a\odot a/m$, $m = |\det M|$. It follows from Theorem 3.2 that the cascade algorithm with mask a converges in $W_2^k(\mathbb{R}^s)$. □

4. Computation of integrals of refinable functions and wavelets

In this section we apply the results of Section 3 to the evaluation of integrals of products of refinable functions and wavelets. For simplicity we shall assume that the dilation matrix $M = 2I$. The evaluation of such an integral requires the computation of integrals of the form

$$\int_{\mathbb{R}^s}\prod_{j=0}^{\ell} D^{\mu^j}\phi_j(2^{k_j}x+\alpha^j)\,dx, \tag{4.1}$$

where for $j = 0,1,\dots,\ell$, ϕ_j are refinable functions, $\mu^j\in\mathbb{N}_0^s$, $k_j\in\mathbb{N}_0$ and $\alpha^j\in\mathbb{Z}^s$. It was shown in [4] that the integral in (4.1) can be determined explicitly in terms of integrals of the form

$$\int_{\mathbb{R}^s}\phi_0(x)\prod_{j=1}^{\ell} D^{\mu^j}\phi_j(x+\alpha^j)\,dx, \tag{4.2}$$

where $\alpha^j\in\mathbb{Z}^s$ for $j = 1,2,\dots,\ell$.

Suppose that for each $j = 0,1,\dots,\ell$, $\phi_j\in W_2^k(\mathbb{R}^s)$ is the normalized solution of the refinement equation

$$\phi_j = \sum_{\alpha\in\mathbb{Z}^s} a^j(\alpha)\phi_j(2\cdot-\alpha), \tag{4.3}$$

where a^j, $j = 1,\dots,\ell$, are finitely supported masks. Let

$$\Phi(X) := \Phi(x^1,\dots,x^\ell) := \int_{\mathbb{R}^s}\phi_0(x)\prod_{j=1}^{\ell}\phi_j(x+x^j)\,dx, \tag{4.4}$$

where

$$X = (x^1,\dots,x^\ell)\in\mathbb{R}^{\ell s},\qquad x^j\in\mathbb{R}^s,\quad j = 1,2,\dots,\ell.$$

It is easy to see that Φ belongs to $C^k(\mathbb{R}^{\ell s})$, is refinable and is the normalized solution of the refinement equation

$$\Phi(X) = \sum_{\gamma \in \mathbb{Z}^{\ell s}} c(\gamma)\Phi(2X - \gamma), \tag{4.5}$$

where

$$c(\gamma) := c(\gamma^1, \dots, \gamma^\ell) := 2^{-s} \sum_{\alpha^0 \in \mathbb{Z}^s} a^0(\alpha^0) \prod_{j=1}^{\ell} a^j(\alpha^0 + \gamma^j), \tag{4.6}$$

and

$$\gamma = (\gamma^1, \dots, \gamma^\ell) \in \mathbb{Z}^{\ell s}, \qquad \gamma^j \in \mathbb{Z}^s, \quad j = 1, 2, \dots, \ell.$$

For multi-indices $\mu^j \in \mathbb{N}_0^s$, $j = 1, 2, \dots, \ell$, with $|\mu^1| + \cdots + |\mu^\ell| \le k$, let

$$v_\mu(\alpha) := \int_{\mathbb{R}^s} \phi_0(x) \prod_{j=1}^{\ell} D^{\mu^j} \phi_j(x + \alpha^j)\, dx, \tag{4.7}$$

where $\mu := (\mu^1, \cdots, \mu^\ell)$. Then (4.4) and (4.5) give

$$T_c v_\mu = 2^{-(|\mu^1| + \cdots + |\mu^\ell|)} v_\mu. \tag{4.8}$$

Further,

$$\sum_{\alpha \in \mathbb{Z}^{\ell s}} (-\alpha)^\nu v_\mu(\alpha) = (\mu^1 + \cdots + \mu^\ell)!\, \delta_{\nu, \mu^1 + \cdots + \mu^\ell}, \qquad |\nu| \le k. \tag{4.9}$$

This follows from the fact that Φ satisfies the moment conditions of order $k+1$, or equivalently for any polynomial $p \in \Pi_k$,

$$\sum_{\alpha \in \mathbb{Z}^{\ell s}} p(\alpha)\Phi(\cdot - \alpha) - p$$

is a polynomial of degree less than $\deg p$.

It was shown in ([4], Theorem 3.5) that if the refinable functions ϕ_j belong to $C^k(\mathbb{R}^s)$, $j = 0, 1, \dots, \ell$, and are stable, then the vector v_μ is the unique eigenvector of T_c satisfying (4.8) and (4.9). As a simple consequence of the results in the previous section we shall show that if the cascade algorithms with mask a^j and dilation $2I$ converge in $W_2^k(\mathbb{R}^s)$ for $j = 0, 1, \dots, \ell$, then v_μ is the unique eigenvector of T_c satisfying (4.8) and (4.9). We remark that the stability of ϕ_j implies the convergence of the cascade algorithms in $W_2^k(\mathbb{R}^s)$ with mask a^j, $j = 0, \dots, \ell$. By Theorem 3.2 the convergence of cascade algorithms with masks a^j can be determined by computing the spectral radii of the transition matrices $T_{b^j}|_{V_{2k}}$, where $b^j = a^j \odot a^j / 2^s$.

We shall first obtain the result based on the convergence of a cascade algorithm in $C^k(\mathbb{R}^s)$. Let ϕ be the normalized solution of the refinement equation

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(2x - \alpha), \qquad x \in \mathbb{R}^s, \tag{4.10}$$

where a is a finitely supported sequence satisfying $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = 2^s$. Suppose that ϕ lies in $C^k(\mathbb{R}^s)$ for some non-negative integer k. For $|\gamma| = k$, applying D^γ to both sides of (4.10) yields

$$D^\gamma\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)2^k D^\gamma\phi(2x-\alpha), \qquad x \in \mathbb{R}^s.$$

It follows that, for $\beta \in \mathbb{Z}^s$,

$$D^\gamma\phi(\beta) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)2^k D^\gamma\phi(2\beta-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} a(2\beta-\alpha)2^k D^\gamma\phi(\alpha).$$

Let v_γ be the sequence given by

$$v_\gamma(\beta) := D^\gamma\phi(\beta), \qquad \beta \in \mathbb{Z}^s.$$

Then v_γ is an eigenvector of the transition operator T_a corresponding to the eigenvalue 2^{-k}, i.e.,

$$T_a v_\gamma = 2^{-k} v_\gamma, \tag{4.11}$$

and v_γ is supported on $K = \sum_{n=1}^\infty M^{-n}\operatorname{supp} a$. Further,

$$\sum_{\beta\in\mathbb{Z}^s} (-\beta)^\nu v_\gamma(\beta) = \gamma!\,\delta_{\gamma,\nu} \qquad \forall\,|\nu| \le k. \tag{4.12}$$

Theorem 4.1. *Let ϕ_0 be a compactly supported stable function in $C^k(\mathbb{R}^s)$. If the cascade sequence $(Q_a^n\phi_0)_{n\in\mathbb{N}}$ converges in the space $C^k(\mathbb{R}^s)$ to the normalized solution of the refinement equation* (1.1), *then there exists a unique sequence $v_\gamma \in \ell_0(\mathbb{Z}^s)$ satisfying conditions* (4.11) *and* (4.12).

Proof. Suppose there are two sequences u and v in $\ell_0(\mathbb{Z}^s)$ satisfying both (4.11) and (4.12). Consider $w := u - v$. It follows from (4.12) that

$$\sum_{\beta\in\mathbb{Z}^s} (-\beta)^\nu w(\beta) = 0 \qquad \forall\,|\nu| \le k.$$

In other words, $w \in V_k$. Since the cascade algorithm associated with a converges in $C^k(\mathbb{R}^s)$, we have $\rho(T_a|_{V_k}) < 2^{-k}$. Hence, w must be 0, for otherwise (4.11) would imply $\rho(T_a|_{V_k}) \ge 2^{-k}$, which is a contradiction. This proves the uniqueness of the solution of (4.11) and (4.12). □

Theorem 4.2. *Let c and v_μ be as in* (4.6) *and* (4.7) *respectively and suppose that $\phi_j \in C^k(\mathbb{R}^s)$ for $j = 1, \dots, \ell$. If the cascade algorithms with mask a^j converge in $W_2^k(\mathbb{R}^s)$, $j = 0, 1, \dots, \ell$, then v_μ is the unique eigenvector of T_c satisfying* (4.8) *and* (4.9).

Proof. Suppose that the cascade algorithms with masks a^j, $j = 1, \dots, \ell$, converge in $W_2^k(\mathbb{R}^s)$. For each $j = 0, \dots, \ell$, let $\phi_{j,0}$ be the tensor product of one-dimensional central B-splines of degree $\ge k$. Then

$$\Phi_0(X) := \int_{\mathbb{R}^s} \phi_{0,0}(x) \prod_{j=1}^{\ell} \phi_{j,0}(x + x^j)\, dx \tag{4.13}$$

is a compactly supported stable function in $C^k(\mathbb{R}^{\ell s})$.

For $j = 0, 1 \ldots$, and $n = 1, 2 \ldots$, define

$$\phi_{j,n} = \sum_{\alpha \in \mathbb{Z}^s} a^j(\alpha) \phi_{j,n-1}(2 \cdot - \alpha), \tag{4.14}$$

and let

$$\Phi_n(X) := \int_{\mathbb{R}^s} \phi_{0,n}(x) \prod_{j=1}^{\ell} \phi_{j,n}(x + x^j)\ dx. \tag{4.15}$$

Then $(\Phi_n)_{n \in \mathbb{N}}$ is a cascade sequence with mask c, i.e.,

$$\Phi_n(X) = \sum_{\gamma \in \mathbb{Z}^{\ell s}} c(\gamma) \Phi_{n-1}(2X - \gamma), \tag{4.16}$$

where c is defined in (4.6). Since $\phi_0 \in W_2^k(\mathbb{R}^s)$ and for $j = 1, \ldots, \ell$, $\phi_j \in C^k(\mathbb{R}^s)$, the convergence of $(\phi_{j,n})_{n \in \mathbb{N}}$ in $W_2^k(\mathbb{R}^s)$ to ϕ_j for $j = 0, \ldots, n$, implies that $(\Phi_n)_{n \in \mathbb{N}}$ converges to Φ in $C^k(\mathbb{R}^s)$ as $n \to \infty$. By Theorem 4.1, we conclude that v_μ is the unique eigenvector of T_c satisfying (4.8) and (4.9). $\square$

Remark 4.1. *In Theorem 4.2 we do not require ϕ_0 to belong to $C^k(\mathbb{R}^s)$. If $\ell = 1$ we also do not require ϕ_1 to belong to $C^k(\mathbb{R}^s)$.*

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