Multiwavelets on the Interval

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Smooth orthogonal and biorthogonal multiwavelets on the real line with their scaling function vectors being supported on $[-1,1]$ are of interest in constructing wavelet bases on the interval $[0,1]$ due to their simple structure. In this paper, we shall present a symmetric C^2 orthogonal multiwavelet with multiplicity 4 such that its orthogonal scaling function vector is supported on $[-1,1]$, has accuracy order 4 and belongs to the Sobolev space $W^{2.56288}$. Biorthogonal multiwavelets with multiplicity 4 and vanishing moments of order 4 are also constructed such that the primal scaling function vector is supported on $[-1,1]$, has the Hermite interpolation properties and belongs to $W^{3.63298}$ while the dual scaling function vector is supported on $[-1,1]$ and belongs to $W^{1.75833}$. A continuous dual scaling function vector of the cardinal Hermite interpolant with multiplicity 4 and support $[-1,1]$ is also given. All the wavelet filters constructed in this paper have closed form expressions. Based on the above constructed orthogonal and biorthogonal multiwavelets on the real line, both orthogonal and biorthogonal multiwavelet bases on the interval $[0,1]$ are presented. Such multiwavelet bases on the interval $[0,1]$ have symmetry, small support, high vanishing moments, good smoothness and simple structures. Furthermore, the sequence norms for the coefficients based on such orthogonal and biorthogonal multiwavelet expansions characterize Sobolev norms $\|\cdot\|_{W^s([0,1])}$ for $s \in (-2.56288, 2.56288)$ and for $s \in (-1.75833, 3.63298)$, respectively.

Key Words: accuracy, sum rules, smoothness, scaling function, cardinal Hermite interpolant, orthogonal multiwavelet, biorthogonal multiwavelet, multiwavelet on the interval

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1. INTRODUCTION

With high vanishing moments, both orthogonal and biorthogonal wavelets on the real line prove to be very useful in various applications. However, in many applications, one is interested in problems confined to an interval such as solutions to differential equations with boundary conditions and image processing. One excellent construction of orthogonal wavelet bases on the interval $[0,1]$ was given by Cohen, Daubechies and Vial [4] by adapting the famous Daubechies orthogonal wavelets on the real line to the interval $[0,1]$ (see also [1, 3]). The motivation to construct wavelets on the interval and the fast wavelet transforms associated with wavelets on the interval were explained in detail in [4]. In the literature, several other approaches were also reported in [6, 9, 12, 20] to obtain wavelets on the interval by adapting the wavelets on the real line. Recently, the theory of multiwavelets has been extensively studied in the literature. As a generalization of scalar wavelets, multiwavelets have several promising features such as short support and relatively high vanishing moments.

Before proceeding further, let us recall some definitions related to multiwavelets. A *refinable function vector* $\phi = (\phi_1, \dots, \phi_r)^T$, where the positive integer r is the multiplicity, satisfies the following refinement equation

$$\phi = \sum_{k\in\mathbb{Z}} H_k \phi(2\cdot -k), \tag{1.1}$$

where $H = \{H_k\}_{k\in\mathbb{Z}}$ is a finitely supported sequence on $\mathbb{Z}$ called the *(matrix refinement) mask.*

If the matrix $H_s := \sum_{k\in\mathbb{Z}} H_k$ has a simple eigenvalue 2 and all the other eigenvalues in modulus are less than 2, then up to a scalar multiplication there is a unique vector ϕ of distributions which is a solution to the refinement equation (1.1) and $H_s\widehat{\phi}(0) = 2\widehat{\phi}(0)$. Such a solution ϕ is called the *normalized solution* to the refinement equation with mask H. In this paper the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by

$$\widehat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{-it\omega}\,dt, \qquad \omega \in \mathbb{R}.$$

For $\nu \geq 0$, W^ν denotes the Sobolev space consisting of all functions f in $L^2(\mathbb{R})$ with $\widehat{f}(\omega)(1+|\omega|^2)^{\frac{\nu}{2}} \in L^2(\mathbb{R})$.

We say that a function vector $\phi = (\phi_1, \dots, \phi_r)^T$ has *accuracy order* n if

$$x^j = \sum_{k\in\mathbb{Z}} \lambda_k^j \phi(x-k) \qquad \forall\, j = 0, \dots, n-1$$

for some sequences λ^j of $1\times r$ vectors on $\mathbb{Z}$. Accuracy order of a refinable function vector has a close relation to the order of vanishing moments of a (bi)orthogonal multiwavelet.

Biorthogonal multiwavelets come from a *primal scaling function vector* $\phi = (\phi_1, \dots, \phi_r)^T \in \left(L^2(\mathbb{R})\right)^r$ and a *dual scaling function vector* $\widetilde{\phi} = (\widetilde{\phi}_1, \dots, \widetilde{\phi}_r)^T \in \left(L^2(\mathbb{R})\right)^r$ such that

$$\phi = \sum_{k\in\mathbb{Z}} H_k \phi(2\cdot -k) \qquad \text{and} \qquad \widetilde{\phi} = \sum_{k\in\mathbb{Z}} \widetilde{H}_k \widetilde{\phi}(2\cdot -k),$$

where H and $\widetilde{H}$ are finitely supported masks, and ϕ and $\widetilde{\phi}$ satisfy the following biorthogonal relation:

$$\int_{\mathbb{R}} \phi(t)\widetilde{\phi}(t-k)^* \, dt = \delta_k I_r \qquad \forall\, k \in \mathbb{Z}, \tag{1.2}$$

where $\widetilde{\phi}(t-k)^* := \overline{\widetilde{\phi}(t-k)^T}$ and δ is the Dirac sequence such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. If ϕ satisfies (1.2) with $\widetilde{\phi} = \phi$, then ϕ is called an *orthogonal scaling function vector*.

If ϕ is a primal scaling function vector with mask H and $\widetilde{\phi}$ is a dual scaling function vector with mask $\widetilde{H}$, then it is necessary that

$$\sum_{k\in\mathbb{Z}} H_{k+2j}\widetilde{H}_k^* = 2\delta_j \, I_r \qquad \forall\, j \in \mathbb{Z}. \tag{1.3}$$

where $\widetilde{H}_k^* := \overline{\widetilde{H}_k^T}$. Let H be a finitely supported mask on $\mathbb{Z}$. If there is a finitely supported mask $\widetilde{H}$ such that (1.3) holds, then the mask H is called a *primal mask* and any such mask $\widetilde{H}$ in (1.3) is called a *dual mask* of H. H and $\widetilde{H}$ are also called wavelet filters in the literature. If H and $\widetilde{H}$ satisfy (1.3) and the subdivision schemes associated with H and $\widetilde{H}$ converge in the L^2 norm, then it was proved in Dahmen and Micchelli [7] that ϕ and $\widetilde{\phi}$ satisfy (1.2) where ϕ and $\widetilde{\phi}$ are the normalized solutions to the refinement equations (1.1) with the masks H and $\widetilde{H}$, respectively.

It is known that there are symmetric smooth orthogonal multiwavelet bases while there is no symmetric continuous orthogonal scalar wavelet (see [8, 10]). The properties of short support, reasonably high vanishing moments and certain smoothness of a multiwavelet are particularly attractive for the construction of wavelet bases on the interval. For example, multiwavelet bases on the interval were reported in [6, 9, 12]. In the current literature, many (bi)orthogonal multiwavelets with multiplicity 2 were reported, see e.g. [6, 10, 12, 17] and references therein. To obtain multiwavelets with short support and high vanishing moments, it is quite natural to consider multiwavelets with higher multiplicity. However, with higher multiplicity, the construction of multiwavelets and the algorithm to perform the associated wavelet transform will be much more complicated. Therefore, it is natural and in its own right to consider multiwavelets with multiplicity 4. In this paper, we shall construct both orthogonal and biorthogonal smooth multiwavelets with multiplicity 4 and support $[-1, 1]$. Then the multiwavelets on the interval $[0, 1]$ are obtained from them. The multiwavelets on the interval in this paper have four boundary wavelets at each level with simple structure and small support. All the wavelet filters reported in this paper have closed form expressions and the multiwavelet bases enjoy good smoothness, short support, symmetry and high vanishing moments. For example, we construct a symmetric C^2 orthogonal multiwavelet with multiplicity 4, while a C^2 orthogonal multiwavelet given in [9] has multiplicity 11 and some components of the (piecewise polynomial) refinable function vector lack symmetry. In comparison with many other constructions of wavelet bases on the interval $[0, 1]$, besides many good properties such as good smoothness, short support, symmetry, high vanishing moments and interpolation properties, multiwavelet bases constructed in this paper have very simple structure with explicit expressions and therefore, can be implemented efficiently.

The structure of this paper is as follows. In Section 2, we shall obtain a symmetric C^2 orthogonal scaling function vector with multiplicity 4 such that it is supported on $[-1,1]$, has accuracy order 4 and belongs to the Sobolev space $W^{2.56288}$. Here and in the following, a function f being in the Sobolev space $W^{2.56288}$ means $f \in W^s$, where s is approximately 2.56288. In Section 3, we shall discuss how to construct biorthogonal multiwavelets by the parameterization method in [18] and the coset by coset (CBC) algorithm in [11]. In particular, a C^3 primal scaling function vector ϕ and its C^1 dual scaling function vector $\widetilde{\phi}$ are constructed such that both of them are supported on $[-1,1]$ and have accuracy order 4 while $\phi \in W^{3.63298}$ and $\widetilde{\phi} \in W^{1.75833}$. Moreover, the primal function vector ϕ is a Hermite interpolant. In Section 3, based on the CBC algorithm in [11] we construct a continuous dual scaling function vector $\widetilde{\phi}$ for the following cardinal Hermite interpolant $\phi = (\phi_1, \dots, \phi_4)^T$ given by

$$\begin{cases} \phi_1(t) = (t+1)^4(1-4t+10t^2-20t^3)\chi_{[-1,0)}(t) \\ \qquad\qquad\qquad\qquad +(t-1)^4(1+4t+10t^2+20t^3)\chi_{[0,1]}(t), \\ \phi_2(t) = (t+1)^4(t-4t^2+10t^3)\chi_{[-1,0)}(t) \\ \qquad\qquad\qquad\qquad\quad +(t-1)^4(t+4t^2+10t^3)\chi_{[0,1]}(t), \\ \phi_3(t) = (t+1)^4(t^2/2-2t^3)\chi_{[-1,0)}(t) + (t-1)^4(t^2/2+2t^3)\chi_{[0,1]}(t), \\ \phi_4(t) = (t+1)^4t^3/6\chi_{[-1,0)}(t) + (t-1)^4t^3/6\chi_{[0,1]}(t). \end{cases} \tag{1.4}$$

It is easy to check that ϕ is supported on $[-1,1]$, has accuracy order 8 and belongs to $W^{4.5}$. Therefore, $\phi \in C^{4-\epsilon}$ for any $\epsilon > 0$. The dual scaling function vector $\widetilde{\phi}$ we construct in Section 3 is supported on $[-4,4]$, has accuracy order 8 and belongs to $W^{1.13762}$.

In Section 4, we discuss how to obtain multigenerators on the interval $[0,1]$ from the orthogonal scaling function vector ϕ, and from primal and dual scaling function vectors ϕ, $\widetilde{\phi}$ constructed in the preceding sections. The orthogonal multigenerators Φ_j and biorthogonal multigenerators Φ_j and $\widetilde{\Phi}_j$ at each level $j \in \mathbb{Z}_+$ are explicitly constructed with a simple expression. The matrices $\mathbb{H}_j$ and $\widetilde{\mathbb{H}}_j$, for the refinement equations $\Phi_j = \mathbb{H}_j \Phi_{j+1}$ and $\widetilde{\Phi}_j = \widetilde{\mathbb{H}}_j \widetilde{\Phi}_{j+1}$, are explicitly given. Finally, in Section 5, the orthogonal multiwavelets Ψ_j, and biorthogonal multiwavelets Ψ_j, $\widetilde{\Psi}_j$ at each level j are explicitly given. Moreover, The matrices $\mathbb{G}_j$ and $\widetilde{\mathbb{G}}_j$ for $\Psi_j = \mathbb{G}_j \Phi_{j+1}$ and $\widetilde{\Psi}_j = \widetilde{\mathbb{G}}_j \widetilde{\Phi}_{j+1}$ are explicitly given. The sequence norms for the coefficients based on such orthogonal and biorthogonal multiwavelet expansions characterize Sobolev norms $\|\cdot\|_{W^s([0,1])}$ for $s \in (-2.56288, 2.56288)$ and for $s \in (-1.75833, 3.63298)$, respectively. The orthogonal and biorthogonal multiwavelet bases on the interval constructed in Section 5 have good smoothness and vanishing moments. At each level, two boundary multiwavelets with simple expressions are used for each end point of the interval $[0,1]$. Moreover, the method of constructing (bi)orthogonal multiwavelets in this paper can be used to construct multiwavelets which are supported on $[-1,1]$ with multiplicity other than 4.

2. ORTHONORMAL MULTIWAVELETS ON THE REAL LINE

To construct symmetric/antisymmetric orthogonal multiwavelets, we take the mask H to be the following form:

$$\begin{aligned} &H_{-1} = \begin{bmatrix} a & -aw \\ -b & bw \end{bmatrix}, \quad H_0 = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad H_1 = \begin{bmatrix} a & aw \\ b & bw \end{bmatrix} \quad \text{and} \\ &H_k = 0 \qquad \forall\, k \neq -1, 0, 1, \end{aligned} \tag{2.1}$$

where a, b, c, d and w are 2×2 real-valued matrices and $w^* w = I_2$, i.e., w is an orthogonal matrix.

In order to obtain an orthogonal scaling function vector, the mask H necessarily satisfies

$$\sum_{k \in \mathbb{Z}} H_{k+2j} H_k^* = 2\delta_j I_4 \qquad \forall\, j \in \mathbb{Z}. \tag{2.2}$$

From (2.1), it is easy to verify that the discrete orthogonal condition (2.2) is equivalent to

$$4aa^* + cc^* = 2I_2 \qquad \text{and} \qquad 4bb^* + dd^* = 2I_2. \tag{2.3}$$

With the expression of (2.1), we want to construct an orthogonal scaling function vector with good smoothness and approximation properties. Let us first recall some results about accuracy order of a refinable function vector. Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ is a compactly supported refinable function vector with mask H and $\phi_j \in L^2(\mathbb{R})$. For any nonnegative integer m, denote

$$E_m := \frac{1}{m!} \sum_{k \in \mathbb{Z}} (2k)^m H_{2k} \qquad \text{and} \qquad O_m := \frac{1}{m!} \sum_{k \in \mathbb{Z}} (1+2k)^m H_{1+2k}. \tag{2.4}$$

If $\big(\widehat{\phi}_j(\omega+2k\pi)\big)_{k \in \mathbb{Z}}, j = 1, \ldots, r$, are linearly independent for $\omega = 0$ and $\omega = \pi$, then ϕ has accuracy order n if and only if there exist $1 \times r$ row vectors $y_m, m = 0, \ldots, n-1$ with $y_0 \neq 0$ such that (see [14])

$$\begin{gathered} \sum_{m=0}^{s} (-1)^m 2^{s-m} y_{s-m} E_m = y_s \quad \text{and} \quad \sum_{m=0}^{s} (-1)^m 2^{s-m} y_{s-m} O_m = y_s \\ \forall\, s = 0, \ldots, n-1. \end{gathered} \tag{2.5}$$

Furthermore, under the condition $y_0 \widehat{\phi}(0) = 1$, we have

$$\frac{x^s}{s!} = \sum_{k \in \mathbb{Z}} \sum_{m=0}^{s} \frac{k^m}{m!} y_{s-m} \phi(x-k), \qquad x \in \mathbb{R},\ s = 0, 1, \ldots, n-1. \tag{2.6}$$

If a mask H satisfies (2.5), then we say that H satisfies the *sum rules* of order n with $\{y_m : m = 0, \ldots, n-1\}$. See also [11, 13, 15, 19] about the relation between the accuracy of a refinable function vector and the sum rules of the corresponding mask.

In the following we shall construct a mask H of the form given in (2.1) such that (2.3) holds and H satisfies the sum rules of order 4 with vectors $\{y_0, y_1, y_2, y_3\}$. By an appropriate orthogonal transform and the special structure of the mask H, without loss of generality, we may assume

$$y_0 = [1,0,0,0], \quad y_1 = [0,0,x_1,0], \quad y_2 = [x_2,x_3,0,0], \quad \text{and} \quad y_3 = [0,0,x_4,x_5],$$

for some real numbers $x_j, j = 1, \ldots, 5$. If the refinable function vector $\phi \in C^2$, then by (2.6), we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 \\ x_2 & x_3 & 0 & 0 \end{bmatrix} [\phi(0)\ \phi'(0)\ \phi''(0)] = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} [\phi(0)\ \phi'(0)\ \phi''(0)] = I_3. \tag{2.7}$$

Therefore $x_1 \neq 0$ and $x_3 \neq 0$. In the following we also assume that $x_5 \neq 0$.

It is easy to see that the mask H of the form given in (2.1) satisfies the sum rules of order 1 if and only if

$$[1,0]a = [1/2, 0] \quad \text{and} \quad [1,0]c = [1,0]. \tag{2.8}$$

Under the condition (2.8), H satisfies the sum rules of order 2 if and only if

$$[x_1, 0]b = [1/4, 0] + [x_1/4, 0]w^* \quad \text{and} \quad [1,0]d = [1/2, 0]. \tag{2.9}$$

Under the conditions (2.8) and (2.9), H satisfies the sum rules of order 3 if and only if

$$[x_2, x_3]a = [x_2/8 + 1/16, x_3/8] + [x_1/8, 0]w^* \quad \text{and} \quad [x_2, x_3]c = [x_2/4, x_3/4]. \tag{2.10}$$

Finally, under the conditions (2.8), (2.9) and (2.10), H satisfies the sum rules of order 4 if and only if

$$\begin{aligned} &[x_4, x_5]b = [x_2/16 + 1/96, x_3/16] + [x_1/32 + x_4/16, x_5/16]w^*, \\ &[x_4, x_5]d = [x_4/8, x_5/8]. \end{aligned} \tag{2.11}$$

The equations (2.8) and (2.10) together are equivalent to

$$a = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1-6x_2}{16x_3} & \frac{1}{8} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{x_1}{8x_3} & 0 \end{bmatrix} w^* \qquad \text{and} \qquad c = \begin{bmatrix} 1 & 0 \\ -\frac{3x_2}{4x_3} & \frac{1}{4} \end{bmatrix}. \tag{2.12}$$

The equations (2.9) and (2.11) together are equivalent to

$$b = \begin{bmatrix} \frac{1}{4x_1} & 0 \\ \frac{x_1+6x_1x_2-24x_4}{96x_1x_5} & \frac{x_3}{16x_5} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{x_1-6x_4}{32x_5} & \frac{1}{16} \end{bmatrix} w^* \quad \text{and} \quad d = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3x_4}{8x_5} & \frac{1}{8} \end{bmatrix}. \tag{2.13}$$

Let $w_{i,j}$ denote the (i,j) entry of the orthogonal matrix w. By the first equation of (2.3), one can obtain

$$\begin{cases} x_2 = \frac{1}{6}w_{1,1}x_1 + \frac{1}{12}, \quad x_3 = \frac{1}{30}(x_1 w_{2,1} \pm \sqrt{x_1^2 w_{2,1}^2 - 30c_0}), \text{where} \\ c_0 = 30x_2^2 - x_1^2 - 2x_1 w_{1,1}(x_2 + \frac{1}{2}) - x_2 - \frac{1}{4}. \end{cases} \tag{2.14}$$

By the second equation of (2.3), we have

$$\begin{cases} x_1 = \frac{1}{6}\left(w_{1,1} \pm \sqrt{w_{1,1}^2 + 6}\right), \\ x_4 = \frac{-(252\alpha\beta - \eta\alpha - \eta_1) \pm \sqrt{(252\alpha\beta - \eta\alpha - \eta_1)^2 - 504(1+\alpha^2)(126\beta^2 - \gamma - \beta\eta)}}{252(1+\alpha^2)}, \\ x_5 = \alpha x_4 + \beta, \end{cases} \tag{2.15}$$

where

$$\begin{cases} \alpha = (30x_1 - w_{1,1})/w_{1,2}, \\ \beta = -\frac{x_1}{w_{1,2}}((x_2 + \frac{1}{6})/x_1 + w_{1,1}(x_2 + \frac{2}{3}) + x_3 w_{2,1} + \frac{x_1}{2}), \\ \eta = 2(x_2 + \frac{1}{6})w_{1,2} + 2x_3 w_{2,2}, \\ \eta_1 = 2(x_2 + \frac{1}{6})w_{1,1} + 2x_3 w_{2,1} + x_1, \\ \gamma = (x_2 + \frac{1}{6})^2 + x_3^2 + \frac{\eta_1 x_1}{2} - \frac{x_1^2}{4}. \end{cases}$$

From the above expressions, we compute the parameters in the order x_1, x_2, x_3, x_4 and x_5. For an orthogonal 2×2 matrix w, it is given by one parameter. Thus there is one free parameter for the mask H given in (2.1) such that H satisfies the discrete orthogonal condition (2.2) and the sum rules of order 4.

By choosing $t = 477/512$ and

$$w := \begin{bmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{bmatrix} = \begin{bmatrix} t & -\sqrt{1-t^2} \\ -\sqrt{1-t^2} & -t \end{bmatrix},$$

we obtain the orthogonal scaling function vector ϕ such that $\phi \in W^{2.56288}$ and ϕ has accuracy order 4 where ϕ is the normalized solution to the refinement equation (1.1) with the mask H. Here and in the following we use the smoothness estimate for scaling functions provided in [16]. Thus $\phi \in C^{2.06288}$. Moreover, the orthogonal scaling function vector $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ satisfies

$$\begin{aligned} &\phi_1(-x) = \phi_1(x), \quad \phi_2(-x) = \phi_2(x), \quad \phi_3(-x) = -\phi_3(x), \\ &\phi_4(-x) = -\phi_4(x) \qquad \forall\, x \in \mathbb{R}. \end{aligned}$$

In [9], a C^2 scaling function vector supported on $[-1, 1]$ was constructed. Each component of the scaling function vector is a piecewise polynomial. However its multiplicity is 11 and some components of the refinable function vector lack symmetry.

In the following we shall derive the corresponding multiwavelets on the real line from the mask H and ϕ. Set $G_k = 0$ for all $k \in \mathbb{Z}\backslash\{-1, 0, 1\}$ and set

$$G_{-1} = \begin{bmatrix} e & -ew \\ -f & fw \end{bmatrix}, \; G_0 = \begin{bmatrix} -4ea^*(c^{-1})^* & 0 \\ 0 & -4fb^*(d^{-1})^* \end{bmatrix}, \; G_1 = \begin{bmatrix} e & ew \\ f & fw \end{bmatrix}, \tag{2.16}$$

where the 2×2 matrices e and f are given by

$$e^*e = [2I_2 + 8a^*(c^{-1})^*c^{-1}a]^{-1} \quad \text{and} \quad f^*f = [2I_2 + 8b^*(d^{-1})^*d^{-1}b]^{-1}. \tag{2.17}$$

It is not difficult to verify that

$$\sum_{k\in\mathbb{Z}} H_{k+2j}G_k^* = \sum_{k\in\mathbb{Z}} G_{k+2j}H_k^* = 0 \quad \text{and} \quad \sum_{k\in\mathbb{Z}} G_{k+2j}G_k^* = 2\delta_j I_4 \qquad \forall\, j \in \mathbb{Z}.$$

Define $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ as follows:

$$\psi(x) = G_{-1}\phi(2x+1) + G_0\phi(2x) + G_1\phi(2x-1). \tag{2.18}$$

Then for all $x \in \mathbb{R}$,

$$\psi_1(-x) = \psi_1(x), \quad \psi_2(-x) = \psi_2(x), \quad \psi_3(-x) = -\psi_3(x), \quad \psi_4(-x) = -\psi_4(x),$$

and

$$\int_{\mathbb{R}} \phi(x)\psi(x+k)^* \, dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} \psi(x)\psi(x+k)^* \, dx = \delta_k I_4 \qquad \forall\, k \in \mathbb{Z}.$$

The reader is referred to Figures 2 and 3 in the Appendix for the graphs of ϕ and ψ.

In the rest of this section, we discuss the interpolation properties of ϕ. Since ϕ is supported on $[-1,1]$ and $\phi \in C^{2.06288}$, we have $\phi^{(j)}(k) = 0$ for all $j = 0,1,2$ and $k \in \mathbb{Z}\backslash\{0\}$. From $\phi(x) = H_{-1}\phi(2x+1) + H_0\phi(2x) + H_1\phi(2x-1)$, we have $\phi'(0) = 2H_0\phi'(0)$. By the symmetry of ϕ and $\phi'(0) = 2H_0\phi'(0)$, from (2.7) and (2.13), we have

$$\phi(0) = [1, -x_2/x_3, 0, 0]^T, \quad \phi'(0) = [0, 0, 1/x_1, -x_4/(x_1x_5)]^T \quad \text{and}$$
$$\phi''(0) = [0, 1/x_3, 0, 0]^T.$$

Assume that $\{f_k^j\}_{k\in\mathbb{Z}}$, $j = 0,1,2$, are three arbitrary sequences. Let f be the function defined by

$$f(x) := \sum_{k\in\mathbb{Z}} (f_k^0 + x_2 f_k^2)\phi_1(x+k) + x_3 f_k^2 \phi_2(x+k) + x_1 f_k^1 \phi_3(x+k), \qquad x \in \mathbb{R}.$$

Then we have

$$f(k) = f_k^0, \quad f'(k) = f_k^1 \quad \text{and} \quad f''(k) = f_k^2 \qquad \forall\, k \in \mathbb{Z}.$$

For ϕ constructed above, the corresponding parameters x_1, x_2, x_3 are

$$x_1 = (477 - \sqrt{1800393})/3072, \qquad x_2 = (337987 - 159\sqrt{1800393})/3145728,$$
$$x_3 = \frac{\sqrt{34615}(\sqrt{1800393} - 477) - 4\sqrt{1483904377950 - 970842870\sqrt{1800393}}}{47185920}.$$

3. BIORTHOGONAL MULTIWAVELETS ON THE REAL LINE

In this section we shall construct biorthogonal scaling function vectors and biorthogonal multiwavelets with good approximation and smoothness properties.

3.1. The primal Hermite interpolatory mask with the sum rules of order 4

In this section, we want to obtain a refinable function vector ϕ with mask H such that

$$\phi(k) = \delta_k[1,0,0,0]^T, \quad \phi'(k) = \delta_k[0,0,1,0]^T, \quad \phi''(k) = \delta_k[0,1,0,0]^T,$$
$$\phi'''(k) = \delta_k[0,0,0,1]^T \quad \forall\, k \in \mathbb{Z}.$$

That is, ϕ is a refinable Hermite interpolant. By Lemma 4.1 in [11], it is necessary that

$$H_{2k} = \delta_k \mathrm{diag}(1, 1/4, 1/2, 1/8) \qquad \forall\, k \in \mathbb{Z}$$

and H satisfies the sum rules of order 4 with

$$y_0 = [1, 0, 0, 0], \quad y_1 = [0, 0, 1, 0], \quad y_2 = [0, 1, 0, 0], \quad y_3 = [0, 0, 0, 1]. \tag{3.1}$$

We take the primal Hermite interpolatory mask H to be the following form: $H_k = 0$ for all $k \in \mathbb{Z}\backslash\{-1, 0, 1\}$ and

$$H_{-1} = \begin{bmatrix} a & -aw \\ -b & bw \end{bmatrix}, \quad H_0 = \mathrm{diag}(1, 1/4, 1/2, 1/8), \quad \text{and} \quad H_1 = \begin{bmatrix} a & aw \\ b & bw \end{bmatrix}, \tag{3.2}$$

where a, b and w are 2×2 matrices and w is nonsingular.

Let $O_m, m \in \mathbb{Z}_+$ be defined in (2.4). It is easy to verify that H of the form given in (3.2) satisfies the sum rules of order 4 with $\{y_0, y_1, y_2, y_3\}$ given in (3.1) if and only if

$$y_0 O_0 = y_0; \tag{3.3}$$
$$2y_1 O_0 - y_0 O_1 = y_1; \tag{3.4}$$
$$4y_2 O_0 - 2y_1 O_1 + y_0 O_2 = y_2; \tag{3.5}$$
$$8y_3 O_0 - 4y_2 O_1 + 2y_1 O_2 - y_0 O_3 = y_3. \tag{3.6}$$

One can obtain that (3.3) is equivalent to

$$[1, 0]a = [1/2, 0]. \tag{3.7}$$

Under condition (3.7), (3.4) is equivalent to

$$[1, 0]b = [1/4, 0] + [1/4, 0]w^{-1}. \tag{3.8}$$

Under conditions (3.7) and (3.8), (3.5) is equivalent to

$$[0, 1]a = [1/16, 1/8] + [1/8, 0]w^{-1}. \tag{3.9}$$

Under conditions (3.7), (3.8) and (3.9), (3.6) is equivalent to

$$[0, 1]b = [1/96, 1/16] + [1/32, 1/16]w^{-1}. \tag{3.10}$$

Combine (3.7) and (3.9) together, we have

$$a = \begin{bmatrix} 1/2 & 0 \\ 1/16 & 1/8 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/8 & 0 \end{bmatrix} w^{-1}. \tag{3.11}$$

Combine (3.8) and (3.10) together, we have

$$b = \begin{bmatrix} 1/4 & 0 \\ 1/96 & 1/16 \end{bmatrix} + \begin{bmatrix} 1/4 & 0 \\ 1/32 & 1/16 \end{bmatrix} w^{-1}. \tag{3.12}$$

3.2. The dual mask with the sum rules of order 4

In the following, we shall recall the coset by coset (CBC) algorithm proposed in [11] to construct dual masks with arbitrary order of sum rules for any given interpolatory mask. The reader is referred to [11] for a proof of the following result.

THEOREM 3.1. (*Coset By Coset* (*CBC*) *Algorithm*) *Let H be a finitely supported mask on $\mathbb{Z}$ with multiplicity r such that the matrix $\sum_{k\in\mathbb{Z}} H_k$ has a simple eigenvalue 2 and all the other eigenvalues in modulus less than 2. Suppose H_0 is invertible and $H_{2k}=0$ for all $k\in\mathbb{Z}\backslash\{0\}$. Let n be any positive integer.*

1.*Let $\widetilde{y}_0$ be a nonzero $1\times r$ row vector such that $\widetilde{y}_0 J_0 = 2\widetilde{y}_0$ (Since $J_0 := \sum_{k\in\mathbb{Z}} H_k$, by the assumption, $\widetilde{y}_0$ is unique up to a constant multiplication). Compute the $1\times r$ row vectors $\widetilde{y}_m, 0<m<n$ by the following recursive formula:*

$$\widetilde{y}_m = \Big(\sum_{0\leqslant s<m} 2^{m-s}\widetilde{y}_s J_{m-s}\Big)\big(2^{m+1}I_r - J_0\big)^{-1}, \qquad 0<m<n, \tag{3.13}$$

where the matrices J_m are defined as follows

$$J_m := \frac{2^{-m}}{m!}\sum_{k\in\mathbb{Z}} H_k\, k^m, \qquad m\in\mathbb{Z}_+;$$

2.*Choose an appropriate subset E of $\mathbb{Z}$ (e.g., any subset E of $\mathbb{Z}$ such that the cardinality of E is no less than n) such that after setting $\widetilde{H}_{1+2k}=0$ for all $k\in\mathbb{Z}\backslash E$, the following linear system*

$$\sum_{m=0}^{s}(-1)^m 2^{s-m}\widetilde{y}_{s-m}\widetilde{O}_m = \widetilde{y}_s \qquad \forall\, s=0,\dots,n-1 \tag{3.14}$$

has at least one solution for $\{\widetilde{H}_{1+2j}\ :\ j\in E\}$ where the matrices $\widetilde{O}_m$ are defined by

$$\widetilde{O}_m := \frac{1}{m!}\sum_{k\in E}\widetilde{H}_{1+2k}(1+2k)^m, \qquad m\in\mathbb{Z}_+;$$

3.*Construct the coset of $\widetilde{H}$ at 0 as follows:*

$$\widetilde{H}_{2j} = \Big[2\delta_j I_r - \sum_{k\in E}\widetilde{H}_{1+2k}\, H^*_{1+2(k-j)}\Big](H_0^{-1})^*, \qquad j\in\mathbb{Z}.$$

Then the mask $\widetilde{H}$ is a dual mask of the primal mask H and $\widetilde{H}$ satisfies the sum rules of order n.

From (3.13) and the special structure of the primal mask H in (3.2), we have

$$\widetilde{y}_0=[1,c_0,0,0],\quad \widetilde{y}_1=[0,0,c_1,c_2],\quad \widetilde{y}_2=[c_3,c_4,0,0],\quad \widetilde{y}_3=[0,0,c_5,c_6], \tag{3.15}$$

where c_0,c_1,c_2,c_3,c_4,c_5 and c_6 are uniquely determined by (3.13) and depend only on w.

By a simple computation, we take the dual mask to be the following form: $\widetilde{H}_k = 0$ for all $k \in \mathbb{Z}\backslash\{-1, 0, 1\}$, and

$$\widetilde{H}_{-1} = \begin{bmatrix} \widetilde{a} & -\widetilde{a}(w^{-1})^* \\ -\widetilde{b} & \widetilde{b}(w^{-1})^* \end{bmatrix}, \qquad \widetilde{H}_0 = \begin{bmatrix} 2(I_2 - 2\widetilde{a}a^*)D & 0 \\ 0 & 4(I_2 - 2\widetilde{b}b^*)D \end{bmatrix}, \qquad (3.16)$$
$$\widetilde{H}_1 = \begin{bmatrix} \widetilde{a} & \widetilde{a}(w^{-1})^* \\ \widetilde{b} & \widetilde{b}(w^{-1})^* \end{bmatrix},$$

where $\widetilde{a}$ and $\widetilde{b}$ are 2×2 matrices and $D := \mathrm{diag}(1, 4)$. It is easy to verify that H and $\widetilde{H}$ satisfy the discrete biorthogonal relation (1.3). The reader is referred to [18] for the parametric expressions of $H, \widetilde{H}$ satisfying (1.3) and the expressions of the matrix filters for the multiwavelets. Now we want to construct $\widetilde{H}$ of the form given in (3.16) such that $\widetilde{H}$ satisfies the sum rules of order 4.

With the expression of $\widetilde{H}$ given in (3.16), the equations in (3.14) with $n = 4$ in Theorem 3.1 are equivalent to

$$[1, c_0]\widetilde{a} = [1/2, c_0/2]; \qquad (3.17)$$
$$[c_1, c_2]\widetilde{b} = [1/4, c_0/4] + [c_1/4, c_2/4]w^*; \qquad (3.18)$$
$$[c_3, c_4]\widetilde{a} = [c_3/8 + 1/16, c_0/16 + c_4/8] + [c_1/8, c_2]w^*; \qquad (3.19)$$
$$[c_5, c_6]\widetilde{b} = [c_3/16 + 1/96, c_0/96 + c_4/16] + [c_1/32 + c5/16, c_2/32 + c_6/16]w^*. \qquad (3.20)$$

The equations (3.17) and (3.19) together are equivalent to

$$\widetilde{a} = \begin{bmatrix} 1 & c_0 \\ c_3 & c_4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1/2 & c_0/2 \\ c_3/8 + 1/16 & c_0/16 + c_4/8 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_1/8 & c_2/8 \end{bmatrix} w^* \right). \qquad (3.21)$$

The equations (3.18) and (3.20) together are equivalent to

$$\widetilde{b} = \begin{bmatrix} c_1 & c_2 \\ c_5 & c_6 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1/4 & c_0/4 \\ c_3/16 + 1/96 & c_0/96 + c_4/16 \end{bmatrix} + \begin{bmatrix} c_1/4 & c_2/4 \\ c_1/32 + c_5/16 & c_2/32 + c_6/16 \end{bmatrix} w^* \right). \qquad (3.22)$$

With the matrices $\widetilde{a}$ and $\widetilde{b}$ given in (3.21) and (3.22), by Theorem 3.1, $\widetilde{H}_0$ is given in (3.16) and $\widetilde{H}$ satisfies the sum rules of order 4. The reader is referred to [11] for more detail on the CBC algorithm.

For the primal and dual masks given above, with the choice of

$$w = \begin{bmatrix} -443/128 & 221 \\ 25/1024 & -1989/64 \end{bmatrix}, \qquad (3.23)$$

the corresponding primal scaling function vector $\phi \in W^{3.63298}$, ϕ has accuracy order 4, while the dual scaling function vector $\tilde{\phi}$ has accuracy order 4 and $\tilde{\phi} \in W^{1.75833}$. Therefore, $\phi \in C^3$ and $\tilde{\phi} \in C^1$. See Figures 4 and 5 in the Appendix for the graphs of the scaling function vectors. For our choice w in (3.23), from (3.13), we have

$$c_0=\frac{1483}{61828},\qquad c_1=\frac{6900768640}{91534081821},\qquad c_2=\frac{906791128}{2889861726063},$$
$$c_3=\frac{18820966561}{366136327284},\qquad c_4=\frac{22776774817267}{11149034233961442},$$
$$c_5=\frac{105623702245419799136}{4252338953597035742307 3},\qquad c_6=\frac{1403308886084382119774}{1221696981368428368764887 29}.$$

One can choose other nonsingular matrix w such that ϕ is more smooth while $\widetilde{\phi}$ is less smooth, or ϕ is less smooth while $\widetilde{\phi}$ is more smooth.

In the following we shall derive the multiwavelets and dual multiwavelets from the masks H and $\widetilde{H}$.

Set $G_k=0$ and $\widetilde{G}_k=0$ for all $k\in\mathbb{Z}\backslash\{-1,0,1\}$ and set

$$\begin{aligned}
G_{-1}&=\begin{bmatrix} e & -ew\\ -f & fw\end{bmatrix},\qquad G_1=\begin{bmatrix} e & ew\\ f & fw\end{bmatrix},\\
G_0&=\begin{bmatrix} 2e\widetilde{a}^*(2a\widetilde{a}^*-I_2)^{-1}D^{-1} & 0\\ 0 & f\widetilde{b}^*(2b\widetilde{b}^*-I_2)^{-1}D^{-1}\end{bmatrix}
\end{aligned}\tag{3.24}$$

and

$$\widetilde{G}_{-1}=\begin{bmatrix} \widetilde{e} & -\widetilde{e}(w^{-1})^*\\ -\widetilde{f} & \widetilde{f}(w^{-1})^*\end{bmatrix},\ \widetilde{G}_0=\begin{bmatrix} -4\widetilde{e}a^*D & 0\\ 0 & -8\widetilde{f}b^*D\end{bmatrix},\ \widetilde{G}_1=\begin{bmatrix} \widetilde{e} & \widetilde{e}(w^{-1})^*\\ \widetilde{f} & \widetilde{f}(w^{-1})^*\end{bmatrix},\tag{3.25}$$

where $D:=\operatorname{diag}(1,4)$ and the matrices e, $\widetilde{e}$, f and $\widetilde{f}$ satisfy the following relation:

$$e^*\widetilde{e}=[2I_2+4a^*(I_2-2\widetilde{a}a^*)^{-1}\widetilde{a}]^{-1}\quad\text{and}\quad f^*\widetilde{f}=[2I_2+4b^*(I_2-2\widetilde{b}b^*)^{-1}\widetilde{b}]^{-1}.\tag{3.26}$$

It is not difficult to verify that for all $j\in\mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}}H_{k+2j}\widetilde{G}_k^*=\sum_{k\in\mathbb{Z}}\widetilde{H}_{k+2j}G_k^*=0\quad\text{and}\quad\sum_{k\in\mathbb{Z}}G_{k+2j}\widetilde{G}_k^*=\sum_{k\in\mathbb{Z}}\widetilde{G}_{k+2j}G_k^*=2\delta_jI_4.$$

Define $\psi,\widetilde{\psi}$ as follows:

$$\psi=G_{-1}\phi(2x+1)+G_0\phi(2x)+G_1\phi(2x-1)$$

and

$$\widetilde{\psi}=\widetilde{G}_{-1}\widetilde{\phi}(2x+1)+\widetilde{G}_0\widetilde{\phi}(2x)+\widetilde{G}_1\widetilde{\phi}(2x-1).$$

Then for all $k\in\mathbb{Z}$,

$$\int_{\mathbb{R}}\phi(x)\widetilde{\psi}(x+k)^*\,dx=\int_{\mathbb{R}}\widetilde{\phi}(x)\psi(x+k)^*\,dx=0\quad\text{and}\quad\int_{\mathbb{R}}\psi(x)\widetilde{\psi}(x+k)^*\,dx=\delta_kI_4.$$

Moreover, the first two components of $\psi,\widetilde{\psi}$ are symmetric about the origin and the other two components of $\psi,\widetilde{\psi}$ are antisymmetric about the origin.

3.3. Cardinal Hermite Interpolation

In the rest of this section, we are interested in constructing a continuous dual scaling function vector for the cardinal Hermite interpolant given in (1.4). A refinable function vector $\phi = (\phi_1, \dots, \phi_r)^T$ is said to be a *refinable Hermite interpolant* if $\phi_j \in C^{r-1}$ for all $j = 1, \dots, r$ and

$$\phi_j^{(l)}(k) = \delta_k \delta_{j-l-1} \qquad \forall\, j = 1, \dots, r,\ l = 0, \dots, r-1,\ k \in \mathbb{Z}.$$

It is easy to see that the cardinal Hermite interpolant given in (1.4) satisfies the following refinement equation

$$\phi = H_{-1}\phi(2\cdot +1) + H_0\phi(2\cdot) + H_1\phi(2\cdot -1),$$

where the mask H is given by $H_0 = diag(1, 1/2, 1/4, 1/8)$ and

$$H_{-1} = \frac{1}{128}\begin{bmatrix} 64 & 140 & 0 & -840 \\ -22 & -38 & 60 & 420 \\ 3 & 4 & -14 & -60 \\ -1/6 & -1/6 & 1 & 3 \end{bmatrix}, \quad H_1 = \frac{1}{128}\begin{bmatrix} 64 & -140 & 0 & 840 \\ 22 & -38 & -60 & 420 \\ 3 & -4 & -14 & 60 \\ 1/6 & -1/6 & -1 & 3 \end{bmatrix}.$$

Therefore, ϕ is a refinable Hermite interpolant. Both this refinable Hermite interpolant and the piecewise Hermite cubics belong to the family of refinable Hermite interpolants constructed in Theorems 4.2 and 4.3 of [11].

By employing the CBC algorithm in Theorem 3.1, we have

$$\begin{aligned} &\widetilde{y}_0 = [42, 0, 1, 0], \quad \widetilde{y}_1 = [0, 10/3, 0, 1/45], \quad \widetilde{y}_2 = [7/3, 0, 1/10, 0], \\ &\widetilde{y}_3 = [0, 7/55, 0, 1/990], \quad \widetilde{y}_4 = [49/990, 0, 1/396, 0], \\ &\widetilde{y}_5 = [0, 3/1430, 0, 7/386100], \quad \widetilde{y}_6 = [1/1716, 0, 1/30888, 0], \\ &\widetilde{y}_7 = [0, 1/49140, 0, 1/5405400]. \end{aligned}$$

We find a dual scaling function vector $\widetilde{\phi}$ of the cardinal Hermite interpolant in (1.4) such that $\widetilde{\phi}$ satisfies the following refinement equation

$$\widetilde{\phi} = \sum_{k=-4}^{4} \widetilde{H}_k \widetilde{\phi}(2\cdot -k),$$

where the matrices $\widetilde{H}_k$ are given by

$$\widetilde{H}_0 = \begin{bmatrix} \frac{169991055281}{73383542784} & 0 & \frac{24027875971}{513684799488} & 0 \\ 0 & \frac{1030147673162020993}{218124545427505152} & 0 & \frac{71818411656475751}{3271868181412577280} \\ -\frac{22191993992813965}{308085516140544} & 0 & -\frac{1901125625545013}{1232342064562176} & 0 \\ 0 & -\frac{15490427520365900927}{18177045452292096} & 0 & -\frac{896579329192526443}{272655681784381440} \end{bmatrix},$$

$$\widetilde{H}_1 = \begin{bmatrix} -\frac{33683707}{509607936} & \frac{3674703181}{32105299968} & \frac{27}{4096} & 0 \\ \frac{90291809}{339738624} & \frac{5005522001}{9172942848} & \frac{303577211}{2474639360} & -\frac{5303296415}{2535683260416} \\ \frac{3953600465}{169869312} & -\frac{34718219293}{1528823808} & \frac{1131593}{4194304} & -\frac{88734645931}{1267841630208} \\ \frac{65634021059}{169869312} & -\frac{745486058039}{1528823808} & -\frac{14990207793}{1237319680} & -\frac{489554719329}{422613876736} \end{bmatrix},$$

$$\widetilde{H}_2 = \begin{bmatrix} -\frac{3337913993}{36691771392} & \frac{20345104379}{513684799488} & -\frac{5266413727}{1027369598976} & \frac{76450861}{770527199232} \\ -\frac{5955365400962245}{7394052387373056} & \frac{120271049835895153}{436249090855010304} & -\frac{230213273223029917}{4362490908550103040} & \frac{794944485478807}{3271868181412577280} \\ \frac{7115204694626665}{616171032281088} & -\frac{7384072244519443}{1232342064562176} & \frac{1166055435816953}{2464684129124352} & -\frac{26679945678437}{1848513096843264} \\ \frac{394243385262887461}{1232342064562176} & -\frac{10558051260308115955}{72708181809168384} & \frac{11269664992509816943}{727081818091683840} & -\frac{136098273547007401}{545311363568762880} \end{bmatrix},$$

$$\widetilde{H}_3 = \begin{bmatrix} -\frac{4875}{4194304} & \frac{2625}{4194304} & -\frac{1125}{8388608} & \frac{125}{8388608} \\ \frac{183}{8192} & -\frac{24771}{524288} & -\frac{123}{32768} & -\frac{139}{1048576} \\ \frac{2255}{4096} & -\frac{3533}{4096} & -\frac{21}{512} & -\frac{245}{65536} \\ \frac{6539}{1024} & -\frac{4549}{512} & -\frac{181540107}{618659840} & -\frac{9053017641}{211306938368} \end{bmatrix},$$

$$\widetilde{H}_4 = \begin{bmatrix} -\frac{375}{67108864} & 0 & 0 & 0 \\ \frac{332499}{8388608} & -\frac{134091}{8388608} & \frac{16113}{8388608} & 0 \\ \frac{674875}{1048576} & -\frac{272531}{1048576} & \frac{16385}{524288} & \frac{15}{1048576} \\ \frac{21109340991791}{3380911013888} & -\frac{251733163699759}{99736874909696} & \frac{302423546916659}{997368749096960} & \frac{170360752669}{498684374548480} \end{bmatrix}$$

and $\widetilde{H}_k = diag(1,-1,1,-1)\widetilde{H}_{-k}diag(1,-1,1,-1)$ for $k = -4,-3,-2,-1$. It is easy to check that $\widetilde{H}$ satisfies the sum rules of order 8 and $\widetilde{\phi} \in W^{1.13762}$. Therefore, $\widetilde{\phi}$ has accuracy order 8 and $\widetilde{\phi}$ is a continuous dual scaling function vector of the cardinal Hermite interpolant ϕ in (1.4). See Figure 1 for the graphs of ϕ and $\widetilde{\phi}$.

4. ORTHOGONAL AND BIORTHOGONAL MULTIGENERATORS ON $[0,1]$

In this section we construct multigenerators on $[0,1]$ based on the scaling function vectors constructed in the preceding sections. Since the refinable function vectors ϕ are supported on $[-1,1]$, the refinable function vectors (multigenerators) on $[0,1]$ are constructed by just restricting $\phi_{[j,k]}$ to $[0,1]$ and using a (bi)orthogonal procedure. Here for a function vector ϕ, denote

$$\phi_{[j,k]} = 2^{j/2}\phi(2^j \cdot -k), \quad j \in \mathbb{Z}_+, k \in \mathbb{Z}.$$

Suppose ϕ and $\widetilde{\phi}$ are primal and dual scaling function vectors on the real line satisfying

$$\phi(x) = H_{-1}\phi(2x+1) + H_0\phi(2x) + H_1\phi(2x-1) \tag{4.1}$$

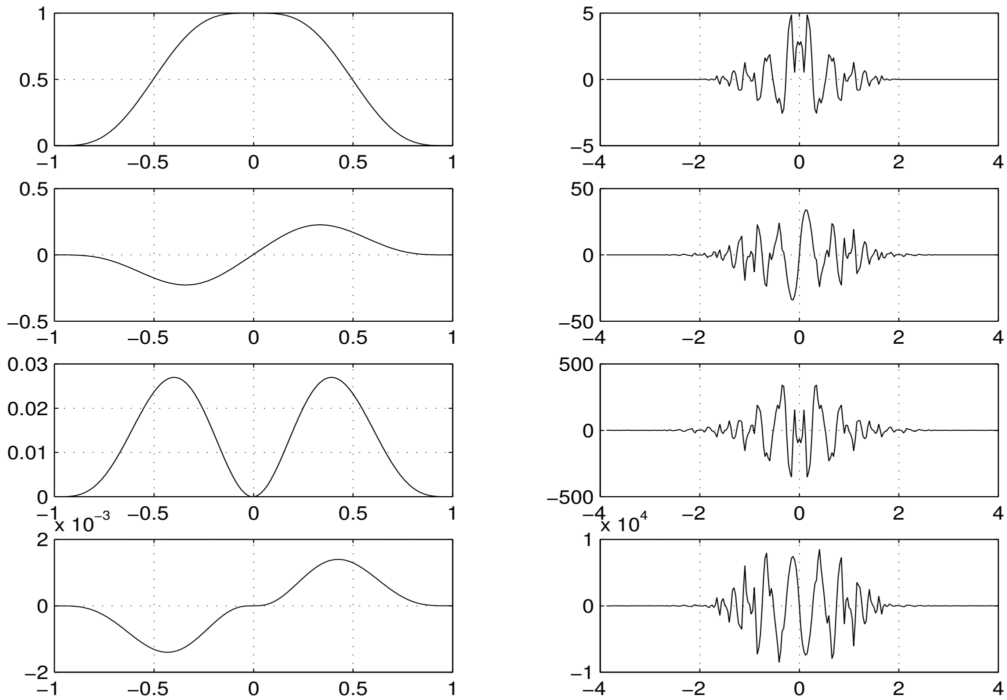

FIG. 1. The cardinal Hermite interpolant (the left column) and its dual scaling function vector (the right column)

and

$$\widetilde{\phi}(x) = \widetilde{H}_{-1}\widetilde{\phi}(2x+1) + \widetilde{H}_0\widetilde{\phi}(2x) + \widetilde{H}_1\widetilde{\phi}(2x-1) \tag{4.2}$$

for some refinement masks $\{H_k\}_{k=-1}^{1}, \{\widetilde{H}_k\}_{k=-1}^{1}$ such that the refinable function vectors ϕ and $\widetilde{\phi}$ satisfy $\int_{\mathbb{R}} \phi(x)\widetilde{\phi}(x+k)^* \, dx = \delta_k I_{4\times 4}$ for all $k \in \mathbb{Z}$. For simplicity, in this section, we assume

$$S_0\phi(-x) = \phi(x), \quad S_0\widetilde{\phi}(-x) = \widetilde{\phi}(x) \ \forall\, x \in \mathbb{R}, \quad \text{where} \quad S_0 := \begin{bmatrix} I_2 & \\ & -I_2 \end{bmatrix}. \tag{4.3}$$

In other words, the first two components of ϕ and $\widetilde{\phi}$ are symmetric about the origin and the other two components of ϕ and $\widetilde{\phi}$ are antisymmetric about the origin.

First we have the following lemma.

LEMMA 4.1. *Let ϕ and $\widetilde{\phi}$ in $L^2(\mathbb{R})$ be given in* (4.1) *and* (4.2) *such that the biorthogonal relation* (1.2) *holds. Denote $C := \int_0^1 \widetilde{\phi}(x)\phi(x)^* \, dx$. If 2 is not the eigenvalue of the matrix $\overline{H_0} \otimes \widetilde{H}_0$ where $\otimes$ denotes the Kronecker product of two matrices, then the matrix C is uniquely determined by the following relation:*

$$2C = \widetilde{H}_0 C H_0^* + \widetilde{H}_1 H_1^*. \tag{4.4}$$

Proof. Since $\phi(x)|_{[0,1]} = H_0\phi(2x)|_{[0,1]} + H_1\phi(2x-1)$ and $\widetilde{\phi}(x)|_{[0,1]} = \widetilde{H}_0\widetilde{\phi}(2x)|_{[0,1]} + \widetilde{H}_1\widetilde{\phi}(2x-1)$, we obtain

$$\begin{aligned}
2C &= 2\int_0^1 \widetilde{\phi}(x)\phi(x)^*\,dx \\
&= 2\int_0^1 \left[\widetilde{H}_0\widetilde{\phi}(2x) + \widetilde{H}_1\widetilde{\phi}(2x-1)\right]\left[H_0\phi(2x) + H_1\phi(2x-1)\right]^*\,dx \\
&= 2\widetilde{H}_0\int_0^1 \widetilde{\phi}(2x)\phi(2x)^*\,dx H_0^* + 2\widetilde{H}_1\int_0^1 \widetilde{\phi}(2x-1)\phi(2x-1)^*\,dx H_1^* \\
&= \widetilde{H}_0\int_0^1 \widetilde{\phi}(x)\phi(x)^*\,dx H_0^* + \widetilde{H}_1\int_{-1}^1 \widetilde{\phi}(x)\phi(x)^*\,dx H_1^* \\
&= \widetilde{H}_0 C H_0^* + \widetilde{H}_1 H_1^*.
\end{aligned}$$

Thus

$$(2I_{16} - \overline{H_0}\otimes\widetilde{H}_0)\mathrm{vec}(C) = \mathrm{vec}(\widetilde{H}_1 H_1^*).$$

Here for a matrix $B = (B_1, \ldots, B_n)$ with B_j being its jth column, $\mathrm{vec}(B)$ denotes the column vector $(B_1^T, \ldots, B_n^T)^T$. Therefore, C is uniquely determined by (4.4). This completes the proof. ∎

For the purpose of graphing the biorthogonal multiwavelet on the interval, we shall adjust the refinable function vectors so that they are balanced and can be plotted together at a comparable scale. That is, we shall consider the primal scaling function vector $\phi^{new} := diag(1, 2^6, 2^5, 2^{12})\phi$ and its dual scaling function vector $\widetilde{\phi}^{new} := diag(1, 2^{-6}, 2^{-5}, 2^{-12})\widetilde{\phi}$ with ϕ, $\widetilde{\phi}$ constructed in Sections 3.1 and 3.2. For the scaling function vectors $\phi^{new} = diag(1, 2^6, 2^5, 2^{12})\phi$ and $\widetilde{\phi}^{new} = diag(1, 2^{-6}, 2^{-5}, 2^{-12})\widetilde{\phi}$ with ϕ, $\widetilde{\phi}$ constructed in Sections 3.1 and 3.2, the matrix C is

$$C = \begin{bmatrix} 1/2 & 0 & .58592278017697 & -.30202156474400 \\ 0 & 1/2 & .73688152900109 & 1.22479094459656 \\ .23970858883924 & .09038744399097 & 1/2 & 0 \\ -.12013864250355 & .14687265801947 & 0 & 1/2 \end{bmatrix}.$$

Now let us give the main theorem of this section.

THEOREM 4.1. *Let ϕ and $\widetilde{\phi}$ be the scaling function vectors given in* (4.1) *and* (4.2) *such that the biorthogonal relation* (1.2) *holds. Assume that $C := \int_0^1 \widetilde{\phi}(x)\phi(x)^*\,dx$ is invertible. For $j \in \mathbb{Z}_+$, define*

$$\begin{aligned}
\phi_j^L(x) &:= C_1^{-1}\phi_{[j,0]}(x)|_{[0,1]}, \quad \phi_j^R(x) := S_0\phi_j^L(1-x) = S_0 C_1^{-1} S_0 \phi_{[j,2^j]}(x)|_{[0,1]}, \\
\widetilde{\phi}_j^L(x) &:= \widetilde{C}_1^{-1}\widetilde{\phi}_{[j,0]}(x)|_{[0,1]}, \quad \widetilde{\phi}_j^R(x) := S_0\widetilde{\phi}_j^L(1-x) = S_0 \widetilde{C}_1^{-1} S_0 \widetilde{\phi}_{[j,2^j]}(x)|_{[0,1]},
\end{aligned}$$

where $C_1, \widetilde{C}_1$ *are two matrices satisfying* $\widetilde{C}_1 C_1^* = C$*, and* S_0 *is defined in* (4.3). *Then*

$$\begin{aligned}
\sqrt{2}\phi_j^L &= C_1^{-1} H_0 C_1 \phi_{j+1}^L + C_1^{-1} H_1 \phi_{[j+1,1]},\\
\sqrt{2}\phi_j^R &= S_0 C_1^{-1} H_{-1} \phi_{[j+1,2^{j+1}-1]} + S_0 C_1^{-1} H_0 C_1 S_0 \phi_{j+1}^R,\\
\sqrt{2}\widetilde{\phi}_j^L &= \widetilde{C}_1^{-1} \widetilde{H}_0 \widetilde{C}_1 \widetilde{\phi}_{j+1}^L + \widetilde{C}_1^{-1} \widetilde{H}_1 \widetilde{\phi}_{[j+1,1]},\\
\sqrt{2}\widetilde{\phi}_j^R &= S_0 \widetilde{C}_1^{-1} \widetilde{H}_{-1} S_0 \widetilde{\phi}_{[j+1,2^{j+1}-1]} + S_0 \widetilde{C}_1^{-1} \widetilde{H}_0 \widetilde{C}_1 S_0 \widetilde{\phi}_{j+1}^R.
\end{aligned}$$

Denote

$$\Phi_j := \{\phi_j^L; \quad \phi_{[j,k]},\ k = 1, \ldots, 2^j - 1; \quad \phi_j^R\}$$

and

$$\widetilde{\Phi}_j := \{\widetilde{\phi}_j^L; \quad \widetilde{\phi}_{[j,k]},\ k = 1, \ldots, 2^j - 1; \quad \widetilde{\phi}_j^R\}.$$

Then Φ_j *and* $\widetilde{\Phi}_j$ *are biorthogonal to each other with respect to the* $L^2([0,1])$ *norm,* $\Phi_j = \mathbb{H}_j \Phi_{j+1}$ *and* $\widetilde{\Phi}_j = \widetilde{\mathbb{H}}_j \widetilde{\Phi}_{j+1}$ *with the* $(2^{j+2}+4) \times (2^{j+3}+4)$ *matrices* $\mathbb{H}_j$ *and* $\widetilde{\mathbb{H}}_j$ *given by*

$$\mathbb{H}_j = \frac{\sqrt{2}}{2} \begin{bmatrix}
C_1^{-1} H_0 C_1 & C_1^{-1} H_1 & & & & & & & \\
& H_{-1} & H_0 & H_1 & & & & & \\
& & & H_{-1} & H_0 & H_1 & & & \\
& & & & & \ddots & & & \\
& & & & & H_{-1} & H_0 & H_1 & \\
& & & & & & & S_0 C_1^{-1} H_{-1} S_0 & S_0 C_1^{-1} H_0 C_1 S_0
\end{bmatrix}$$

and

$$\widetilde{\mathbb{H}}_j = \frac{\sqrt{2}}{2} \begin{bmatrix}
\widetilde{C}_1^{-1} \widetilde{H}_0 \widetilde{C}_1 & \widetilde{C}_1^{-1} H_1 & & & & & & & \\
& \widetilde{H}_{-1} & \widetilde{H}_0 & \widetilde{H}_1 & & & & & \\
& & & \widetilde{H}_{-1} & \widetilde{H}_0 & \widetilde{H}_1 & & & \\
& & & & & \ddots & & & \\
& & & & & \widetilde{H}_{-1} & \widetilde{H}_0 & \widetilde{H}_1 & \\
& & & & & & & S_0 \widetilde{C}_1^{-1} \widetilde{H}_{-1} S_0 & S_0 \widetilde{C}_1^{-1} \widetilde{H}_0 \widetilde{C}_1 S_0
\end{bmatrix}.$$

Moreover, if ϕ *and* $\widetilde{\phi}$ *have accuracy order* k *and* $\widetilde{k}$ *respectively, then* Φ_j *and* $\widetilde{\Phi}_j$ *can reproduce all polynomials on* $[0,1]$ *of degrees up to* $k-1$ *and* $\widetilde{k}-1$*, respectively.*

Proof. It suffices to prove that $\int_0^1 \widetilde{\phi}_j^L(x) \phi_j^L(x)^* \, dx = I_4$. By a simple computation, we have

$$\begin{aligned}
\int_0^1 \widetilde{\phi}_j^L(x) \phi_j^L(x)^* \, dx &= 2^j \widetilde{C}_1^{-1} \int_0^1 \widetilde{\phi}(2^j x) \phi(2^j x)^* \, dx (C_1^*)^{-1}\\
&= \widetilde{C}_1^{-1} \int_0^1 \widetilde{\phi}(x) \phi(x)^* \, dx (C_1^*)^{-1} = I_4.
\end{aligned}$$

The other statements can be easily verified. ∎

In Theorem 4.1, if we choose $C_1 = I_4$, then the boundary multigenerators ϕ_j^L, ϕ_j^R are just the restrictions of $\phi_{[j,0]}, \phi_{[j,2^j]}$ to $[0,1]$, respectively. The choices of $C_1, \tilde{C}_1$ provide some flexibility for the construction of the boundary multigenerators. Here we would like to choose $C_1, \tilde{C}_1$ such that three components of both ϕ_j^L and ϕ_j^R are continuous on $\mathbb{R}$.

For the scaling function vectors ϕ and $\widetilde{\phi}$ constructed in Sections 3.1 and 3.2, Φ_j and $\widetilde{\Phi}_j$ defined in Theorem 4.1 by $C_1 = I_4$ and $\widetilde{C}_1 = C$ reproduce all polynomials on $[0,1]$ of degrees up to three on $[0,1]$. Furthermore, Φ_j has the Hermite interpolation properties at knots $k/2^j, 0 \leqslant k \leqslant 2^j$:

$$
\begin{aligned}
&\big[(\phi_j^L)\ \ (\phi_j^L)''\ \ (\phi_j^L)'\ \ (\phi_j^L)'''\big]\,(\frac{k}{2^j}) = \delta_k J_j \qquad\qquad \forall\, k = 0, \ldots, 2^j;\\
&\big[(\phi_j^R)\ \ (\phi_j^R)''\ \ (\phi_j^R)'\ \ (\phi_j^R)'''\big]\,(\frac{k}{2^j}) = \delta_{2^j-k} J_j \qquad\qquad \forall\, k = 0, \ldots, 2^j;\\
&\big[(\phi_{[j,m]})\ \ (\phi_{[j,m]})''\ \ (\phi_{[j,m]})'\ \ (\phi_{[j,m]})'''\big]\,(\frac{k}{2^j}) = \delta_{m-k} J_j\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall\, m = 1, \ldots, 2^j-1, k = 0, \ldots, 2^j,
\end{aligned}
$$

where $J_j := \operatorname{diag}(2^{j/2}, 2^{5j/2}, 2^{3j/2}, 2^{7j/2})$ since our primal scaling function vector is $\phi^{new} = diag(1, 2^6, 2^5, 2^{12})\phi$ and its dual scaling function vector is $\widetilde{\phi}^{new} = diag(1, 2^{-6}, 2^{-5}, 2^{-12})\widetilde{\phi}$ with ϕ, $\widetilde{\phi}$ constructed in Sections 3.1 and 3.2.

Similarly, for the orthogonal case, we have the following theorem.

THEOREM 4.2. *Assume that ϕ is an orthogonal scaling function vector satisfying* (4.1). *Suppose $C := \int_0^1 \phi(x)\phi(x)^*\,dx$ is invertible. For $j \in \mathbb{Z}_+$, define*

$$
\phi_j^L(x) := C_1^{-1}\phi_{[j,0]}(x)|_{[0,1]}, \quad \phi_j^R(x) := S_0\phi_j^L(1-x) = S_0 C_1^{-1} S_0 \phi_{[j,2^j]}(x)|_{[0,1]},
$$

where C_1 is an invertible matrix satisfying $C_1 C_1^* = C$. Then

$$
\begin{aligned}
\sqrt{2}\phi_j^L &= C_1^{-1} H_0 C_1 \phi_{j+1}^L + C_1^{-1} H_1 \phi_{[j+1,1]},\\
\sqrt{2}\phi_j^R &= S_0 C_1^{-1} H_0 C_1 S_0 \phi_{j+1}^R + S_0 C_1^{-1} H_{-1} S_0 \phi_{[j+1,2^{j+1}-1]},
\end{aligned}
$$

where S_0 is defined in (4.3). *Denote $\Phi_j := \{\phi_j^L;\ \phi_{[j,k]}, k = 1, \ldots, 2^j-1;\ \phi_j^R\}$. Then elements in Φ_j are orthonormal to each other with respect to the $L^2([0,1])$ norm, and $\Phi_j = \mathbb{H}_j \Phi_{j+1}$ with the $(2^{j+1}+4) \times (2^{j+3}+4)$ matrix $\mathbb{H}_j$ given by*

$$
\mathbb{H}_j = \frac{\sqrt{2}}{2}
\begin{bmatrix}
C_1^{-1}H_0C_1 & C_1^{-1}H_1 & & & & & & \\
 & H_{-1} & H_0 & H_1 & & & & \\
 & & & H_{-1} & H_0 & H_1 & & \\
 & & & & & \ddots & & \\
 & & & & & H_{-1} & H_0 & H_1 \\
 & & & & & & S_0C_1^{-1}H_{-1}S_0 & S_0C_1^{-1}H_0C_1S_0
\end{bmatrix}.
$$

Moreover, if ϕ has accuracy order k, then Φ_j can reproduce all polynomials on $[0,1]$ *of degrees up to* $k-1$.

For the orthogonal scaling function vector ϕ constructed in Section 2, C is given below. The matrix C_1 is not unique. We choose a C_1 such that the second, third and fourth components of ϕ_j^L are continuous on $\mathbb{R}$.

$$C = \begin{bmatrix} 1/2 & 0 & -.43677970083296 & .17535239421305 \\ 0 & 1/2 & .23763810755456 & .38168609250669 \\ -.43677970083296 & .23763810755456 & 1/2 & 0 \\ .17535239421305 & .38168609250669 & 0 & 1/2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} .14574297280612 & -.55915098774268 & .39502847421501 & .10030784288073 \\ .17671071874332 & .43276809059487 & .51325396998223 & .13437062157172 \\ 0 & .69442476791991 & -.13012323474392 & .02902043209824 \\ 0 & .13329254635566 & .67450586007773 & -.16515126946486 \end{bmatrix}.$$

5. ORTHOGONAL AND BIORTHOGONAL MULTIWAVELETS ON $[0,1]$

In this section, we construct orthogonal and biorthogonal multiwavelets on $[0,1]$ from the orthogonal and biorthogonal multiwavelets on the real line constructed in Sections 2 and 3.

5.1. Orthogonal multiwavelets on $[0,1]$

Let the masks H and G be given in (2.1) and (2.16) with the matrices a, b, c and d given in (2.12) and (2.13). Then we have the following result.

THEOREM 5.1. *Let B be the 2×2 matrix given by*

$$B^*B = \Big[I_2 + \frac{1}{2}[w^*, I_2]H_1^*(H_0^{-1})^*C^{-1}H_0^{-1}H_1[w^*, I_2]^*\Big]^{-1}.$$

Define the 2×1 function vectors ψ_j^L and ψ_j^R as follows:

$$\sqrt{2}\psi_j^L := B_1\phi_{j+1}^L + B_2\phi_{[j+1,1]} \qquad \textit{and} \qquad \psi_j^R(x) := \psi_j^L(1-x),$$

where $B_1 := -B[w^*, I_2]H_1^*(H_0^{-1})^*(C_1^{-1})^*$ and $B_2 := B[w^*, I_2]$. Let

$$\Psi_j := \{\psi_j^L;\quad \psi_{[j,k]},\ k = 1, \ldots, 2^j - 1;\quad \psi_j^R\},$$

where

$$\psi(x) := G_{-1}\phi(2x+1) + G_0\phi(2x) + G_1\phi(2x-1).$$

Let Φ_j be given in Theorem 4.2. Then $\{\Psi_j, \Phi_j\}$ is an orthogonal system, span$\{\Phi_j, \Psi_j\}$ = span$\{\Phi_{j+1}\}$ and $\Psi_j = \mathbb{G}_j\Phi_{j+1}$ where the $2^{j+2}\times(2^{j+3}+4)$ matrix $\mathbb{G}_j$ is given by

$$\mathbb{G}_j = \frac{\sqrt{2}}{2}\begin{bmatrix} B_1 & B_2 & & & & & & \\ & G_{-1} & G_0 & G_1 & & & & \\ & & & G_{-1} & G_0 & G_1 & & \\ & & & & & \ddots & & \\ & & & & & G_{-1} & G_0 & G_1 \\ & & & & & & B_2S_0 & B_1 \end{bmatrix}, \tag{5.1}$$

where S_0 is defined in (4.3).

Proof. It is equivalent to verifying the following equalities.

$$\int_0^1 \psi_j^L(x)\phi_j^L(x)^*\,dx = 0, \quad \int_0^1 \psi_j^L(x)\phi_{[j,1]}(x)^*\,dx = 0,$$
$$\int_0^1 \psi_j^L(x)\psi_{[j,1]}(x)^*\,dx = 0, \quad \int_0^1 \psi_j^L(x)\psi_j^L(x)^*\,dx = I_2.$$

From Theorem 4.2, the above equalities are equivalent to

$$B_1C_1^*H_0^*(C_1^{-1})^* + B_2H_1^*(C_1^{-1})^* = 0, \qquad B_2H_{-1}^* = 0,$$
$$B_2G_{-1}^* = 0, \qquad B_1B_1^* + B_2B_2^* = 2I_2$$

which can be easily verified by the assumptions. ■

5.2. Biorthogonal multiwavelets on $[0,1]$

In the following, we shall construct biorthogonal multiwavelets on $[0,1]$ from the biorthogonal multiwavelets on the real line constructed in Sections 3.1 and 3.2. The method used in [6] can be employed to obtain a multiwavelet basis on the interval from the biorthogonal multiwavelet constructed in Section 3.3.

Let the masks H, G and $\widetilde{H}, \widetilde{G}$ be given in (3.2), (3.24), (3.16) and (3.25), respectively. Then we have the following result.

THEOREM 5.2. *Let B and $\widetilde{B}$ be 2×2 matrices such that*

$$\widetilde{B}^*B = \Big[I_2 + \frac{1}{2}[w^{-1}, I_2]\widetilde{H}_1^*(\widetilde{H}_0^{-1})^*(C^{-1})^*H_0^{-1}H_1[w^*, I_2]^*\Big]^{-1}.$$

Define the boundary wavelets $\psi_j^L, \widetilde{\psi}_j^L$ and $\psi_j^R, \widetilde{\psi}_j^R$ as follows:

$$\sqrt{2}\psi_j^L := B_1\phi_{j+1}^L + B_2\phi_{[j+1,1]} \qquad \textit{and} \qquad \psi_j^R(x) := \psi_j^L(1-x),$$
$$\sqrt{2}\widetilde{\psi}_j^L := \widetilde{B}_1\widetilde{\phi}_{j+1}^L + \widetilde{B}_2\widetilde{\phi}_{[j+1,1]} \qquad \textit{and} \qquad \widetilde{\psi}_j^R(x) := \widetilde{\psi}_j^L(1-x),$$

where

$$B_1 := -B[w^{-1}, I_2]\widetilde{H}_1^*(\widetilde{H}_0^{-1})^*(\widetilde{C}_1^{-1})^* \qquad \textit{and} \qquad B_2 := B[w^{-1}, I_2],$$
$$\widetilde{B}_1 := -\widetilde{B}[w^*, I_2]H_1^*(H_0^{-1})^*(C_1^{-1})^* \qquad \textit{and} \qquad \widetilde{B}_2 := \widetilde{B}[w^*, I_2].$$

Let

$$\Psi_j := \{\psi_j^L; \quad \psi_{[j,k]},\ k = 1, \dots, 2^j - 1; \quad \psi_j^R\},$$
$$\widetilde{\Psi}_j := \{\widetilde{\psi}_j^L; \quad \widetilde{\psi}_{[j,k]},\ k = 1, \dots, 2^j - 1; \quad \widetilde{\psi}_j^R\},$$

where

$$\psi(x) := G_{-1}\phi(2x+1) + G_0\phi(2x) + G_1\phi(2x+1),$$
$$\widetilde{\psi}(x) := \widetilde{G}_{-1}\widetilde{\phi}(2x+1) + \widetilde{G}_0\widetilde{\phi}(2x) + \widetilde{G}_1\widetilde{\phi}(2x+1).$$

Let Φ_j and $\widetilde{\Phi}_j$ be given in Theorem 4.1. *Then $\{\Phi_j, \Psi_j\}$ and $\{\widetilde{\Phi}_j, \widetilde{\Psi}_j\}$ form a biorthogonal system, $\Psi_j = \mathbb{G}_j \Phi_{j+1}$ and $\widetilde{\Psi}_j = \widetilde{\mathbb{G}}_j \widetilde{\Phi}_{j+1}$ where $\mathbb{G}_j$ takes the form in* (5.1) *and the $2^{j+2} \times (2^{j+3}+4)$ matrix $\widetilde{\mathbb{G}}_j$ is given by*

$$\widetilde{\mathbb{G}}_j = \frac{\sqrt{2}}{2} \begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_2 & & & & & & & \\ & \widetilde{G}_{-1} & \widetilde{G}_0 & \widetilde{G}_1 & & & & & \\ & & & \widetilde{G}_{-1} & \widetilde{G}_0 & \widetilde{G}_1 & & & \\ & & & & & \ddots & & & \\ & & & & & \widetilde{G}_{-1} & \widetilde{G}_0 & \widetilde{G}_1 & \\ & & & & & & & \widetilde{B}_2 S_0 & \widetilde{B}_1 \end{bmatrix},$$

where S_0 is defined in (4.3).

The matrices B and $\widetilde{B}$ in both Theorem 5.1 and Theorem 5.2 are used to obtain boundary wavelets on the interval $[0,1]$. Though they are not uniquely determined by the relation in Theorems 5.1 and 5.2, different choices of B and $\widetilde{B}$ will not affect the symmetry and smoothness of the boundary wavelets. In this paper we are choosing such matrices B and $\widetilde{B}$ such that some of the corresponding boundary wavelets are continuous.

The orthogonal and biorthogonal multiwavelet bases characterize the Sobolev norm $\|\cdot\|_{W^s([0,1])}$, where $W^s([0,1])$ is the restriction of W^s to $[0,1]$. For the multiwavelets on $[0,1]$ constructed in the above theorem based on $\phi, \psi, \widetilde{\phi}, \widetilde{\psi}$ constructed in the preceding section, we have (see [2], [5], [6]) the following theorem.

THEOREM 5.3. *For any $f \in W^s([0,1])$,*

$$\|(f, \widetilde{\Phi}_{j_0})_{[0,1]}\|^2_{\ell_2(\triangle_{j_0})} + \sum_{j=j_0}^{\infty} \|(f, \widetilde{\Psi}_j)_{[0,1]}\|^2_{\ell_2(\bigtriangledown_j)}$$
$$\approx \begin{cases} \|f\|^2_{W^s([0,1])}, & s \in [0, 3.63298), \\ \|f\|^2_{(W^{-s}([0,1]))^*}, & s \in (-1.75833, 0), \end{cases}$$

where $\triangle_j = \{0, 1, 2, \dots, 2^j\}, \bigtriangledown_j = \{2^j+1, \dots, 2^{j+1}\}$, and for $s < 0$, $W^s([0,1])$ means $(W^{-s}([0,1]))^*$, the dual of $W^{-s}([0,1])$ with respect to the $L^2([0,1])$ norm.

For the orthogonal multiwavelets on $[0,1]$, we have the similar result to that in Theorem 5.3, and details are not provided here.

6. APPENDIX: NUMERICAL RESULTS AND FIGURES

The matrix filters $H_j, G_j, j = -1, 0, 1$ for the orthogonal scaling function vector and multiwavelet are given by (2.1) and (2.16) with

$$a = \begin{bmatrix} 1/2 & 0 \\ -.45468071806728 & -.26628159762860 \end{bmatrix},$$

$$b = \begin{bmatrix} -.65516955124945 & -.09084524817293 \\ -.15478756558925 & -.40706090807325 \end{bmatrix},$$

$$c = \begin{bmatrix} 1 & 0 \\ .90936143613457 & 1/4 \end{bmatrix}, \qquad d = \begin{bmatrix} 1/2 & 0 \\ -1.10713319281209 & 1/4 \end{bmatrix},$$

$$w = \begin{bmatrix} \frac{477}{512} & -\sqrt{\frac{34615}{262144}} \\ -\sqrt{\frac{34615}{262144}} & -\frac{477}{512} \end{bmatrix},$$

$$e = \begin{bmatrix} -.08817966722265 & -.02537874177881 \\ .18838734274447 & -.65456094462631 \end{bmatrix},$$

$$f = \begin{bmatrix} -.02396881534188 & -.00902548880016 \\ .21498642874753 & -.57093528403364 \end{bmatrix}.$$

The matrices B_1, B_2 for the boundary multiwavelets are not unique. Here we choose B_1, B_2 such that the second components ψ_j^L are continuous on $\mathbb{R}$.

$$B_1 = \begin{bmatrix} 1.00000000001851 & -.09411319357463 & -.13884178466033 & -.96673409066909 \\ 0 & -.78022635514195 & -.51400379675387 & .34983279839230 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -.04654898832974 & .12836909088437 & -.09001381625714 & -.10267884247414 \\ -.26830917494375 & .65600469438857 & -.48834736449517 & -.51366216914375 \end{bmatrix}.$$

The matrix filters $H_j, G_j, \widetilde{H}_j, \widetilde{G}_j, j = -1, 0, 1$ for the primal scaling function $\phi^{new} := diag(1, 2^6, 2^5, 2^{12})\phi$, the dual scaling function $\widetilde{\phi}^{new} := diag(1, 2^6, 2^5, 2^{12})\psi$, the wavelet function $\psi^{new} := diag(1, 2^{-6}, 2^{-5}, 2^{-12})\widetilde{\phi}$, and the dual wavelet function $\widetilde{\psi}^{new} := diag(1, 2^{-6}, 2^{-5}, 2^{-12})\widetilde{\psi}$ with $\phi, \widetilde{\phi}, \psi, \widetilde{\psi}$ constructed in Sections 3.1 and 3.2 are given by the expressions in (3.2), (3.24), (3.16) and (3.25) with

$$a = \begin{bmatrix} 1/2 & 0 \\ 5932/3787 & -4405/30296 \end{bmatrix}, \qquad b = \begin{bmatrix} 10540/3787 & -512/3787 \\ 9209728/2510781 & -386564/836927 \end{bmatrix},$$

$$\widetilde{a} = \begin{bmatrix} -.07085112284136 & .48497457820580 \\ .37186639437622 & .18407568883484 \end{bmatrix},$$

$$\widetilde{b} = \begin{bmatrix} .03043186232144 & .51210505841283 \\ .12868875149091 & -.28051043186050 \end{bmatrix},$$

$$\widetilde{e} = \begin{bmatrix} -.17651325291088 & -1.99059580472762 \\ .04562063640325 & .44445743874147 \end{bmatrix},$$

$$\widetilde{f} = \begin{bmatrix} -.44326459211764 & -3.09657013747741 \\ .19860552584308 & 1.37397727410741 \end{bmatrix},$$

$$e = diag(4/15, 32/27), \qquad f = diag(16/125, 8/25),$$

$$w = \begin{bmatrix} -443/2048 & 221/4096 \\ 25/256 & -1989/4096 \end{bmatrix}.$$

The matrices C_1 and $\widetilde{C}_1$ are not unique. We choose $C_1, \widetilde{C}_1$ such that the second, third and fourth components of $\phi_j^L, \widetilde{\phi}_j^L$ and the second and fourth components of $(\phi_j^L)', (\widetilde{\phi}_j^L)'$ are continuous on $\mathbb{R}$.

$$C_1 = \begin{bmatrix} 1 & .12213868011413 & 23.55798401371176 & 2.20138413721209 \\ 0 & .30982146776962 & 0 & .69017853223038 \\ 0 & .59992163085174 & 16 & 4.70871839801238 \\ 0 & .68497680513495 & 0 & -.68497680513495 \end{bmatrix},$$

$$\widetilde{C}_1 = \begin{bmatrix} .05203894626657 & -.30431512234326 & .00782769758447 & .13660720156594 \\ -.02045938821446 & 1.73409203069902 & -.00307750089492 & -.05398449308234 \\ 0 & .09038744399097 & .00126034979518 & .09038744399097 \\ 0 & .65066966768202 & -.00106462060293 & -.07928205659789 \end{bmatrix}.$$

The matrices $B_1, B_2, \widetilde{B}_1, \widetilde{B}_2$ for the boundary multiwavelets are not unique. We choose $B_1, B_2, \widetilde{B}_1, \widetilde{B}_2$ such that the second components of $\psi_j^L, \widetilde{\psi}_j^L$ are continuous on $\mathbb{R}$.

$$B_1 = \begin{bmatrix} 6.96195979650562 & .08108497396158 & -82.00486879511919 & 1.39090847840064 \\ 0 & .07283055384749 & -1.38478266392370 & -.68720021815292 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} .21559817693860 & -.24145277751466 & -.07021511127282 & .12888104774905 \\ .17151328069360 & -.19673018205524 & -.05631172839761 & .10478534354032 \end{bmatrix},$$

$$\widetilde{B}_1 = \begin{bmatrix} .14363771541799 & -.07889677818832 & -.01080299159265 & .05903520751900 \\ 0 & 9.82435846789804 & .01573952134476 & -.49420187843209 \end{bmatrix},$$

$$\widetilde{B}_2 = \begin{bmatrix} -.00946383319422 & -.08794190565746 & .09362091265444 & .19992881379324 \\ -.48279739269897 & -2.87774612632776 & 3.90613826483934 & 6.71176643508005 \end{bmatrix}.$$

In the following we provide graphs of multigenerators ϕ_2^L, $\phi_{[2,1]}$, $\widetilde{\phi}_2^L$, $\widetilde{\phi}_{[2,1]}$, and multiwavelets ψ_2^L, $\psi_{[2,1]}$, $\widetilde{\psi}_2^L$, $\widetilde{\psi}_{[2,1]}$with level $j = 2$.

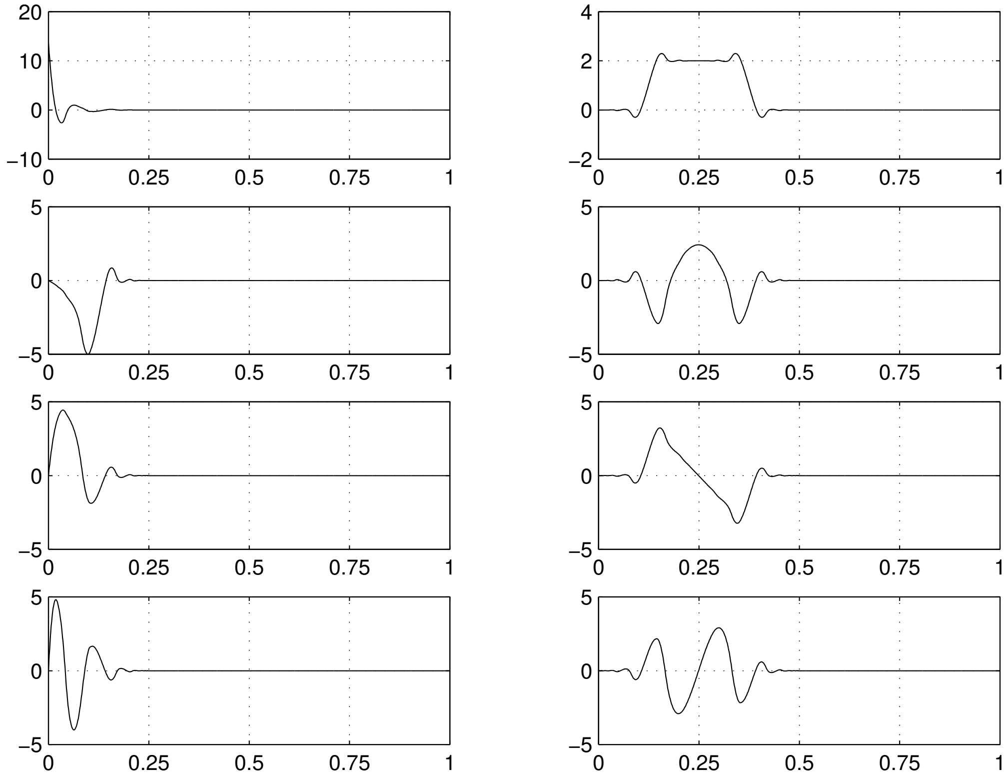

FIG. 2. Orthonormal boundary multigenerator ϕ_2^L (the left column) and orthonormal mid multigenerator $\phi_{[2,1]}$ (the right column)

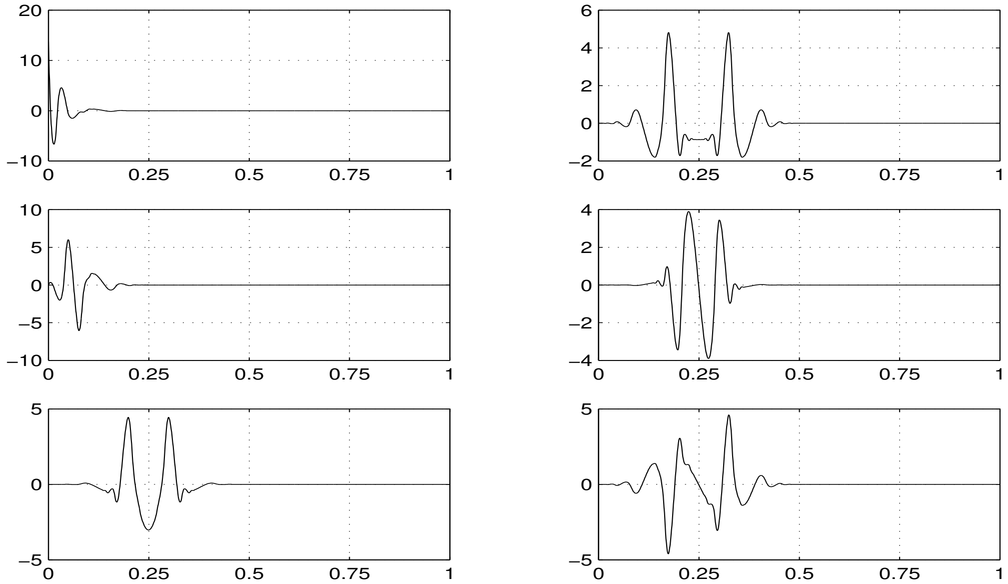

FIG. 3. Orthonormal boundary multiwavelet ψ_2^L (the first two in the left column) and orthonormal mid multiwavelet $\psi_{[2,1]}$ (the others)

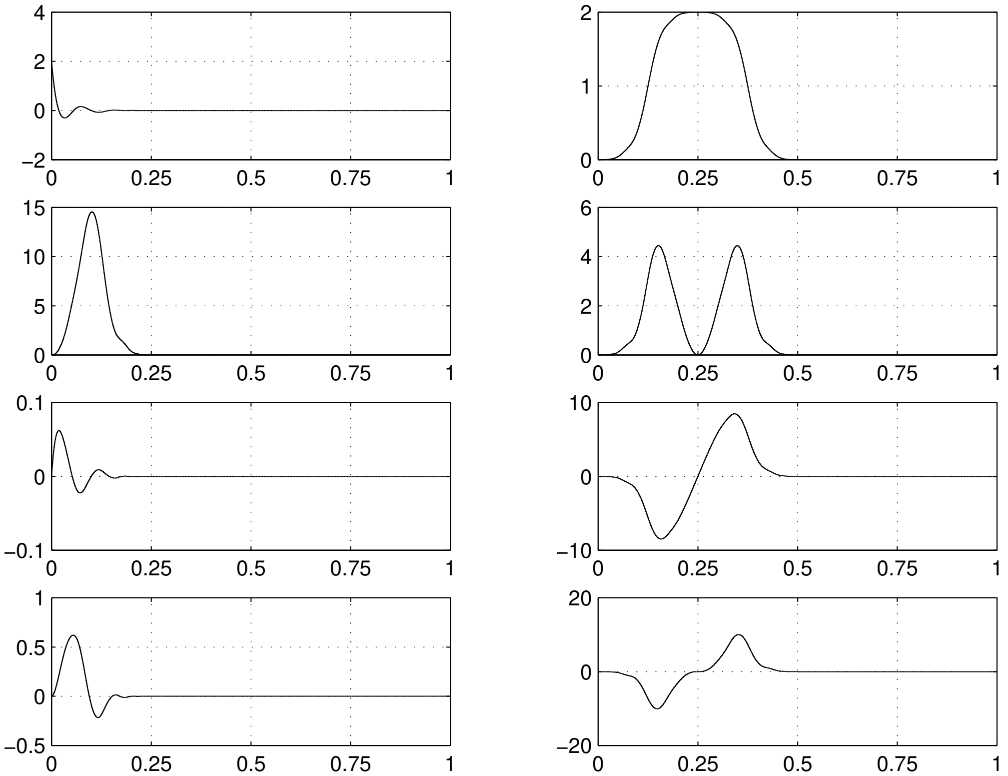

FIG. 4. Primal boundary multigenerator ϕ_2^L (the left column) and primal mid multigenerator $\phi_{[2,1]}$ (the right column)

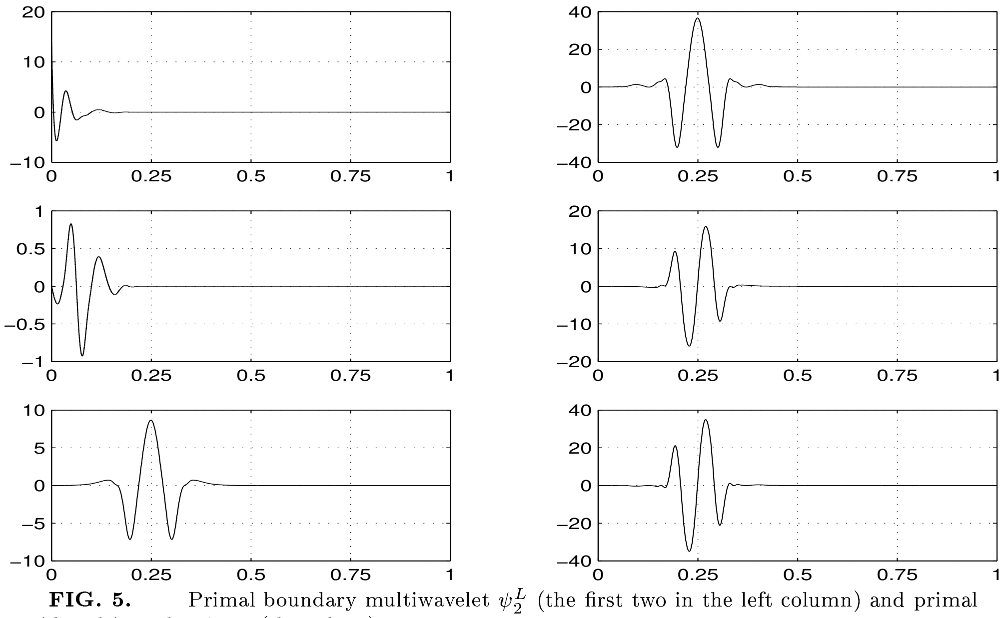

FIG. 5. Primal boundary multiwavelet ψ_2^L (the first two in the left column) and primal mid multiwavelet $\psi_{[2,1]}$ (the others)

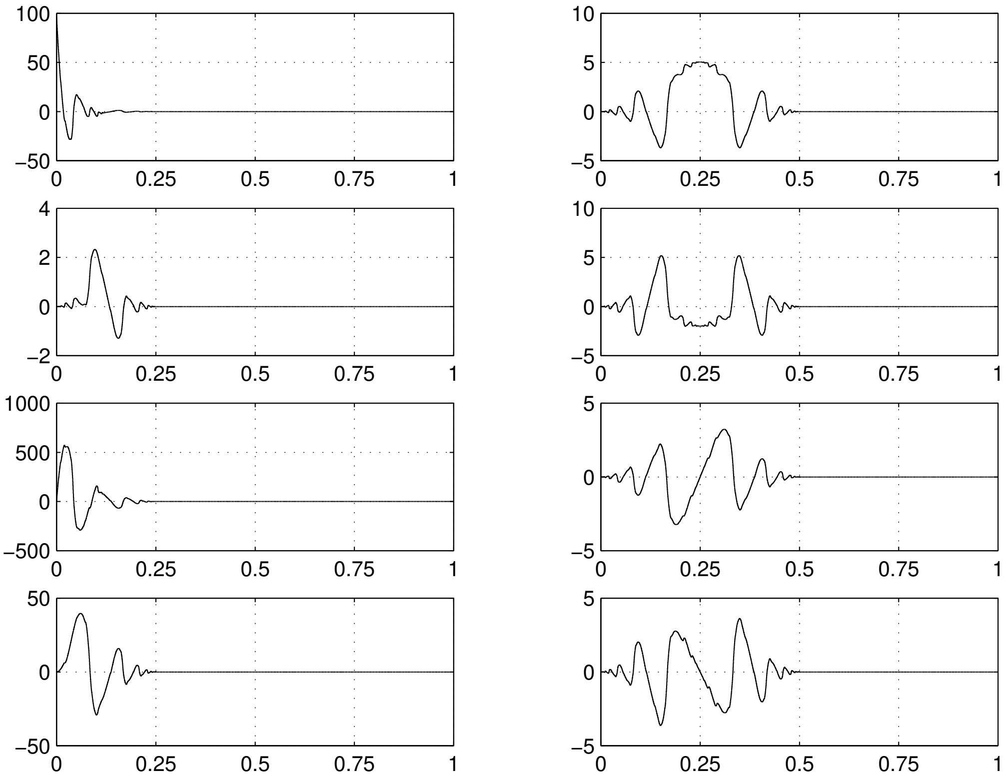

FIG. 6. Dual boundary multigenerator $\widetilde{\phi}_2^L$ (the left column) and dual mid multigenerator $\widetilde{\phi}_{[2,1]}$ (the right column)

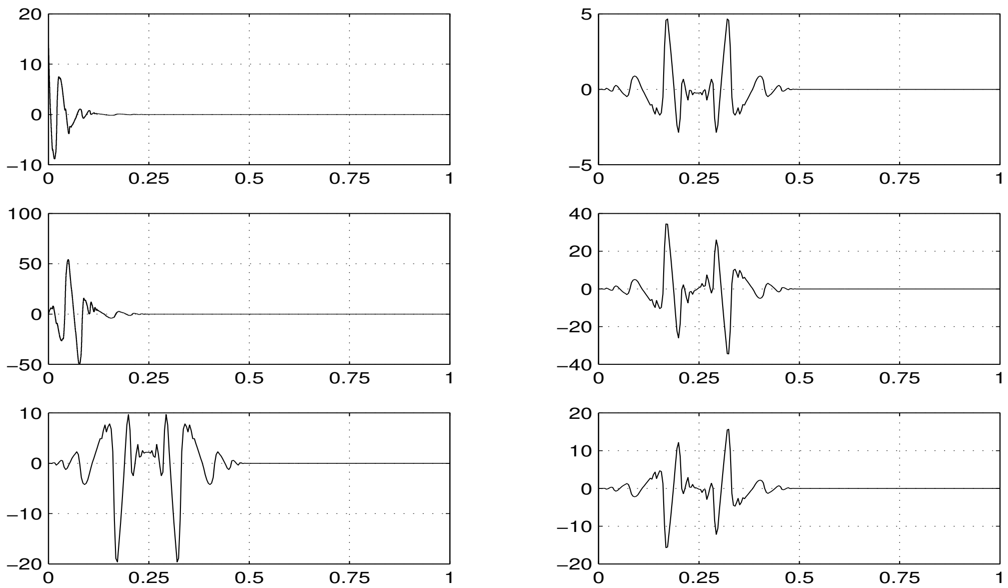

FIG. 7. Dual boundary multiwavelet $\widetilde{\psi}_2^L$ (the first two in the left column) and dual mid multiwavelet $\widetilde{\psi}_{[2,1]}$ (the others)

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