Hexagonal tight frame filter banks with idealized high-pass filters

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Abstract

This paper studies the construction of hexagonal tight wavelet frame filter banks which contain three "idealized" high-pass filters. These three high-pass filters are suitable spatial shifts and frequency modulations of the associated low-pass filter, and they are used by Simoncelli and Adelson in [37] for the design of hexagonal filter banks and by Riemenschneider and Shen in [30, 31] for the construction of 2-dimensional orthogonal filter banks. For an idealized low-pass filter, these three associated high-pass filters separate high frequency components of a hexagonal image in 3 different directions in the frequency domain. In this paper we show that an idealized tight frame, a frame generated by a tight frame filter bank containing the "idealized" high-pass filters, has at least 7 frame generators. We provide an approach to construct such tight frames based on the method by Lai and Stöckler in [24] to decompose non-negative trigonometric polynomials as the summations of the absolute squares of other trigonometric polynomials. In particular, we show that if the non-negative trigonometric polynomial associated with the low-pass filter p can be written as the summation of the absolute squares of other 3 or less than 3 trigonometric polynomials, then the idealized tight frame associated with p requires exact 7 frame generators. We also discuss the symmetry of frame filters. In addition, we present in this paper several examples, including that with the scaling functions to be the Courant element B_{111} and the box-spline B_{222}. The tight frames constructed in this paper will have potential applications to hexagonal image processing.

1. Introduction

Images are conventionally sampled at the points (notes) on a square or rectangular lattice (array) and therefore, traditional image processing is carried out on such a lattice. See a square lattice on the left of Fig. 1. The hexagonal lattice (on the right of Fig. 1) was proposed four decades ago in [28] as an alternative method for image sampling.

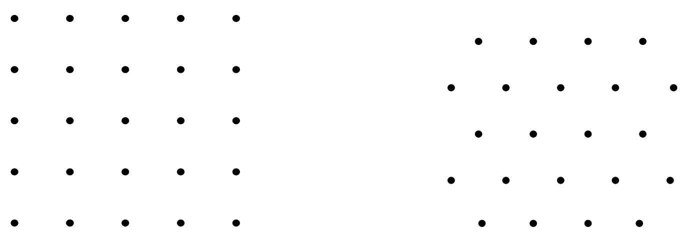

Figure 1: *Square lattice (left) and hexagonal lattice (right)*

For square images (sampled on a square lattice), it is assumed that each element (pixel) on the square lattice represents a (small) square cell with this element as its center. See the left part of Fig. 2 for an element a and the shadowed square it represents. For hexagonal images (hexagonally sampled images), each element on the hexagonal lattice represents a (small) hexagonal cell with that element as its center. An element b and the hexagonal cell (shadowed) it represents are shown in the right part of

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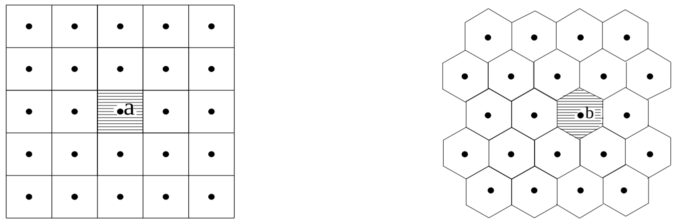

Figure 2: *Square tessellation (left) and hexagonal tessellation (right)*

Fig. 2. All the square cells form a square tessellation of the plane, while all the hexagonal cells form a hexagonal tessellation of the plane (see Fig. 2). A square image is displayed by square cells and a hexagonal image is displayed by hexagonal cells.

Compared with the square lattice, the hexagonal lattice has certain advantages. For example, the hexagonal lattice needs less number of sampling points to maintain equally high frequency information than a square lattice; a regular hexagonal lattice has 6-fold line (axial) symmetry while a square lattice has 4-fold line symmetry; the hexagonal structure has better consistent connectivity, and it is more visually pleasing to human eyes, see e.g. [1, 20, 27, 28, 38, 39, 40]. Therefore, the hexagonal lattice has been used in many fields such as medical imaging, computer vision, computer graphics, and geoscience, see e.g. [14, 25, 26, 5, 3, 34].

A regular hexagonal lattice can be described by two vectors with angles $\frac{\pi}{3}$ or $\frac{2\pi}{3}$. For example, vectors

$$\mathbf{v}_1 = (1, 0), \quad \mathbf{v}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$$

generate the unit regular hexagonal lattice $\mathcal{G}$:

$$\mathcal{G} = \{n_1\mathbf{v}_1 + n_2\mathbf{v}_2 : \quad (n_1, n_2) \in \mathbb{Z}^2\}. \tag{1.1}$$

The modulation (or dual) lattice of $\mathcal{G}$, denoted by $\mathcal{G}^*$, is the subset of $\mathbb{R}^2$ such that the dot product of any $\mathbf{g} \in \mathcal{G}, \mathbf{g}^* \in \mathcal{G}^*$ is an integer [37, 40]. One can show that $\mathcal{G}^*$ is a hexagonal lattice given by

$$\mathcal{G}^* = \{k_1\mathbf{u}_1 + k_2\mathbf{u}_2 : (k_1, k_2) \in \mathbb{Z}^2\}, \tag{1.2}$$

where

$$\mathbf{u}_1 = (1, \frac{\sqrt{3}}{3}), \mathbf{u}_2 = (0, \frac{2\sqrt{3}}{3}). \tag{1.3}$$

For a sequence $(H_\mathbf{g})_{\mathbf{g}\in\mathcal{G}}$ of real numbers associated with $\mathcal{G}$, let $H(\omega)$ denote the filter with its impulse response coefficients being $H_\mathbf{g}$ (here a factor $\frac{1}{4}$ is added for convenience)

$$H(\omega) = \frac{1}{4} \sum_{\mathbf{g}\in\mathcal{G}} H_\mathbf{g} e^{-i\mathbf{g}\cdot\omega}.$$

Here and below $\mathbf{x} \cdot \mathbf{y}$ denotes the dot product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. $H(\omega)$ is invariant under $2\pi G^*$, namely $H(\omega + 2\pi\mathbf{g}^*) = H(\omega)$ for any $\mathbf{g}^* \in G^*$. Such a filter $H(\omega)$ will be called a **hexagonal filter**.

For a hexagonal lattice, there are three types of interesting refinements: dyadic, $\sqrt{3}$ and $\sqrt{7}$ refinements (see [15, 4]). Here we consider the dyadic refinement. The refinement of the hexagonal lattice makes it possible that the multiresolution (multiscale) method can be used to process hexagonal images. The 4-subband (dyadic refinement) multiresolution processing of hexagonal images is studied in [37]. For a low-pass FIR (finite impulse response) hexagonal filter $P(\omega)$, [37] uses the high-pass filters (up to some modulations) given by

$$\begin{cases} F_1(\omega) = e^{-i\frac{1}{2}(\omega_1+\sqrt{3}\omega_2)} P(\omega + \pi\mathbf{u}_1), \\ F_2(\omega) = e^{i\omega_1} P(\omega + \pi(\mathbf{u}_1 + \mathbf{u}_2)), \\ F_3(\omega) = e^{i\frac{1}{2}(-\omega_1+\sqrt{3}\omega_2)} P(\omega + \pi\mathbf{u}_2), \end{cases} \tag{1.4}$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are the vectors given in (1.3). The choice of F_1, F_2, F_3 is based on the observation that if $P(\omega)$ is an ideal low-pass filter, namely, restricted to the (bigger) hexagon in Fig. 3, $|P(\omega)|$ is the characteristic function of the smaller hexagon (shadowed region) in the first picture of Fig. 3, then $|F_1|, |F_2|, |F_3|$ are respectively the characteristic functions of the shadowed regions in the second to fourth pictures of Fig. 3. Therefore F_1, F_2, F_3 are ideal high-pass filters to separate high frequency components of an image in 3 different directions in the frequency domain. Here we call F_1, F_2, F_3 **idealized high-pass filters** associated with P.

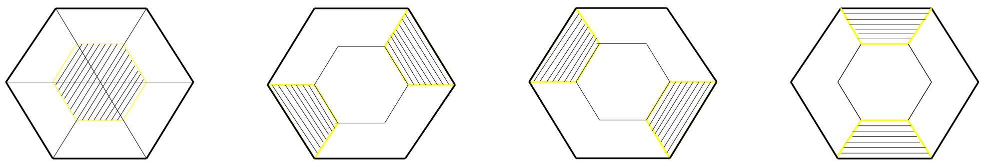

Figure 3: *Idealized partition of frequency domain: idealized low-pass filter (1st from left), idealized high-pass filters (2nd, 3rd, 4th from left)*

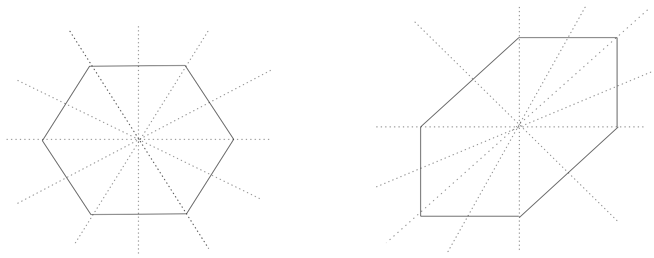

Figure 4: *Lines (directions) of symmetry in hexagonal grid (left) and their counterparts in 3-directional grid of* $\mathbb{Z}^2$ *(right)*

To design filter banks for hexagonal data, [37] starts with a low-pass FIR filter P with 6-fold line (axial) symmetry, namely its impulse response coefficients $P_{\mathbf{g}}$ are symmetric around 6 lines shown on the left of Fig. 4. $P(\omega)$ is given by some parameters. Then with F_1, F_2, F_3 defined by (1.4), [37] chooses the parameters for $P(\omega)$ by minimizing a filter bank error and intra-band aliasing error function. The filter bank $\{P, F_1, F_2, F_3\}$ designed in this way is not a perfect reconstruction filter bank. Actually, we can show that for an FIR filter P with 6-fold line symmetry, there is no FIR filter bank biorthogonal to $\{P, F_1, F_2, F_3\}$, see Remark 1 below.

The design of filter banks for hexagonal data is also discussed in [35]. The construction of biorthogonal FIR hexagonal filter banks is fully investigated in [12] and a few biorthogonal filter banks are constructed there. A structure of orthogonal and biorthogonal FIR hexagonal filter banks is obtained [1, 2]. In [23], by transforming the hexagonal lattice into the square lattice of $\mathbb{Z}^2$, we show rigorously mathematically that the filter banks considered in [1, 2] have 3-fold line symmetry. [23] also obtains a new structure of biorthogonal FIR hexagonal filter banks with 3-fold line symmetry and 3-fold rotational symmetry. Furthermore, [23] presents some orthogonal and biorthogonal FIR hexagonal filters with scaling functions having optimal Sobolev smoothness.

When we construct orthogonal and bio-orthogonal hexagonal filters, we encounter such difficulties that in order to have smooth scaling functions/waveletes, one needs to use filters with many non-zero impulse response coefficients. One the other hand, since there are nice refinable functions ϕ such as some box-splines along the lattice $\mathcal{G}$ which have 6-fold line symmetry, small supports and high smoothness orders, it is desirable to have such functions ϕ as primal scaling functions. However, the corresponding dual scaling functions $\tilde{\phi}$ must have large supports if $\tilde{\phi}$ have reasonable smoothness orders. Furthermore, even when the desired $\phi, \tilde{\phi}$ are constructed, it is difficult to construct the associated high-pass filters. In addition, as mentioned above, for filters P of these nice scaling functions, there are no FIR filter banks $\{\tilde{P}, \tilde{Q}^{(1)}, \tilde{Q}^{(2)}, \tilde{Q}^{(3)}\}$ biorthogonal to $\{P, F_1, F_2, F_3\}$ with F_1, F_2, F_3 given in (1.4). Because of these reasons, we consider other type of filter banks.

The study of frames is of an recently active research field (see e.g. [6]-[11], [13, 16, 18, 21, 22, 29, 32, 33, 36]). The frame theory provides the flexibility for the construction of filter banks. We call a hexagonal filter bank $\{P, Q^{(1)}, \cdots, Q^{(L)}\}$ a **hexagonal tight frame filter bank** if

$$\begin{cases} |P(\omega)|^2 + \sum_{\ell=1}^L |Q^{(\ell)}(\omega)|^2 = 1, \\ P(\omega)\overline{P(\omega+\tilde{\eta}_k)} + \sum_{\ell=1}^L Q^{(\ell)}(\omega)\overline{Q^{(\ell)}(\omega+\tilde{\eta}_k)} = 0, 1 \le k \le 3, \omega \in \mathbb{R}^2, \end{cases} \tag{1.5}$$

where

$$\tilde{\eta}_0 = (0,0),\ \tilde{\eta}_1 = (\pi, \pi\sqrt{3}),\ \tilde{\eta}_2 = (\pi, \frac{\pi\sqrt{3}}{3}),\ \tilde{\eta}_3 = (0, \frac{2\pi\sqrt{3}}{3}). \tag{1.6}$$

Observe that $\tilde{\eta}_1 = \pi(\mathbf{u}_1 + \mathbf{u}_2), \tilde{\eta}_2 = \pi\mathbf{u}_1, \tilde{\eta}_3 = \pi\mathbf{u}_2$, where $\mathbf{u}_1, \mathbf{u}_2$ are the vectors defined in (1.3) generating $\mathcal{G}^*$. If the scaling function Φ associated with the low-pass filter P has certain smoothness, then (1.5) implies that $\{\Psi^{(\ell)}_{j,\mathbf{g}} : 1 \le \ell \le L, j \in \mathbb{Z}, \mathbf{g} \in \mathcal{G}\}$ is a hexagonal tight frame (refer to [32]):

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}, \mathbf{g}\in\mathcal{G}} |\langle f, \Psi^{(\ell)}_{j,\mathbf{g}}\rangle_{L^2(\mathbb{R}^2)}|^2 = A\|f\|^2,\ \forall f \in L^2(\mathbb{R}^2),$$

for some constant $A \neq 0$, where $\Psi^{(\ell)}, 1 \le \ell \le L$ are defined by

$$\widehat{\Psi}^{(\ell)}(\omega) = Q^{(\ell)}(\frac{\omega}{2})\widehat{\Phi}(\frac{\omega}{2}),$$

and $\Psi^{(\ell)}_{j,\mathbf{g}}(\mathbf{x}) = 2^j\Psi^{(\ell)}(2^j\mathbf{x} - \mathbf{g}), \mathbf{x} \in \mathbb{R}^2$. $\Psi^{(1)}, \cdots, \Psi^{(L)}$ are called hexagonal tight frame generators (framelets). The tight frame filter bank $\{P, Q^{(1)}, \cdots, Q^{(L)}\}$ can be used as the analysis and synthesis filter banks for decomposition/reconstruction algorithms for multiresolution hexagonal image processing. A hexagonal tight frame filter bank $\{P, Q^{(1)}, \cdots, Q^{(L)}\}$ is called an **idealized hexagonal tight frame filter bank** if $Q^{(1)}, Q^{(2)}, Q^{(3)}$ are the high-pass filters F_1, F_2, F_3 defined in (1.4). The objective of this paper is about the construction of idealized hexagonal tight frame filter banks. Since their first three high-pass frame filters $Q^{(j)}, j = 1, 2, 3$ are F_j which can separate high frequency components of a hexagonal image in three different directions, the tight frames considered in this paper will have potential applications to hexagonal image processing.

The rest of this paper is organized as follows. In Section 2, we discuss briefly about how to transform the construction of filters along the hexagonal lattice into that along the square lattice of $\mathbb{Z}^2$. In Section 3, we study the construction of the idealized tight frame filter bank for a given low-pass filter with certain symmetry. In the last section of this paper, Section 4, we present a few examples.

2. Transforming hexagonal lattice to square lattice

Since most multiresolution analysis theory and algorithms for image processing are developed along square lattice $\mathbb{Z}^2$, to design hexagonal filter banks, here we will transform the hexagonal lattice into the square lattice $\mathbb{Z}^2$ so that we can use the well-developed integer-shift multiresolution analysis theory and methods. To this regard, as in [23], we will use the following matrix for the transformation

$$U = \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{bmatrix}. \tag{2.1}$$

To connect each point on the unit regular hexagonal lattice $\mathcal{G}$ to its nearest 6 neighbors, one has a grid, which is called the 3-directional grid (mesh), see the left figure of Fig. 5. With matrix U, this grid is transformed into the 3-directional grid with notes of $\mathbb{Z}^2$ shown on the right of Fig. 5. In the following

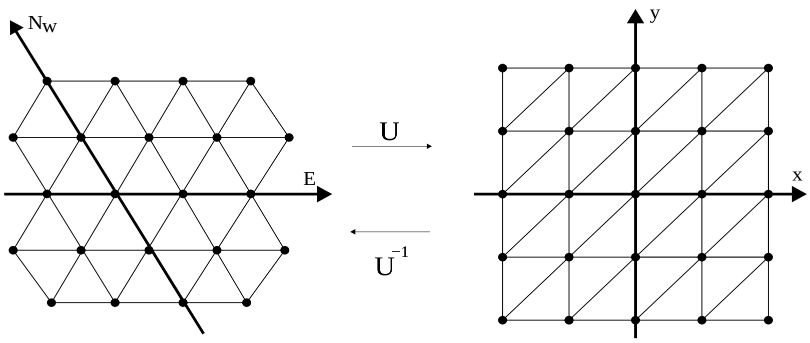

Figure 5: *Hexagonal grid (left) and 3-directional grid of $\mathbb{Z}^2$ (right)*

we will call the grid on the left of Fig. 5 the **hexagonal grid** to distinguish it from the 3-directional grid of $\mathbb{Z}^2$ on the right of Fig. 5.

By the transformation with U, a hexagonal filter becomes a square filter. More precisely, for a hexagonal filter $H(\omega) = \frac{1}{4}\sum_{\mathbf{g}\in\mathcal{G}} H_{\mathbf{g}} e^{-i\mathbf{g}\cdot\omega}$ with its impulse response coefficients $H_{\mathbf{g}}$, by the transformation with the matrix U, we have a corresponding filter $h(\omega) = \frac{1}{4}\sum_{\mathbf{k}\in\mathbb{Z}^2} h_{\mathbf{k}} e^{-i\mathbf{k}\cdot\omega}$ for the square lattice data with impulse response coefficients $h_{\mathbf{k}} = H_{U^{-1}\mathbf{k}}$. Conversely, for a square filter $h(\omega) = \frac{1}{4}\sum_{\mathbf{k}\in\mathbb{Z}^2} h_{\mathbf{k}} e^{-i\mathbf{k}\cdot\omega}$, by the transformation with U^{-1}, we have a hexagonal filter $H(\omega) = \frac{1}{4}\sum_{\mathbf{g}\in\mathcal{G}} h_{U\mathbf{g}} e^{-i\mathbf{g}\cdot\omega}$. In the frequency domain, the relationship between $H(\omega)$ and $h(\omega)$ (with $h_{\mathbf{k}} = H_{U^{-1}\mathbf{k}}$) is given by

$$H(\omega) = h(U^{-T}\omega). \tag{2.2}$$

Here and below U^{-T} denotes $(U^{-1})^T$, the transpose of U^{-1}.

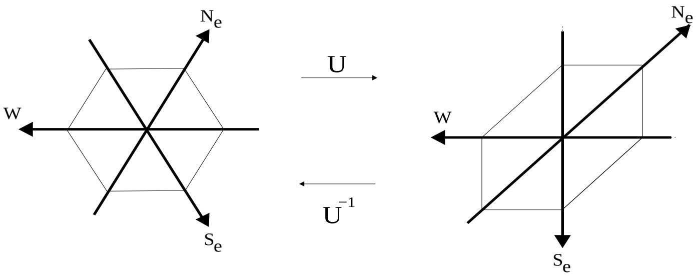

Figure 6: *Symmetry lines (directions) N_e, W, S_e in hexagonal grid (left) and their counterparts in 3-directional grid of $\mathbb{Z}^2$ (right)*

Since the hexagonal lattice/grid has 6-fold line symmetry, it is desirable that the low-pass filter has 6-fold line symmetry or at least has 3-fold line symmetry (namely, it is symmetry around 3 lines shown on the left of Fig. 6), and that the associated high-pass filters have certain rotation invariant property. The matrix U also transforms the symmetry structure of the hexagonal grid and the symmetry of filters. See symmetry lines in the hexagonal grid and their counterparts in the 3-directional grid of $\mathbb{Z}^2$ in Fig. 4. More specifically, the right figure in Fig. 6 shows three symmetry lines N_e, W, S_e in the 3-directional grid of $\mathbb{Z}^2$ corresponding to the symmetry lines in the hexagonal grid in the left figure. In the 3-directional grid of $\mathbb{Z}^2$, the symmetry of filters around the line W (the x-axis) and the line S_e (the y-axis) should be understood in a slight different way. They can be described as follows. We say a square filter $p(\omega) = \frac{1}{4}\sum_{\mathbf{k}\in\mathbb{Z}^2} p_{\mathbf{k}} e^{-i\mathbf{k}\cdot\omega}$ is symmetric around the line W and the line S_e resp. if it satisfies $p_{W\mathbf{k}} = p_{\mathbf{k}}$ and $p_{S_e\mathbf{k}} = p_{\mathbf{k}}$, where

$$W = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad S_e = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{2.3}$$

As an example, Fig. 7 shows the symmetry of the impulse response coefficients $P_{\mathbf{g}}$ of a hexagonal filter $P(\omega) = \frac{1}{4}\sum_{g\in\mathcal{G}} P_{\mathbf{g}} e^{-i\mathbf{g}\cdot\omega}$ around S_e in the hexagonal grid and the symmetry of the corresponding impulse response coefficients $p_{\mathbf{k}}$ (with $p_{\mathbf{k}} = P_{U^{-1}\mathbf{k}}$) in the 3-directional grid of $\mathbb{Z}^2$. The symmetry of p around N_e (the line $y = x$) in the 3-directional grid of $\mathbb{Z}^2$ has the ordinary meaning: it satisfies $p_{N_e\mathbf{k}} = p_{\mathbf{k}}$, where

$$N_e = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.4}$$

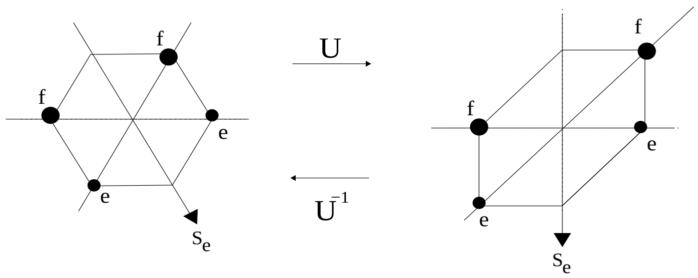

Figure 7: *Symmetry of impulse response coefficients about line S_e in hexagonal grid (left) and in 3-directional grid of $\mathbb{Z}^2$ (right)*

The matrix U transforms the scaling functions, wavelets and frame generators along the hexagonal grid to that along the 3-directional grid of $\mathbb{Z}^2$. For example, assume that Φ is refinable along the hexagonal grid with refinement mask $(P_{\mathbf{g}})_{\mathbf{g}\in\mathcal{G}}$, namely, it satisfies $\Phi(\mathbf{x}) = \sum_{\mathbf{g}\in\mathcal{G}} P_{\mathbf{g}}\Phi(2\mathbf{x}-\mathbf{g}), \mathbf{x}\in\mathbb{R}^2$. Let ϕ be the function defined by $\phi(\mathbf{x}) = \Phi(U^{-1}\mathbf{x})$. Then ϕ is refinable with mask $(p_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ corresponding to $(P_{\mathbf{g}})_{\mathbf{g}\in\mathcal{G}}$ (with $p_{\mathbf{k}} = P_{U^{-1}\mathbf{k}}$). Conversely, if ϕ is refinable with $(p_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$, then $\Phi(\mathbf{x}) = \phi(U\mathbf{x})$ is refinable with $(p_{U\mathbf{g}})_{\mathbf{g}\in\mathcal{G}}$.

By the transformation with U, the tight frame condition (1.5) becomes the well-known condition called the unitary extension principle:

$$\begin{cases} |p(\omega)|^2 + \sum_{\ell=1}^{L} |q^{(\ell)}(\omega)|^2 = 1, \\ p(\omega)\overline{p(\omega+\eta_k)} + \sum_{\ell=1}^{L} q^{(\ell)}(\omega)\overline{q^{(\ell)}(\omega+\eta_k)} = 0, 1\le k\le 3, \omega\in\mathbb{R}^2, \end{cases} \tag{2.5}$$

where $p, q^{(\ell)}$ are the square filters corresponding to $P, Q^{(\ell)}, 1\le\ell\le L$, namely, $p(\omega) = P(U^T\omega), q^{(\ell)}(\omega) = Q^{(\ell)}(U^T\omega)$, and $\eta_0, \cdots, \eta_3$ are

$$\eta_0 = (0,0), \quad \eta_1 = (\pi,\pi), \quad \eta_2 = (\pi,0), \quad \eta_3 = (0,\pi). \tag{2.6}$$

For the scaling function Φ and frame generators $\Psi^{(\ell)}$ along the hexagonal lattice $\mathcal{G}$, let $\phi, \psi^{(\ell)}$ be the corresponding scaling function, frame generators along $\mathbb{Z}^2$, namely, $\phi(\omega) = \Phi(U^{-1}\omega), \psi^{(\ell)}(\omega) = \Psi^{(\ell)}(U^{-1}\omega)$, then

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^2} |\langle f, \psi^{(\ell)}_{j,\mathbf{k}}\rangle_{L^2(\mathbb{R}^2)}|^2 = A_1\|f\|^2, \quad \forall f\in L^2(\mathbb{R}^2),$$

for some $A_1 \neq 0$, where

$$\psi^{(\ell)}_{j,\mathbf{k}}(\mathbf{x}) = 2^j\psi^{(\ell)}(2^j\mathbf{x}-\mathbf{k}), \quad j\in\mathbb{Z}, \mathbf{k}\in\mathbb{Z}^2.$$

Therefore $\psi^{(1)}, \cdots, \psi^{(L)}$ are the traditional frame generators for square data processing. Clearly, $\phi, \psi^{(\ell)}$ have the conventional relationship:

$$\widehat{\psi^{(\ell)}}(\omega) = q^{(\ell)}(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}), \quad 1\le\ell\le L.$$

Next, we consider the square filters corresponding to the idealized high-pass filters F_1, F_2, F_3 defined (1.4). We have the following proposition.

Proposition 1 *Suppose P is a low-pass hexagonal filter and F_1, F_2, F_3 are the hexagonal filters defined by (1.4). Let $p, q^{(1)}, q^{(2)}, q^{(3)}$ be the square filters corresponding to P, F_1, F_2, F_3. Then*

$$\begin{cases} q^{(1)}(\omega) = e^{-i(\omega_1+\omega_2)}p(\omega_1+\pi, \omega_2), \\ q^{(2)}(\omega) = e^{i\omega_1}p(\omega_1+\pi, \omega_2+\pi), \\ q^{(3)}(\omega) = e^{i\omega_2}p(\omega_1, \omega_2+\pi). \end{cases} \tag{2.7}$$

Proof. We give the proof of the formula for $q^{(1)}$. The proof for others is similar and it is omitted here. For $q^{(1)}$, we have

$$\begin{aligned} q^{(1)}(\omega) = F_1(U^T\omega) &= e^{-\frac{i}{2}(\omega_1+\sqrt{3}\frac{\omega_1+2\omega_2}{\sqrt{3}})}P(U^T\omega+\pi\mathbf{u}_1) \\ &= e^{-i(\omega_1+\omega_2)}p(U^{-T}(U^T\omega+\pi\mathbf{u}_1)) = e^{-i(\omega_1+\omega_2)}p(\omega+(\pi,0)).\P \end{aligned}$$

The filters $q^{(1)}, q^{(2)}, q^{(3)}$ (up to some modulations) are the orthogonal high-pass filters used in [31] for the construction of 2-D orthogonal high-pass filters. It is shown in [31] that if p is a (square) quadrature mirror filter (QMF) and satisfies $\overline{p(\omega)} = p(\omega)$, then $\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$ is an orthogonal filter bank. [19] obtains conditions on p such that the functions $\psi^{(1)}, \psi^{(2)}, \psi^{(3)}$ defined by $q^{(1)}, q^{(2)}, q^{(3)}$ generate Riesz bases of $L^2(\mathbb{R}^2)$. Here we call $q^{(1)}, q^{(2)}$ and $q^{(3)}$ the idealized high-pass filters associated with p.

For a filter bank $\{p, q^{(1)}, q^{(2)}, \cdots, q^{(L)}\}$, if it satisfies (2.5), then we call it a **tight frame filter bank**. If in addition, $q^{(1)}, q^{(2)}, q^{(3)}$ are defined by (2.7), then we also call $\{p, q^{(1)}, q^{(2)}, \cdots, q^{(L)}\}$ an **idealized tight frame filter bank**. For a given p and its associated $q^{(1)}, q^{(2)}, q^{(3)}$ defined by (2.7), in case we construct $q^{(4)}, \cdots, q^{(L)}$ such that (2.5) is satisfied, then we have an idealized hexagonal tight frame filter bank $\{P(\omega), Q^{(1)}(\omega), \cdots, Q^{(L)}(\omega)\}$ with $P_\mathbf{g} = p_{U\mathbf{g}}, Q^{(\ell)}_\mathbf{g} = q^{(\ell)}_{U\mathbf{g}}, 1 \le \ell \le L$.

3. Idealized tight frame filter banks

In this section we discuss the construction of idealized frame (square) filter banks for given low-pass FIR filters with certain symmetry. A low-pass filter p is said to have **3-fold line symmetry** if its impulse response coefficients $p_\mathbf{k}$ satisfy

$$p_{N_e\mathbf{k}} = p_\mathbf{k},\ p_{W\mathbf{k}} = p_\mathbf{k},\ p_{S_e\mathbf{k}} = p_\mathbf{k}, \quad \mathbf{k} \in \mathbb{Z}^2, \tag{3.1}$$

or equivalently $p(\omega)$ satisfies

$$p(\omega) = p(N_e\omega) = p(W^{-T}\omega) = p({S_e}^{-T}\omega), \quad \omega \in \mathbb{R}^2, \tag{3.2}$$

where N_e, W, S_e are the matrices given in (2.4) and (2.3). Square filters $q^{(2)}(\omega)$ and $q^{(3)}(\omega)$ are said to be the "$\frac{2}{3}\pi$" and "$\frac{4}{3}\pi$" **"rotations"** of a filter $q^{(1)}(\omega)$ resp. if their impulse response coefficients satisfy

$$q^{(2)}_\mathbf{k} = q^{(1)}_{R_1\mathbf{k}}, \quad q^{(3)}_\mathbf{k} = q^{(1)}_{R_2\mathbf{k}}, \tag{3.3}$$

where

$$R_1 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}. \tag{3.4}$$

If $Q^{(1)}(\omega), Q^{(2)}(\omega), Q^{(3)}(\omega)$ are the hexagonal filters corresponding to $q^{(1)}(\omega), q^{(2)}(\omega)$, $q^{(3)}(\omega)$ with impulse response coefficients to be $q^{(1)}_{U\mathbf{g}}, q^{(2)}_{U\mathbf{g}}$, $q^{(3)}_{U\mathbf{g}}$ resp., then $Q^{(2)}(\omega)$ and $Q^{(3)}(\omega)$ are indeed the $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$ rotations of $Q^{(1)}(\omega)$, resp. Fig. 8 displays the impulse response coefficients of a square filter (on the left) and its "$\frac{2\pi}{3}$ rotation" (in the middle) and "$\frac{4\pi}{3}$ rotations" (on the right). Clearly, (3.3) is equivalent to that

$$q^{(2)}(\omega) = q^{(1)}(R_1^{-T}\omega),\ q^{(3)}(\omega) = q^{(1)}(R_2^{-T}\omega). \tag{3.5}$$

Observe that $R_1 = WN_e$, $R_2 = S_eN_e$. Thus if p has 3-fold line symmetry, namely, it satisfies (3.1), then $p_\mathbf{k} = p_{R_1\mathbf{k}}$, $p_\mathbf{k} = p_{R_2\mathbf{k}}$, that is, p is invariant under "$\frac{2}{3}\pi$" and "$\frac{4}{3}\pi$" "rotations". In the next proposition we show that if p has 3-fold line symmetry, then its associated idealized high-pass filters $q^{(1)}, q^{(2)}$ and $q^{(3)}$ possess "rotation" invariant property.

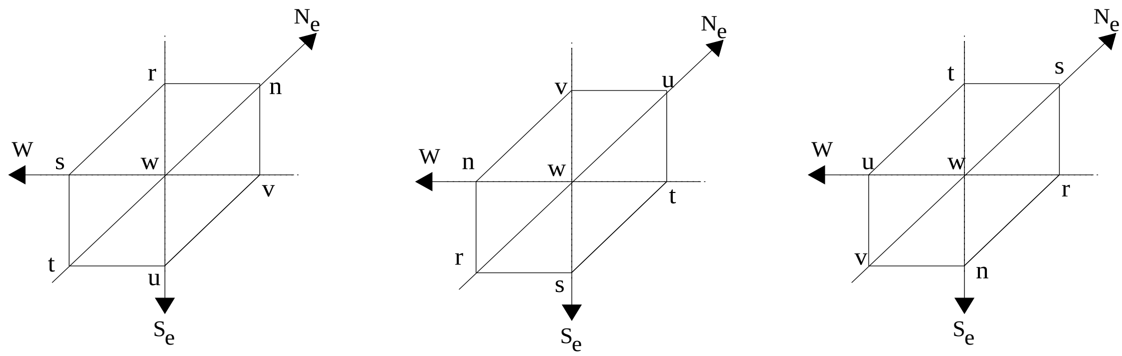

Figure 8: *"$\frac{2\pi}{3}$ rotation" (middle) and "$\frac{4\pi}{3}$ rotation" (right) of filter in left picture*

Proposition 2 *Suppose $q^{(1)}, q^{(2)}$, and $q^{(3)}$ are defined by (2.7). If p has 3-fold line symmetry, then $q^{(2)}$ and $q^{(3)}$ are the "$\frac{2}{3}\pi$" and "$\frac{4}{3}\pi$" "rotations" of $q^{(1)}$.*

Proof. We need to show that (3.5) holds. Indeed, we have

$$\begin{aligned} q^{(1)}(R_1^{-T}\omega) &= q^{(1)}(\omega_2, -\omega_1-\omega_2) = e^{i\omega_1}p(\omega_2+\pi, -\omega_1-\omega_2) \\ &= e^{i\omega_1}p(W^{-T}\begin{bmatrix} \omega_2+\pi \\ -\omega_1-\omega_2 \end{bmatrix}) = e^{i\omega_1}p(\omega_2+\pi, \omega_1-\pi) \\ &= e^{i\omega_1}p(\omega_2+\pi, \omega_1+\pi) = e^{i\omega_1}p(\omega_1+\pi, \omega_2+\pi) \\ &= q^{(2)}(\omega), \end{aligned}$$

and

$$\begin{aligned} q^{(1)}(R_2^{-T}\omega) &= q^{(1)}(-\omega_1-\omega_2, \omega_1) = e^{i\omega_2}p(\pi-\omega_1-\omega_2, \omega_1) \\ &= e^{i\omega_2}p({S_e}^{-T}\begin{bmatrix} \pi-\omega_1-\omega_2 \\ \omega_1 \end{bmatrix}) = e^{i\omega_2}p(\omega_2-\pi, \omega_1) \\ &= e^{i\omega_2}p(\omega_2+\pi, \omega_1) = e^{i\omega_2}p(\omega_1, \omega_2+\pi) \\ &= q^{(3)}(\omega).\P \end{aligned}$$

In the following, we will consider low-pass filters $p(\omega)$ that satisfy

$$\overline{p(\omega)} = p(\omega), \quad \omega \in \mathbb{R}^2. \tag{3.6}$$

First we have the following lemma which is the modified version of Lemma 2.12 in [31] for 2-D filters. We will give the proof of this lemma for the self-containing purpose. In the following we denote $q^{(0)}(\omega) = p(\omega)$. Write $q^{(\ell)}(\omega), 0 \le \ell \le 3$, as

$$q^{(\ell)}(\omega) = \frac{1}{2}(q_{ee}^{(\ell)}(2\omega) + q_{oo}^{(\ell)}(2\omega)z_1^{-1}z_2^{-1} + q_{oe}^{(\ell)}(2\omega)z_1 + q_{eo}^{(\ell)}(2\omega)z_2), \tag{3.7}$$

where $q_{ee}^{(\ell)}, q_{oo}^{(\ell)}, q_{oe}^{(\ell)}, q_{eo}^{(\ell)}$ are trigonometric polynomials, and

$$z_1 = e^{-i\omega_1}, \quad z_2 = e^{-i\omega_2}. \tag{3.8}$$

Lemma 1 *Suppose $q^{(1)}, q^{(2)}), q^{(3)}$ are defined by (2.7). If p satisfies (3.6), then*

$$p(\omega)\overline{p(\omega+\eta_k)} + \sum_{\ell=1}^{3} q^{(\ell)}(\omega)\overline{q^{(\ell)}(\omega+\eta_k)} = \begin{cases} \sum_{j=0}^{3}|p(\omega+\eta_j)|^2, & \text{if } k=0, \\ 0, & \text{if } k=1,2,3, \end{cases} \tag{3.9}$$

where $\eta_k, k = 0, \cdots, 3$ are defined by (2.6).

Proof. With $q^{(0)} = p$, write $q^{(\ell)}, 0 \le \ell \le 3$ as in (3.7). Then

$$\left[q^{(\ell)}(\omega + \eta_j)\right]_{0\le j,\ell \le 3} = U(\omega) W_0(2\omega),$$

where

$$U(\omega) = \frac{1}{2}\begin{bmatrix} 1 & z_1^{-1}z_2^{-1} & z_1 & z_2 \\ 1 & z_1^{-1}z_2^{-1} & -z_1 & -z_2 \\ 1 & -z_1^{-1}z_2^{-1} & -z_1 & z_2 \\ 1 & -z_1^{-1}z_2^{-1} & z_1 & -z_2 \end{bmatrix}, \; W_0(\omega) = \begin{bmatrix} p_{ee}(\omega) & q_{ee}^{(1)}(\omega) & \cdots & q_{ee}^{(3)}(\omega) \\ p_{oo}(\omega) & q_{oo}^{(1)}(\omega) & \cdots & q_{oo}^{(3)}(\omega) \\ p_{oe}(\omega) & q_{oe}^{(1)}(\omega) & \cdots & q_{oe}^{(3)}(\omega) \\ p_{eo}(\omega) & q_{eo}^{(1)}(\omega) & \cdots & q_{eo}^{(3)}(\omega) \end{bmatrix}. \tag{3.10}$$

Clearly, $U(\omega)U(\omega)^* = I_4$. From (2.7), we have

$$\begin{aligned} &q_{ee}^{(1)}(\omega) = -p_{oo}(\omega), \quad q_{oo}^{(1)}(\omega) = z_1 z_2 p_{ee}(\omega), \quad q_{oe}^{(1)}(\omega) = z_2 p_{eo}(\omega), \quad q_{eo}^{(1)}(\omega) = -z_1 p_{oe}(\omega), \\ &q_{ee}^{(2)}(\omega) = -p_{oe}(\omega), \quad q_{oo}^{(2)}(\omega) = -z_2 p_{eo}(\omega), \quad q_{oe}^{(2)}(\omega) = z_1^{-1} p_{ee}(\omega), \quad q_{eo}^{(2)}(\omega) = z_1^{-1}z_2^{-1} p_{oo}(\omega), \\ &q_{ee}^{(3)}(\omega) = -p_{eo}(\omega), \quad q_{oo}^{(3)}(\omega) = z_1 p_{oe}(\omega), \quad q_{oe}^{(3)}(\omega) = -z_1^{-1}z_2^{-1} p_{oo}(\omega), \quad q_{eo}^{(3)}(\omega) = z_2^{-1} p_{ee}(\omega), \end{aligned}$$

while the condition $\overline{p(\omega)} = p(\omega)$ implies that

$$\overline{p_{ee}(\omega)} = p_{ee}(\omega), \quad \overline{p_{oo}(\omega)} = z_1^{-1}z_2^{-1} p_{oo}(\omega), \quad \overline{p_{oe}(\omega)} = z_1 p_{oe}(\omega), \quad \overline{p_{eo}(\omega)} = z_2 p_{eo}(\omega).$$

Thus

$$\begin{aligned} &W_0(\omega)W_0(\omega)^* \\ &= \begin{bmatrix} p_{ee} & -p_{oo} & -p_{oe} & -p_{eo} \\ p_{oo} & z_1 z_2 p_{ee} & -z_2 p_{eo} & z_1 p_{oe} \\ p_{oe} & z_2 p_{eo} & z_1^{-1} p_{ee} & -z_1^{-1}z_2^{-1} p_{oo} \\ p_{eo} & -z_1 p_{oe} & z_1^{-1}z_2^{-1} p_{oo} & z_2^{-1} p_{ee} \end{bmatrix} \begin{bmatrix} p_{ee} & z_1^{-1}z_2^{-1} p_{oo} & z_1 p_{oe} & z_2 p_{eo} \\ -z_1^{-1}z_2^{-1} p_{oo} & z_1^{-1}z_2^{-1} p_{ee} & p_{eo} & -p_{oe} \\ -z_1 p_{oe} & -p_{eo} & z_1 p_{ee} & p_{oo} \\ -z_2 p_{eo} & p_{oe} & -p_{oo} & z_2 p_{ee} \end{bmatrix} \\ &= (|p_{ee}(\omega)|^2 + |p_{oo}(\omega)|^2 + |p_{oe}(\omega)|^2 + |p_{eo}(\omega)|^2) I_4 \\ &= (\sum_{j=0}^{3} |p(\frac{\omega}{2} + \eta_j)|^2) I_4. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left[q^{(\ell)}(\omega + \eta_j)\right]_{0\le j,\ell\le 3} (\left[q^{(\ell)}(\omega + \eta_j)\right]_{0\le j,\ell\le 3})^* = U(\omega)W_0(2\omega)W_0(2\omega)^* U(\omega)^* \\ &= (\sum_{j=0}^{3} |p(\omega + \eta_j)|^2) I_4, \ \omega \in \mathbb{R}^2, \end{aligned}$$

which leads to (3.9). ¶

Remark 1 *Suppose that an FIR filter p satisfies (3.6). If p is not a QMF (up to a constant), that is $\sum_{j=0}^{3} |p(\omega + \eta_j)|^2 \not\equiv const$, then Lemma 1 implies that $\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$ with $q^{(1)}, q^{(2)}, q^{(3)}$ defined by (2.7) does not have an FIR biorthogonal dual filter bank. This is based on the fact that if it had an FIR biorthogonal dual filter bank, then the determinant of*

$$V(\omega) = \left[q^{(\ell)}(\omega + \eta_j)\right]_{0\le j,\ell\le 3}$$

was $c_0 z_1^{n_1} z_2^{n_2}$ for some $c_0 \ne 0, n_1, n_2 \in \mathbb{Z}$, and hence $|det(V(\omega))| \equiv const$. However the proof of Lemma 1 shows that $|det(V(\omega))| = (\sum_{j=0}^{3} |p(\omega + \eta_j)|^2)^2 \not\equiv const$, a contradiction.

One can show that if an FIR filter p has 6-fold line symmetry (equivalently, p has 3-fold line symmetry and $\overline{p(\omega)} = p(\omega)$), then p is not a QMF. Therefore, for an FIR filter p of 6-fold line symmetry, there is no FIR filter bank biorthogonal to the idealized filter bank $\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$. Since the FIR low-pass filters P considered in [37] have 6-fold line symmetry, by the above discussion and by the transformation with U, we conclude that the idealized hexagonal filter banks $\{P, F_1, F_2, F_3\}$ designed in [37] do not have FIR biorthogonal dual filter banks.

In the following we study the construction of idealized tight frame filter banks $\{p, q^{(1)}, \cdots, q^{(L)}\}$ for a given p with the assumption that p is not a QMF (up to a constant), p satisfies (3.6) and

$$\sum_{j=0}^{3} |p(\omega + \eta_j)|^2 \le 1, \quad \omega \in \mathbb{R}^2. \tag{3.11}$$

To construct other high-pass filters $q^{(4)}, \cdots, q^{(L)}$, we write them in the form of (3.7). Let $W_1(\omega)$ be the polyphase matrix of $q^{(4)}, \cdots, q^{(L)}$ defined by

$$W_1(\omega) = \begin{bmatrix} q_{ee}^{(4)}(\omega) & q_{ee}^{(5)}(\omega) & \cdots & q_{ee}^{(L)}(\omega) \\ q_{oo}^{(4)}(\omega) & q_{oo}^{(5)}(\omega) & \cdots & q_{oo}^{(L)}(\omega) \\ q_{oe}^{(4)}(\omega) & q_{oe}^{(5)}(\omega) & \cdots & q_{oe}^{(L)}(\omega) \\ q_{eo}^{(4)}(\omega) & q_{eo}^{(5)}(\omega) & \cdots & q_{eo}^{(L)}(\omega) \end{bmatrix}. \tag{3.12}$$

The next lemma provides the conditions on W_1 such that $\{p, q^{(1)}, \cdots, q^{(L)}\}$ is a tight frame filter bank.

Lemma 2 *Assume that p satisfies (3.6) and (3.11). Let $q^{(1)}, q^{(2)}, q^{(3)}$ be the high-pass filters defined by (2.7). Then $\{p, q^{(1)}(\omega), \cdots, q^{(L)}(\omega)\}$ is a tight frame filter bank, namely it satisfies (2.5), for some FIR filters $q^{(4)}, \cdots, q^{(L)}$ if and only if $W_1(\omega)$ defined by (3.12) satisfies*

$$W_1(2\omega)W_1(2\omega)^* = (1 - \sum_{j=0}^{3} |p(\omega + \eta_j)|^2) I_4, \quad \omega \in \mathbb{R}^2. \tag{3.13}$$

Proof. Let W_0 and W_1 be the matrices defined by (3.10) and (3.12). Denote $W(\omega) = [W_0(\omega), W_1(\omega)]$. Then

$$\left[q^{(\ell)}(\omega + \eta_j)\right]_{0 \le j \le 3, 0 \le \ell \le L} = U(\omega)W(2\omega),$$

where $U(\omega)$ is the unitary matrix defined in (3.10). Thus (2.5) or equivalently

$$\left[q^{(\ell)}(\omega + \eta_j)\right]_{0 \le j \le 3, 0 \le \ell \le L} (\left[q^{(\ell)}(\omega + \eta_j)\right]_{0 \le j \le 3, 0 \le \ell \le L})^* = I_4, \ \omega \in \mathbb{R}^2$$

holds if and only if

$$W(\omega)W(\omega)^* = I_4, \ \omega \in \mathbb{R}^2. \tag{3.14}$$

Since $W_0(2\omega)W_0(2\omega)^* = (\sum_{j=0}^{3} |p(\omega + \eta_j)|^2) I_4$ as shown in the proof of Lemma 1, we know that (3.14) and (3.13) are equivalent. ¶

From Lemma 2, we know that for a given p, to construct an idealized frame filter bank $\{p, q^{(1)}, \cdots, q^{(L)}\}$, we need to construct W_1 to satisfy (3.13). In case where we construct such a W_1, then $q^{(4)}(\omega), \cdots, q^{(L)}(\omega)$ defined by

$$[q^{(4)}(\omega), \cdots, q^{(L)}(\omega)] = \frac{1}{2}[1, e^{i(\omega_1 + \omega_2)}, e^{-i\omega_1}, e^{-i\omega_2}] W_1(2\omega),$$

are the desired filters. From the computation point of view, one might hope that L is as small as possible. However, the next proposition tells us that L cannot be smaller than 7.

Proposition 3 *Suppose p is not a QMF, and p satisfies (3.6) and (3.11). If $\{p, q^{(1)}, \cdots, q^{(L)}\}$ is a tight frame filter bank with $q^{(1)}, q^{(2)}, q^{(3)}$ defined by (2.7), then $L \ge 7$.*

Proof. Let $W_1(\omega)$ be the polyphase matrix of $q^{(4)}(\omega), \cdots, q^{(L)}(\omega)$ defined by (3.12). Then by Lemma 2, W_1 satisfies (3.13). Thus $\mathrm{rank}(W_1(\omega)) = 4$. Therefore, $L-3$, the number of the columns of $W_1(\omega)$, is not smaller than 4. That is $L \geq 7$, as desired. ¶

From Proposition 3, we know to have a tight frame filter bank with $q^{(1)}, q^{(2)}, q^{(3)}$ defined by (2.7), we need at least 7 tight frame generators (including 3 generators defined by $q^{(1)}, q^{(2)}, q^{(3)}$). The next proposition gives us the sufficient conditions on p such that the idealized tight frame filter bank requires exact 7 generators.

Proposition 4 *Suppose p satisfies (3.6). Let $q^{(1)}, q^{(2)}, q^{(3)}$ be the filters defined by (2.7). If*

$$1 - \sum_{j=0}^{3} |p(\omega + \eta_j)|^2 = \sum_{k=1}^{K} |g_k(2\omega)|^2, \quad \omega \in \mathbb{R}^2, \tag{3.15}$$

for some FIR filters $g_k(\omega)$ with (real) impulse response coefficients and $1 \leq K \leq 3$, then there are FIR filters $q^{(4)}, \cdots, q^{(7)}$ such that $\{p, q^{(1)}, \cdots, q^{(7)}\}$ is a tight frame filter bank. More precisely, $q^{(4)}, \cdots, q^{(7)}$ may be given by

$$[q^{(4)}(\omega), \cdots, q^{(7)}(\omega)] = \frac{1}{2}[1, e^{i(\omega_1+\omega_2)}, e^{-i\omega_1}, e^{-i\omega_2}]\, diag(g_1(2\omega), g_1(2\omega), g_1(2\omega), g_1(2\omega)), \quad \textit{when } K = 1, \tag{3.16}$$

$$[q^{(4)}(\omega), \cdots, q^{(7)}(\omega)] = \frac{1}{2}[1, e^{i(\omega_1+\omega_2)}, e^{-i\omega_1}, e^{-i\omega_2}] \begin{bmatrix} 0 & -g_1(2\omega) & -g_2(2\omega) & 0 \\ g_1(2\omega) & 0 & 0 & \overline{g_2(2\omega)} \\ g_2(2\omega) & 0 & 0 & -\overline{g_1(2\omega)} \\ 0 & -\overline{g_2(2\omega)} & \overline{g_1(2\omega)} & 0 \end{bmatrix}, \quad \textit{when } K = 2, \tag{3.17}$$

or

$$[q^{(4)}(\omega), \cdots, q^{(7)}(\omega)] = \frac{1}{2}[1, e^{i(\omega_1+\omega_2)}, e^{-i\omega_1}, e^{-i\omega_2}] \begin{bmatrix} 0 & -g_1(2\omega) & -g_2(2\omega) & -g_3(2\omega) \\ g_1(2\omega) & 0 & -\overline{g_3(2\omega)} & \overline{g_2(2\omega)} \\ g_2(2\omega) & \overline{g_3(2\omega)} & 0 & -\overline{g_1(2\omega)} \\ g_3(2\omega) & -\overline{g_2(2\omega)} & \overline{g_1(2\omega)} & 0 \end{bmatrix}, \quad \textit{when } K = 3. \tag{3.18}$$

Proof. Let $W_1(\omega)$ be the polyphase matrix of $q^{(4)}(\omega), \cdots, q^{(7)}(\omega)$ defined by (3.12) with $L = 7$. Then for $q^{(4)}(\omega), \cdots, q^{(7)}(\omega)$ given by (3.16), (3.17) or (3.18), the corresponding $W_1(2\omega)$ is the 4×4 matrix on the right side of (3.16), (3.17) or (3.18), resp. One can easily verify that this $W_1(\omega)$ satisfies (3.13). Thus, Lemma 2 implies that $\{p, q^{(1)}, \cdots, q^{(7)}\}$ is a tight frame filter bank. ¶

The reader refers to [24] for the conditions on writing a non-negative trigonometric polynomial in the form of $\sum_{k=1}^{K} |g_k(\omega)|^2$ for some trigonometric polynomials g_k. In the next proposition, we concern whether $q^{(4)}, \cdots, q^{(7)}$ given in Proposition 4 have certain rotation invariant property. Here we consider the case that $K = 3$ in (3.15).

Proposition 5 *Suppose p satisfies (3.6) and (3.15) with $K = 3$ for some FIR (real coefficients) filters $g_1(\omega), g_2(\omega), g_3(\omega)$. Let $q^{(5)}(\omega), q^{(6)}(\omega), q^{(7)}(\omega)$ be the filters defined by (3.18). If*

$$g_2(\omega_1, \omega_2) = g_1(\omega_2, -\omega_1 - \omega_2), \quad g_3(\omega_1, \omega_2) = g_1(-\omega_1 - \omega_2, \omega_1), \quad (\omega_1, \omega_2) \in \mathbb{R}^2, \tag{3.19}$$

then $q^{(6)}(\omega)$ and $q^{(7)}(\omega)$ are the "$\frac{2}{3}\pi$" and "$\frac{4}{3}\pi$" "rotations" of $q^{(5)}(\omega)$, resp.

Proof. We need to show that $q^{(5)}(R_1^{-T}\omega) = q^{(6)}(\omega)$ and $q^{(5)}(R_2^{-T}\omega) = q^{(7)}(\omega)$, where R_1 and R_2 are the matrices defined by (3.4). Indeed, we have

$$
\begin{aligned}
&q^{(5)}(R_1^{-T}\omega) = q^{(5)}(\omega_2, -\omega_1-\omega_2)\\
&= \frac{1}{2}(-g_1(2\omega_2, -2\omega_1-2\omega_2) + e^{-i\omega_2}\overline{g}_3(2\omega_2, -2\omega_1-2\omega_2) - e^{i(\omega_1+\omega_2)}\overline{g}_2(2\omega_2, -2\omega_1-2\omega_2))\\
&= \frac{1}{2}(-g_1(2\omega_2, -2\omega_1-2\omega_2) + e^{-i\omega_2}g_3(-2\omega_2, 2\omega_1+2\omega_2) - e^{i(\omega_1+\omega_2)}g_2(-2\omega_2, 2\omega_1+2\omega_2))\\
&= \frac{1}{2}(-g_2(2\omega_1, 2\omega_2) + e^{-i\omega_2}g_1(-2\omega_1, -2\omega_2) - e^{i(\omega_1+\omega_2)}g_1(2\omega_1+2\omega_2, -2\omega_1))\\
&= \frac{1}{2}(-g_2(2\omega_1, 2\omega_2) + e^{-i\omega_2}\overline{g_1(2\omega_1, 2\omega_2)} - e^{i(\omega_1+\omega_2)}\overline{g_3(2\omega_1, 2\omega_2)})\\
&= q^{(6)}(\omega),
\end{aligned}
$$

and

$$
\begin{aligned}
&q^{(5)}(R_2^{-T}\omega) = q^{(5)}(-\omega_1-\omega_2, \omega_1)\\
&= \frac{1}{2}(-g_1(-2\omega_1-2\omega_2, 2\omega_1) + e^{i(\omega_1+\omega_2)}\overline{g}_3(-2\omega_1-2\omega_2, 2\omega_1) - e^{-i\omega_1}\overline{g}_2(-2\omega_1-2\omega_2, 2\omega_1))\\
&= \frac{1}{2}(-g_1(-2\omega_1-2\omega_2, 2\omega_1) + e^{i(\omega_1+\omega_2)}g_3(2\omega_1+2\omega_2, -2\omega_1) - e^{-i\omega_1}g_2(2\omega_1+2\omega_2, -2\omega_1))\\
&= \frac{1}{2}(-g_3(2\omega_1, 2\omega_2) + e^{i(\omega_1+\omega_2)}g_1(-2\omega_2, 2\omega_1+2\omega_2) - e^{-i\omega_1}g_1(-2\omega_1, -2\omega_2))\\
&= \frac{1}{2}(-g_3(2\omega_1, 2\omega_2) + e^{i(\omega_1+\omega_2)}\overline{g_2(2\omega_1, 2\omega_2)} - e^{-i\omega_1}\overline{g_1(2\omega_1, 2\omega_2)})\\
&= q^{(7)}(\omega).\P
\end{aligned}
$$

Remark 2 *If the low-pass filter p satisfies $1-\sum_{j=0}^{3}|p(\omega+\eta_j)|^2 = \sum_{k=1}^{K}|g_k(2\omega)|^2$ for some $K > 3$, then one can construct $q^{(4)}, \cdots, q^{(L)}$ with $L = 3+4L'$ for some L' such that they, together with $p, q^{(1)}, q^{(2)}, q^{(3)}$, form an idealized tight frame filter bank. More precisely, one may choose*

$$
[q^{(4)}(\omega), \cdots, q^{(L)}(\omega)] = \frac{1}{2}[1, e^{i(\omega_1+\omega_2)}, e^{-i\omega_1}, e^{-i\omega_2}][P_1(2\omega), P_2(2\omega), \cdots, P_{L'}(2\omega)],
$$

where each P_ℓ is a 4×4 matrix given on the right side of (3.16), (3.17) or (3.18) for some g_k, $1 \le k \le K$. If one's main concern of frame systems is not about the number of the frame generators, but about their supports, then one may simply choose to have $4K+3$ frame generators with $q^{(4)}, \cdots, q^{(3+4K)}$ given by

$$
[q^{(4)}(\omega), \cdots, q^{(3+4K)}(\omega)] = \frac{1}{2}[1, e^{i(\omega_1+\omega_2)}, e^{-i\omega_1}, e^{-i\omega_2}][P_1(2\omega), P_2(2\omega), \cdots, P_K(2\omega)],
$$

where $P_k(\omega) = diag(g_k(\omega), g_k(\omega), g_k(\omega), g_k(\omega)), 1 \le k \le K$.

4. Examples

In this section we present a few examples to illustrate the general theory.

Example 1. Let $p(\omega) = \frac{1}{4}(1 + \frac{1}{2}(z_1 + z_1 z_2 + z_2 + z_1^{-1} + z_1^{-1}z_2^{-1} + z_2^{-1}))$ be the refinement mask (symbol) for the Courant element B_{111} on the 3-directional mesh of $\mathbb{Z}^2$. Then

$$
1 - \sum_{j=0}^{3}|p(\omega+\eta_j)|^2 = \sum_{k=1}^{3}|g_k(2\omega)|^2,
$$

with

$$g_1(\omega) = \frac{1}{4}(1 - z_1 z_2),\ g_2(\omega) = \frac{1}{4}(1 - z_1^{-1}),\ g_3(\omega) = \frac{1}{4}(1 - z_2^{-1}).$$

The filters $q^{(4)}, \cdots, q^{(7)}$ defined by (3.18) are

$$\begin{aligned}
q^{(4)}(\omega) &= \frac{1}{8}(z_1^{-1}z_2^{-1} - z_1 z_2 + z_1 - z_1^{-1} + z_2 - z_2^{-1}),\\
q^{(5)}(\omega) &= \frac{1}{8}(z_1^2 z_2^2 - 1 + z_1 - z_1 z_2^2 - z_2 + z_1^2 z_2),\\
q^{(6)}(\omega) &= \frac{1}{8}(z_1^{-2} - 1 + z_1^{-1} z_2 - z_1^{-1} z_2^{-1} + z_2 - z_1^{-2} z_2^{-1}),\\
q^{(7)}(\omega) &= \frac{1}{8}(z_2^{-2} - 1 + z_1^{-1} z_2^{-1} - z_1 z_2^{-1} - z_1 + z_1^{-1} z_2^{-2}).
\end{aligned}$$

Since g_1, g_2, g_3 satisfy (3.19), Proposition 5 implies that $q^{(6)}$ and $q^{(7)}$ are the "$\frac{2}{3}\pi$" and "$\frac{4}{3}\pi$" "rotations" of $q^{(5)}$, resp. Actually, one can also easily verify this property directly from the expressions of $q^{(5)}, q^{(6)}$ and $q^{(7)}$.

Let $\{P, Q^{(1)}, \cdots, Q^{(7)}\}$ be the corresponding hexagonal tight frame filter bank. The non-zero impulse response coefficients of P and $Q^{(1)}$ are displayed in the 1st and 2nd pictures (from the left) in Fig. 9, while $Q^{(2)}, Q^{(3)}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(1)}$. The non-zero impulse responses of $Q^{(4)}$ and $Q^{(5)}$ are displayed the 3rd and 4th pictures in Fig. 9 with $Q^{(6)}, Q^{(7)}$ being $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(5)}$. ¶

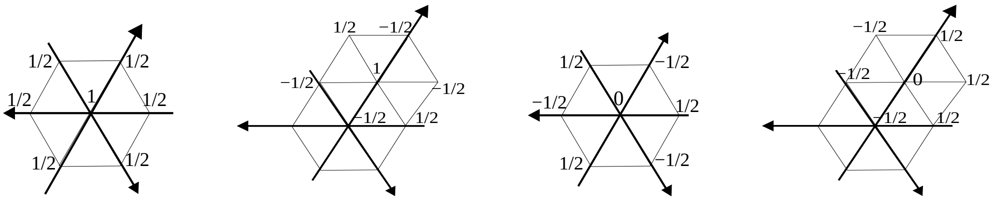

Figure 9: *Low-pass filter $P(\omega)$ (1st from left), high-pass filter $Q^{(1)}$ (2nd), high-pass filter $Q^{(4)}$ (3rd), high-pass filter $Q^{(5)}$ (4th), while $Q^{(2)}, Q^{(3)}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(1)}$, and $Q^{(6)}, Q^{(7)}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(5)}$*

Up to $cz_1^{n_1} z_2^{n_2}$, this tight frame filter bank is actually the one constructed in [8] via the Kroneckler products. In [24], for this function B_{111}, another tight frame filter bank is constructed with 6 frame generators. The frame generators in [24] have bigger supports and their frame filter bank does not include the idealized high-pass filters.

Example 2. Let $p(\omega) = \frac{1}{4}\sum_{\mathbf{k}} p_{\mathbf{k}} z_1^{k_1} z_2^{k_2}$ be the refinement mask with non-zero coefficients $p_{\mathbf{k}}$ given by

$$\begin{aligned}
&p_{00} = 1, \quad p_{10} = p_{11} = p_{01} = p_{-10} = p_{-1-1} = p_{0-1} = \frac{1}{2} - c,\\
&p_{21} = p_{12} = p_{-11} = p_{-2-1} = p_{-1-2} = p_{1-1} = c,
\end{aligned}$$

where c is a real number with $-\frac{1}{6} \le c \le \frac{1}{2}$. Then we have

$$1 - \sum_{j=0}^{3} |p(\omega + \eta_j)|^2 = \sum_{k=1}^{3} |g_k(2\omega)|^2,$$

where

$$g_1(\omega) = e_0 + e_1 z_1 z_2 + e_2 z_1^{-1},\ g_2(\omega) = e_0 + e_1 z_1^{-1} + e_2 z_2^{-1},\ g_3(\omega) = e_0 + e_1 z_2^{-1} + e_2 z_1 z_2, \tag{4.1}$$

with

$$
\begin{aligned}
e_2 &= \frac{1}{8}(-(2c+1) \pm \sqrt{1+4c-12c^2}),\\
e_1 &= \frac{1}{8}((2c+1) \pm \sqrt{1+4c-12c^2}),\\
e_0 &= \frac{(1+4c-12c^2)e_2}{4(2c^2+(2c+1)e_2)}.
\end{aligned}
$$

Let $q^{(4)}, \cdots, q^{(7)}$ be the filter defined by (3.18) with g_1, g_2, g_3 defined in (4.1). Then we have an idealized tight frame filter banks $\{p, q^{(1)}, \cdots, q^{(7)}\}$ with $q^{(1)}, q^{(2)}, q^{(3)}$ given by (2.7). Since g_1, g_2, g_3 in (4.1) satisfy (3.19), we know from Proposition 5 that $q^{(6)}$ and $q^{(7)}$ are the "$\frac{2}{3}\pi$" and "$\frac{4}{3}\pi$" "rotations" of $q^{(5)}$, resp. Here we would not consider the particular choice of the parameter c. ¶

Example 3. Let $p(\omega) = \frac{1}{4}\sum_{\mathbf{k}} p_{\mathbf{k}} z_1^{k_1} z_2^{k_2}$ be the refinement mask with non-zero coefficients $p_{\mathbf{k}}$ given by

$$
\begin{aligned}
&p_{00} = 1-6d, \quad p_{10} = p_{11} = p_{01} = p_{-10} = p_{-1-1} = p_{0-1} = \frac{1}{2} - c,\\
&p_{21} = p_{12} = p_{-11} = p_{-2-1} = p_{-1-2} = p_{1-1} = c,\\
&p_{22} = p_{-20} = p_{0-2} = p_{20} = p_{02} = p_{-2-2} = d,
\end{aligned}
$$

for some $c, d \in \mathbb{R}$. Then

$$
1 - \sum_{j=0}^{3} |p(\omega + \eta_j)|^2 = \sum_{k=1}^{3} |g_k(2\omega)|^2,
$$

with

$$
\begin{aligned}
g_1(\omega) &= e + f z_1 z_2 + h z_1^{-1} + s z_1 z_2^2 + u z_2,\\
g_2(\omega) &= e + f z_1^{-1} + h z_2^{-1} + s z_1^{-2} z_2^{-1} + u z_1^{-1} z_2^{-1},\\
g_3(\omega) &= e + f z_2^{-1} + h z_1 z_2 + s z_1 z_2^{-1} + u z_1,
\end{aligned}
$$

where e, f, s, u are given by

$$
\begin{aligned}
e &= \frac{h}{d^2}(2d^2 + 4E_0 + c^2),\ f = \frac{E_0}{h},\ s = -\frac{d^2}{4s},\\
u &= \Big(4(16h^4 - d^4)(8d^2h^2 + 4c^2h^2 - 16h^4 - d^4)E_0 + h^2\{256(-2d^2 - c^2)h^6\\
&\quad +16(4c^4 - d^2 - 8d^3 + 60d^4 + 28c^2d^2 - 4cd^2)h^4\\
&\quad +8(24d^6 - d^4 + 6d^4c^2 - 8d^5 - 4d^4c)h^2 - 8d^7 + 12d^6c^2 - 4d^6c + 44d^8 - d^6\}\Big)\\
&\div\Big(4h(d^4 + 4d^2h^2 + 16h^4)(12d^2h^2 + 4(d^2 + 4h^2)E_0 + 4c^2h^2 - d^4)\Big)
\end{aligned}
$$

with h being a real number,

$$
E_0 = \frac{-B_0 \pm \sqrt{B_0^2 - 4A_0C_0}}{2A_0},
$$

and

$$
\begin{aligned}
A_0 &= 16(16h^4 + 4d^2h^2 + d^4),\\
B_0 &= 64h^4(2c^2 + 5d^2) + 16d^2h^2(c^2 + 2d^2) - 4d^6,\\
C_0 &= 16(7d^4 + 5c^2d^2 + c^4)h^4 + (16d^6 - 8d^5 + 4d^4c^2 - d^4 - 4d^4c)h^2 + d^8.
\end{aligned}
$$

With filters $q^{(4)}, \cdots, q^{(7)}$ defined by (3.18), we have a tight frame filter bank $\{p, q^{(1)}, \cdots, q^{(7)}\}$ with $q^{(1)}, q^{(2)}, q^{(3)}$ being the idealized high-pass filters given by (2.7). Since the above g_1, g_2, g_3 satisfy (3.19), we know from Proposition 5 that $q^{(6)}$ and $q^{(7)}$ are the "$\frac{2}{3}\pi$" and "$\frac{4}{3}\pi$" "rotations" of $q^{(5)}$, resp.

The particular choices of c, d are

$$c = \frac{1}{8}, \ d = \frac{1}{16}.$$

For such c, d, the corresponding p is the refinement mask for the box-spline B_{222}. For this particular mask, we may choose $h = \frac{1}{32}$. Then the corresponding e, f, s, u are (with the choice of $+$ from "$\pm$" in E_0)

$$e = \frac{9 + \sqrt{309}}{96}, \ f = \frac{-9 + \sqrt{309}}{96}, \ s = -\frac{1}{32}, \ u = -\frac{\sqrt{309}}{48}.$$

Let $\{P, Q^{(1)}, \cdots, Q^{(7)}\}$ be the hexagonal tight frame filter bank corresponding to $\{p, q^{(1)}, \cdots, q^{(7)}\}$ with these special choices of c, d, h. The non-zero impulse response coefficients of P and $Q^{(1)}$ are displayed in the top-left and top-right pictures in Fig. 10 while $Q^{(2)}, Q^{(3)}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(1)}$. The non-zero impulse responses of $Q^{(4)}$ and $Q^{(5)}$ are displayed in the bottom-left and bottom-right pictures in Fig. 10 with $Q^{(6)}, Q^{(7)}$ being $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(5)}$. ¶

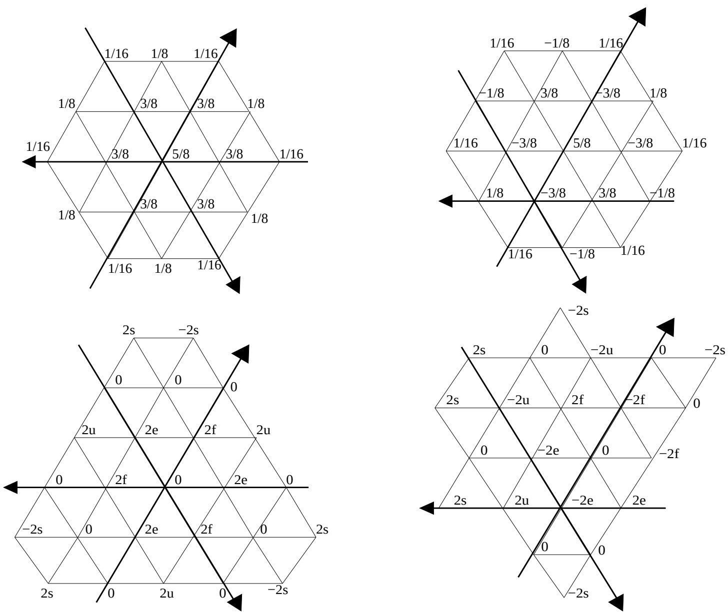

Figure 10: *Low-pass filter P (top-left), high-pass filter $Q^{(1)}$ (top-right), high-pass filter $Q^{(4)}$ (bottom-left), high-pass filter $Q^{(5)}$ (bottom-right), while $Q^{(2)}, Q^{(3)}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(1)}$, and $Q^{(6)}, Q^{(7)}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ rotations of $Q^{(5)}$*

A tight frame filter bank corresponding to the mask of the box-spline B_{222} is constructed in [24]. The construction in [24] also leads to 7 frame generators associated with B_{222}. The frame generators there have bigger supports. For B_{222}, it is shown in [17] that B_{222} has no biorthogonal dual $\tilde{\phi}$ supported in $[-4, 4] \times [-4, 4]$. From the above example, we know that the frame system does provide the flexility for the construction of filter banks. In addition, the approach of construction introduced in this paper leads to that the first few frame filters $q^{(1)}, q^{(2)}, q^{(3)}$ are defined by (2.7).

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