# Bi-frames with 4-fold Axial Symmetry for Quadrilateral Surface Multiresolution Processing 

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#### Abstract

When bivariate filter banks and wavelets are used for surface multiresolution processing, it is required that the decomposition and reconstruction algorithms for regular vertices derived from them have high symmetry. This symmetry requirement makes it possible to design the corresponding multiresolution algorithms for extraordinary vertices. Recently lifting-scheme based biorthogonal bivariate wavelets with high symmetry have been constructed for surface multiresolution processing. If biorthogonal wavelets have certain smoothness, then the analysis or synthesis scaling function, or both have big supports in general. In particular, when the synthesis lowpass filter is a commonly used scheme such as Loop's scheme or Catmull-Clark's scheme, the corresponding analysis lowpass filter has a big support and the corresponding analysis scaling function and wavelets have poor smoothness. Big supports of scaling functions, or in other words, big templates of multiresolution algorithms are undesirable for surface processing. On the other hand, frame provides a flexibility for the construction of "basis" systems. This paper concerns the construction of wavelet (or affine) bi-frames with high symmetry.

In this paper we study the construction of wavelet bi-frames with 4 -fold symmetry for quadrilateral surface multiresolution processing, with both the dyadic and $\sqrt{2}$ refinements considered. The constructed bi-frames have 4 framelets (or frame generators) for the dyadic refinement, and 2 framelets for the $\sqrt{2}$ refinement. Namely, with either the dyadic or $\sqrt{2}$ refinement, a frame system constructed in this paper has one more generator only than a wavelet system. The constructed bi-frames have better smoothness and smaller supports than biorthogonal wavelets. Furthermore, all the frame algorithms considered in this paper are given by templates so that one can easily implement them.


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## 1 Introduction

Multiresolution resolution surface processing has been studied in [30, 31, 37, 25]. One of key issues is the construction of wavelets. Since the surface (mesh) to be processed is an object in 3-D space and the mesh in general consists of not only regular vertices but also extraordinary vertices, it requires that wavelets and the corresponding multiresolution algorithms have high symmetry. Researchers have made efforts to construct such wavelets or/and multiresolution algorithms. For example, Doo's subdivision scheme based wavelets for quadrilateral (quad) surfaces are constructed in [36]. Recently with the idea of lifting scheme [38, 10], biorthogonal wavelets with high symmetry for surface multiresolution processing have been constructed in [1, 2, 41, 42].

[^0]The smoothness and approximation property of highly symmetric wavelets (for regular vertices) have been analyzed in $[21,22]$ and new symmetric wavelets are also constructed there.

If the biorthogonal wavelets have certain smoothness, then they will have big supports, namely, the multiresolution algorithms have big sizes of templates. Compared with (bi)orthogonal wavelet systems, the elements in a frame system may be linearly dependent, namely, frames can be redundant, which provides a flexibility for the construction of framelets with high symmetry and smaller supports than biorthogonal wavelets.

Let $A$ be a dilation matrix, a 2 by 2 integer matrix with each of its two eigenvalues $\lambda$ satisfying $|\lambda|>1$. For a function $f$ on $\mathbb{R}^{2}$, denote $f_{j, \mathbf{k}}(\mathbf{x})=|\operatorname{det} A|^{j / 2} f\left(A^{j} \mathbf{x}-\mathbf{k}\right)$. We call functions $\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(L)}$ on $\mathbb{R}^{2}$, where $L \geq|\operatorname{det} A|-1$, wavelet (or affine) framelets, or wavelet frame generators, just called framelets in this paper for short, if $\left\{\psi_{j, \mathbf{k}}^{(1)}, \psi_{j, \mathbf{k}}^{(2)}, \cdots, \psi_{j, \mathbf{k}}^{(L)}\right\}_{j \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^{2}}$ is a wavelet frame of $L^{2}\left(\mathbb{R}^{2}\right)$, namely, there are two positive constants $B$ and $C$ such that

$$
B\|f\|_{2}^{2} \leq \sum_{\ell=1}^{L} \sum_{j \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^{2}}\left|\left\langle f, \psi_{j, \mathbf{k}}^{(\ell)}\right\rangle\right|^{2} \leq C\|f\|_{2}^{2}, \quad \forall f \in L^{2}\left(\mathbb{R}^{2}\right),
$$

where $\langle\cdot, \cdot\rangle$ and $\|\cdot\|_{2}:=\langle\cdot, \cdot\rangle^{\frac{1}{2}}$ denote the inner product and the norm of $L^{2}\left(\mathbb{R}^{2}\right)$. We say that $\psi^{(\ell)}, \widetilde{\psi}^{(\ell)}, \ell=1, \cdots, L$, generate biorthogonal wavelet frames (bi-frames for short) of $L^{2}\left(\mathbb{R}^{2}\right)$ or dual wavelet frames of $L^{2}\left(\mathbb{R}^{2}\right)$ if $\left\{\psi_{j, \mathbf{k}}^{(1)}, \cdots, \psi_{j, \mathbf{k}}^{(L)}\right\}_{j \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^{2}}$ and $\left\{\widetilde{\psi}_{j, \mathbf{k}}^{(1)}, \cdots, \widetilde{\psi}_{j, \mathbf{k}}^{(L)}\right\}_{j \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^{2}}$ are frames of $L^{2}\left(\mathbb{R}^{2}\right)$ and that for any $f \in L^{2}\left(\mathbb{R}^{2}\right), f$ can be written as (in $L^{2}$-norm)

$$
f=\sum_{1 \leq \ell \leq L} \sum_{j \in \mathbf{Z}, \mathbf{k} \in \mathbf{Z}^{2}}\left\langle f, \widetilde{\psi}_{j, \mathbf{k}}^{(\ell)}\right\rangle \psi_{j, \mathbf{k}}^{(\ell)} .
$$

The reader refers to [7] and references therein for the properties on frames.
Let $A$ be a dilation matrix. For a sequence $\left\{p_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ of real numbers with finitely many $p_{\mathbf{k}}$ nonzero, let $p(\boldsymbol{\omega})$ denote the corresponding finite impulse response (FIR) filter (also called symbol of $\left.\left\{p_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}\right)$ :

$$
p(\boldsymbol{\omega})=\frac{1}{|\operatorname{det} A|} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}} e^{-i \mathbf{k} \omega}
$$

A pair of FIR filter banks $\left\{p, q^{(1)}, \cdots, q^{(L)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\right\}$, each is called a frame filter bank in this paper, is said to be biorthogonal (with dilation matrix $A$ ) if

$$
\overline{p(\boldsymbol{\omega})} \widetilde{p}\left(\boldsymbol{\omega}+2 \pi\left(A^{-1}\right)^{T} \boldsymbol{\eta}_{j}\right)+\sum_{\ell=1}^{L} \overline{q^{(\ell)}(\boldsymbol{\omega})} \widetilde{q}^{(\ell)}\left(\boldsymbol{\omega}+2 \pi\left(A^{-1}\right)^{T} \boldsymbol{\eta}_{j}\right)= \begin{cases}1, & j=0, \\ 0, & 1 \leq j<|\operatorname{det} A|,\end{cases}
$$

where $\boldsymbol{\eta}_{j}, 0 \leq j<|\operatorname{det} A|$ are the representatives of the group $\mathbf{Z}^{2} /\left(A^{T} \mathbf{Z}^{2}\right)$ with $\boldsymbol{\eta}_{0}=(0,0)$.
Let $\left\{p, q^{(1)}, \cdots, q^{(L)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(2)}\right\}$ be frame filter banks. Suppose $\phi$ and $\widetilde{\phi}$ are the associated refinable (or scaling) functions (with dilation matrix $A$ ) which satisfy the refinement equations

$$
\phi(\mathbf{x})=\sum_{\mathbf{k}} p_{\mathbf{k}} \phi(A \mathbf{x}-\mathbf{k}), \widetilde{\phi}(\mathbf{x})=\sum_{\mathbf{k}} \widetilde{p}_{\mathbf{k}} \widetilde{\phi}(A \mathbf{x}-\mathbf{k})
$$

Let $\psi^{(\ell)}, \widetilde{\psi}^{(\ell)}, \ell=1, \cdots, L$, be the functions defined by

$$
\psi(\mathbf{x})=\sum_{\mathbf{k}} q_{\mathbf{k}}^{(\ell)} \phi(A \mathbf{x}-\mathbf{k}), \widetilde{\psi}(\mathbf{x})=\sum_{\mathbf{k}} \widetilde{q}_{\mathbf{k}}^{(\ell)} \widetilde{\phi}(A \mathbf{x}-\mathbf{k})
$$

Then the Mixed Unitary Extension Principle of [33] states that if $\left\{p, q^{(1)}, \cdots, q^{(L)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots\right.$, $\left.\widetilde{q}^{(L)}\right\}$ are biorthogonal, $\phi, \widetilde{\phi} \in L^{2}\left(\mathbb{R}^{2}\right)$ with $\widehat{\phi}(0,0) \widehat{\tilde{\phi}}(0,0) \neq 0$, and that $p(0,0)=\widetilde{p}(0,0)=1$, $p\left(2 \pi\left(A^{-1}\right)^{T} \boldsymbol{\eta}_{j}\right)=\widetilde{p}\left(2 \pi\left(A^{-1}\right)^{T} \boldsymbol{\eta}_{j}\right)=q^{(\ell)}(0,0)=\widetilde{q}^{(\ell)}(0,0)=0$ for $1 \leq j<|\operatorname{det} A|, 1 \leq \ell \leq L$, then $\psi^{(\ell)}, \widetilde{\psi}^{(\ell)}, \ell=1, \cdots, L$, generate bi-frames of $L^{2}\left(\mathbb{R}^{2}\right)$.


Figure 1: Left: Quad mesh; Middle: Coarse mesh of dyadic refinement; Right: Coarse mesh of $\sqrt{2}$ refinement

In this paper we consider frames for quad mesh (surface) multiresolution processing. The quad mesh near a regular vertex (with valence 4) can be represented locally as a 2-D mesh shown on the left of Fig. 1. The quad surface subdivision allows not only the dyadic refinement but also other refinements such as $\sqrt{2}$ and $\sqrt{5}$ refinements, see $[9,17,29,39,40]$. The dyadic refinement is the most commonly used refinement for multiresolution data processing and for surface subdivision. The nodes with circles $\bigcirc$ in the middle of Fig. 1 form the coarse quad mesh of the dyadic refinement. From a finer mesh to its coarse mesh of the dyadic refinement, the nodes are reduced by a factor $\frac{1}{4}$.

The right part of Fig. 1 shows the $\sqrt{2}$ refinement with the nodes of circles $\bigcirc$ forming the coarse quad mesh of the $\sqrt{2}$ refinement. The nodes on the $\sqrt{2}$-refinement coarse mesh are reduced by a factor $\frac{1}{2}$. Compared with the dyadic refinement, the $\sqrt{2}$ refinement generates more resolution levels within a prescribed number of quads.

The construction of $\sqrt{2}$-refinement wavelets (also called quincunx wavelets) are studied in some papers, see e.g. [26], [27] and [16], and the $\sqrt{2}$ subdivision has been investigated in [29, 39, 40].

In this paper we study the construction of dyadic and $\sqrt{2}$ refinement bi-frames with high symmetry. For the dyadic refinement, the dilation matrix $A$ is $2 I_{2}$, where $I_{2}$ is the 2 by 2 identity matrix, while for the $\sqrt{2}$ refinement, we can choose $A$ to be one of the following matrices

$$
A_{1}=\left[\begin{array}{cc}
1 & -1  \tag{1}\\
1 & 1
\end{array}\right], A_{2}=\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] .
$$

In this paper, we construct dyadic bi-frames with 4 framelets and $\sqrt{2}$-refinement bi-frames with 2 framelets. Namely, a dyadic frame filter bank $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ has four highpass filters, and a $\sqrt{2}$ frame filter bank $\left\{p, q^{(1)}, q^{(2)}\right\}$ has two highpass filters.

The construction of multivariate wavelet frames has been studied in some papers, see e.g., $[13,28,34,35,5,6,11,12]$. Compared with the framelets in these papers, our framelets possess simultaneously the properties of high symmetry, small support and few generators (just one more generator than biorthogonal wavelets). Furthermore, our construction starts with symmetric templates of small sizes with the templates given by some parameters, then we select the
parameters such that the constructed framelets gain (numerically) optimal smoothness and/or vanishing moments.

For quad surface processing, biorthogonal wavelets are required to have 4 -fold symmetry, see [22]. In the following we introduce the 4 -fold symmetry of framelets.


Figure 2: 4 symmetric axes (lines)
Definition 1. Let $T_{k}, 0 \leq k \leq 3$ be the axes in Fig. 2. A dyadic refinement frame filter bank $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ is said to have $\mathbf{4}$-fold axial (or line) symmetry if (i) its lowpass filter $p(\boldsymbol{\omega})$ and first highpass filter $q^{(1)}(\boldsymbol{\omega})$ are symmetric around $T_{k}, 0 \leq k \leq 3$, (ii) $e^{-i\left(\omega_{1}+\omega_{2}\right)} q^{(2)}(\boldsymbol{\omega})$ is symmetric around the axes $T_{k}, 0 \leq k \leq 3$, (iii) $e^{i \omega_{1}} q^{(3)}(\boldsymbol{\omega})$ is symmetric around the axes $T_{1}$ and $T_{3}$, and (iv) $q^{(4)}(\boldsymbol{\omega})$ is the reflection of $q^{(3)}(\boldsymbol{\omega})$ around the line $\omega_{1}=\omega_{2}$, i.e., $q^{(4)}\left(\omega_{1}, \omega_{2}\right)=q^{(3)}\left(\omega_{2}, \omega_{1}\right)$.
Definition 2. Let $T_{k}, 0 \leq k \leq 3$ be the axes in Fig. 2. A $\sqrt{2}$-refinement frame filter bank $\left\{p, q^{(1)}, q^{(2)}\right\}$ is said to have $\mathbf{4}$-fold axial (or line) symmetry if (i) its lowpass filter $p(\boldsymbol{\omega})$ and first highpass filter $q^{(1)}(\boldsymbol{\omega})$ are symmetric around $T_{k}, 0 \leq k \leq 3$, (ii) $e^{-i \omega_{1}} q^{(2)}(\boldsymbol{\omega})$ is symmetric around the axes $T_{k}, 0 \leq k \leq 3$.

In this paper we construct 4 -fold symmetric dyadic and $\sqrt{2}$ refinement biorthogonal FIR frame filter banks and the associated framelets. The work on 4 -fold symmetric dyadic bi-frames and that on 4 -fold symmetric $\sqrt{2}$-refinement bi-frames are carried out in $\S 2$ and $\S 3$ respectively. In each of these two sections, first we show that frame multiresolution analysis and synthesis algorithms can be represented as templates by associating the outputs appropriately with the nodes of $\mathbf{Z}^{2}$, with which the original quad mesh is represented. Then we construct symmetric bi-frames with 3 - and 4 -step algorithms based on symmetric templates. In this paper we consider bi-frames for regular vertices only. The corresponding frame multiresolution algorithms for extraordinary vertices will be presented elsewhere. Symmetric 1-D bi-frames are considered in [23] with the corresponding frame multiresolution algorithms also given by iterative templates. Those 1-D frame algorithms can be used as boundary algorithms for multiresolution processing of open surfaces.

As in [22], here we also use bold-faced letters such as $\mathbf{k}, \mathbf{x}$ to denote elements of $\mathbf{Z}^{2}$ and $\mathbb{R}^{2}$. For $\mathbf{k}$ and $\mathbf{x}$ in $\mathbb{R}^{2}$, they will be written as row vectors

$$
\mathbf{k}=\left(k_{1}, k_{2}\right), \mathbf{x}=\left(x_{1}, x_{2}\right) .
$$

When we consider $A \mathbf{k}$ and $A \mathbf{x}$, where $A$ is a $2 \times 2$ matrix, $\mathbf{k}$ and $\mathbf{x}$ should be understood as column vectors $\left[k_{1}, k_{2}\right]^{T}$ and $\left[x_{1}, x_{2}\right]^{T}$. We also use the following notations. For a positive integer $n, I_{n}$ denotes the $n \times n$ identity matrix; for a matrix $M, M^{*}$ denotes its complex conjugate and transpose; and for a nonsingular matrix $B$, we use $B^{-T}$ to denote $\left(B^{-1}\right)^{T}$.

## 2 Dyadic bi-frames with 4 -fold symmetry

In this section we study dyadic bi-frames with 4 -fold symmetry. This section consists of three subsections. In the first subsection, we show how frame multiresolution analysis and synthesis algorithms can be represented as templates and discuss the symmetry of frame filter banks. We construct 4 -fold symmetric bi-frames with 3 - and 4 -step algorithms in the second and third subsections.

### 2.1 4-fold symmetric dyadic bi-frames and associated templates

As mentioned above, for dyadic frames, the dilation matrix $A$ is $2 I_{2}$. For $A=2 I_{2}$, we may choose the representatives $\boldsymbol{\eta}_{j}, 0 \leq j \leq 3$ of $\mathbf{Z}^{2} /\left(A^{T} \mathbf{Z}^{2}\right)$ to be

$$
\begin{equation*}
\boldsymbol{\eta}_{0}=(0,0), \boldsymbol{\eta}_{1}=(-1,-1), \boldsymbol{\eta}_{2}=(1,0), \boldsymbol{\eta}_{3}=(0,1) . \tag{2}
\end{equation*}
$$

For an FIR dyadic frame filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}\right\}$, with notation $q^{(0)}(\boldsymbol{\omega})=p(\boldsymbol{\omega})$, write $q^{(\ell)}(\boldsymbol{\omega}), 0 \leq \ell \leq 4$ as

$$
q^{(\ell)}(\boldsymbol{\omega})=\frac{1}{2}\left(q_{0}^{(\ell)}(2 \boldsymbol{\omega})+q_{1}^{(\ell)}(2 \boldsymbol{\omega}) e^{i\left(\omega_{1}+\omega_{2}\right)}+q_{2}^{(\ell)}(2 \boldsymbol{\omega}) e^{-i \boldsymbol{\omega}_{1}}+q_{3}^{(\ell)}(2 \boldsymbol{\omega}) e^{-i \boldsymbol{\omega}_{2}}\right),
$$

where $q_{k}^{(\ell)}(\boldsymbol{\omega}), 0 \leq k \leq 3$ are trigonometric polynomials. Then the polyphase matrix of frame filter bank $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ is the $5 \times 4$ matrix $V(\boldsymbol{\omega})$ defined by

$$
\begin{equation*}
V(\boldsymbol{\omega})=\left[q_{k}^{(\ell)}(\boldsymbol{\omega})\right]_{0 \leq \ell \leq 4,0 \leq k \leq 3} . \tag{3}
\end{equation*}
$$

From

$$
\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), \cdots, q^{(4)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{2} V(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})
$$

where $I_{00}(\boldsymbol{\omega})$ is defined by

$$
I_{00}(\boldsymbol{\omega})=\left[1, e^{i\left(\omega_{1}+\omega_{2}\right)}, e^{-i \omega_{1}}, e^{-i \omega_{2}}\right]^{T},
$$

one can easily find that two frame filter banks $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ are biorthogonal if and only if their polyphase matrices $V(\boldsymbol{\omega})$ and $\widetilde{V}(\boldsymbol{\omega})$ satisfy

$$
V(\boldsymbol{\omega})^{*} \tilde{V}(\boldsymbol{\omega})=I_{4}, \quad \boldsymbol{\omega} \in \mathbb{R}^{2}
$$

The multiresolution decomposition algorithm with a dyadic analysis frame filter bank $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ for input data or a regular quad mesh $\mathcal{C}=\left\{c_{\mathbf{k}}^{0}\right\}$ is

$$
\begin{equation*}
c_{\mathbf{n}}^{j+1}=\frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}-2 \mathbf{n}} c_{\mathbf{k}}^{j}, d_{\mathbf{n}}^{(\ell, j+1)}=\frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} q_{\mathbf{k}-2 \mathbf{n}}^{(\ell)} c_{\mathbf{k}}^{j}, \tag{4}
\end{equation*}
$$

with $\ell=1, \cdots, 4, \mathbf{n} \in \mathbf{Z}^{2}$ for $j=0,1, \cdots, J-1$, where $J$ is a positive integer. If $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ is biorthogonal to $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$, then $c_{\mathbf{k}}^{0}$ can be recovered by the multiresolution reconstruction algorithm:

$$
\begin{equation*}
c_{\mathbf{k}}^{j}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{\mathbf{k}-2 \mathbf{n}} c_{\mathbf{n}}^{j+1}+\sum_{1 \leq \ell \leq 4} \sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{q}_{\mathbf{k}-2 \mathbf{n}}^{(\ell)} d_{\mathbf{n}}^{(\ell, j+1)} \tag{5}
\end{equation*}
$$

with $\mathbf{k} \in \mathbf{Z}^{2}$ for $j=J-1, J-2, \cdots, 0$. $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ is called the synthesis frame filter bank, and $\left\{c_{\mathbf{k}}^{j}\right\}$ and $\left\{d_{\mathbf{k}}^{(\ell, j)}\right\}$ are called respectively the "approximation" and the "details" of $\mathcal{C}$. $\left\{c_{\mathbf{k}}^{j}\right\}$ and $\left\{d_{\mathbf{k}}^{(\ell, j)}\right\}$ are also called respectively lowpass and highpass outputs of $\mathcal{C}$, and $p, \widetilde{p}$ and $q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$ lowpass and highpass filters.

Next, we show that decomposition and reconstruction algorithms can be represented as templates by associating appropriately lowpass and highpass outputs to the nodes of $\mathbf{Z}^{2}$ with which a regular quad mesh is represented. First, as in [22], we separate the nodes of $\mathbf{Z}^{2}$ into four groups.

We call the nodes of $\mathbf{Z}^{2}$ with labels ( $2 k_{1}, 2 k_{2}$ ) type $V$ nodes (or vertex nodes). Thus all type $V$ nodes, which have indices of $2 \mathbf{Z}^{2}=\left\{\left(2 k_{1}, 2 k_{2}\right),\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2}\right\}$, form the coarse mesh of the dyadic refinement. Next, we separate $\mathbf{Z}^{2} \backslash\left(2 \mathbf{Z}^{2}\right)$ into type $F$ nodes (or face nodes) with indices in $\{2 \mathbf{k}-(1,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and type $E$ nodes (or edge nodes) with indices in $\{2 \mathbf{k}+(1,0), 2 \mathbf{k}+(0,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$. The type $E$ nodes are further separated into two groups with indices in $\{2 \mathbf{k}+(1,0)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and $\{2 \mathbf{k}+(0,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ respectively. See the left of Fig. 3, where the big circles, squares, $\Delta$ and $\nabla$ denote type $V$ nodes, type $F$ nodes, and two groups of type $E$ nodes respectively.

Let $\mathcal{C}=\left\{c_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ be the data sampled on $\mathbf{Z}^{2}$ or a regular quad mesh with vertices $c_{\mathbf{k}}$. Then $\left\{c_{2 \mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of data/vertices associated with type $V$ nodes, $\left\{c_{2 \mathbf{k}-(1,1)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of data/vertices associated with type $F$ nodes, and $\left\{c_{2 \mathbf{k}+(1,0)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and $\left\{c_{2 \mathbf{k}+(0,1)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ are the sets of data/vertices associated with the above two groups of type $E$ nodes. Denote

$$
\begin{equation*}
v_{\mathbf{k}}=c_{2 \mathbf{k}}, f_{\mathbf{k}}=c_{2 \mathbf{k}-(1,1)}, e_{\mathbf{k}}^{(2)}=c_{2 \mathbf{k}+(1,0)}, e_{\mathbf{k}}^{(3)}=c_{2 \mathbf{k}+(0,1)}, \mathbf{k} \in \mathbf{Z}^{2} \tag{6}
\end{equation*}
$$

Refer to the middle picture of Fig. 3 for these four groups of data/vertices.


Figure 3: Left: Type V nodes, type F nodes, and two types of type E nodes; Middle: Original data/vertices associated with four groups of nodes; Right: "Approximation" and "details" associated with four groups of nodes

Let $c_{\mathbf{k}}^{1}$ and $d_{\mathbf{k}}^{(\ell, 1)}, 1 \leq \ell \leq 4$ be the lowpass and highpass outputs with initial input $\mathcal{C}=\left\{c_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ after the decomposition algorithm with lowpass and highpass filters $p$ and $q^{(\ell)}$. Denote

$$
\widetilde{v}_{\mathbf{k}}=c_{\mathbf{k}}^{1}, \widetilde{g}_{\mathbf{k}}=d_{\mathbf{k}}^{(1,1)}, \widetilde{f}_{\mathbf{k}}=d_{\mathbf{k}}^{(2,1)}, \widetilde{e}_{\mathbf{k}}^{(2)}=d_{\mathbf{k}}^{(3,1)}, \widetilde{e}_{\mathbf{k}}^{(3)}=d_{\mathbf{k}}^{(4,1)}
$$

Then, the decomposition algorithm (4) is

$$
\left\{\begin{array}{l}
\widetilde{v}_{\mathbf{k}}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} p_{\mathbf{k}^{\prime}-2 \mathbf{k}} c_{\mathbf{k}^{\prime}}, \widetilde{g}_{\mathbf{k}}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-2 \mathbf{k}^{(1)}}^{(1)} c_{\mathbf{k}^{\prime}},  \tag{7}\\
\widetilde{f}_{\mathbf{k}}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-2 \mathbf{k}}^{(2)} c_{\mathbf{k}^{\prime}}, \widetilde{e}_{\mathbf{k}}^{(2)}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-2 \mathbf{k}} c_{\mathbf{k}^{\prime}}, \widetilde{e}_{\mathbf{k}}^{(3)}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-2 \mathbf{k}}^{(4)} c_{\mathbf{k}^{\prime}}
\end{array}\right.
$$

for $\mathbf{k} \in \mathbf{Z}^{2}$, and the reconstruction algorithm (5) is

$$
\begin{equation*}
c_{\mathbf{k}}=\sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}}\{\widetilde{p}_{\mathbf{k}-2 \mathbf{k}^{\prime}} \widetilde{v}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-2 \mathbf{k}^{\prime}}^{(1)} \widetilde{g}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-2 \mathbf{k}^{\prime}}^{(2)} \widetilde{f}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-2 \mathbf{k}^{\prime}}^{(3)}{\widetilde{\mathbf{k}^{\prime}}}_{(2)}+\widetilde{q}_{\mathbf{k}-2 \mathbf{k}^{\prime}}^{(4)} \overbrace{\mathbf{k}^{\prime}}^{(3)}\}, \mathbf{k} \in \mathbf{Z}^{2} \tag{8}
\end{equation*}
$$

When $\mathbf{k}$ is respectively $2 \mathbf{j}, 2 \mathbf{j}-(1,1), 2 \mathbf{j}+(1,0)$, and $2 \mathbf{j}+(0,1)$, then accordingly, $c_{\mathbf{k}}$ is $v_{\mathbf{j}}, f_{\mathbf{j}}, e_{\mathbf{j}}^{(2)}, e_{\mathbf{j}}^{(3)}$ as defined in (6). Thus the reconstruction algorithm (8) can be further written as

$$
\begin{align*}
& v_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{\mathbf{2}} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}}^{(1)} \widetilde{g}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}}^{(2)} \widetilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{\mathbf{q} \mathbf{n}}^{(3)} \widetilde{\mathrm{n}}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}}^{(4)} \widetilde{\rho}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\}, \\
& f_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{\mathbf{2}}}\left\{\widetilde{p}_{\mathbf{n}-(1,1)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}-(1,1)}^{(1)} \widetilde{g}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}-(1,1)}^{(2)} \widetilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}-(1,1)}^{(3)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}-(1,1)}^{(4)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\}, \\
& e_{\mathbf{k}}^{(2)}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{\mathbf{n}+(1,0)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{\mathbf{2 n}+(1,0)}^{(1)} \widetilde{g}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(1,0)}^{(2)} \widetilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(1,0)}^{(3)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}+(1,0)}^{(4)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\},  \tag{9}\\
& e_{\mathbf{k}}^{(3)}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{\mathbf{2}+(0,1)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(0,1)}^{(1)} \widetilde{g}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(0,1)}^{(2)} \widetilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(0,1)}^{(3)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}+(0,1)}^{(4)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\} .
\end{align*}
$$

Next, we associate both the lowpass output $\widetilde{v}_{\mathbf{k}}$ and the first highpass output $\widetilde{g}_{\mathbf{k}}$ with type $V$ nodes with labels $2 \mathbf{k}$, and the second highpass output $\widetilde{f}_{\mathbf{k}}$ with type $F$ nodes with labels $2 \mathbf{k}-(1,1)$ , and third and fourth highpass output $\widetilde{e}_{\mathbf{k}}^{(2)}, \widetilde{e}_{\mathbf{k}}^{(3)}$ with type $E$ nodes with labels $2 \mathbf{k}+(1,0)$ and $2 \mathbf{k}+(0,1)$ respectively. In this way, both analysis and synthesis algorithms can be represented as templates.

If we set "details" $\widetilde{g}_{\mathbf{k}}, \widetilde{f}_{\mathbf{k}}, \widetilde{e}_{\mathbf{k}}^{(1)}, \widetilde{e}_{\mathbf{k}}^{(2)}$ to be zero, then (9) is reduced to the subdivision algorithm:

$$
v_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{2 \mathbf{n}} \widetilde{v}_{\mathbf{k}-\mathbf{n}}, f_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{2 \mathbf{n}-(1,1)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}, e_{\mathbf{k}}^{(2)}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{2 \mathbf{n}+(1,0)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}, e_{\mathbf{k}}^{(3)}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{2 \mathbf{n}+(0,1)} \widetilde{v}_{\mathbf{k}-\mathbf{n}},
$$

which can be represented as the subdivision templates. For example, if $\widetilde{p}$ is the filter for $C^{2}$ bi-spline supported on $[-2,2]^{2}$ :

$$
\begin{equation*}
\widetilde{p}(\boldsymbol{\omega})=\frac{e^{2 i\left(\omega_{1}+\omega_{2}\right)}}{256}\left(1+e^{-i \omega_{1}}\right)^{4}\left(1+e^{-i \omega_{2}}\right)^{4}, \tag{10}
\end{equation*}
$$

then the subdivision templates are shown in Fig. 4. This subdivision scheme is called the CatmullClark scheme [4] (for regular vertices).


Figure 4: Catmull-Clark scheme
As the subdivision templates, when analysis and synthesis algorithm templates are used for surface processing, these templates must have certain symmetry. Firstly, because both $\widetilde{e}_{\mathbf{k}}^{(2)}$ and $\widetilde{e}_{\mathbf{k}}^{(3)}$ associate with type $E$ vertices and they should be treated equally, the templates to obtain $\widetilde{e}_{\mathbf{k}}^{(2)}$ and $\widetilde{e}_{\mathbf{k}}^{(3)}$ must be same. With the same reason, the templates to recover $e_{\mathbf{k}}^{(2)}$ and $e_{\mathbf{k}}^{(3)}$ by (9) should also be the same. Secondly, the templates to obtain $\widetilde{v}_{\mathbf{k}}, \widetilde{g}_{\mathbf{k}}$ and $\widetilde{f}_{\mathbf{k}}$ by $(7)$, and that to recover
to $v_{\mathbf{k}}$ and $f_{\mathbf{k}}$ by (9) must rotational and reflective invariant. Thirdly, the template to obtain $\widetilde{e}_{\mathbf{k}}^{(2)}$ and $\widetilde{e}_{\mathbf{k}}^{(3)}$ and the template to recover $e_{\mathbf{k}}^{(2)}$ and $e_{\mathbf{k}}^{(3)}$ have certain symmetry. The 4 -fold symmetric biorthogonal frame filter banks result in templates with such desired symmetry.


Figure 5: Decomposition and reconstruction algorithms
Since in this paper we consider such algorithms that the templates to obtain $\widetilde{e}_{\mathbf{k}}^{(2)}, \widetilde{e}_{\mathbf{k}}^{(3)}$ are the same, and those to recover $e_{\mathbf{k}}^{(2)}, e_{\mathbf{k}}^{(3)}$ are also identical, in the following we may simply let $e$ denote the original data associated with type $E$ nodes, and use $\widetilde{e}$ to denote the third and fourth highpass outputs after the decomposition algorithm. Thus, the decomposition algorithm is to decompose the original data $\{v\} \cup\{f\} \cup\{e\}$ into $\{\widetilde{v}\},\{\widetilde{g}\},\{\widetilde{f}\}$ and $\{\widetilde{e}\}$, and the reconstruction algorithm to recover $\{v\} \cup\{f\} \cup\{e\}$ from $\{\widetilde{v}\},\{\widetilde{g}\}\{\widetilde{f}\}$ and $\{\widetilde{e}\}$, see Fig. 5. Therefore, we may simply use $v, f, e$ and $\widetilde{v}, \widetilde{g}, \widetilde{f}, \widetilde{e}$ describe frame algorithms. For a dyadic multiresolution algorithm with 4-fold symmetric frame filter banks, the decomposition algorithm (reconstruction algorithm resp.) can be represented as four templates (three templates resp.).

In the following we provide a characterization of the 4 -fold symmetry of a frame filter bank $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$. As mentioned above, in this paper to construct bi-frames, we start with symmetric templates. We can use the characterization of symmetry to verify that symmetric templates do yield frame filter banks with 4 -fold symmetry. This characterization will also be useful if one uses a different approach to construct bi-frames with 4-fold axial symmetry.

Proposition 1. A dyadic frame filter bank $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ has 4-fold axial symmetry if and only if its polyphase matrix $V(\boldsymbol{\omega})$ satisfies

$$
\begin{equation*}
V\left(J_{0} \boldsymbol{\omega}\right)=M_{01} V(\boldsymbol{\omega}) M_{02}, V\left(O_{1} \boldsymbol{\omega}\right)=M_{1}(\boldsymbol{\omega}) V(\boldsymbol{\omega}) M_{2}(\boldsymbol{\omega}) \tag{11}
\end{equation*}
$$

where

$$
\begin{gather*}
O_{1}=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right], J_{0}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], M_{01}=\left[\begin{array}{cc}
I_{3} & \mathbf{0} \\
\mathbf{0} & J_{0}
\end{array}\right], M_{02}=\left[\begin{array}{cc}
I_{2} & \mathbf{0} \\
\mathbf{0} & J_{0}
\end{array}\right],  \tag{12}\\
M_{1}(\boldsymbol{\omega})=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & e^{-i \omega_{1}} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & e^{i \omega_{1}} & 0
\end{array}\right], \quad M_{2}(\boldsymbol{\omega})=\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & e^{i \omega_{1}} & 0 & 0 \\
0 & 0 & 0 & e^{-i \omega_{1}} \\
0 & 0 & 1 & 0
\end{array}\right] .
\end{gather*}
$$

A similar characterization of the 4 -fold symmetry of dyadic wavelet filter banks is provided in [22]. We say a wavelet filter bank $\left\{p, q^{(2)}, q^{(3)}, q^{(4)}\right\}$ to have 4-fold symmetry if $p, q^{(2)}, q^{(3)}, q^{(4)}$
satisfy the conditions in Definition 1 (ignore the condition for $q^{(1)}$ ). Compared with a 4 -fold symmetric wavelet filter bank, a 4 -fold symmetric frame filter bank has only one extra highpass filter $q^{(1)}(\boldsymbol{\omega})$ which has the same symmetry as the lowpass filter $p(\boldsymbol{\omega})$. Thus, one can give the proof of Proposition 1 similarly to that for the characterization of the 4 -fold symmetry of dyadic wavelet filter banks given in [22]. The details are omitted here.

Assume that two FIR frame filter banks $\left\{p_{0}, q_{0}^{(1)}, \cdots, q_{0}^{(4)}\right\}$ and $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ are related by $\left[p, q^{(1)}, \cdots, q^{(4)}\right]^{T}(\boldsymbol{\omega})=B(2 \boldsymbol{\omega})\left[p_{0}, q_{0}^{(1)}, \cdots, q_{0}^{(4)}\right]^{T}(\boldsymbol{\omega})$, where $B(\boldsymbol{\omega})$ is a $4 \times 4$ trigonometric polynomial matrix. Then their polyphase matrices, denoted as $V_{0}(\boldsymbol{\omega})$ and $V(\boldsymbol{\omega})$, satisfy

$$
V(\boldsymbol{\omega})=B(\boldsymbol{\omega}) V_{0}(\boldsymbol{\omega}) .
$$

From Proposition 1, we know if $\left\{p_{0}, q_{0}^{(1)}, \cdots, q_{0}^{(4)}\right\}$ has 4-fold symmetry, then $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ has 4-fold symmetry if and only if $B(\boldsymbol{\omega})$ satisfies

$$
B\left(J_{0} \boldsymbol{\omega}\right)=M_{01} B(\boldsymbol{\omega}) M_{01}, B\left(O_{1} \boldsymbol{\omega}\right)=M_{1}(\boldsymbol{\omega}) B(\boldsymbol{\omega}) M_{1}(\boldsymbol{\omega})^{-1}
$$

This observation enables us to construct a 4 -fold symmetric frame filter bank from another one by choosing a $B(\boldsymbol{\omega})$ satisfying the above condition.

To construct 4 -fold symmetric bi-frames, we start with symmetric templates of decomposition and reconstruction algorithms. The algorithm templates are given by several iterative steps with each step given by a template. With the templates and decomposition and reconstruction algorithms (7)(9), we then obtain the corresponding bi-frame filter banks which are given by some parameters. Then we select the parameters based on the smoothness and vanishing moments of framelets.

For the smoothness of framelets, which is determined by the smoothness of the corresponding scaling functions, in this paper we consider the Sobolev smoothness. We say a function $f$ on $\mathbb{R}^{2}$ to be in the Sobolev space $W^{s}$ for some $s>0$ if its Fourier transform $\hat{f}$ satisfies $\int_{\mathbb{R}^{2}}(1+$ $\left.|\boldsymbol{\omega}|^{2}\right)^{s}|\hat{f}(\boldsymbol{\omega})|^{2} d \boldsymbol{\omega}<\infty$. The Sobolev smoothness of a scaling function $\phi$ can be given by the eigenvalues of the transition operator matrix associated with the corresponding lowpass filter $p$, see [19, 20].

To construct a smooth wavelet basis, the corresponding scaling function $\phi$ must have certain approximation power, which can be described by the sum rule order of the associated subdivision mask $p(\boldsymbol{\omega})$, see e.g. [18]. For the dyadic refinement, we say $p(\boldsymbol{\omega})$ to have sum rule order $K$ if it satisfies that $p(0,0)=1$ and

$$
\left.\frac{\partial^{\alpha_{1}+\alpha_{2}}}{\partial \omega_{1}^{\alpha_{1}} \partial \omega_{2}^{\alpha_{2}}} p\left(\omega_{1}, \omega_{2}\right)\right|_{\left(\omega_{1}, \omega_{2}\right)=\pi \eta_{j}}=0,1 \leq j \leq 3,
$$

for all $\left(\alpha_{1}, \alpha_{2}\right) \in \mathbf{Z}_{+}^{2}$ with $\alpha_{1}+\alpha_{2}<K$, where $\boldsymbol{\eta}_{j}, 1 \leq j \leq 3$ are defined by (2).
For an FIR (highpass) filter $q(\boldsymbol{\omega})$, we say it has the vanishing moments of order $J$ if

$$
\left.\frac{\partial^{\alpha_{1}+\alpha_{2}}}{\partial \omega_{1}^{\alpha_{1}} \partial \omega_{2}^{\alpha_{2}}} q\left(\omega_{1}, \omega_{2}\right)\right|_{\left(\omega_{1}, \omega_{2}\right)=(0,0)}=0
$$

for all $\left(\alpha_{1}, \alpha_{2}\right) \in \mathbf{Z}_{+}^{2}$ with $\alpha_{1}+\alpha_{2}<J$. It is important in signal/image processing and other applications that highpass filters have vanishing moments.

When we construct bi-frames, in general we choose the parameters such that the synthesis scaling function $\widetilde{\phi}$ is smoother than the analysis scaling function $\phi$, the synthesis lowpass filter
$\widetilde{p}(\boldsymbol{\omega})$ has a higher sum rule order than the analysis lowpass filter $p(\boldsymbol{\omega})$, and that the analysis highpass filters $q^{(\ell)}(\boldsymbol{\omega})$ have higher vanishing moments. In case we can choose the parameters such that $\widetilde{\phi}$ is a compactly supported bi-spline or box-spline with certain smoothness, then we will first consider such choices of the parameters.

### 2.2 3-step dyadic bi-frame multiresolution algorithm



Figure 6: Top-left: Template to obtain $v^{\prime \prime}, g^{\prime \prime}$ in Decomposition Alg. Step 1; Top-right: Decomposition Alg. Step 2; Bottom-left: Template to obtain lowpass output $\widetilde{v}$ in Decomposition Alg. Step 3 (template to obtain first highpass output $\widetilde{g}$ is similar with $v^{\prime \prime}$ replaced by $g^{\prime \prime}$ )

This subsection is about a 3 -step dyadic multiresolution algorithm. The decomposition algorithm is given by (13)-(15) and shown in Fig. 6, where $b, d, s, n, m, a, c, h, j, k, r, d_{1}, s_{1}, n_{1}, m_{1}$ are some constants. Namely, first we replace all $v$ associated with type $V$ nodes of $2 \mathbf{Z}^{2}$ by $v^{\prime \prime}, g^{\prime \prime}$ given by (13). Then, with $v^{\prime \prime}, g^{\prime \prime}$ obtained, we replace all $f$ associated with type $F$ nodes and all $e$ associated with type $E$ nodes by $\widetilde{f}$ and $\widetilde{e}$ respectively with the formulas in (14). Finally, based on $\widetilde{f}, \tilde{e}$ obtained, all $v^{\prime \prime}, g^{\prime \prime}$ in Step 1 are updated by $\widetilde{v}$ and $\widetilde{g}$ given in (15).

## 3-step Decomposition Algorithm:

Step 1. $\left\{\begin{array}{l}v^{\prime \prime}=\frac{1}{b}\left\{v-d\left(e_{0}+e_{1}+e_{2}+e_{3}\right)-s\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}, \\ g^{\prime \prime}=v-n\left(e_{0}+e_{1}+e_{2}+e_{3}\right)-m\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\end{array}\right.$
Step 2. $\left\{\begin{array}{l}\tilde{e}=e-a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}\right)-c\left(v_{2}^{\prime \prime}+v_{3}^{\prime \prime}+v_{4}^{\prime \prime}+v_{5}^{\prime \prime}\right)-h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}\right)-j\left(g_{2}^{\prime \prime}+g_{3}^{\prime \prime}+g_{4}^{\prime \prime}+g_{5}^{\prime \prime}\right)(14) \\ \tilde{f}=f-k\left(v_{6}^{\prime \prime}+v_{7}^{\prime \prime}+v_{8}^{\prime \prime}+v_{9}^{\prime \prime}\right)-r\left(g_{6}^{\prime \prime}+g_{7}^{\prime \prime}+g_{8}^{\prime \prime}+g_{9}^{\prime \prime}\right)\end{array}\right.$

Step 3. $\left\{\begin{array}{l}\widetilde{v}=v^{\prime \prime}-d_{1}\left(\widetilde{e}_{0}+\widetilde{e}_{1}+\widetilde{e}_{2}+\widetilde{e}_{3}\right)-s_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right), \\ \widetilde{g}=g^{\prime \prime}-n_{1}\left(\widetilde{e}_{0}+\widetilde{e}_{1}+\widetilde{e}_{2}+\widetilde{e}_{3}\right)-m_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right) .\end{array}\right.$


Figure 7: Top-right: Template to obtain $v^{\prime \prime}$ in Reconstruction Alg. Step 1 (template to obtain $g^{\prime \prime}$ is similar with $\widetilde{v}$ replaced by $\widetilde{g}$ ); Bottom-left: Reconstruction Alg. Step 2; Bottom-right: Reconstruction Alg. Step 3

The reconstruction algorithm of this 3 -step algorithm is the reverse algorithm of the decomposition algorithm. It is given by (16)-(18) and shown in Fig. 7, where $b, d, s, n, m, a, c, h, j$, $k, r, d_{1}, s_{1}, n_{1}, m_{1}$ are the same constants in the decomposition algorithm and $t \in \mathbb{R}$. More precisely, first we replace the lowpass output $\widetilde{v}$ and the first highpass output $\widetilde{g}$, both associated with type $V$ nodes of $2 \mathbf{Z}^{2}$, by $v^{\prime \prime}$ and $g^{\prime \prime}$ respectively given by the formulas in (16). After that, with $v^{\prime \prime}, g^{\prime \prime}$ obtained, we replace all $\widetilde{f}$ and $\widetilde{e}$ by $f$ and $e$ respectively with the formulas in (17). This step recovers the original data (vertices) associated with type $F$ and type $E$ nodes of $\mathbf{Z}^{2} \backslash\left(2 \mathbf{Z}^{2}\right)$. Finally, based on $f, e$ obtained in Step 2, all $v^{\prime \prime}, g^{\prime \prime}$ in Step 1 are replaced by $v$ with the resulting $v$ given by formula (18). The final step recovers the original data (vertices) associated with type $V$ nodes of $2 \mathbf{Z}^{2}$.

## 3-step Reconstruction Algorithm:

Step 1. $\left\{\begin{aligned} v^{\prime \prime} & =\widetilde{v}+d_{1}\left(\widetilde{e}_{0}+\widetilde{e}_{1}+\widetilde{e}_{2}+\widetilde{e}_{3}\right)+s_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right), \\ g^{\prime \prime} & =\widetilde{g}+n_{1}\left(\widetilde{e}_{0}+\widetilde{e}_{1}+\widetilde{e}_{2}+\widetilde{e}_{3}\right)+m_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right)\end{aligned}\right.$
Step 2. $\left\{\begin{array}{l}e=\tilde{e}+a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}\right)+c\left(v_{2}^{\prime \prime}+v_{3}^{\prime \prime}+v_{4}^{\prime \prime}+v_{5}^{\prime \prime}\right)+h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}\right)+j\left(g_{2}^{\prime \prime}+g_{3}^{\prime \prime}+g_{4}^{\prime \prime}+g_{5}^{\prime \prime}\right)(17) \\ f=\widetilde{f}+k\left(v_{6}^{\prime \prime}+v_{7}^{\prime \prime}+v_{8}^{\prime \prime}+v_{9}^{\prime \prime}\right)+r\left(g_{6}^{\prime \prime}+g_{7}^{\prime \prime}+g_{8}^{\prime \prime}+g_{9}^{\prime \prime}\right) .\end{array}\right.$

Step 3.

$$
\begin{align*}
v= & t\left\{b v^{\prime \prime}+d\left(e_{0}+e_{1}+e_{2}+e_{3}\right)+s\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}+  \tag{18}\\
& (1-t)\left\{g^{\prime \prime}+n\left(e_{0}+e_{1}+e_{2}+e_{3}\right)+m\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\} .
\end{align*}
$$

To obtain the filters corresponding to this 3 -step algorithm, let us first consider the case when $d_{1}=s_{1}=n_{1}=m_{1}=0$. In this case, the 3 -step algorithm is a 2 -step algorithm with the decomposition algorithm given by (13)(14) (with $\widetilde{v}=v^{\prime \prime}, \widetilde{g}=g^{\prime \prime}$ ) and the reconstruction algorithm given by $(17)(18)\left(\right.$ with $\left.v^{\prime \prime}=\widetilde{v}, g^{\prime \prime}=\widetilde{g}\right)$.

With the formulas in (7) and (9), one can obtain that the filter banks $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ corresponding to this 2-step algorithm are

$$
\begin{aligned}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), \cdots, q^{(4)}(\boldsymbol{\omega})\right]^{T}=E_{1}(2 \boldsymbol{\omega}) E_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega}),} \\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \cdots, \widetilde{q}^{(4)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{4} \widetilde{E}_{1}(2 \boldsymbol{\omega}) \widetilde{E}_{0}(2 \omega) I_{00}(\boldsymbol{\omega}),}
\end{aligned}
$$

where

$$
\begin{align*}
& E_{0}(\boldsymbol{\omega})=\left[\begin{array}{cccc}
\frac{1}{b} & -\frac{s}{b}(1+x)(1+y) & -\frac{d}{b}\left(1+\frac{1}{x}\right) & -\frac{d}{b}\left(1+\frac{1}{y}\right) \\
1 & -m(1+x)(1+y) & -n\left(1+\frac{1}{x}\right) & -n\left(1+\frac{1}{y}\right) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right],  \tag{19}\\
& E_{1}(\boldsymbol{\omega})=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-k\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & -r\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & 1 & 0 & 0 \\
-(1+x)\left(a+c y+\frac{c}{y}\right) & -(1+x)\left(h+j y+\frac{j}{y}\right) & 0 & 1 & 0 \\
-(1+y)\left(a+c x+\frac{c}{x}\right) & -(1+y)\left(h+j x+\frac{1}{x}\right) & 0 & 0 & 1
\end{array}\right],  \tag{20}\\
& \widetilde{E}_{0}(\boldsymbol{\omega})=\left[\begin{array}{cccc}
t b & 0 & 0 & 0 \\
1-t & 0 & 0 & 0 \\
(t s+(1-t) n)\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & 1 & 0 & 0 \\
(t d+(1-t) n)(1+x) & 0 & 1 & 0 \\
(t d+(1-t) n)(1+y) & 0 & 0 & 1
\end{array}\right],  \tag{21}\\
& \widetilde{E}_{1}(\boldsymbol{\omega})=\left[\begin{array}{ccccc}
1 & 0 & k(1+x)(1+y) & \left(1+\frac{1}{x}\right)\left(a+c y+\frac{c}{y}\right) & \left(1+\frac{1}{y}\right)\left(a+c x+\frac{c}{x}\right) \\
0 & 1 & r(1+x)(1+y) & \left(1+\frac{1}{x}\right)\left(h+j y+\frac{1}{y}\right) & \left(1+\frac{1}{y}\right)\left(h+j x+\frac{j}{x}\right) \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right] . \tag{22}
\end{align*}
$$

Through this paper, we use the notations:

$$
x=e^{-i \omega_{1}}, y=e^{-i \omega_{2}} .
$$

Observe that the polyphase matrices $V(\boldsymbol{\omega})$ and $\widetilde{V}(\boldsymbol{\omega})$ of $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ are $2 E_{1}(\boldsymbol{\omega}) E_{0}(\boldsymbol{\omega})$ and $\frac{1}{2} \widetilde{E}_{1}(\boldsymbol{\omega}) \widetilde{E}_{0}(\boldsymbol{\omega})$ respectively. One can easily show that $E_{0}(\boldsymbol{\omega})^{*} \widetilde{E}_{0}(\boldsymbol{\omega})=$ $I_{4}, E_{1}(\boldsymbol{\omega})^{*} \widetilde{E}_{1}(\boldsymbol{\omega})=I_{5}, \boldsymbol{\omega} \in \mathbb{R}^{2}$. Thus, $V(\boldsymbol{\omega})^{*} \widetilde{V}(\boldsymbol{\omega})=I_{4}, \boldsymbol{\omega} \in \mathbb{R}^{2}$, and hence, $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ are indeed biorthogonal. Furthermore, one can also easily show that $V(\boldsymbol{\omega})$ and $\widetilde{V}(\boldsymbol{\omega})$ satisfy (11). Thus both $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ are 4 -fold symmetric.

Solving the system of equations for sum rule order 1 of $\widetilde{p}$ and $p$, and for vanishing moment order 1 of $q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$, we have

$$
b=4, d=-\frac{1}{2}, s=-\frac{1}{4}, k=\frac{1}{4}, a=\frac{1}{2}-2 c, m=\frac{1}{4}-n, t=1 .
$$

The resulting $p(\boldsymbol{\omega})=\frac{1}{16 x y}(1+x)^{2}(1+y)^{2}$. Thus, the corresponding $\phi$ is the tensor product of continuous linear splines supported on $[-1,1]$. The resulting $\widetilde{\sim}(\boldsymbol{\omega})$ depends on $c$. If $c=-\frac{1}{8}$ we have the (numerically) best Sobolev smooth $\widetilde{\phi}$ with $\widetilde{\phi} \in W^{0.44076}$. To construct smoother framelets we need consider algorithms with more iterative steps such as the 3-step algorithm.

Before we discuss the 3-step algorithm, here we remark that if we choose

$$
\begin{equation*}
b=4, d=-\frac{1}{2}, s=-\frac{1}{4}, m=\frac{1}{4}-n, a=\frac{3}{8}, c=\frac{1}{16}, k=\frac{1}{4}, t=\frac{1-2 n}{7-2 n} \tag{23}
\end{equation*}
$$

then $p$ and $\widetilde{p}$ have sum rule orders 2 and 4 , and $q^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moment order 2 with $p(\omega)=\frac{1}{16 x y}(1+x)^{2}(1+y)^{2}$. Furthermore, if $n=\frac{3}{10}$, then the resulting $\widetilde{p}$ is the filter given by (10); while if $n=\frac{1}{2}$, the resulting $\widetilde{p}$ is

$$
\begin{equation*}
\widetilde{p}(\boldsymbol{\omega})=\frac{1}{64}(1+x)^{2}(1+y)^{2}\left(\frac{1}{x}+\frac{1}{y}\right)\left(1+\frac{1}{x y}\right) \tag{24}
\end{equation*}
$$

Thus the corresponding $\widetilde{\phi}$ in the former case is the $C^{2}$ cubic bi-spline supported on $[-2,2]^{2}$, and $\widetilde{\phi}$ in the latter case is the $C^{2}$ box-spline with direction set (refer to [3] for box-splines)

$$
\Theta=\left[\begin{array}{cccccc}
1 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & 1 & -1 & -1
\end{array}\right]
$$

The filter $\widetilde{p}$ in (10) results in Catmull-Clark subdivision scheme [4] as mentioned above, while the subdivision scheme derived from the filter $\widetilde{p}$ in $(24)$ is used in [32]. However, in either case, we cannot choose the remaining parameters $j, h, r$ such that all $\widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moments. In Appendix A, we provide the resulting filters with $n=\frac{3}{10}, j=0, h=\frac{15}{8}, r=\frac{15}{16}$ and that with $n=\frac{1}{2}, j=h=r=0$. When $n=\frac{3}{10}, h=\frac{15}{8}-2 j, r=\frac{15}{16}, q^{(2)}, q^{(3)}, q^{(4)}$ have vanishing moment order 4.

Next let us consider the 3-step algorithm. With the formulas in (7) and (9), and the filters for the 2-step algorithm discussed above, one can obtain the filter banks $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ corresponding to (13)-(18) to be

$$
\begin{aligned}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), \cdots, q^{(4)}(\boldsymbol{\omega})\right]^{T}=E_{2}(2 \boldsymbol{\omega}) E_{1}(2 \boldsymbol{\omega}) E_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})} \\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \cdots, \widetilde{q}^{(4)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{4} \widetilde{E}_{2}(2 \boldsymbol{\omega}) \widetilde{E}_{1}(2 \boldsymbol{\omega}) \widetilde{E}_{0}(2 \omega) I_{00}(\boldsymbol{\omega})}
\end{aligned}
$$

where $E_{0}(\boldsymbol{\omega}), E_{1}(\boldsymbol{\omega}), \widetilde{E}_{0}(\boldsymbol{\omega}), \widetilde{E}_{1}(\boldsymbol{\omega})$ are defined by (19)-(22), and

$$
E_{2}(\boldsymbol{\omega})=\left[\begin{array}{ccccc}
1 & 0 & -s_{1}(1+x)(1+y) & -d_{1}\left(1+\frac{1}{x}\right) & -d_{1}\left(1+\frac{1}{y}\right)  \tag{25}\\
0 & 1 & -m_{1}(1+x)(1+y) & -n_{1}\left(1+\frac{1}{x}\right) & -n_{1}\left(1+\frac{1}{y}\right) \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

and $\widetilde{E}_{2}(\boldsymbol{\omega})=\left(E_{2}(\boldsymbol{\omega})^{-1}\right)^{*}$ :

$$
\widetilde{E}_{2}(\boldsymbol{\omega})=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0  \tag{26}\\
0 & 1 & 0 & 0 & 0 \\
s_{1}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & m_{1}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & 1 & 0 & 0 \\
d_{1}(1+x) & n_{1}(1+x) & 0 & 1 & 0 \\
d_{1}(1+y) & n_{1}(1+y) & 0 & 0 & 1
\end{array}\right]
$$

In the following we choose the parameters such that the resulting $\widetilde{\phi}$ is $C^{2}$ box-spline and $C^{2}$ cubic bi-spline with $\widetilde{p}$ given by (24) and (10) respectively and that all $q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moments.

If

$$
\begin{align*}
& a=\frac{3}{8}, k=\frac{1}{4}, c=\frac{1}{16}, d_{1}=-\frac{3}{16}, s_{1}=-\frac{3}{64}, d=b+n-4 n b, m=\frac{1}{4}-n,  \tag{27}\\
& s=\frac{1}{4}-n-\frac{5}{4} b+4 n b, r=\frac{1}{12 b}(1-4 b-48 b j-24 h-b), t=\frac{1}{4 b},
\end{align*}
$$

then the resulting $\widetilde{p}$ is the filter given by (10), $p$ has sum rule order $2, q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moment order 2 . The resulting $p$ depends on $b, n, h, j$. Next we choose these parameters based on the smoothness of $\phi$. We can choose them such that $\phi$ is in $W^{1.27204}$, the (numerically) best smoothness order $\phi$ can gain. If

$$
\begin{equation*}
b=10, n=\frac{3}{4}, h=1, j=\frac{9}{32} \text {, } \tag{28}
\end{equation*}
$$

then $\phi \in W^{1.2373}$. If $n=\frac{1}{2}$, then all $q^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moment order 4 . However, in this case, we cannot choose other parameters such that $\phi$ is in $L^{2}\left(\mathbb{R}^{2}\right)$. In the following we provide the parameters $d, m, s, r, t$ in (27) when $b, n, h, j$ are given by (28):

$$
d=-\frac{31}{32}, m=-\frac{1}{32}, s=-\frac{41}{32}, r=-\frac{213}{40}, t=\frac{1}{40} .
$$

If

$$
\begin{equation*}
a=\frac{3}{8}, k=\frac{1}{4}, c=\frac{1}{16}, n=\frac{1}{2}, m=d_{1}=-\frac{1}{4}, t=0, h=-\frac{1}{8}-2 j, s=\frac{1}{4}-\frac{1}{4} b-d, \tag{29}
\end{equation*}
$$

then the resulting $\widetilde{p}$ is the filter given by (24), $p$ has sum rule order $2, q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moment order 2 . The resulting $p$ depends on $b, d, j$. We can choose them such that $\phi$ is in $W^{1.25875}$, the (numerically) best smoothness order $\phi$ can gain. If $b=6, d=-\frac{5}{8}, j=\frac{1}{32}$, then $\phi \in W^{1.25492}$. If $d=\frac{1}{2}-b$, then all $q^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moment order 4 . In this case, $p$ depends on $b, j$. However, we cannot choose $b, j$ such that $\phi$ is in $L^{2}\left(\mathbb{R}^{2}\right)$. We provide in Appendix B the coefficients of resulting filters with $b=6, d=-\frac{5}{8}, j=\frac{1}{32}, r=n_{1}=m_{1}=0$ and other parameters given in (29).

The scaling functions $\phi$ constructed above are supported on $[-3,3]^{2}$. From Example 2 in [22], we know if $\widetilde{\phi}$ is the $C^{2}$ bi-spline associated with $\widetilde{p}(\boldsymbol{\omega})$ given by (10), then it is impossible to construct 4-fold symmetric biorthogonal wavelets with the analysis scaling function $\phi$ in $L^{2}\left(\mathbb{R}^{2}\right)$ and supported on $[-5,5]^{2}$. Thus, the frame system does provide the flexibility for the construction of compactly supported biorthogonal system generators.

### 2.3 4-step dyadic bi-frame multiresolution algorithm

In this subsection we discuss a 4 -step multiresolution algorithm. The decomposition algorithm is given by (30)-(33) and shown in Fig. 8, and the multiresolution reconstruction algorithm is given by (34)-(37) and shown in Fig. 9, where $b, d, s, n, m, a, c, h, j, k, r, d_{1}, s_{1}, n_{1}, m_{1}, a_{1}, c_{1}, h_{1}, j_{1}$, $k_{1}, r_{1}, t$ are constants to be determined.

## 4-step Decomposition Algorithm:

Step 1. $\left\{\begin{array}{l}v^{\prime \prime}=\frac{1}{b}\left\{v-d\left(e_{0}+e_{1}+e_{2}+e_{3}\right)-s\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}, \\ g^{\prime \prime}=v-n\left(e_{0}+e_{1}+e_{2}+e_{3}\right)-m\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\end{array}\right.$


Figure 8: Top-left: Template to obtain $v^{\prime \prime}, g^{\prime \prime}$ in Decomposition Alg. Step 1; Top-right: Decomposition Alg. Step 2; Bottom-left: Template to obtain lowpass output $\widetilde{v}$ in Decomposition Alg. Step 3 (template to obtain first highpass output $\widetilde{g}$ is similar with $v^{\prime \prime}$ replaced by $g^{\prime \prime}$ ); Bottom-right: Decomposition Alg. Step 4

Step 2. $\left\{\begin{array}{l}e^{\prime \prime}=e-a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}\right)-c\left(v_{2}^{\prime \prime}+v_{3}^{\prime \prime}+v_{4}^{\prime \prime}+v_{5}^{\prime \prime}\right)-h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}\right)-j\left(g_{2}^{\prime \prime}+g_{3}^{\prime \prime}+g_{4}^{\prime \prime}+g_{5}^{\prime \prime}\right)(31) \\ f^{\prime \prime}=f-k\left(v_{6}^{\prime \prime}+v_{7}^{\prime \prime}+v_{8}^{\prime \prime}+v_{9}^{\prime \prime}\right)-r\left(g_{6}^{\prime \prime}+g_{7}^{\prime \prime}+g_{8}^{\prime \prime}+g_{9}^{\prime \prime}\right)\end{array}\right.$
Step 3. $\left\{\begin{array}{l}\widetilde{v}=v^{\prime \prime}-d_{1}\left(e_{0}^{\prime \prime}+e_{1}^{\prime \prime}+e_{2}^{\prime \prime}+e_{3}^{\prime \prime}\right)-s_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right), \\ \widetilde{g}=g^{\prime \prime}-n_{1}\left(e_{0}^{\prime \prime}+e_{1}^{\prime \prime}+e_{2}^{\prime \prime}+e_{3}^{\prime \prime}\right)-m_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)\end{array}\right.$
Step 4. $\left\{\begin{array}{l}\tilde{e}=e^{\prime \prime}-a_{1}\left(\widetilde{v}_{0}+\widetilde{v}_{1}\right)-c_{1}\left(\widetilde{v}_{2}+\widetilde{v}_{3}+\widetilde{v}_{4}+\widetilde{v}_{5}\right)-h_{1}\left(\widetilde{g}_{0}+\widetilde{g}_{1}\right)-j_{1}\left(\widetilde{g}_{2}+\widetilde{g}_{3}+\widetilde{g}_{4}+\widetilde{g}_{(33}\right)_{3} \\ \widetilde{f}=f^{\prime \prime}-k_{1}\left(\widetilde{v}_{6}+\widetilde{v}_{7}+\widetilde{v}_{8}+\widetilde{v}_{9}\right)-r_{1}\left(\widetilde{g}_{6}+\widetilde{g}_{7}+\widetilde{g}_{8}+\widetilde{g}_{9}\right) .\end{array}\right.$

## 4-step Reconstruction Algorithm:

Step 1. $\left\{\begin{array}{l}\left.e^{\prime \prime}=\widetilde{e}+a_{1}\left(\widetilde{v}_{0}+\widetilde{v}_{1}\right)+c_{1}\left(\widetilde{v}_{2}+\widetilde{v}_{3}+\widetilde{v}_{4}+\widetilde{v}_{5}\right)+h_{1}\left(\widetilde{g}_{0}+\widetilde{g}_{1}\right)+j_{1}\left(\widetilde{g}_{2}+\widetilde{g}_{3}+\widetilde{g}_{4}+\widetilde{g}_{5}\right)_{34}\right) \\ f^{\prime \prime}=\widetilde{f}+k_{1}\left(\widetilde{v}_{6}+\widetilde{v}_{7}+\widetilde{v}_{8}+\widetilde{v}_{9}\right)+r_{1}\left(\widetilde{g}_{6}+\widetilde{g}_{7}+\widetilde{g}_{8}+\widetilde{g}_{9}\right)\end{array}\right.$


Figure 9: Top-left: Reconstruction Alg. Step 1; Top-right: Template to obtain v" in Reconstruction Alg. Step 2 (template to obtain $g^{\prime \prime}$ is similar with $\widetilde{v}$ replaced by $\widetilde{g}$ ); Bottom-left: Reconstruction Alg. Step 3; Bottom-right: Reconstruction Alg. Step 4

Step 2. $\left\{\begin{array}{l}v^{\prime \prime}=\widetilde{v}+d_{1}\left(e_{0}^{\prime \prime}+e_{1}^{\prime \prime}+e_{2}^{\prime \prime}+e_{3}^{\prime \prime}\right)+s_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right), \\ g^{\prime \prime}=\widetilde{g}+n_{1}\left(e_{0}^{\prime \prime}+e_{1}^{\prime \prime}+e_{2}^{\prime \prime}+e_{3}^{\prime \prime}\right)+m_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)\end{array}\right.$
Step 3. $\left\{\begin{array}{l}e=e^{\prime \prime}+a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}\right)+c\left(v_{2}^{\prime \prime}+v_{3}^{\prime \prime}+v_{4}^{\prime \prime}+v_{5}^{\prime \prime}\right)+h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}\right)+j\left(g_{2}^{\prime \prime}+g_{3}^{\prime \prime}+g_{4}^{\prime \prime}+g_{5}^{\prime \prime}\right)(3) \\ f=f^{\prime \prime}+k\left(v_{6}^{\prime \prime}+v_{7}^{\prime \prime}+v_{8}^{\prime \prime}+v_{9}^{\prime \prime}\right)+r\left(g_{6}^{\prime \prime}+g_{7}^{\prime \prime}+g_{8}^{\prime \prime}+g_{9}^{\prime \prime}\right)\end{array}\right.$
Step 4.

$$
\begin{equation*}
v=t\left\{b v^{\prime \prime}+d\left(e_{0}+e_{1}+e_{2}+e_{3}\right)+s\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}+ \tag{36}
\end{equation*}
$$

$$
\begin{equation*}
(1-t)\left\{g^{\prime \prime}+n\left(e_{0}+e_{1}+e_{2}+e_{3}\right)+m\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\} \tag{37}
\end{equation*}
$$

With the formulas in (7) and (9), and the filters for the 3 -step algorithm given in the above subsection, we obtain the filter banks $\left\{p, q^{(1)}, \cdots, q^{(4)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(4)}\right\}$ corresponding to the algorithms (30)-(37) to be

$$
\begin{aligned}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), \cdots, q^{(4)}(\boldsymbol{\omega})\right]^{T}=E_{3}(2 \boldsymbol{\omega}) E_{2}(2 \boldsymbol{\omega}) E_{1}(2 \boldsymbol{\omega}) E_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})} \\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \cdots, \widetilde{q}^{(4)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{4} \widetilde{E}_{3}(2 \boldsymbol{\omega}) \widetilde{E}_{2}(2 \boldsymbol{\omega}) \widetilde{E}_{1}(2 \boldsymbol{\omega}) \widetilde{E}_{0}(2 \omega) I_{00}(\boldsymbol{\omega})}
\end{aligned}
$$

where $E_{0}(\boldsymbol{\omega}), E_{1}(\boldsymbol{\omega}), \widetilde{E}_{0}(\boldsymbol{\omega}), \widetilde{E}_{1}(\boldsymbol{\omega}), E_{2}(\boldsymbol{\omega}), \widetilde{E}_{2}(\boldsymbol{\omega})$ are defined by (19)-(22), (25) and (26) and

$$
E_{3}(\boldsymbol{\omega})=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-k_{1}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & -r_{1}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & 1 & 0 & 0 \\
-(1+x)\left(a_{1}+c_{1} y+\frac{c_{1}}{y}\right) & -(1+x)\left(h_{1}+j_{1} y+\frac{j_{1}}{y}\right) & 0 & 1 & 0 \\
-(1+y)\left(a_{1}+c_{1} x+\frac{c_{1}}{x}\right) & -(1+y)\left(h_{1}+j_{1} x+\frac{j_{1}}{x}\right) & 0 & 0 & 1
\end{array}\right],
$$

and $\widetilde{E}_{3}(\boldsymbol{\omega})=\left(E_{3}(\boldsymbol{\omega})^{-1}\right)^{*}:$

$$
\widetilde{E}_{3}(\boldsymbol{\omega})=\left[\begin{array}{ccccc}
1 & 0 & k_{1}(1+x)(1+y) & \left(1+\frac{1}{x}\right)\left(a_{1}+c_{1} y+\frac{c_{1}}{y}\right) & \left(1+\frac{1}{y}\right)\left(a_{1}+c_{1} x+\frac{c_{1}}{x}\right) \\
0 & 1 & r_{1}(1+x)(1+y) & \left(1+\frac{1}{x}\right)\left(h_{1}+j_{1} y+\frac{y_{1}}{y}\right) & \left(1+\frac{1}{y}\right)\left(h_{1}+j_{1} x+\frac{y_{1}}{x}\right) \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right] .
$$

For this 4 -step algorithm, we can choose the parameters such that $\phi$ is a $C^{4}$ bi-spline with $\phi$ in $L^{2}\left(\mathbb{R}^{2}\right)$. For example,

$$
\begin{aligned}
& {\left[b, d, s, n, m, a, c, h, j, k, r, d_{1}, s_{1}, n_{1}, m_{1}, a_{1}, c_{1}, k_{1}, h_{1}, j_{1}, r_{1}, t\right]=} \\
& {\left[\frac{107}{128}, \frac{1}{64},-\frac{47}{256}, \frac{1}{4},-\frac{1}{16}, \frac{1}{12}, \frac{1}{24}, \frac{7}{6}, \frac{1}{12}, 1,-3,-\frac{3}{16},-\frac{3}{64}, 0,-\frac{1}{64},-\frac{1}{12},-\frac{1}{24},-1,0,0,0,0\right],}
\end{aligned}
$$

then the resulting $\widetilde{p}(\boldsymbol{\omega})$ is

$$
\widetilde{p}(\boldsymbol{\omega})=\frac{1}{4096 x^{4} y^{4}}(1+x)^{4}\left(1+x^{2}\right)^{2}(1+y)^{4}\left(1+y^{2}\right)^{2}
$$

$p$ has sum rule order 2 , and $q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$ have vanishing moment order 2 . Thus the corresponding $\tilde{\phi}$ is a $C^{4}$ bi-spline supported on $[-4,4]^{2}$. In this case $\phi \in W^{0.44915}$.

We may choose other parameters such that $\phi$ has a higher smoothness order (but $\tilde{\phi}$ is not a spline). For example, we can select a set of parameters such that $\phi \in W^{1.26798}, \widetilde{\phi} \in W^{3.99999}$, and another set of parameters with $\phi \in W^{1.70745}, \widetilde{\phi} \in W^{3.33270}$, both with the resulting $p$ and $\widetilde{p}$ having sum rule of orders 2 and 4 respectively, and $q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 4$ having vanishing moment order 2 . Here we would not provide the selected parameters.

In the above two subsections we consider 3 -step and 4 -step algorithms. If we use algorithms with more iterative steps, or use templates with bigger sizes, then we can construct symmetric framelets with higher smoothness and/or vanishing moment orders. Here we will not discuss more algorithms.

## $3 \sqrt{2}$-refinement bi-frames with 2-fold symmetry

In this section we study 4 -fold symmetric $\sqrt{2}$-refinement bi-frames. The 4 -fold symmetry and the templates of $\sqrt{2}$-refinement frame filter banks are discussed in $\S 3.1$, and symmetric bi-frames with 3 - and 4 -step algorithms are constructed in $\S 3.2$ and $\S 3.3$ respectively.

## $3.1 \sqrt{2}$-refinement bi-frame multiresolution algorithms and associated templates

Let $A=A_{1}$ in (1) be the dilation matrix for $\sqrt{2}$ refinement. (Since the lowpass $p(\boldsymbol{\omega})$ has 4 -fold symmetry, dilation matrices $A_{1}$ and $A_{2}$ yield the same scaling function, see [15].) For this $A$, we choose the representatives $\boldsymbol{\eta}_{0}, \boldsymbol{\eta}_{1}$ of $\mathbf{Z}^{2} /\left(A^{T} \mathbf{Z}^{2}\right)$ to be

$$
\boldsymbol{\eta}_{0}=(0,0), \boldsymbol{\eta}_{1}=(1,0) .
$$

For an FIR $\sqrt{2}$ frame filter bank $\left\{p, q^{(1)}, q^{(2)}\right\}$, with notation $q^{(0)}(\boldsymbol{\omega})=p(\boldsymbol{\omega})$, write $q^{(\ell)}(\boldsymbol{\omega}), 0 \leq$ $\ell \leq 2$ as

$$
q^{(\ell)}(\boldsymbol{\omega})=\frac{1}{\sqrt{2}}\left(q_{0}^{(\ell)}\left(A^{T} \boldsymbol{\omega}\right)+q_{1}^{(\ell)}\left(A^{T} \boldsymbol{\omega}\right) e^{-i \omega_{1}}\right),
$$

where $q_{k}^{(\ell)}(\boldsymbol{\omega}), k=0,1$ are trigonometric polynomials. Then we define the polyphase matrix $V(\boldsymbol{\omega})$ of a $\sqrt{2}$ frame filter bank $\left\{p, q^{(1)}, q^{(2)}\right\}$ to be

$$
V(\boldsymbol{\omega})=\left[\begin{array}{ll}
p_{0}(\boldsymbol{\omega}) & p_{1}(\boldsymbol{\omega})  \tag{38}\\
q_{0}^{(1)}(\boldsymbol{\omega}) & q_{1}^{(1)}(\boldsymbol{\omega}) \\
q_{0}^{(2)}(\boldsymbol{\omega}) & q_{1}^{(2)}(\boldsymbol{\omega})
\end{array}\right] .
$$

We have

$$
\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{\sqrt{2}} V\left(A^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega}),
$$

where $I_{0}(\omega)$ defined by

$$
I_{0}(\boldsymbol{\omega})=\left[1, e^{-i \omega_{1}}\right]^{T}, \boldsymbol{\omega}=\left(\omega_{1}, \omega_{2}\right) \in \mathbb{R}^{2}
$$

Again, two $\sqrt{2}$-refinement frame filter banks $\left\{p, q^{(1)}, q^{(2)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ are biorthogonal if and only if

$$
V(\boldsymbol{\omega})^{*} \tilde{V}(\boldsymbol{\omega})=I_{2}, \quad \boldsymbol{\omega} \in \mathbb{R}^{2}
$$

where $V(\boldsymbol{\omega})$ and $\tilde{V}(\boldsymbol{\omega})$ are their polyphase matrices defined by (38).
For a $\sqrt{2}$ frame filter bank $\left\{p, q^{(1)}, q^{(2)}\right\}$, the multiresolution decomposition with dilation matrix $A$ for input data or a regular quad mesh $\mathcal{C}=\left\{c_{\mathbf{k}}^{0}\right\}$ is

$$
\begin{equation*}
c_{\mathbf{n}}^{j+1}=\frac{1}{2} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}-A \mathbf{n}} c_{\mathbf{k}}^{j}, d_{\mathbf{n}}^{(\ell, j+1)}=\frac{1}{2} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} q_{\mathbf{k}-A \mathbf{n}} c_{\mathbf{k}}^{j}, \mathbf{n} \in \mathbf{Z}^{2}, \quad \ell=1,2, \mathbf{n} \in \mathbf{Z}^{2} \tag{39}
\end{equation*}
$$

for $j=0,1, \cdots, J-1$. If $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ is biorthogonal to $\left\{p, q^{(1)}, q^{(2)}\right\}$, then the initial data/vertices $c_{\mathbf{k}}^{0}$ can be recovered by the multiresolution reconstruction algorithm:

$$
\begin{equation*}
c_{\mathbf{k}}^{j}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{\mathbf{k}-A \mathbf{n}} c_{\mathbf{n}}^{j+1}+\sum_{\ell=1}^{2} \sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{q}_{\mathbf{k}-A \mathbf{n}}^{(\ell)} \mathbf{n}_{\mathbf{n}}^{(\ell, j+1)}, \quad \mathbf{k} \in \mathbf{Z}^{2}, j=1-J, \cdots,-1,0 \tag{40}
\end{equation*}
$$

Analogously, $\left\{p, q^{(1)}, q^{(2)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ are respectively called the analysis frame filter bank and the synthesis frame filter bank, $\left\{c_{\mathbf{k}}^{j}\right\}$ and $\left\{d_{\mathbf{k}}^{(\ell, j)}\right\}$ are respectively called the "approximation" and the "details", or the lowpass and highpass outputs.


Figure 10: Left: Type $V$ nodes and type $F$ nodes; Middle: Original data/vertices associated with type $V$ nodes and type $F$ nodes; Right: "Approximation" $\widetilde{v}$ and "details" $\widetilde{g}, \widetilde{f}$ associated with type $V$ nodes and type $F$ nodes

Next, we show that $\sqrt{2}$ frame decomposition and reconstruction algorithms can be represented as templates by associating appropriately lowpass and highpass outputs to the nodes of $\mathbf{Z}^{2}$. To this regard, we first separate the nodes of $\mathbf{Z}^{2}$ into two groups as in [22].

Suppose a regular quad mesh is represented as the quad mesh with the square lattice $\mathbf{Z}^{2}$. Then $A \mathbf{Z}^{2}=\left\{A \mathbf{k}=\left(k_{1}-k_{2}, k_{1}+k_{2}\right),\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2}\right\}$ is the set of the labels for the vertices of the coarse mesh. The nodes with labels $A \mathbf{k}$ are called type $V$ nodes (or vertex nodes) for the $\sqrt{2}$ refinement, and the other nodes with labels $A \mathbf{Z}^{2}+(1,0)$ are called type $F$ nodes (or face nodes). See the left picture of Fig. 10 for these two groups of nodes, where big circles $\bigcirc$ denote type $V$ nodes.

Let $\mathcal{C}=\left\{c_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ be the initial data (mesh). Thus, $\left\{c_{A \mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of data (vertices) associated with type $V$ nodes, $\left\{c_{A \mathbf{k}+(1,0)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of data (vertices) associated with type $F$ nodes. Denote

$$
\begin{equation*}
v_{\mathbf{k}}=c_{A \mathbf{k}}, f_{\mathbf{k}}=c_{A \mathbf{k}+(1,0)}, \mathbf{k} \in \mathbf{Z}^{2} \tag{41}
\end{equation*}
$$

See the middle picture of Fig. 10 for these two groups of data/vertices.
Let $\left\{c_{\mathbf{k}}^{1}\right\}_{\mathbf{k}}$ and $\left\{d_{\mathbf{k}}^{(1,1)}\right\}_{\mathbf{k}},\left\{d_{\mathbf{k}}^{(2,1)}\right\}_{\mathbf{k}}$ be the lowpass and highpass outputs with an analysis frame filter bank $\left\{p, q^{(1)}, q^{(2)}\right\}$. Denote

$$
\widetilde{v}_{\mathbf{k}}=c_{\mathbf{k}}^{1}, \widetilde{g}_{\mathbf{k}}=d_{\mathbf{k}}^{(1,1)}, \widetilde{f}_{\mathbf{k}}=d_{\mathbf{k}}^{(2,1)}
$$

Then, the decomposition algorithm can be written as

$$
\begin{equation*}
\widetilde{v}_{\mathbf{k}}=\frac{1}{2} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} p_{\mathbf{k}^{\prime}-A \mathbf{k}} c_{\mathbf{k}^{\prime}}, \widetilde{g}_{\mathbf{k}}=\frac{1}{2} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-A \mathbf{k}}^{(1)} c_{\mathbf{k}^{\prime}}, \widetilde{f}_{\mathbf{k}}=\frac{1}{2} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-A \mathbf{k}}^{(2)} c_{\mathbf{k}^{\prime}}, \mathbf{k} \in \mathbf{Z}^{2}, \tag{42}
\end{equation*}
$$

and the reconstruction algorithm with a synthesis filter bank $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ is

$$
\begin{equation*}
c_{\mathbf{k}}=\sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{\mathbf{k}-A \mathbf{k}^{\prime}} \widetilde{v}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-A \mathbf{k}^{\prime}}^{(1)} \widetilde{g}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-A \mathbf{k}^{\prime}}^{(2)} \widetilde{f}_{\mathbf{k}^{\prime}}\right\} . \tag{43}
\end{equation*}
$$

Considering $c_{\mathbf{k}}$ in (43) with $\mathbf{k}$ in two different cases: $A \mathbf{j}, A \mathbf{j}+(1,0)$, and using the definitions for $v_{\mathbf{k}}, f_{\mathbf{k}}$ in (41), we can write the reconstruction algorithm (43) as

$$
\begin{align*}
& v_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{A \mathbf{n}} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{A \mathbf{n}}^{(1)} \widetilde{g}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{A \mathbf{n}}^{(2)} \widetilde{f}_{\mathbf{k}-\mathbf{n}}\right\} \\
& f_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{A \mathbf{n}+(1,0)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{A \mathbf{n}+(1,0)}^{(1)} \widetilde{g}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{A \mathbf{n}+(1,0)}^{(2)} \widetilde{f}_{\mathbf{k}-\mathbf{n}}\right\} . \tag{44}
\end{align*}
$$

Next we associate both the "approximation" $\left\{\widetilde{v}_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and the first highpass output $\widetilde{g}_{\mathbf{k}}$ with type $V$ nodes with labels $A \mathbf{k}$, and associate the second highpass output $\tilde{f}_{\mathbf{k}}$ with type $F$ nodes with labels $A \mathbf{k}+(1,0)$, see the right picture of Fig. 10 for these three groups of data/vertices. In this way, we can represent both analysis and synthesis algorithms as templates.

Again when these algorithm templates are used for surface processing, they must have certain symmetry. The templates to obtain $\widetilde{v}_{\mathbf{k}}, \widetilde{g}_{\mathbf{k}}$ and $\widetilde{f_{\mathbf{k}}}$ by (42), and those to recover $v_{\mathbf{k}}, f_{\mathbf{k}}$ by (44) must rotational and reflective invariant. $\sqrt{2}$ frame filter banks with 4 -fold symmetry yield the templates with the desired symmetry.

In the following we provide a characterization of the 4 -fold symmetry of $\sqrt{2}$ frame filter banks. We can use this characterization to check that symmetric templates result in $\sqrt{2}$ frame filter banks with 4 -fold symmetry.
Proposition 2. $A \sqrt{2}$ frame filter bank $\left\{p, q^{(1)}, q^{(2)}\right\}$ has 4-fold axial symmetry if and only if its polyphase matrix $V(\boldsymbol{\omega})$ satisfies

$$
\begin{align*}
& V\left(J_{0} \boldsymbol{\omega}\right)=\operatorname{diag}\left(1,1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right) V(\boldsymbol{\omega}) \operatorname{diag}\left(1, e^{i\left(\omega_{2}-\omega_{1}\right)}\right),  \tag{45}\\
& V\left(O_{1} \boldsymbol{\omega}\right)=\operatorname{diag}\left(1,1, e^{-i \omega_{2}}\right) V(\boldsymbol{\omega}) \operatorname{diag}\left(1, e^{i \omega_{2}}\right),
\end{align*}
$$

where $J_{0}, O_{1}$ are the matrices defined by (12).
The proof can be carried out similarly to that for the characterization of the 4 -fold symmetry of $\sqrt{2}$ wavelet filters in [22]. The details are omitted here.


Figure 11: $\sqrt{2}$-refinement decomposition and reconstruction algorithms
From the above discussion, we know a $\sqrt{2}$ decomposition algorithm is to decompose the original data $\{v\} \cup\{e\}$ into $\{\widetilde{v}\},\{\widetilde{g}\}$ and $\{\widetilde{e}\}$, and the reconstruction algorithm to recover $\{v\} \cup\{e\}$ from $\{\widetilde{v}\},\{\tilde{g}\}$ and $\{\widetilde{e}\}$, see Fig. 11. For a $\sqrt{2}$ frame multiresolution algorithm, the decomposition algorithm (reconstruction algorithm resp.) can be represented as three templates (two templates resp.).

To construct 4-fold symmetric bi-frames, again, we start with symmetric templates of decomposition and reconstruction algorithms. With the templates and decomposition and reconstruction algorithms (42)(44), we then obtain the corresponding $\sqrt{2}$ bi-frame filter banks which are given by some parameters. Then we select the parameters based on sum rule order of the lowpass filters, the smoothness and vanishing moments of framelets. For a $\sqrt{2}$ lowpass filter $p(\boldsymbol{\omega})$, we say it has sum rule order $K$ (with $A=A_{1}$ ) if it satisfies that $p(0,0)=1$ and

$$
\left.\frac{\partial^{\alpha_{1}+\alpha_{2}}}{\partial \omega_{1}^{\alpha_{1}} \partial \omega_{2}^{\alpha_{2}}} p\left(\omega_{1}, \omega_{2}\right)\right|_{\left(\omega_{1}, \omega_{2}\right)=(\pi, \pi)}=0, \quad \forall\left(\alpha_{1}, \alpha_{2}\right) \in \mathbf{Z}_{+}^{2}, \alpha_{1}+\alpha_{2}<K .
$$

### 3.2 3-step $\sqrt{2}$-refinement bi-frame multiresolution algorithm



Figure 12: Left: Template to obtain $v^{\prime \prime}, g^{\prime \prime}$ in Decomposition Alg. Step 1; Middle: Decomposition Alg. Step 2; Right: Template to obtain lowpass output $\widetilde{v}$ in Decomposition Alg. Step 3 (template to obtain first highpass output $\widetilde{g}$ is similar with $v^{\prime \prime}$ replaced by $g^{\prime \prime}$ )

In this subsection we discuss a 3 -step $\sqrt{2}$ multiresolution algorithm. The decomposition algorithm is to obtain $\widetilde{v}, \widetilde{g}$ and $\tilde{f}$ with certain rules, and the reconstruction algorithm is to recover $v$ and $e$ from $\widetilde{v}, \widetilde{g}$ and $\widetilde{f}$. More precisely, the decomposition algorithm is given by (46)-(48) and shown in Fig. 12, where $b, d, n, a, h, d_{1}, n_{1}$ are constants to be determined. Namely, first we replace all $v$ associated with type $V$ nodes of $A \mathbf{Z}^{2}$ by $v^{\prime \prime}, g^{\prime \prime}$ given by formulas in (46). Then, with $v^{\prime \prime}, g^{\prime \prime}$ obtained, we replace all $f$ associated with type $F$ nodes in $\mathbf{Z}^{2} \backslash\left(A \mathbf{Z}^{2}\right)$ by $\tilde{f}$ given in formula (47). Finally, based on $\tilde{f}$ obtained in Step 2, all $v^{\prime \prime}, g^{\prime \prime}$ in Step 1 are updated by $\widetilde{v}$ and $\widetilde{g}$ with the formulas in (48).

## 3-step $\sqrt{2}$-refinement Decomposition Algorithm:

$$
\text { Step 1. }\left\{\begin{array}{l}
v^{\prime \prime}=\frac{1}{b}\left\{v-d\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}  \tag{46}\\
g^{\prime \prime}=v-n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)
\end{array}\right.
$$

Step 2. $\tilde{f}=f-a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}+v_{2}^{\prime \prime}+v_{3}^{\prime \prime}\right)-h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}+g_{2}^{\prime \prime}+g_{3}^{\prime \prime}\right)$
Step 3. $\left\{\begin{array}{l}\widetilde{v}=v^{\prime \prime}-d_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right), \\ \widetilde{g}=g^{\prime \prime}-n_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right) .\end{array}\right.$
The multiresolution reconstruction algorithm is given by (49)-(51) and shown in Fig. 13, where $b, d, n, a, h, d_{1}, n_{1}$ are the same constants in the multiresolution decomposition algorithm and $t \in \mathbb{R}$. More precisely, first we replace the lowpass output $\widetilde{v}$ and the first highpass output $\widetilde{g}$ both associated with type $V$ nodes of $A \mathbf{Z}^{2}$ by $v^{\prime \prime}$ and $g^{\prime \prime}$ respectively given by formulas in (49). After that, with $v^{\prime \prime}, g^{\prime \prime}$ obtained, we replace all $\widetilde{f}$ by $f$ given in (50). Finally, based on $f$ obtained in Step 2, all $v^{\prime \prime}, g^{\prime \prime}$ in Step 1 are replaced $v$ given by (51).

## 3-step $\sqrt{2}$-refinement Reconstruction Algorithm:

Step 1. $\left\{\begin{array}{l}v^{\prime \prime}=\widetilde{v}+d_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right), \\ g^{\prime \prime}=\widetilde{g}+n_{1}\left(\widetilde{f}_{0}+\widetilde{f}_{1}+\widetilde{f}_{2}+\widetilde{f}_{3}\right)\end{array}\right.$
Step 2. $f=\tilde{f}+a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}+v_{2}^{\prime \prime}+v_{3}^{\prime \prime}\right)+h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}+g_{2}^{\prime \prime}+g_{3}^{\prime \prime}\right)$
Step 3. $v=t\left\{b v^{\prime \prime}+d\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}+(1-t)\left\{g^{\prime \prime}+n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}$.


Figure 13: Left: Template to obtain $v^{\prime \prime}$ in Reconstruction Alg. Step 1 (template to obtain $g^{\prime \prime}$ is similar with $\widetilde{v}$ replaced by $\widetilde{g}$ ); Middle: Reconstruction Alg. Step 2; Right: Reconstruction Alg. Step 3

To obtain the filters corresponding to this 3 -step algorithm, we first consider the case with $d_{1}=n_{1}=0$. In this case, the 3 -step algorithm is reduced to be a 2 -step algorithm with the analysis algorithm given by $(46)(47)$ (with $\widetilde{v}=v^{\prime \prime}, \widetilde{g}=g^{\prime \prime}$ ) and the synthesis algorithm given by $(50)(51)$ (with $\left.v^{\prime \prime}=\widetilde{v}, g^{\prime \prime}=\widetilde{g}\right)$. With the formulas in (42) and (44), one can obtain the corresponding filter banks $\left\{p, q^{(1)}, q^{(2)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ to be

$$
\begin{aligned}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega})\right]^{T}=C_{1}\left(A^{T} \boldsymbol{\omega}\right) C_{0}\left(A^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})} \\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{2} \widetilde{C}_{1}\left(A^{T} \boldsymbol{\omega}\right) \widetilde{C}_{0}\left(A^{T} \omega\right) I_{0}(\boldsymbol{\omega}),}
\end{aligned}
$$

where, with $x=e^{-i \omega_{1}}, y=e^{-i \omega_{2}}, C_{0}(\boldsymbol{\omega}), C_{1}(\boldsymbol{\omega})$ and $\widetilde{C}_{0}(\boldsymbol{\omega}), \widetilde{C}_{1}(\boldsymbol{\omega})$ are given by

$$
\begin{align*}
& C_{0}(\boldsymbol{\omega})=\left[\begin{array}{cc}
\frac{1}{b} & -\frac{d}{b}\left(1+\frac{1}{x}\right)(1+y) \\
1 & -n\left(1+\frac{1}{x}\right)(1+y) \\
0 & 1
\end{array}\right], C_{1}(\boldsymbol{\omega})=\left[\begin{array}{cc}
1 & 0 \\
0 & 1 \\
-a(1+x)\left(1+\frac{1}{y}\right) & -h(1+x)\left(1+\frac{1}{y}\right) \\
0
\end{array}\right]  \tag{52}\\
& \widetilde{C}_{0}(\boldsymbol{\omega})=\left[\begin{array}{cc}
t b & 0 \\
1-t & 0 \\
(t d+(1-t) n)(1+x)\left(1+\frac{1}{y}\right) & 1
\end{array}\right], \widetilde{C}_{1}(\boldsymbol{\omega})=\left[\begin{array}{ccc}
1 & 0 & a\left(1+\frac{1}{x}\right)(1+y) \\
0 & 1 & h\left(1+\frac{1}{x}\right)(1+y) \\
0 & 0 & 1
\end{array}\right] . \tag{53}
\end{align*}
$$

Observe that the polyphase matrices $V(\boldsymbol{\omega})$ and $\widetilde{V}(\boldsymbol{\omega})$ for $\left\{p, q^{(1)}, q^{(2)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ are $\sqrt{2} C_{1}(\boldsymbol{\omega}) C_{0}(\boldsymbol{\omega})$ and $\frac{1}{\sqrt{2}} \widetilde{C}_{1}(\boldsymbol{\omega}) \widetilde{C}_{0}(\boldsymbol{\omega})$ respectively. One can easily show that $C_{0}(\boldsymbol{\omega})^{*} \widetilde{C}_{0}(\boldsymbol{\omega})=$ $I_{2}, C_{1}(\boldsymbol{\omega})^{*} \widetilde{C}_{1}(\boldsymbol{\omega})=I_{3}, \boldsymbol{\omega} \in \mathbb{R}^{2}$, which implies $V(\boldsymbol{\omega})^{*} \widetilde{V}(\boldsymbol{\omega})=I_{2}$. Thus, $\left\{p, q^{(1)}, q^{(2)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ are indeed biorthogonal. Furthermore, one can easily check that $V(\boldsymbol{\omega}), \widetilde{V}(\boldsymbol{\omega})$ satisfy (45). Thus these two filter banks have 4 -fold symmetry.

Solving the system of equations for sum rule order 3 of $\widetilde{p}$ and sum rule order 1 of $p$, we have

$$
b=2, d=-\frac{1}{4}, n=a=t=\frac{1}{4}
$$

The resulting $p, \widetilde{p}, q^{(1)}$ and $\widetilde{q}^{(2)}$ are

$$
\begin{aligned}
& p(\boldsymbol{\omega})=\frac{1}{8}\left(4+e^{-i \omega_{1}}+e^{i \omega_{1}}+e^{-i \omega_{2}}+e^{i \omega_{2}}\right), q^{(1)}(\boldsymbol{\omega})=\frac{1}{4}\left(4-e^{-i \omega_{1}}-e^{i \omega_{1}}-e^{-i \omega_{2}}-e^{i \omega_{2}}\right), \\
& \widetilde{p}(\boldsymbol{\omega})=\frac{1}{64}\left(4+e^{-i \omega_{1}}+e^{i \omega_{1}}+e^{-i \omega_{2}}+e^{i \omega_{2}}\right)^{2}, \widetilde{q}^{(2)}(\boldsymbol{\omega})=\frac{1}{16} e^{-i \omega_{1}}\left(8+e^{-i \omega_{1}}+e^{i \omega_{1}}+e^{-i \omega_{2}}+e^{i \omega_{2}}\right),
\end{aligned}
$$

while $q^{(2)}, \widetilde{q}^{(1)}$ depend on the remaining parameter $h$. The resulting $p$ and $\widetilde{p}$ are are actually have sum rule orders 4 and 2 respectively with resulting $\phi$ and $\widetilde{\phi}$ are in $W^{1.57764}$ and $W^{3.91803}$ respectively. The corresponding $q^{(1)}, q^{(2)}$ automatically have vanishing moment order 2. Furthermore, if $h=\frac{3}{8}$, then $q^{(2)}$ has vanishing moment order 4. In the following we provide $q^{(2)}, \widetilde{q}^{(1)}$ with $h=0$ and $h=\frac{3}{8}$. When $h=0$,

$$
\begin{aligned}
& q^{(2)}(\boldsymbol{\omega})=\frac{1}{32} e^{-i \omega_{1}}\left(4-e^{-i \omega_{1}}-e^{i \omega_{1}}-e^{-i \omega_{2}}-e^{i \omega_{2}}\right)\left(8+e^{-i \omega_{1}}+e^{i \omega_{1}}+e^{-i \omega_{2}}+e^{i \omega_{2}}\right), \\
& \widetilde{q}^{(1)}(\boldsymbol{\omega})=\frac{3}{8}
\end{aligned}
$$

and when $h=\frac{3}{8}$,

$$
\begin{aligned}
& q^{(2)}(\boldsymbol{\omega})=\frac{1}{16} e^{-i \omega_{1}}\left(4-e^{-i \omega_{1}}-e^{i \omega_{1}}-e^{-i \omega_{2}}-e^{i \omega_{2}}\right)^{2}, \\
& \widetilde{q}^{(1)}(\boldsymbol{\omega})\left(=\frac{3}{2} \widetilde{p}(\boldsymbol{\omega})\right)=\frac{3}{128}\left(4+e^{-i \omega_{1}}+e^{i \omega_{1}}+e^{-i \omega_{2}}+e^{i \omega_{2}}\right)^{2} .
\end{aligned}
$$

Observe that the above resulting $\widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega})$ has no vanishing moment. Solving the system of equations for sum rule order 1 of $p$ and $\widetilde{p}$ and for vanishing moment order 1 of $q^{(\ell)}, \widetilde{q}^{(\ell)}, \ell=1,2$, we have

$$
b=2, d=-\frac{1}{4}, n=a=\frac{1}{4}, t=1 .
$$

The resulting $\phi$ is in $W^{1.57764}$. However, the resulting $\widetilde{\phi}$ is not in $L^{2}\left(\mathbb{R}^{2}\right)$. Thus, to construct framelets both having vanishing moment, we need to consider algorithms with more iterative steps.

Next, let us consider the 3 -step algorithm. With the formulas in (42) and (44), and the filters for the 2-step algorithm discussed above, one can obtain that the filter banks $\left\{p, q^{(1)}, q^{(2)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ corresponding to (46)-(51) are

$$
\begin{aligned}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega})\right]^{T}=C_{2}\left(A^{T} \boldsymbol{\omega}\right) C_{1}\left(A^{T} \boldsymbol{\omega}\right) C_{0}\left(A^{T} \boldsymbol{\omega}\right) I_{00}(\boldsymbol{\omega}),} \\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{2} C_{2}\left(A^{T} \boldsymbol{\omega}\right) \widetilde{C}_{1}\left(A^{T} \boldsymbol{\omega}\right) \widetilde{C}_{0}\left(A^{T} \omega\right) I_{0}(\boldsymbol{\omega}),}
\end{aligned}
$$

where $C_{0}(\boldsymbol{\omega}), C_{1}(\boldsymbol{\omega})$ and $\widetilde{C}_{0}(\boldsymbol{\omega}), \widetilde{C}_{1}(\boldsymbol{\omega})$ are defined by (52) and (53) respectively, and

$$
C_{2}(\boldsymbol{\omega})=\left[\begin{array}{ccc}
1 & 0 & -d_{1}\left(1+\frac{1}{x}\right)(1+y)  \tag{54}\\
0 & 1 & -n_{1}\left(1+\frac{1}{x}\right)(1+y) \\
0 & 0 & 1
\end{array}\right]
$$

and $\widetilde{C}_{2}(\boldsymbol{\omega})=\left(C_{2}(\boldsymbol{\omega})^{-1}\right)^{*}:$

$$
\widetilde{C}_{2}(\boldsymbol{\omega})=\left[\begin{array}{ccc}
1 & 0 & 0  \tag{55}\\
0 & 1 & 0 \\
d_{1}(1+x)\left(1+\frac{1}{y}\right) & n_{1}(1+x)\left(1+\frac{1}{y}\right) & 1
\end{array}\right] .
$$

Solving the system of equations for sum rule orders 2 and 4 for $p$ and $\widetilde{p}$ respectively, and vanishing moment order 2 for $q^{(\ell)}, \widetilde{q}^{(\ell)}, \ell=1,2$, we have

$$
n=\frac{1}{4}, a=\frac{1}{4}, d=\frac{1}{4}-\frac{b}{4}, h=\frac{1}{12 b}-\frac{1}{6}, d_{1}=-\frac{3}{16}, t=\frac{1}{2 b} .
$$

The resulting $\widetilde{p}$ is the filter given below with corresponding $\widetilde{\phi} \in W^{3.91803}$ :

$$
\begin{equation*}
\widetilde{p}(\boldsymbol{\omega})=\frac{1}{64}\left(4+e^{-i \omega_{1}}+e^{i \omega_{1}}+e^{-i \omega_{2}}+e^{i \omega_{2}}\right)^{2} . \tag{56}
\end{equation*}
$$

The resulting $p$ depends on $b$. Furthermore if $n_{1}=\frac{3 b}{16(2 b-1)}$, then $q^{(1)}$ has vanishing moment order 4. If $b=\frac{4}{3}$, then $p$ has sum rule order 4 with resulting $\phi \in W^{1.19688}$. We can also choose other $b$ such that $\phi$ is smoother. It seems the bigger $b$ is, the smoother $\phi$. For example, if $b=8$ then $\phi \in W^{1.57796}$ and if $b=16$, then $\phi \in W^{1.70047}$. Next we provide other filters with the choices of $b=8, n_{1}\left(=\frac{3 b}{16(2 b-1)}\right)=\frac{1}{10}$ :

$$
\begin{aligned}
p(\boldsymbol{\omega})= & \frac{9}{256}\left(4+x+y+\frac{1}{x}+\frac{1}{y}\right)\left\{-\frac{10}{9}+\frac{8}{3}\left(x+y+\frac{1}{x}+\frac{1}{y}\right)\right. \\
& \left.-x y-\frac{1}{x y}-\frac{x}{y}-\frac{y}{x}-\frac{1}{2}\left(x^{2}+y^{2}+\frac{1}{x^{2}}+\frac{1}{y^{2}}\right)\right\}, \\
q^{(1)}(\boldsymbol{\omega})= & \frac{1}{320}\left\{20+3\left(x+y+\frac{1}{x}+\frac{1}{y}\right)\right\}\left(4-x-y-\frac{1}{x}-\frac{1}{y}\right)^{2}, \\
q^{(2)}(\boldsymbol{\omega})= & \frac{x}{32}\left\{8+3\left(x+y+\frac{1}{x}+\frac{1}{y}\right)\right\}\left(4-x-y-\frac{1}{x}-\frac{1}{y}\right), \\
\widetilde{q}^{(1)}(\boldsymbol{\omega})= & \frac{5}{512}\left(12+x+y+\frac{1}{x}+\frac{1}{y}\right)\left(4-x-y-\frac{1}{x}-\frac{1}{y}\right), \\
\widetilde{q}^{(2)}(\boldsymbol{\omega})= & \frac{x}{256}\left(8+x+y+\frac{1}{x}+\frac{1}{y}\right)\left(4+x+y+\frac{1}{x}+\frac{1}{y}\right)\left(4-x-y-\frac{1}{x}-\frac{1}{y}\right) .
\end{aligned}
$$

The coefficients of the resulting filters are also provided in Appendix C.

### 3.3 4-step $\sqrt{2}$-refinement bi-frame multiresolution algorithm

In this subsection, we consider a 4 -step $\sqrt{2}$-refinement algorithm. The decomposition algorithm is given by (57)-(60) and shown in Fig. 14, and the reconstruction algorithm is given by (61)-(64) and shown in Fig. 15, where $b, d, n, a, h, d_{1}, n_{1}, a_{1}, h_{1}, t$ are some constants.

## 4-step $\sqrt{2}$-refinement Decomposition Algorithm:

Step 1. $\left\{\begin{array}{l}v^{\prime \prime}=\frac{1}{b}\left\{v-d\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}, \\ g^{\prime \prime}=v-n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\end{array}\right.$
Step 2. $f^{\prime \prime}=f-a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}+v_{2}^{\prime \prime}+v_{3}^{\prime \prime}\right)-h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}+g_{2}^{\prime \prime}+g_{3}^{\prime \prime}\right)$
Step 3. $\left\{\begin{array}{l}\widetilde{v}=v^{\prime \prime}-d_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right), \\ \widetilde{g}=g^{\prime \prime}-n_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)\end{array}\right.$
Step 4. $\widetilde{f}=f^{\prime \prime}-a_{1}\left(\widetilde{v}_{0}+\widetilde{v}_{1}+\widetilde{v}_{2}+\widetilde{v}_{3}\right)-h_{1}\left(\widetilde{g}_{0}+\widetilde{g}_{1}+\widetilde{g}_{2}+\widetilde{g}_{3}\right)$.

## 4-step $\sqrt{2}$-refinement Reconstruction Algorithm:

Step 1. $f^{\prime \prime}=\widetilde{f}+a_{1}\left(\widetilde{v}_{0}+\widetilde{v}_{1}+\widetilde{v}_{2}+\widetilde{v}_{3}\right)+h_{1}\left(\widetilde{g}_{0}+\widetilde{g}_{1}+\widetilde{g}_{2}+\widetilde{g}_{3}\right)$
Step 2. $\left\{\begin{array}{l}v^{\prime \prime}=\widetilde{v}+d_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right), \\ g^{\prime \prime}=\widetilde{g}+n_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)\end{array}\right.$
Step 3. $f=f^{\prime \prime}+a\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}+v_{2}^{\prime \prime}+v_{3}^{\prime \prime}\right)+h\left(g_{0}^{\prime \prime}+g_{1}^{\prime \prime}+g_{2}^{\prime \prime}+g_{3}^{\prime \prime}\right)$
Step 4. $v=t\left\{b v^{\prime \prime}+d\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}+(1-t)\left\{g^{\prime \prime}+n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}$.


Figure 14: Top-left: Template to obtain $v^{\prime \prime}, g^{\prime \prime}$ in Decomposition Alg. Step 1; Top-right: Decomposition Alg. Step 2; Bottom-left: Template to obtain lowpass output $\widetilde{v}$ in Decomposition Alg. Step 3 (template to obtain first highpass output $\widetilde{g}$ is similar with $v^{\prime \prime}$ replaced by $g^{\prime \prime}$ ); Bottom-right: Decomposition Alg. Step 4

With the formulas in (42) and (44), and the filter banks for the 3 -step algorithm obtained in $\S 3.2$, we obtain the filter banks $\left\{p, q^{(1)}, q^{(2)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}\right\}$ corresponding to the algorithms (57)-(64):

$$
\begin{aligned}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega})\right]^{T}=C_{3}\left(A^{T} \boldsymbol{\omega}\right) C_{2}\left(A^{T} \boldsymbol{\omega}\right) C_{1}\left(A^{T} \boldsymbol{\omega}\right) C_{0}\left(A^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega}),} \\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{2} \widetilde{C}_{3}\left(A^{T} \boldsymbol{\omega}\right) \widetilde{C}_{2}\left(A^{T} \boldsymbol{\omega}\right) \widetilde{C}_{1}\left(A^{T} \boldsymbol{\omega}\right) \widetilde{C}_{0}\left(A^{T} \omega\right) I_{0}(\boldsymbol{\omega}),}
\end{aligned}
$$

where $C_{0}(\boldsymbol{\omega}), C_{1}(\boldsymbol{\omega}), \widetilde{C}_{0}(\boldsymbol{\omega}), \widetilde{C}_{1}(\boldsymbol{\omega}), C_{2}(\boldsymbol{\omega}), \widetilde{C}_{2}(\boldsymbol{\omega})$ are defined by (52), (53), (54) and (55) respectively, and

$$
C_{3}(\boldsymbol{\omega})=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-a_{1}(1+x)\left(1+\frac{1}{y}\right) & -h_{1}(1+x)\left(1+\frac{1}{y}\right) & 1
\end{array}\right],
$$

and $\widetilde{C}_{3}(\boldsymbol{\omega})=\left(C_{3}(\boldsymbol{\omega})^{-1}\right)^{*}:$

$$
\widetilde{C}_{3}(\boldsymbol{\omega})=\left[\begin{array}{ccc}
1 & 0 & a_{1}\left(1+\frac{1}{x}\right)(1+y) \\
0 & 1 & h_{1}\left(1+\frac{1}{x}\right)(1+y) \\
0 & 0 & 1
\end{array}\right]
$$

For this 4 -step algorithm, we can choose the parameters such that both $\phi$ and $\tilde{\phi}$ are smooth. For example, if

$$
b=8, d=\frac{17}{20}, n=\frac{1}{20}, a=\frac{5}{32}, h=-\frac{35}{256}, d_{1}=-\frac{1}{5}, n_{1}=\frac{8}{65}, a_{1}=\frac{13}{32}, t=\frac{1}{8},
$$



Figure 15: Top-left: Reconstruction Alg. Step 1; Top-right: Template to obtain $v^{\prime \prime}$ in Reconstruction Alg. Step 2 (template to obtain $g^{\prime \prime}$ is similar with $\widetilde{v}$ replaced by $\widetilde{g}$ ); Bottom-left: Reconstruction Alg. Step 3; Bottom-right: Reconstruction Alg. Step 4
then both $p$ and $\widetilde{p}$ have sum rule order 6 , and $q^{(\ell)}, \widetilde{q}^{(\ell)}, \ell=1,2$ have vanishing moment order 2 . The corresponding $\phi$ and $\tilde{\phi}$ are in $W^{5.99082}$ and $W^{3.07544}$ respectively. Furthermore, if $h_{1}=\frac{455}{768}$, then $q^{(2)}$ has vanishing moment order 4.

For this 4 -step algorithm, we can also choose the parameters such that both $q^{(\ell)}, \ell=1,2$, have vanishing moment order $4, \widetilde{q}^{(\ell)}, \ell=1,2$, have vanishing moment order 2 , and $p$ and $\widetilde{p}$ have sum rule orders of 2 and 4 respectively. But the resulting $\phi, \widetilde{\phi}$ cannot have nice smoothness. Here we do not provide the selected parameters.

To construct 4 -fold symmetric $\sqrt{2}$ framelets with higher smoothness and/or vanishing moment orders, we need to use algorithms with more iterative steps or templates with bigger sizes.

## Appendix A

1. Resulting symmetric frame filter banks for 2-step algorithm in $\S 2.2$ with $n=\frac{3}{10}, j=0, h=$ $\frac{15}{8}, r=\frac{15}{16}$ and other parameters given by (23):

$$
\begin{aligned}
p(\boldsymbol{\omega})= & \frac{1}{16 x y}(1+x)^{2}(1+y)^{2}, q^{(1)}(\boldsymbol{\omega})=1-\frac{3}{10}\left(x+\frac{1}{x}+y+\frac{1}{y}\right)+\frac{1}{20}\left(x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right), \\
q^{(2)}(\boldsymbol{\omega})= & \frac{1}{x y}\left\{\frac{3}{4}+\frac{1}{2}\left(x+y+\frac{1}{x}+\frac{1}{y}\right)-\left(x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right)-\frac{1}{8}\left(x^{2}+y^{2}+\frac{1}{x^{2}}+\frac{1}{y^{2}}\right)\right. \\
& \left.+\frac{1}{4}\left(x^{2} y+x y^{2}+\frac{1}{x y^{2}}+\frac{1}{x^{2} y}+\frac{x^{2}}{y}+\frac{y^{2}}{x}+\frac{x}{y^{2}}+\frac{y}{x^{2}}\right)-\frac{1}{16}\left(x^{2} y^{2}+\frac{1}{x^{2} y^{2}}+\frac{x^{2}}{y^{2}}+\frac{y^{2}}{x^{2}}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
q^{(3)}(\boldsymbol{\omega})= & \frac{x}{64}\left\{130-126\left(x+\frac{1}{x}\right)-\frac{31}{2}\left(y+\frac{1}{y}\right)+\frac{65}{2}\left(x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right)+33\left(x^{2}+\frac{1}{x^{2}}\right)\right. \\
& -\frac{31}{4}\left(x^{2} y+\frac{1}{x^{2} y}+\frac{x^{2}}{y}+\frac{y}{x^{2}}\right)-y^{2}-\frac{1}{y^{2}}-x y^{2}-\frac{1}{x y^{2}}-\frac{x}{y^{2}}-\frac{y^{2}}{x} \\
& \left.-\frac{1}{2}\left(y^{3}+\frac{1}{y^{3}}+x^{2} y^{2}+\frac{1}{x^{2} y^{2}}+\frac{y^{2}}{x^{2}}+\frac{x^{2}}{y^{2}}+\frac{x}{y^{3}}+\frac{y^{3}}{x}+x y^{3}+\frac{1}{x y^{3}}\right)-\frac{1}{4}\left(x^{2} y^{3}+\frac{1}{x^{2} y^{3}}+\frac{x^{2}}{y^{3}}+\frac{y^{3}}{x^{2}}\right)\right\},
\end{aligned}
$$

$$
q^{(4)}(\boldsymbol{\omega})=q^{(3)}\left(\omega_{2}, \omega_{1}\right)
$$

$$
\widetilde{p}(\boldsymbol{\omega})=\frac{1}{256 x^{2} y^{2}}(1+x)^{4}(1+y)^{4}
$$

$$
\widetilde{q}^{(1)}(\boldsymbol{\omega})=\frac{15}{64}\left\{\frac{11}{4}+2\left(x+\frac{1}{x}+y+\frac{1}{y}\right)+x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right.
$$

$$
\left.+\frac{3}{8}\left(x^{2}+\frac{1}{x^{2}}+y^{2}+\frac{1}{y^{2}}\right)-\frac{1}{16}\left(x^{2} y^{2}+\frac{1}{x^{2} y^{2}}+\frac{x^{2}}{y^{2}}+\frac{y^{2}}{x^{2}}\right)\right\}
$$

$$
\widetilde{q}^{(2)}(\boldsymbol{\omega})=\frac{1}{4 x y}-\frac{1}{64}\left(1+x^{2}+\frac{1}{y^{2}}+\frac{1}{x^{2} y^{2}}\right), \widetilde{q}^{(3)}(\boldsymbol{\omega})=\frac{x}{4}+\frac{1}{16}\left(1+x^{2}\right), \widetilde{q}^{(4)}(\boldsymbol{\omega})=\frac{y}{4}+\frac{1}{16}\left(1+y^{2}\right) .
$$

2. Resulting symmetric frame filter banks for 2-step algorithm in $\S 2.2$ with $n=\frac{1}{2}, j=h=$ $r=0$ and other parameters given by (23):

$$
\begin{aligned}
p(\boldsymbol{\omega})= & \frac{1}{16 x y}(1+x)^{2}(1+y)^{2}, q^{(1)}(\boldsymbol{\omega})=\frac{1}{4 x y}(1-x)^{2}(1-y)^{2}, \\
q^{(2)}(\boldsymbol{\omega})= & \frac{1}{x y}\left\{\frac{15}{16}-\frac{1}{16}\left(x+y+\frac{1}{x}+\frac{1}{y}+x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right)-\frac{1}{64}\left(x^{2} y^{2}+\frac{1}{x^{2} y^{2}}+\frac{x^{2}}{y^{2}}+\frac{y^{2}}{x^{2}}\right)\right. \\
& \left.-\frac{1}{32}\left(x^{2}+y^{2}+\frac{1}{x^{2}}+\frac{1}{y^{2}}+x^{2} y+x y^{2}+\frac{1}{x^{2} y}+\frac{1}{x y^{2}}+\frac{x^{2}}{y}+\frac{y}{x^{2}}+\frac{x}{y^{2}}+\frac{y^{2}}{x}\right)\right\}, \\
q^{(3)}(\boldsymbol{\omega})= & \frac{x}{64}\left\{58-6\left(x+\frac{1}{x}\right)-\frac{7}{2}\left(y+\frac{1}{y}+x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right)-3\left(x^{2}+\frac{1}{x^{2}}\right)\right. \\
& -\left(y^{2}+\frac{1}{y^{2}}+x y^{2}+\frac{1}{x y^{2}}+\frac{x}{y^{2}}+\frac{y^{2}}{x}\right)-\frac{7}{4}\left(x^{2} y+\frac{1}{x^{2} y}+\frac{x^{2}}{y}+\frac{y}{x^{2}}\right) \\
& \left.-\frac{1}{2}\left(x^{2} y^{2}+\frac{1}{x^{2} y^{2}}+\frac{x^{2}}{y^{2}}+\frac{y^{2}}{x^{2}}+y^{3}+\frac{1}{y^{3}}+x y^{3}+\frac{1}{x y^{3}}+\frac{x}{y^{3}}+\frac{y^{3}}{x}\right)-\frac{1}{4}\left(x^{2} y^{3}+\frac{1}{x^{2} y^{3}}+\frac{x^{2}}{y^{3}}+\frac{y^{3}}{x^{2}}\right)\right\}
\end{aligned}
$$

$q^{(4)}(\boldsymbol{\omega})=q^{(3)}\left(\omega_{2}, \omega_{1}\right) ;$
$\widetilde{p}(\boldsymbol{\omega})=\frac{1}{64}(1+x)^{2}(1+y)^{2}\left(\frac{1}{x}+\frac{1}{y}\right)\left(1+\frac{1}{x y}\right), \widetilde{q}^{(1)}(\boldsymbol{\omega})=\frac{1}{4}, \widetilde{q}^{(2)}(\boldsymbol{\omega})=\frac{1}{4 x y}-\frac{1}{16}\left(1+\frac{1}{x^{2}}+\frac{1}{y^{2}}+\frac{1}{x^{2} y^{2}}\right)$,
$\widetilde{q}^{(3)}(\boldsymbol{\omega})=\frac{x}{4}+\frac{1}{8}\left(1+x^{2}\right), \widetilde{q}^{(4)}(\boldsymbol{\omega})=\frac{y}{4}+\frac{1}{8}\left(1+y^{2}\right)$.

## Appendix B

Coefficients of 4 -fold symmetric frame filter banks in $\S 2.2$ with $\phi \in W^{1.25492}$ and $\widetilde{\phi}$ being $C^{2}$ box-spline:

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
p_{-3,-3} & \cdots & p_{-3,3} \\
\vdots & \vdots & \vdots \\
\cdots & \mathbf{p}_{\mathbf{0}, \mathbf{0}} & \cdots \\
\vdots & \vdots & \vdots \\
p_{3,-3} & \cdots & p_{3,3}
\end{array}\right]=\frac{1}{96}\left[\begin{array}{ccccccc}
-\frac{11}{4} & \frac{7}{4} & -\frac{19}{4} & -11 & -\frac{19}{4} & \frac{7}{4} & -\frac{11}{4} \\
\frac{7}{4} & -8 & -\frac{37}{4} & 4 & -\frac{37}{4} & -8 & \frac{7}{4} \\
-\frac{19}{4} & -\frac{37}{4} & \frac{145}{4} & 74 & \frac{145}{4} & -\frac{37}{4} & -\frac{19}{4} \\
-11 & 4 & 74 & \mathbf{1 1 2} & 74 & 4 & -11 \\
-\frac{19}{4} & -\frac{37}{4} & \frac{145}{4} & 74 & \frac{145}{4} & -\frac{37}{4} & -\frac{19}{4} \\
\frac{7}{4} & -8 & -\frac{37}{4} & 4 & -\frac{37}{4} & -8 & \frac{7}{4} \\
-\frac{11}{4} & \frac{7}{4} & -\frac{19}{4} & -11 & -\frac{19}{4} & \frac{7}{4} & -\frac{11}{4}
\end{array}\right],} \\
& \begin{array}{l}
{\left[\begin{array}{ccc}
q_{-1,-1}^{(1)} & q_{-1,0}^{(1)} & q_{-1,1}^{(1)} \\
q_{0,-1}^{(1)} & \mathbf{q}_{\mathbf{0 , 0}}^{(1)} & q_{0,1}^{(1)} \\
q_{1,-1}^{(1)} & q_{1,0}^{(1)} & q_{1,1}^{(1)}
\end{array}\right]=\left[\begin{array}{ccc}
1 & -2 & 1 \\
-2 & \mathbf{4} & -2 \\
1 & -2 & 1
\end{array}\right],} \\
{\left[\begin{array}{ccc}
q_{-3,-3}^{(2)} & \cdots & q_{-3,1}^{(2)} \\
\cdots & \ldots & \cdots \\
\cdots & \mathbf{q}_{-\mathbf{1},-\mathbf{1}}^{(2)} & \cdots \\
\ldots & \cdots & \cdots \\
q_{1,-3}^{(2)} & \cdots & q_{1,1}^{(2)}
\end{array}\right]=-\frac{5}{48}\left[\begin{array}{ccccc}
1 & 1 & 2 & 1 & 1 \\
1 & \frac{8}{5} & 2 & \frac{8}{5} & 1 \\
2 & 2 & -\frac{\mathbf{1 7 2}}{\mathbf{5}} & 2 & 2 \\
1 & \frac{8}{5} & 2 & \frac{8}{5} & 1 \\
1 & 1 & 2 & 1 & 1
\end{array}\right],}
\end{array} \\
& {\left[\begin{array}{ccc}
q_{-2,-3}^{(3)} & \cdots & q_{-2,3}^{(3)} \\
\vdots & \vdots & \vdots \\
\cdots & \mathbf{q}_{\mathbf{1}, \mathbf{0}}^{(\mathbf{3})} & \cdots \\
\vdots & \vdots & \vdots \\
q_{4,-3}^{(3)} & \cdots & q_{4,3}^{(3)}
\end{array}\right]=\frac{1}{192}\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-11 & 7 & -5 & -102 & -5 & 7 & -11 \\
7 & -32 & -95 & 96 & -95 & -32 & 7 \\
-22 & 14 & -10 & \mathbf{5 6 4} & -10 & 14 & -22 \\
7 & -32 & -95 & 96 & -95 & -32 & 7 \\
-11 & 7 & -5 & -102 & -5 & 7 & -11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],} \\
& q_{k_{1}, k_{2}}^{(4)}=q_{k_{2}, k_{1}}^{(3)},\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2} ;
\end{aligned}
$$

and

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
\widetilde{p}_{-2,-2} & \cdots & \widetilde{p}_{-2,2} \\
\cdots & \cdots & \cdots \\
\cdots & \widetilde{\mathbf{p}}_{\mathbf{0 , 0}} & \cdots \\
\cdots & \cdots & \ldots \\
\widetilde{p}_{2,-2} & \cdots & \widetilde{p}_{2,2}
\end{array}\right]=\frac{1}{16}\left[\begin{array}{ccccc}
0 & 1 & 2 & 1 & 0 \\
1 & 4 & 6 & 4 & 1 \\
2 & 6 & 8 & 6 & 2 \\
1 & 4 & 6 & 4 & 1 \\
0 & 1 & 2 & 1 & 0
\end{array}\right],} \\
& {\left[\begin{array}{ccc}
\widetilde{q}_{-2,-2}^{(1)} & \cdots & \widetilde{q}_{-2,2}^{(1)} \\
\cdots & \cdots & \cdots \\
\cdots & \mathbf{q}_{0, \mathbf{0}}^{(1)} & \cdots \\
\ldots & \cdots & \ldots \\
\widetilde{q}_{2,-2}^{(1)} & \cdots & \widetilde{q}_{2,2}^{(1)}
\end{array}\right]=\frac{1}{32}\left[\begin{array}{ccccc}
1 & 1 & -2 & 1 & 1 \\
1 & 0 & -6 & 0 & 1 \\
-2 & -6 & \mathbf{2 0} & -6 & -2 \\
1 & 0 & -6 & 0 & 1 \\
1 & 1 & -2 & 1 & 1
\end{array}\right],}
\end{aligned}
$$

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
\widetilde{q}_{-2,-2}^{(2)} & \widetilde{q}_{-2,-1}^{(2)} & \widetilde{q}_{-2,0}^{(2)} \\
\widetilde{q}_{-1,-2}^{(2)} & \widetilde{\mathbf{q}}_{-1,-1}^{(2)} & \widetilde{q}_{-1,0}^{(2)} \\
\widetilde{q}_{0,-2}^{(2)} & \widetilde{q}_{0,-1}^{(2)} & \widetilde{q}_{0,0}^{(2)}
\end{array}\right]=\frac{1}{4}\left[\begin{array}{ccc}
-1 & 0 & -1 \\
0 & \mathbf{4} & 0 \\
-1 & 0 & -1
\end{array}\right],} \\
& {\left[\begin{array}{ccc}
\widetilde{q}_{-2,-3}^{(3)} & \cdots & \widetilde{q}_{-2,3}^{(3)} \\
\vdots & \vdots & \vdots \\
\cdots & \widetilde{\mathbf{q}}_{\mathbf{1}, \mathbf{0}}^{(3)} & \cdots \\
\vdots & \vdots & \vdots \\
\widetilde{q}_{4,-3}^{(3)} & \cdots & \widetilde{q}_{4,3}^{(3)}
\end{array}\right]=-\frac{1}{32}\left[\begin{array}{cccccc}
0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 2 & 3 & 2 & \frac{1}{2} \\
0 & 1 & \frac{7}{2} & -11 & \frac{7}{2} & 1 \\
0 \\
0 & 1 & 4 & -\mathbf{2 6} & 4 & 1 \\
0 \\
0 & 1 & \frac{7}{2} & -11 & \frac{7}{2} & 1 \\
0 & \frac{1}{2} & 2 & 3 & 2 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\
0
\end{array}\right],} \\
& \widetilde{q}_{k_{1}, k_{2}}^{(4)}=\widetilde{q}_{k_{2}, k_{1}}^{(3)},\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2} .
\end{aligned}
$$

## Appendix C

Coefficients of 4-fold symmetric $\sqrt{2}$ frame filter banks in $\S 3.2$ with $\phi \in W^{1.57796}$ and $\widetilde{\phi} \in W^{3.91803}$ :

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
p_{-3,-3} & \cdots & p_{-3,3} \\
\vdots & \vdots & \vdots \\
\cdots & \mathbf{p}_{\mathbf{0}, \mathbf{0}} & \cdots \\
\vdots & \vdots & \vdots \\
p_{3,-3} & \cdots & p_{3,3}
\end{array}\right]=\frac{1}{256}\left[\begin{array}{ccccccc}
0 & 0 & 0 & -9 & 0 & 0 & 0 \\
0 & 0 & -27 & 12 & -27 & 0 & 0 \\
0 & -27 & 24 & 127 & 24 & -27 & 0 \\
-9 & 12 & 127 & \mathbf{1 1 2} & 127 & 12 & -9 \\
0 & -27 & 24 & 127 & 24 & -27 & 0 \\
0 & 0 & -27 & 12 & -27 & 0 & 0 \\
0 & 0 & 0 & -9 & 0 & 0 & 0
\end{array}\right],} \\
& {\left[\begin{array}{ccc}
q_{-3,-3}^{(1)} & \cdots & q_{-3,3}^{(1)} \\
\vdots & \vdots & \vdots \\
\cdots & \mathbf{q}_{\mathbf{0}, \mathbf{0}}^{(\mathbf{1})} & \cdots \\
\vdots & \vdots & \vdots \\
q_{3,-3}^{(1)} & \cdots & q_{3,3}^{(1)}
\end{array}\right]=\frac{1}{160}\left[\begin{array}{ccccccc}
0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 9 & -4 & 9 & 0 & 0 \\
0 & 9 & -8 & -85 & -8 & 9 & 0 \\
3 & -4 & -85 & \mathbf{3 0 4} & -85 & -4 & 3 \\
0 & 9 & -8 & -85 & -8 & 9 & 0 \\
0 & 0 & 9 & -4 & 9 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0
\end{array}\right],} \\
& {\left[\begin{array}{ccc}
q_{-1,-2}^{(2)} & \cdots & q_{-1,2}^{(2)} \\
\cdots & \cdots & \cdots \\
\cdots & \mathbf{q}_{1, \mathbf{0}}^{(2)} & \cdots \\
\ldots & \cdots & \cdots \\
q_{3,-2}^{(2)} & \cdots & q_{3,2}^{(2)}
\end{array}\right]=\frac{1}{16}\left[\begin{array}{ccccc}
0 & 0 & -3 & 0 & 0 \\
0 & -6 & 4 & -6 & 0 \\
-3 & 4 & \mathbf{2 0} & 4 & -3 \\
0 & -6 & 4 & -6 & 0 \\
0 & 0 & -3 & 0 & 0
\end{array}\right],}
\end{aligned}
$$

and

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
\widetilde{p}_{-2,-2} & \cdots & \widetilde{p}_{-2,2} \\
\ldots & \cdots & \cdots \\
\ldots & \tilde{\mathbf{p}}_{\mathbf{0}, \mathbf{0}} & \cdots \\
\ldots & \cdots & \ldots \\
\widetilde{p}_{2,-2} & \cdots & \widetilde{p}_{2,2}
\end{array}\right]=\frac{1}{32}\left[\begin{array}{ccccc}
0 & 0 & 1 & 0 & 0 \\
0 & 2 & 8 & 2 & 0 \\
1 & 8 & \mathbf{2 0} & 8 & 1 \\
0 & 2 & 8 & 2 & 0 \\
0 & 0 & 1 & 0 & 0
\end{array}\right],} \\
& {\left[\begin{array}{ccc}
\widetilde{q}_{-2,-2}^{(1)} & \cdots & \widetilde{q}_{-2,2}^{(1)} \\
\ldots & \cdots & \cdots \\
\ldots & \mathbf{q}_{\mathbf{0 , 0}}^{(\mathbf{1})} & \cdots \\
\ldots & \cdots & \ldots \\
\widetilde{q}_{2,-2}^{(1)} & \cdots & \widetilde{q}_{2,2}^{(1)}
\end{array}\right]=-\frac{5}{256}\left[\begin{array}{ccccc}
0 & 0 & 1 & 0 & 0 \\
0 & 2 & 8 & 2 & 0 \\
1 & 8 & -\mathbf{4 4} & 8 & 1 \\
0 & 2 & 8 & 2 & 0 \\
0 & 0 & 1 & 0 & 0
\end{array}\right],}
\end{aligned}
$$

$$
\left[\begin{array}{ccc}
\widetilde{q}_{-2,-3}^{(2)} & \cdots & \widetilde{q}_{-2,3}^{(2)} \\
\vdots & \vdots & \vdots \\
\cdots & \widetilde{\mathbf{q}}_{\mathbf{1}, \mathbf{0}}^{(\mathbf{2})} & \cdots \\
\vdots & \vdots & \vdots \\
\widetilde{q}_{4,-3}^{(2)} & \cdots & \widetilde{q}_{4,3}^{(2)}
\end{array}\right]=-\frac{1}{128}\left[\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 3 & 8 & 3 & 0 & 0 \\
0 & 3 & 16 & -7 & 16 & 3 & 0 \\
1 & 8 & -7 & -\mathbf{9 6} & -7 & 8 & 1 \\
0 & 3 & 16 & -7 & 16 & 3 & 0 \\
0 & 0 & 3 & 8 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right]
$$

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