SYMMETRIC PARAUNITARY MATRIX EXTENSION AND PARAMETRIZATION OF SYMMETRIC ORTHOGONAL MULTIFILTER BANKS*

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Abstract. This paper is devoted to a study of symmetric paraunitary matrix extensions. The problem for a given compactly supported orthonormal scaling vector with some symmetric property, to construct a corresponding multiwavelet which also has the symmetric property is equivalent to the symmetric paraunitary extension of a given matrix. In this paper we study symmetric paraunitary extensions of two types of matrices which correspond to two different cases for the symmetry of the scaling vector: the components of the scaling vector have or don't have the same symmetric center. In this paper we also discuss parametrizations of symmetric orthogonal multifilter banks.

Key words. symmetric extension, paraunitary, parametrization, factorization, orthogonality, symmetry, multifilter bank, scaling vector, multiwavelet

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1. Introduction. Unlike one dimensional scalar filters, the matrix filter for a multiwavelet of $L_2(R)$, cannot in general be given in terms of the matrix filter for the scaling vector (except some special cases). So for a given $r\times r$ ($r\geq 2$) FIR matrix filter $H(z)=\sum_{k\in Z} h_k z^{-k}$ (called low-pass filter) for a compactly supported orthonormal (o.n.) scaling vector $\phi=(\phi_1,\cdots,\phi_r)^T$, one needs an algorithm to construct another $r\times r$ FIR matrix filter $G(z)=\sum_{k\in Z} g_k z^{-k}$ (called high-pass filter) such that

$$H(z)G(z)^*+H(-z)G(-z)^*=0_r, G(z)G(z)^*+G(-z)G(-z)^*=I_r, \tag{1.1}$$

for all $z\in C\{0\}$. With such a filter G, the vector $\psi=(\psi_1,\cdots,\psi_r)^T$ defined by

$$\widehat{\psi}(\omega):=G(e^{\frac{i\omega}{2}})\widehat{\phi}(\frac{\omega}{2}), \tag{1.2}$$

is a compactly supported multiwavelet, i.e., the collection $\{2^{\frac{j}{2}}\psi_\ell(2^jx-k), 1\leq \ell\leq r, j,k\in Z\}$ forms an o.n. basis of $L_2(R)$ (see [3]). We call a vector of functions $\phi=(\phi_1,\cdots,\phi_r)^T$ an o.n. scaling vector if ϕ is refinable (that is ϕ satisfies $\widehat{\phi}(\omega)=H(e^{\frac{i\omega}{2}})\widehat{\phi}(\frac{\omega}{2})$ for some FIR H), $\phi_j\in L_2(R)$ and

$$\int \phi_j(x-k)\overline{\phi_i(x)}dx=\delta(j-i)\delta(k), \quad 1\leq j,i\leq r, k\in Z.$$

For a matrix filter $P(z)=\sum_{k\in Z} p_k z^{-k}$, it is said to be a *finite impulse response (FIR)* filter if each entry of $P(z)$ is a Laurent polynomial of z^{-1}, i.e., there exist integers k_1,k_2 such that $p_k=0, k<k_1, k>k_2$. If $p_{k_1}\neq 0, p_{k_2}\neq 0$, we use $\mathrm{len}(P):=k_2-k_1+1$ to denote its filter length. An FIR $P(z)$ is said to be *causal* if each entry of $P(z)$ is a polynomial of z^{-1}, i.e., $p_k=0$ for $k<0$. Throughout this paper, P^T (P^* resp.) denotes the transpose (the complex conjugate and transpose resp.) of P, I_r and 0_r denote the $r\times r$ identity matrix and zero matrix respectively. We also let $0_{j\times l}$ denote the j by l zero matrix, and we would drop the subscript $j\times l$ when it

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does not cause any confusion. A necessary condition for H generating an o.n. scaling vector ϕ is that H is a matrix *Conjugate Quadrature Filter (CQF)* (see e.g., [12], [5], [6] for the necessary and sufficient conditions), i.e.,

$$H(z)H(z)^* + H(-z)H(-z)^* = I_r, \quad z \in C. \tag{1.3}$$

A pair $\{H, G\}$ of matrix filters is called a multifilter bank, and it is said to be orthogonal if H, G satisfy (1.1) and (1.3).

For an FIR matrix filter H, write

$$\begin{aligned} H(z) &= \sum h_{2k} z^{-2k} + (\sum h_{2k+1} z^{-2k}) z^{-1} \\ &=: \frac{\sqrt{2}}{2} H_e(z^2) + \frac{\sqrt{2}}{2} H_o(z^2) z^{-1}. \end{aligned} \tag{1.4}$$

Then H satisfies (1.3) if and only if $[H_e(z), H_o(z)]$ is paraunitary. A $j \times l$ $(j \le l)$ matrix filter $P(z)$ with real coefficients p_k is called paraunitary if

$$P(z)P(z^{-1})^T = I_j, \quad z \neq 0,$$

that is $P(e^{i\omega})$ is a matrix of orthonormal rows for all $\omega \in R$. Throughout this paper we assume that the coefficients of the matrix filters are real.

Let G be another FIR matrix filter, and G_e, G_o be the corresponding filters defined in the way of (1.4). Then G satisfies (1.1) if and only if

$$[G_e(z), G_o(z)]E(z^{-1})^T = [0, I_r],$$

where $E(z)$ is the polyphase matrix of the multifilter bank $\{H, G\}$ defined by

$$E(z) := \begin{bmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{bmatrix}. \tag{1.5}$$

Thus given an H satisfying (1.3), to find G to satisfy (1.1) is equivalent to the paraunitary extension problem of a paraunitary matrix: Given an $r \times 2r$ paraunitary matrix $[H_e(z), H_o(z)]$, to find $[G_e(z), G_o(z)]$ such that $E(z)$ defined by (1.5) is paraunitary. It was shown in [9] and [10] that this paraunitary extension problem is always solvable, i.e., given a paraunitary matrix $[H_e(z), H_o(z)]$, one can always find its paraunitary extension $[G_e(z), G_o(z)]$.

The problem considered in this paper is that given an FIR matrix filter H generating an o.n. scaling vector ϕ with some symmetric property, is there a corresponding compactly supported multiwavelet ψ with some symmetric property, and if it exits, how to construct the high-pass filter G? Equivalently, the problem we consider is for a given paraunitary matrix $[H_e(z), H_o(z)]$ with some symmetry, to decide if there is its paraunitary extension $[G_e(z), G_o(z)]$ which also has some symmetry and if it exists, how to construct it. Since symmetry is one of the most important properties of multiwavelets, the problem for a given symmetric o.n. scaling vector, to construct a corresponding symmetric multiwavelet deserves our study.

There are two types of symmetric causal filters H. The first one is that H satisfies

$$z^{-\gamma} S_0 H(z^{-1}) S_0 = H(z), \quad S_0 = \mathrm{diag}(I_s, -I_{r-s}), \tag{1.6}$$

for a nonnegative integer $s \le r$. In this case if H generates an o.n. scaling vector $\phi = (\phi_1, \cdots, \phi_r)^T$, then $\phi_1, \cdots, \phi_s$ are symmetric about $\gamma/2$ while $\phi_{s+1}, \cdots, \phi_r$ are antisymmetric about $\gamma/2$ (see e.g., [1], [7], [15] about the relationship between the symmetry of ϕ, ψ and the property of H, G). Filters with this type of symmetry are called filters with the same symmetric center.

The second type of the symmetric filter H is that H will generate a symmetric o.n. scaling vector with its components not having the same symmetric center. We call the filter of this type to be a filter with different symmetric centers. In this paper we consider $H(z) = \sum_{k=0}^{2\gamma+1} h_k z^{-k}$ satisfying

$$z^{-(2\gamma+1)}\mathrm{diag}(S_0 z^2, 1)H(z^{-1})\mathrm{diag}(S_0, z) = H(z), S_0 = \mathrm{diag}(I_s, -I_{r-s-1}), \tag{1.7}$$

for a nonnegative integer $s \le r-1$. In this case if H generates an o.n. scaling vector $\phi = (\phi_1, \cdots, \phi_r)^T$, then $\phi_1, \cdots, \phi_s$ are symmetric about $\gamma - \frac{1}{2}$ while $\phi_{s+1}, \cdots, \phi_{r-s-1}$ are antisymmetric about $\gamma - \frac{1}{2}$, and ϕ_r is symmetric about γ. $\phi_j, 1 \le j \le r-1$ are supported on $[0, 2\gamma-1]$ while ϕ_r is supported on $[0, 2\gamma]$ (see [13] about the discussion on the supports of scaling vectors).

The symmetric extensions $[G_e, G_o]$ of the paraunitary matrices $[H_e, H_o]$ related to these two types of filters H are carried out in Section 2 and Section 3, respectively. We will construct their paraunitary extensions $[G_e, G_o]$ such that ψ defined by (1.2) with

$$G(z) := \frac{\sqrt{2}}{2}G_e(z^2) + \frac{\sqrt{2}}{2}G_o(z^2)z^{-1}, \tag{1.8}$$

have symmetry, and $\mathrm{len}(G) \le \mathrm{len}(H)$. More precisely, for H satisfying (1.6), the constructed G satisfies

$$z^{-\gamma} S_0 G(z^{-1}) S_0 = -G(z). \tag{1.9}$$

Thus components of the corresponding multiwavelet ψ are symmetric/antisymmetric about $\gamma/2$. For H satisfying (1.7), the constructed G satisfies

$$z^{-(2\gamma+1)}\mathrm{diag}(\mathcal{S}_1 z^2, \mathcal{S}_2)G(z^{-1})\mathrm{diag}(S_0, z) = G(z), \tag{1.10}$$

where

$$\begin{aligned} &\mathcal{S}_1 := -I_{2s-r}, \mathcal{S}_2 := \mathrm{diag}(I_{r-s}, -I_{r-s}), \text{if } 2s \ge r;\\ &\mathcal{S}_1 := I_{r-2s}, \mathcal{S}_2 := \mathrm{diag}(I_s, -I_s), \text{if } 2s < r. \end{aligned} \tag{1.11}$$

The corresponding multiwavelet ψ has the following symmetric properties: (1) if $2s \ge r$, then $\psi_1, \cdots, \psi_{2s-r}$ are antisymmetric about $\gamma - \frac{1}{2}$, $\psi_{2s-r+1}, \cdots, \psi_s$ and $\psi_{s+1}, \cdots, \psi_r$ are symmetric and antisymmetric about γ respectively; (2) if $2s < r$, then $\psi_1, \cdots, \psi_{r-2s}$ are symmetric about $\gamma - \frac{1}{2}$, $\psi_{r-2s+1}, \cdots, \psi_{r-s}$ and $\psi_{r-s+1}, \cdots, \psi_r$ are symmetric and antisymmetric about γ respectively. Our construction also answers the problem on the existence of symmetric multiwavelets.

In Section 4, we discuss the parametrization of symmetric orthogonal multifilter banks. Parametrizations of FIR orthogonal systems are of fundamental importance to the design of filter banks (see e.g., [14], [16], [17]). Parametrizations of orthogonal filter banks are equivalent to the factorizations of paraunitary matrices. The parametrization of symmetric orthogonal multifilter banks $\{H, G\}$ with the low-pass filter H satisfying (1.6) for $\gamma = 2N+1$ was obtained in [7] (see [11] for the special case). For the case $r = 2, S_0 = (1)$, the parametrization of orthogonal multifilter banks $\{H, G\}$ with H satisfying (1.7) was provided in [8]. In Section 4 we present the parametrization of orthogonal multifilter banks $\{H, G\}$ with H satisfying (1.6) for $\gamma = 2N$ and the parametrization of $\{H, G\}$ with H satisfying (1.7).

In this paper we use N, N_0, Z to denote sets of all natural numbers, nonnegative integers and integers, respectively. For $n \in N$, denote

$$D_n := \mathrm{diag}(I_n, -I_{n-1}). \tag{1.12}$$

We use $O(n)$ to denote the set of all n by n real orthogonal matrices, and use $\mathrm{Tr}(M)$ to denote the trace of a matrix M.

2. Symmetric extension of matrices related to the same symmetric center filters. In this section we discuss the symmetric matrix extension related to low-pass filters H satisfying (1.6). We consider the cases $\gamma = 2N + 1$ and $\gamma = 2N$, $N \in N$ in the following two subsections respectively.

2.1. The case $\gamma = 2N + 1$. Let $H = \sum_{k=0}^{2N+1} h_k z^{-k}$ be a matrix CQF satisfying (1.6) with $\gamma = 2N + 1$, and $h_0 \neq 0, h_{2N+1} \neq 0$. Let H_e, H_o be the filters defined by (1.4). Then (1.6) for $\gamma = 2N + 1$ is equivalent to

$$z^{-N} S_0 [H_e(z^{-1}), H_o(z^{-1})] \begin{bmatrix} & S_0 \\ S_0 & \end{bmatrix} = [H_e(z), H_o(z)]. \tag{2.1}$$

Denote

$$P(z) := [H_e(z), H_o(z)] U_0^T,$$

where

$$U_0 := \frac{\sqrt{2}}{2} \begin{bmatrix} I_r & S_0 \\ -I_r & S_0 \end{bmatrix}. \tag{2.2}$$

One can check that P satisfies

$$z^{-N} S_0 P(z^{-1}) \mathrm{diag}(I_r, -I_r) = P(z). \tag{2.3}$$

Note that $U_0 \in O(2r)$. Thus P satisfies $P(z)P(z^{-1})^T = I_r$, i.e., P is also paraunitary. In the following we give a symmetric paraunitary extension of P.

We need a lemma which will be used here and in the following sections.

LEMMA 2.1. *(i) Suppose an $\ell \times 2k$ ($\ell \geq k$) real matrix A satisfies*

$$A \, diag(I_k, -I_k) A^T = 0. \tag{2.4}$$

Then there exists $u \in O(k)$ such that

$$A \begin{bmatrix} I_k \\ u^T \end{bmatrix} = 0. \tag{2.5}$$

(ii) Suppose an $\ell \times (2k-1)$ ($\ell \geq k-1$) real matrix A satisfies

$$A \, diag(I_k, -I_{k-1}) A^T = 0. \tag{2.6}$$

Then there exists $u \in O(k)$ such that

$$A \begin{bmatrix} u^T \\ (I_{k-1}, 0) \end{bmatrix} = 0. \tag{2.7}$$

Proof. (i). By (2.4), the rank of A, denoted by n, is not greater than k. Let $\{x_1, x_2, \cdots, x_n\}$ be an orthonormal basis for the columns of the matrix A^T (found by Gram-Schmidt process). Write

$$[x_1, \cdots, x_n] =: \begin{bmatrix} Y_1 \\ Z_1 \end{bmatrix}, \quad Y_1, Z_1 \text{ are } k \times n \text{ matrices.}$$

Then $Y_1^T Y_1 + Z_1^T Z_1 = I_n$. By (2.4), we have $Y_1^T Y_1 = Z_1^T Z_1$. Thus

$$Y_1^T Y_1 = Z_1^T Z_1 = \frac{1}{2} I_n.$$

Therefore $\sqrt{2}Y_1, \sqrt{2}Z_1$ are $k \times n$ matrices of orthonormal columns. Let Y_2, Z_2 be the $k \times (k-n)$ matrices such that $\sqrt{2}[Y_1, Y_2], \sqrt{2}[Z_1,\, Z_2] \in O(k)$. Then one has

$$\begin{bmatrix} Y_1^T & -Z_1^T \\ Y_2^T & -Z_2^T \end{bmatrix} x_j = 0, \quad 1 \le j \le n.$$

Since each column of A^T is a linear combinations of $x_j, 1 \le j \le n$, we have

$$\begin{bmatrix} Y_1^T & -Z_1^T \\ Y_2^T & -Z_2^T \end{bmatrix} A^T = 0.$$

Thus (2.5) holds true with $u = -2[Y_1,\, Y_2][Z_1, Z_2]^T \in O(k)$.

(ii). The proof is similar. In this case write

$$[x_1, \cdots, x_n] =: \begin{bmatrix} Y_1 \\ Z_1 \end{bmatrix}, \quad Y_1, Z_1 \text{ are } k \times n \text{ and } (k-1) \times n \text{ matrices,}$$

where $\{x_1, x_2, \cdots, x_n\}$ is an orthonormal basis for the columns of A^T. Then $\sqrt{2}Y_1, \sqrt{2}Z_1$ are $k \times n$ and $(k-1) \times n$ matrices of orthonormal columns, respectively. Let Y_2, Z_2 be the $k \times (k-n)$ and $(k-1) \times (k-1-n)$ matrices such that $\sqrt{2}[Y_1, Y_2] \in O(k), \sqrt{2}[Z_1,\, Z_2] \in O(k-1)$. Then one has

$$\begin{bmatrix} Y_1^T & -Z_1^T \\ Y_2^T & -[Z_2, 0]^T \end{bmatrix} A^T = 0, \quad 1 \le j \le n.$$

Thus

$$A \begin{bmatrix} Y_1 & Y_2 \\ -Z_1 & -[Z_2, 0] \end{bmatrix} = 0,$$

and (2.7) holds with $u = -2\text{diag}([Z_1, Z_2], 1)[Y_1,\, Y_2]^T \in O(k)$. □

From the proof of Lemma 2.1, we know that orthogonal matrices u in (2.5) and (2.7) are constructed by the Gram-Schmidt process of the rows of A.

For $v \in O(r)$, define

$$V(z) := \frac{1}{2} \begin{bmatrix} I_r & -v \\ -v^T & I_r \end{bmatrix} + \frac{1}{2} \begin{bmatrix} I_r & v \\ v^T & I_r \end{bmatrix} z^{-1}, \quad v \in O(r). \tag{2.8}$$

Then one has the following lemma.

LEMMA 2.2. *Let $V(z)$ be the matrix defined by (2.8) with some $v \in O(r)$. Then*

(i) $V(z)^T = V(z^{-1})$, $V(z)V(z^{-1}) = I_{2r}$.
(ii) $z^{-1}diag(I_r, -I_r)V(z^{-1})diag(I_r, -I_r) = V(z)$.

Proof. (i) and (ii) follow from the direct calculations. □

For a causal paraunitary matrix P satisfying (2.3), write

$$P = p_0 + \cdots + p_N z^{-N}.$$

By (2.3), $p_N = S_0 p_0 \mathrm{diag}(I_r, -I_r)$. On the other hand, the paraunitaryness of P implies that $p_0 p_N^T = 0$. Thus

$$p_0 \mathrm{diag}(I_r, -I_r) p_0^T = 0.$$

By Lemma 2.1, we can find $v_N \in O(r)$ such that $p_0[I_r, v_N]^T = 0$.

Let V_N be the matrix defined by (2.8) with $v = v_N$. Then $\tilde{P}$ defined by

$$\tilde{P}(z) := P(z)V_N(z^{-1})$$

is causal. Since $V_N(z)$ is paraunitary and satisfies (ii) of Lemma 2.2, $\tilde{P}$ is also paraunitary and satisfies (2.3) with $N-1$. Continuing this process, we construct $v_{N-1}, \cdots, v_1 \in O(r)$ similarly such that P can be written as

$$P(z) = \tilde{P}(z)V_N(z) = \cdots = P_0 V_1(z) \cdots V_N(z),$$

where V_j are defined by (2.8) with $v = v_j$, $1 \le j \le N$, and P_0 is an $r \times 2r$ matrix of constant entries satisfying

$$P_0 P_0^T = I_r, \quad S_0 P_0 = P_0 \mathrm{diag}(I_r, -I_r).$$

One has that for P_0 satisfying the above conditions, it can be written as

$$P_0 = \begin{bmatrix} a_0 & 0 \\ 0 & b_0 \end{bmatrix},$$

where a_0 and b_0 are $s \times r$ and $(r-s) \times r$ matrices respectively with $a_0 a_0^T = I_s, b_0 b_0^T = I_{r-s}$. Let a_1 and b_1 be such matrices that $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \in O(r)$. Denote

$$Q_0 = \begin{bmatrix} 0 & b_1 \\ a_1 & 0 \end{bmatrix}.$$

Then $\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} \in O(2r)$ and $-S_0 Q_0 = Q_0 \mathrm{diag}(I_r, -I_r)$. Thus Q defined by

$$Q(z) := Q_0 V_1(z) \cdots V_N(z),$$

satisfies that

$$z^{-N} S_0 Q(z^{-1}) \mathrm{diag}(I_r, -I_r) = -Q(z),$$

and $\begin{bmatrix} P \\ Q \end{bmatrix}$ is causal and paraunitary. Therefore Q is a symmetric paraunitary extension of P. We note that the degree of each entry of Q as a polynomial of z^{-1} is not greater than N. Let $[G_e, G_o] = Q(z)U_0$. Then we have the following theorem.

THEOREM 2.3. *Suppose $[H_e, H_o]$ is an $r \times 2r$ paraunitary matrix satisfying (1.6) for $\gamma = 2N+1$. Then $[G_e, G_o]$ obtained by the above algorithm is a symmetric paraunitary extension of $[H_e, H_o]$ with*

$$z^{-N} S_0[G_e(z^{-1}), G_o(z^{-1})] \begin{bmatrix} & S_0 \\ S_0 & \end{bmatrix} = -[G_e(z), G_o(z)].$$

Let G be the filter defined by (1.8). Then G is causal, $\mathrm{len}(G) \le 2N+1$, G satisfies (1.9), and $\{H, G\}$ is an orthogonal multifilter bank. Thus we have the following corollary.

COROLLARY 2.4. *Suppose the causal FIR H generates an o.n. scaling vector $\phi = (\phi_1, \cdots, \phi_r)^T$ supported on $[0, 2N+1]$ with the first s components symmetric, and the other components antisymmetric about $N + \frac{1}{2}$. Let G be the causal matrix filter constructed by the above algorithm. Then ψ defined by (1.2) is a multiwavelet supported on $[0, 2N+1]$ with the first s components antisymmetric, and the other components symmetric about $N + \frac{1}{2}$.*

Example 1. Let $H(z) = \sum_{k=0}^{5} h_k z^{-k}$ be a matrix CQF with

$$h_0 = \frac{1}{101}\begin{bmatrix} 100/101 & 10/101 \\ 10e & e \end{bmatrix}, \quad h_1 = \frac{1}{101}\begin{bmatrix} 100/101 & 1000/101 \\ 10e & 100e \end{bmatrix},$$
$$h_2 = \frac{1}{101}\begin{bmatrix} 9801/202 & 990/101 \\ 101f & 0 \end{bmatrix}, \quad h_j = S_0 h_{5-j} S_0, \quad 3 \le j \le 5,$$

where $S_0 = \mathrm{diag}(1, -1)$, and

$$e := \frac{261}{4}\frac{7\sqrt{1147} - 202}{707\sqrt{1147} - 41282}, \quad f := \frac{101}{4}\frac{14\sqrt{1147} - 143}{707\sqrt{1147} - 41282}.$$

H satisfies (1.6) with $\gamma = 5$, and it generates a symmetric/antisymmetric o.n. scaling vector ϕ with $\phi \in W^{1.87659}(R)$. Here $W^s(R)$ denotes the Sobolev space consisting of all functions with $\hat{f}(\omega)(1 + |\omega|^2)^{\frac{s}{2}} \in L_2(R)$, and we use the smoothness estimate of ϕ provided in [4]. We will construct the corresponding symmetric high-pass filter by the above algorithm.

Let H_e, H_o be the filters defined by (1.4). Then $P(z) := [H_e, H_o]U_0^T$ is $p_0 + p_1 z^{-1} + p_2 z^{-2}$ with

$$p_0 = \frac{1}{101}\begin{bmatrix} 200/101 & -990/101 & 0 & -10 \\ 20e & -99e & 0 & -101e \end{bmatrix},$$
$$p_1 = \frac{1}{101}\begin{bmatrix} 99^2/101 & 1980/101 & 0 & 0 \\ 0 & 0 & -202f & 0 \end{bmatrix},$$
$$p_2 = S_0 p_0 \mathrm{diag}(I_2, -I_2).$$

By the above algorithm, we can construct $v_2 \in O(2)$, then $v_1 \in O(2)$ with

$$v_2 = -\frac{1}{101}\begin{bmatrix} 99 & -20 \\ 20 & 99 \end{bmatrix}, \quad v_1 = \frac{1}{101}\begin{bmatrix} 99 & 20 \\ 20 & -99 \end{bmatrix}$$

such that $P(z)V_2(z^{-1})V_1(z^{-1})$ is

$$P_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2f & -2e \end{bmatrix},$$

where V_j are the matrices defined by (2.8) with $v = v_j, j = 1, 2$. Let Q_0 defined by

$$Q_0 = \begin{bmatrix} 0 & 0 & 2e & -2f \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

be the orthogonal extension of P_0. Then $Q(z) = Q_0 V_1(z) V_2(z)$ is a symmetric extension of $P(z)$. Finally, we get $G(z) = \frac{\sqrt{2}}{2} Q(z^2) U_0 \begin{bmatrix} I_2 \\ z^{-1} I_2 \end{bmatrix} =: \sum_{k=0}^{5} g_k z^{-k}$ with

$$g_0 = \frac{1}{101} \begin{bmatrix} 10f & f \\ -495/101 & -99/202 \end{bmatrix}, \quad g_1 = \frac{1}{101} \begin{bmatrix} 10f & 100f \\ -495/101 & -4950/101 \end{bmatrix},$$
$$g_2 = \frac{1}{101} \begin{bmatrix} -101e & 0 \\ 990/101 & 200/101 \end{bmatrix}, \quad g_j = -S_0 g_{5-j} S_0, \quad 3 \le j \le 5.$$

The corresponding multiwavelet ψ is symmetric/antisymmetric about $5/2$. □

2.2. The case $\gamma = 2N$. Suppose H satisfies (1.6) with $\gamma = 2N, N \in N$. We hope to find such a causal G that each component of the corresponding ψ has the same symmetry center N, i.e., to find G to satisfy

$$z^{-2N} S_1 G(z^{-1}) S_0 = G(z), \quad S_1 = \mathrm{diag}(\pm 1, \cdots, \pm 1). \tag{2.9}$$

First we have the following proposition.

PROPOSITION 2.5. *Suppose $\{H, G\}$ is orthogonal, and H, G satisfy (1.6) for $\gamma = 2N$ and (2.9) respectively. Then r is even and by permutations, S_0 and S_1 are diag$(I_{\frac{r}{2}}, -I_{\frac{r}{2}})$.*

Proof. By (1.6) for $\gamma = 2N$ and (2.9),

$$\begin{bmatrix} S_0 & \\ & S_1 \end{bmatrix} \begin{bmatrix} H(1) & H(-1) \\ G(1) & G(-1) \end{bmatrix} \begin{bmatrix} S_0 & \\ & S_0 \end{bmatrix} = \begin{bmatrix} H(1) & H(-1) \\ G(1) & G(-1) \end{bmatrix}, \tag{2.10}$$

and

$$(-1)^N \begin{bmatrix} S_0 & \\ & S_1 \end{bmatrix} \begin{bmatrix} H(-i) & H(i) \\ G(-i) & G(i) \end{bmatrix} \begin{bmatrix} S_0 & \\ & S_0 \end{bmatrix} = \begin{bmatrix} H(-i) & H(i) \\ G(-i) & G(i) \end{bmatrix} \begin{bmatrix} & I_r \\ I_r & \end{bmatrix}. \tag{2.11}$$

(2.10) implies that diag(S_0, S_1) is similar to diag(S_0, S_0). Thus Tr(S_1)=Tr(S_0). While (2.11) implies that Tr(S_1)+Tr$(S_0) = 0$. Therefore Tr(S_1)=Tr$(S_0) = 0$. Hence r is even, and half diagonal entries of both S_0 and S_1 are 1 and the other half diagonal entries are -1. □

Due to Proposition 2.5, in the rest of this subsection we always assume that $r = 2m$ for some $m \in N$ and

$$S_0 = \mathrm{diag}(I_m, -I_m).$$

We will discuss the following symmetric extension problem: Given a causal H satisfying (1.6) for $\gamma = 2N$, to find G such that G satisfies (2.9) with $S_1 = \mathrm{diag}(-I_m, I_m)$ and $\{H, G\}$ is orthogonal. For this we introduce a paraunitary matrix $U(z)$ defined by

$$U(z) := \frac{1}{2} \begin{bmatrix} S_0 U_1 S_0 & U_1 \\ U_1 z^{-1} & S_0 U_1 S_0 \end{bmatrix}, \quad U_1 = \begin{bmatrix} I_r & u \\ u^T & I_r \end{bmatrix}, \quad u \in O(m). \tag{2.12}$$

LEMMA 2.6. *Let $U(z)$ be the matrix defined by (2.12) for some $u \in O(m)$. Then*
(i) $U(z)U(z^{-1})^T = I_{2m}$.
(ii) $U(z^{-1})diag(S_0 z^{-1}, S_0)U(z^{-1})^T = \begin{bmatrix} & S_0 \\ S_0 & \end{bmatrix}$.

Proof. One can obtain (i) by a direct calculation. For (ii), we have

$$U(z^{-1})\mathrm{diag}(S_0 z^{-1}, S_0) = \frac{1}{2}\begin{bmatrix} S_0 U_1 z^{-1} & U_1 S_0 \\ U_1 S_0 & S_0 U_1 \end{bmatrix} = \begin{bmatrix} & S_0 \\ S_0 & \end{bmatrix} U(z).$$

□

For a causal matrix CQF H satisfying (1.6) for $\gamma = 2N$, let H_e, H_o be the causal filters defined by (1.4). Then (1.6) for $\gamma = 2N$ is equivalent to

$$z^{-(N-1)} S_0 [H_e(z^{-1}), H_o(z^{-1})]\mathrm{diag}(z^{-1}S_0, S_0) = [H_e(z), H_o(z)]. \tag{2.13}$$

By (1.3) and symmetry of H, $h_0 h_{2N}^T = 0$ and $h_{2N} = S_0 h_0 S_0$. Thus

$$h_0 S_0 h_0^T = 0.$$

By Lemma 2.1, we can find $u_0 \in O(m)$ such that $h_0[I_m, u_0]^T = 0$. Thus

$$h_0 \begin{bmatrix} I_m & u_0 \\ u_0^T & I_m \end{bmatrix} = 0. \tag{2.14}$$

Let $U_0(z)$ be the paraunitary matrix defined by (2.12) with $u = u_0$. Equation (2.14) implies that the $r \times 2r$ matrix $[\widetilde{H}_e, \widetilde{H}_o]$ defined by

$$[\widetilde{H}_e(z), \widetilde{H}_o(z)] = [H_e(z), H_o(z)]U_0(z^{-1})^T$$

is causal. The paraunitaryness of $[H_e, H_o]$ and $U_0(z)$ imply that $[\widetilde{H}_e, \widetilde{H}_o]$ is also paraunitary. On the other hand, by (ii) in Lemma 2.6 and (2.13), one has

$$z^{-(N-1)} S_0 [\widetilde{H}_e(z^{-1}), \widetilde{H}_o(z^{-1})] \begin{bmatrix} & S_0 \\ S_0 & \end{bmatrix} = [\widetilde{H}_e(z), \widetilde{H}_o(z)].$$

Thus by Theorem 2.3, there exist causal FIR $\widetilde{G}_e(z), \widetilde{G}_o(z)$ such that $[\widetilde{G}_e(z), \widetilde{G}_o(z)]$ is a symmetric paraunitary extension of $[\widetilde{H}_e, \widetilde{H}_o]$ with

$$z^{-(N-1)} S_0 [\widetilde{G}_e(z^{-1}), \widetilde{G}_o(z^{-1})] \begin{bmatrix} & S_0 \\ S_0 & \end{bmatrix} = -[\widetilde{G}_e(z), \widetilde{G}_o(z)].$$

Define

$$[G_e(z), G_o(z)] := [\widetilde{G}_e(z), \widetilde{G}_o(z)]U_0(z).$$

Then $[G_e(z), G_o(z)]$ is a symmetric paraunitary extension of $[H_e, H_o]$ and it satisfies

$$z^{-(N-1)} S_0 [G_e(z^{-1}), G_o(z^{-1})]\mathrm{diag}(z^{-1}S_0, S_0) = -[G_e(z), G_o(z)]. \tag{2.15}$$

THEOREM 2.7. *Suppose $[H_e, H_o]$ is an $r \times 2r$ causal paraunitary matrix satisfying (2.13). Then $[G_e, G_o]$ obtained by the above algorithm is a symmetric paraunitary extension of $[H_e, H_o]$ with $[G_e(z), G_o(z)]$ satisfying (2.15). Furthermore len(G_e) $\leq N$, len(G_o) $\leq N - 1$.*

Let G be the filter defined by (1.8). Then G is causal, $\mathrm{len}(G) \le 2N$, G satisfies (2.9) with $S_1 = \mathrm{diag}(-I_m, I_m)$, and $\{H, G\}$ is orthogonal.

COROLLARY 2.8. *Suppose the causal filter H generates an o.n. scaling vector $\phi = (\phi_1, \cdots, \phi_{2m})^T$ supported on $[0, 2N]$ with the first m components symmetric, and the other m components antisymmetric about N. Let G be the matrix filter obtained by the above algorithm. Then ψ defined by (1.2) is a multiwavelet supported on $[0, 2N]$ with the first m components antisymmetric, and the other m components symmetric about N.*

3. Symmetric extension of matrices related to different symmetric center filters. Suppose $H = \sum_{k=0}^{2\gamma+1} h_k z^{-k}$ is a matrix CQF satisfying (1.7). Let H_e, H_o be the causal filters defined by (1.4). Then $[H_e, H_o]$ satisfies

$$z^{-\gamma}\mathrm{diag}(S_0 z, 1)[H_e(z^{-1}), H_o(z^{-1})]\mathrm{diag}(J_0, z) = [H_e(z), H_o(z)], \tag{3.1}$$

where

$$J_0 := \begin{bmatrix} & & S_0 \\ & 1 & \\ S_0 & & \end{bmatrix}. \tag{3.2}$$

In this section, we discuss the symmetric extension of $[H_e, H_o]$. We will construct $[G_e, G_o]$ such that it is a paraunitary matrix of $[H_e, H_o]$ and

$$z^{-\gamma}\mathrm{diag}(\mathcal{S}_1 z, \mathcal{S}_2)[G_e(z^{-1}), G_o(z^{-1})]\mathrm{diag}(J_0, z) = [G_e(z), G_o(z)], \tag{3.3}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are defined by (1.11). Then G defined by (1.8) satisfies (1.10).

Define $R_1 \in O(2r-1)$ by

$$R_1 := \frac{\sqrt{2}}{2}\begin{bmatrix} I_{r-1} & 0 & I_{r-1} \\ 0 & \sqrt{2} & 0 \\ -I_{r-1} & 0 & I_{r-1} \end{bmatrix}. \tag{3.4}$$

Then

$$R_1 J_0 R_1^T = \mathrm{diag}(S_0, 1, -S_0).$$

Let M_0 be such a $2r \times 2r$ permutation matrix that

$$M_0 \mathrm{diag}(S_0, 1, -S_0, z) M_0 = \mathrm{diag}(z, I_r, -I_{r-1}) = \mathrm{diag}(z, D_r). \tag{3.5}$$

Recall a matrix is called a permutation matrix if its columns are a permutation of the columns of identity matrix. D_r is the matrix defined by (1.12). Denote

$$P(z) := [H_e(z), H_o(z)]\mathrm{diag}(R_1, 1)M_0,$$

Then P is causal and paraunitary, and $[H_e, H_o]$ satisfies (3.1) if and only if P satisfies

$$z^{-\gamma}\mathrm{diag}(S_0 z, 1)P(z^{-1})\mathrm{diag}(z, D_r) = P(z). \tag{3.6}$$

We now consider the symmetric extension of P. We want to construct a causal filter Q such that Q is a paraunitary extension of P and satisfies

$$z^{-\gamma}\mathrm{diag}(\mathcal{S}_1 z, \mathcal{S}_2)Q(z^{-1})\mathrm{diag}(z, D_r) = Q(z). \tag{3.7}$$

If Q satisfies (3.7), then $[G_e, G_o]$ defined by $[G_e, G_o] = QM_0\mathrm{diag}(R_1^T, 1)$ satisfies (3.3).

First let us consider the case $\gamma = 1$. In this case, (3.6) implies that P can be written in the form of

$$\begin{bmatrix} 0 & a_0 & 0 \\ 0 & 0 & b_0 \\ c_0 & y_1 & y_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y_1 & -y_2 \end{bmatrix} z^{-1},$$

where a_0, b_0 are $s \times r$ and $(r-s-1) \times (r-1)$ matrices, $c_0 \in R$ and y_1, y_2 are $1 \times r$ and $1 \times (r-1)$ row vectors. The paraunitaryness of P implies that

$$a_0 a_0^T = I_s, \quad b_0 b_0^T = I_{r-s-1}, \quad a_0 y_1^T = 0, \quad b_0 y_2^T = 0, \quad y_1 y_1^T = y_2 y_2^T, \quad c_0^2 + 4 y_1 y_1^T = 1.$$

Thus we know a_0, b_0 are $s \times r$ and $(r-s-1) \times (r-1)$ matrices of orthonormal rows.

Let θ be such a real number that

$$\cos\theta = c_0, \quad \sin\theta = \sqrt{1 - c_0^2}.$$

Then y_1, y_2 can be written as

$$y_1 = \frac{1}{2}\sin\theta u_0, \quad y_2 = \frac{1}{2}\sin\theta v_0, \tag{3.8}$$

where u_0 and v_0 are $1 \times r$ and $1 \times (r-1)$ row vectors such that $\begin{bmatrix} a_0 \\ u_0 \end{bmatrix}$ and $\begin{bmatrix} b_0 \\ v_0 \end{bmatrix}$ are $(s+1) \times r$ and $(r-s) \times (r-1)$ matrices of orthonormal rows. Indeed, if $\sin\theta = 0$, then $y_1 = 0, y_2 = 0$ and any unit vectors u_0, v_0 orthonormal to rows of a_0, b_0 respectively will do. If $\sin\theta \neq 0$, $u_0 = 2y_1/\sin\theta$, $v_0 = 2y_2/\sin\theta$.

Consider the case $2s \geq r$. Choose $(r-s-1) \times r$, $(2s-r) \times (r-1)$ and $(r-s-1) \times (r-1)$ matrices $\tilde{u}$, $\tilde{v}_1$, $\tilde{v}$ such that

$$[a_0^T, u_0^T, \tilde{u}^T] \in O(r), \quad [b_0^T, v_0^T, \tilde{v}^T, \tilde{v}_1^T] \in O(r-1),$$

where u_0, v_0 are the vectors satisfying (3.8). Then Q defined by

$$Q(z) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 2\tilde{v}_1 \\ -2\sin\theta & \cos\theta u_0 & \cos\theta v_0 \\ 0 & \tilde{u} & \tilde{v} \\ 0 & u_0 & v_0 \\ 0 & \tilde{u} & \tilde{v} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta u_0 & -\cos\theta v_0 \\ 0 & \tilde{u} & -\tilde{v} \\ 0 & -u_0 & v_0 \\ 0 & -\tilde{u} & \tilde{v} \end{bmatrix} z^{-1},$$

is a symmetric paraunitary extension of P with Q satisfying (3.7) for $\gamma = 1$.

For the case $2s < r$, choose $(r-2s) \times r$, $(s-1) \times (r-1)$ and $(s-1) \times (r-1)$ matrices $\tilde{u}_1$, $\tilde{u}$ and $\tilde{v}$ such that

$$[a_0^T, u_0^T, \tilde{u}^T, \tilde{u}_1^T] \in O(r), \quad [b_0^T, v_0^T, \tilde{v}^T] \in O(r-1),$$

where u_0, v_0 are the vectors satisfying (3.8). Then Q defined by

$$Q(z) = \frac{1}{2}\begin{bmatrix} 0 & 2\tilde{u}_1 & 0 \\ -2\sin\theta & \cos\theta u_0 & \cos\theta v_0 \\ 0 & \tilde{u} & \tilde{v} \\ 0 & u_0 & v_0 \\ 0 & \tilde{u} & \tilde{v} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta u_0 & -\cos\theta v_0 \\ 0 & \tilde{u} & -\tilde{v} \\ 0 & -u_0 & v_0 \\ 0 & -\tilde{u} & \tilde{v} \end{bmatrix} z^{-1},$$

is a symmetric paraunitary extension of P with Q satisfying (3.7) for $\gamma = 1$.

PROPOSITION 3.1. *Suppose P is a causal paraunitary matrix satisfies (3.6) for $\gamma = 1$. Then Q constructed above is a symmetric paraunitary extension of P satisfying (3.7) for $\gamma = 1$.*

Now let us discuss the case $\gamma \geq 2$. First we introduce a paraunitary matrix $W(z)$. For $w =: \begin{bmatrix} \tilde{w} \\ w_r \end{bmatrix} \in O(r)$ with w_r the last row of w, define

$$W(z) := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & \tilde{w} & -I_{r-1} \\ 0 & -\tilde{w} & I_{r-1} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2w_r & 0 \\ 0 & 0 & 0 \\ 0 & \tilde{w} & I_{r-1} \\ 0 & \tilde{w} & I_{r-1} \end{bmatrix} z^{-1}. \tag{3.9}$$

Then by a direct calculation, one has the following lemma.

LEMMA 3.2. *Let $W(z)$ be the matrix defined by (3.9) with some $w \in O(r)$. Then*
(i) $W(z)W(z^{-1})^T = I_{2r}$.
(ii) $z^{-1}diag(z^{-1}, D_r)W(z^{-1})diag(z, D_r) = W(z)$.

Suppose P is a paraunitary matrix satisfying (3.6) for $\gamma \geq 2$. Then P has the form of

$$P = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} z^{-1} + \cdots + \begin{bmatrix} S_0 a_0 & S_0 b_1 D_r \\ c_1 & d_2 D_r \end{bmatrix} z^{-(\gamma-2)} + \begin{bmatrix} 0 & S_0 b_0 D_r \\ c_0 & d_1 D_r \end{bmatrix} z^{-(\gamma-1)} + \begin{bmatrix} 0 & 0 \\ 0 & d_0 D_r \end{bmatrix} z^{-\gamma},$$

for some $c_j \in R$, $(r-1)\times 1$ and $1 \times (2r-1)$ vectors a_j and d_j, and $(r-1)\times(2r-1)$ matrices b_j. The paraunitaryness of P implies that

$$\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & d_0 D_r \end{bmatrix}^T = 0,$$
$$\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}\begin{bmatrix} 0 & S_0 b_0 D_r \\ c_0 & d_1 D_r \end{bmatrix}^T + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & d_0 D_r \end{bmatrix}^T = 0,$$

which leads to

$$\begin{bmatrix} b_0 \\ d_0 \end{bmatrix} D_r[b_0^T, d_0^T] = 0.$$

By Lemma 2.1, we can construct $w_\gamma \in O(r)$ satisfying

$$\begin{bmatrix} b_0 \\ d_0 \end{bmatrix}\begin{bmatrix} w_\gamma^T \\ (I_{r-1}, 0) \end{bmatrix} = 0. \tag{3.10}$$

Write

$$w_\gamma =: \begin{bmatrix} \tilde{w}_\gamma \\ w_{\gamma,r} \end{bmatrix},$$

where $w_{\gamma,r}$ is the last row of w_γ. From (3.10), we have

$$d_0 D_r \begin{bmatrix} w_{\gamma,r}^T & \tilde{w}_\gamma^T \\ 0 & -I_{r-1} \end{bmatrix} = 0. \tag{3.11}$$

Let $W_\gamma(z)$ be the matrix defined by (3.9) with $w = w_\gamma$. Define $\tilde{P}$ by

$$\tilde{P}(z) := P(z) W_\gamma(z^{-1})^T.$$

Then (3.10) and (3.11) imply that $\tilde{P}$ is causal, and it can be written as

$$\tilde{p}_0 + \cdots + \tilde{p}_{N-1} z^{-(\gamma-1)},$$

for some $r \times 2r$ matrices $\tilde{p}_j$. Since $W_\gamma(z)$ is paraunitary and satisfies (ii) of Lemma 3.2, $\tilde{P}$ is also paraunitary and satisfies (3.6) with $\gamma - 1$. In this way, we construct $w_{\gamma-1}, \cdots, w_2 \in O(r)$ similarly such that P can be written as

$$P(z) = \tilde{P}(z) W_\gamma(z) = \cdots = P_1(z) W_2(z) \cdots W_\gamma(z),$$

where W_j is defined by (3.9) with $w = w_j$, and P_1 is an $r \times 2r$ matrix satisfying (3.6) with $\gamma = 1$. By Proposition 3.1, we can construct a causal filter Q_1 such that Q_1 is a symmetric paraunitary extension of P_1. Let

$$Q(z) = Q_1(z) W_2(z) \cdots W_\gamma(z).$$

Then Q is a symmetric extension of P satisfying (3.7). Define

$$[G_e(z), G_o(z)] = Q(z) M_0 \operatorname{diag}(R_1^T, 1).$$

Then $[G_e, G_o]$ is a symmetric paraunitary extension of $[H_e, H_o]$ with $[G_e, G_o]$ satisfying (3.3).

THEOREM 3.3. *Suppose $[H_e, H_o]$ is an $r \times 2r$ causal paraunitary matrix satisfying (3.1). Then $[G_e, G_o]$ obtained by the above algorithm is a symmetric paraunitary extension of $[H_e, H_o]$ satisfying (3.3). Furthermore the filter length of $[G_e, G_o]$ is not greater than γ.*

Let G be the matrix defined by (1.8). Then G is causal, satisfies (1.10) and $\operatorname{len}(G) \leq 2\gamma + 1$.

COROLLARY 3.4. *Assume that the causal FIR H generates an o.n. scaling vector $\phi = (\phi_1, \cdots, \phi_r)^T$ with $\phi_1, \cdots, \phi_s$ and $\phi_{s+1}, \cdots, \phi_{r-s-1}$ symmetric and antisymmetric about $\gamma - \frac{1}{2}$, and ϕ_r is symmetric about γ. Let G be the causal matrix filter obtained by the above algorithm. Then $\psi = (\psi_1, \cdots, \psi_r)^T$ defined by (1.2) is such a multiwavelet that $\psi_1, \cdots, \psi_{2s-r}$ are antisymmetric about $\gamma - \frac{1}{2}$, $\psi_{2s-r+1}, \cdots, \psi_s$ and $\psi_{s+1}, \cdots, \psi_r$ are symmetric and antisymmetric about γ respectively for the case $2s \geq r$; and $\psi_1, \cdots, \psi_{r-2s}$ are symmetric about $\gamma - \frac{1}{2}$, $\psi_{r-2s+1}, \cdots, \psi_{r-s}$ and $\psi_{r-s+1}, \cdots, \psi_r$ are symmetric and antisymmetric about γ respectively for the case $2s < r$.*

Example 2. Let $\phi = (\phi_1, \phi_2)^T$ be the o.n. scaling vector constructed in [2]. The low-pass filter H for ϕ is given by

$$H(z) = \frac{1}{20} \begin{bmatrix} 6 + 6z^{-1} & 8\sqrt{2} \\ (-1 + 9z^{-1} + 9z^{-2} - z^{-3})/\sqrt{2} & -3 + 10z^{-1} - 3z^{-2} \end{bmatrix}.$$

In this case $S_0 = (1)$, and

$$R_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let H_e, H_o be the filters defined by (1.4). Then $P := [H_e, H_o]\mathrm{diag}(R_1, 1)M_0$ is

$$\begin{bmatrix} 0 & \frac{4}{5} & \frac{3}{5} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{20} & \frac{\sqrt{2}}{5} & -\frac{\sqrt{2}}{4} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{3\sqrt{2}}{20} & \frac{\sqrt{2}}{5} & \frac{\sqrt{2}}{4} \end{bmatrix} z^{-1}.$$

By the above algorithm, one can find P's symmetric extension Q:

$$Q(z) = \frac{1}{2}\begin{bmatrix} -\sqrt{2} & -\frac{3\sqrt{2}}{10} & \frac{2\sqrt{2}}{5} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -\frac{3\sqrt{2}}{10} & \frac{2\sqrt{2}}{5} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{3}{5} & -\frac{4}{5} & 1 \end{bmatrix} z^{-1}.$$

Then we get $[G_e, G_o] = Q(z)M_0\mathrm{diag}(R_1^T, 1)$, and finally we have $G(z) = G_e(z^2) + G_o(z^2)z^{-1}$:

$$G(z) = \frac{1}{20}\begin{bmatrix} (9 - z^{-1} - z^{-2} + 9z^{-3})/\sqrt{2} & -3 - 10z^{-1} - 3z^{-2} \\ 9 - z^{-1} + z^{-2} - 9z^{-3} & 3\sqrt{2}(z^{-2} - 1) \end{bmatrix}.$$

The first and the second components of the corresponding $\psi = (\psi_1, \psi_2)^T$ is symmetric and antisymmetric about 1 respectively. ▯

4. Parametrization of symmetric multifilter banks. In this section we discuss parametrizations of symmetric orthogonal filter banks. We consider two types of symmetry filters, having or not having the same symmetric centers, in the following two subsections respectively.

4.1. Filter banks with the same symmetric center. Assume that $\{H, G\}$ is a causal orthogonal filter bank satisfying

$$z^{-\gamma}S_0H(z^{-1})S_0 = H(z), \quad z^{-\gamma}S_1G(z^{-1})S_0 = G(z), \tag{4.1}$$

where

$$S_0 = \mathrm{diag}(I_s, -I_{r-s}), \quad S_1 = \mathrm{diag}(\pm 1, \cdots, \pm 1), \quad s \in N_0.$$

One can show as in Subsection 2.2 that $\mathrm{Tr}(S_1) = \mathrm{Tr}(S_0)$. In this subsection we assume that $S_1 = -S_0$.

THEOREM 4.1. *A causal FIR multifilter bank $\{H, G\}$ is orthogonal and satisfies (4.1) with $\gamma = 2N + 1$ for some $N \in N$, $S_1 = -S_0$ if and only if it can be factorized in the form of*

$$\begin{aligned} H(z) &= \frac{\sqrt{2}}{2}\begin{bmatrix} a_0 & 0 \\ 0 & b_0 \end{bmatrix} V_1(z^2)\cdots V_N(z^2)U_0\begin{bmatrix} I_r \\ I_rz^{-1} \end{bmatrix}, \\ G(z) &= \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & b_1 \\ a_1 & 0 \end{bmatrix} V_1(z^2)\cdots V_N(z^2)U_0\begin{bmatrix} I_r \\ I_rz^{-1} \end{bmatrix}, \end{aligned} \tag{4.2}$$

where V_j are the matrices defined by (2.8) with $v_j \in O(r)$, a_0, b_1 and a_1, b_0 are $s \times r$ and $(r-s) \times r$ matrices respectively with $[a_0^T, a_1^T], [b_0^T, b_1^T] \in O(r)$, and U_0 is the matrix defined by (2.2).

THEOREM 4.2. *A causal FIR multifilter bank $\{H, G\}$ is orthogonal and satisfies (4.1) with $\gamma = 2N$ for some $N \in N$, $S_0 = diag(I_m, -I_m)$, $S_1 = -S_0$ if and only if it can be factorized in the form of*

$$\begin{aligned} H(z) &= \frac{\sqrt{2}}{2}\begin{bmatrix} a_0 & 0 \\ 0 & b_0 \end{bmatrix} V_2(z^2)\cdots V_N(z^2)U_0U(z^2)\begin{bmatrix} I_r \\ I_rz^{-1} \end{bmatrix}, \\ G(z) &= \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & b_1 \\ a_1 & 0 \end{bmatrix} V_2(z^2)\cdots V_N(z^2)U_0U(z^2)\begin{bmatrix} I_r \\ I_rz^{-1} \end{bmatrix}, \end{aligned} \tag{4.3}$$

where V_j are the matrices defined by (2.8) with $v_j \in O(r)$, a_0, b_1 and a_1, b_0 are $s \times r$ and $(r-s) \times r$ matrices respectively with $[a_0^T, a_1^T], [b_0^T, b_1^T] \in O(r)$, and U_0 and $U(z)$ are the matrices defined by (2.2) and (2.12) with $u \in O(m)$ respectively.

Let M_1 be the permutation matrix defined by

$$M_1 := \mathrm{diag}(I_s, \begin{bmatrix} 0 & I_r \\ I_{r-s} & 0 \end{bmatrix}).$$

One can easily show that for H, G given by (4.2) and (4.3) respectively, they can also be written in the forms of

$$\begin{bmatrix} H(z) \\ G(z) \end{bmatrix} = \frac{1}{2} M_1 V_1(z^2) \cdots V_N(z^2) \begin{bmatrix} A & AS_0 \\ B & -BS_0 \end{bmatrix} \begin{bmatrix} I_r \\ I_r z^{-1} \end{bmatrix}, \tag{4.4}$$

and

$$\begin{bmatrix} H(z) \\ G(z) \end{bmatrix} = \frac{1}{2} M_1 V_2(z^2) \cdots V_N(z^2) \begin{bmatrix} A & AS_0 \\ B & -BS_0 \end{bmatrix} U(z^2) \begin{bmatrix} I_r \\ I_r z^{-1} \end{bmatrix}, \tag{4.5}$$

where $A, B \in O(r)$.

Parametric expressions of causal orthogonal multifilter banks (4.4) and (4.5) were provided in [6]. It was shown in [7] that the factorization (4.4) is complete. Theorem 4.2 shows that the factorization (4.5) is also complete. By the completeness of the factorization (4.4) and the equivalence of forms (4.2) and (4.4), Theorem 4.1 is in fact not new. For completeness of this paper, the sketch of its proof is provided here.

Proof of Theorem 4.1. Clearly if $\{H, G\}$ is given by (4.2), then it is a causal symmetric orthogonal filter bank. Conversely, let E be the polyphase matrix of H, G. Then E satisfies

$$z^{-N} \mathrm{diag}(S_0, -S_0) E(z^{-1}) \begin{bmatrix} 0 & S_0 \\ S_0 & 0 \end{bmatrix} = E(z). \tag{4.6}$$

Define $E_1(z) := E(z) U_0^T$, where U_0 is the matrix defined by (2.2). Then E_1 satisfies

$$z^{-N} \mathrm{diag}(S_0, -S_0) E_1(z^{-1}) \mathrm{diag}(I_r, -I_r) = E_1(z).$$

Write

$$E_1(z) = e_0 + \cdots + e_N z^{-N}.$$

By the symmetry of E, $e_N = \mathrm{diag}(S_0, -S_0) e_0 \mathrm{diag}(I_r, -I_r)$. By the paraunitaryness of E, $e_0 e_N^T = 0$. Thus $e_0 \mathrm{diag}(I_r, -I_r) e_0^T = 0$. By Lemma 2.1, we can find $v_N \in O(r)$ such that

$$e_0 \begin{bmatrix} I_r \\ v_N^T \end{bmatrix} = 0.$$

Let $V_N(z)$ be the matrix defined by (2.8) with $v = v_N$. Then $\tilde{E}_1(z) = E_1(z) V_N(z^{-1})$ is causal, paraunitary and satisfies

$$z^{-(N-1)} \mathrm{diag}(S_0, -S_0) \widetilde{E}_1(z^{-1}) \mathrm{diag}(I_r, -I_r) = \widetilde{E}_1(z).$$

Continuing this process, we can find $v_{N-1}, \cdots, v_1 \in O(r)$ such that $E_1(z) V_N(z^{-1}) \cdots V_1(z^{-1})$ is

$$\begin{bmatrix} a_0^T & 0 & 0 & a_1^T \\ 0 & b_0^T & b_1^T & 0 \end{bmatrix}^T,$$

where a_0, b_1 and a_1, b_0 are $s \times r$ and $(r-s) \times r$ matrices respectively satisfying $[a_0^T, a_1^T], [b_0^T, b_1^T] \in O(r)$. Thus E can be factorized into

$$E(z) = \begin{bmatrix} a_0^T & 0 & 0 & a_1^T \\ 0 & b_0^T & b_1^T & 0 \end{bmatrix}^T V_1(z) \cdots V_N(z) U_0. \tag{4.7}$$

Hence H, G can be written in the form of (4.2). $\square$

Proof of Theorem 4.2. Clearly if $\{H, G\}$ is given by (4.3), then it is a causal symmetric orthogonal filter bank. Conversely, let E be the polyphase matrix of H, G. Then E satisfies

$$z^{-(N-1)} \mathrm{diag}(S_0, -S_0) E(z^{-1}) \mathrm{diag}(z^{-1} S_0, S_0) = E(z).$$

Write

$$E(z) = [e_{0,1}, e_{0,2}] + [e_{1,1}, e_{1,2}] z^{-1} + \cdots + [e_{N,1}, e_{N,2}] z^{-N},$$

where $e_{j,1}, e_{j,2}$ are $2r \times r$ matices. Then

$$e_{N,2} = 0, \quad e_{N,1} = \mathrm{diag}(S_0, -S_0) e_{0,1} S_0.$$

By the paraunitaryness of E, $e_{0,1} e_{N,1}^T = 0$. Thus $e_{0,1} S_0 e_{0,1}^T = 0$. By Lemma 2.1, we can find $u_0 \in O(m)$ such that

$$e_{0,1} \begin{bmatrix} I_m \\ u_0^T \end{bmatrix} = 0.$$

Let $U(z)$ be the matrix defined by (2.12) with $u = u_0$. Then $\tilde{E}(z) = E(z) U(z^{-1})^T$ is causal, paraunitary and satisfies (4.6) with $N-1$. By the proof of Theorem 4.1, $\tilde{E}$ can be factorized into the product (4.7) with $N-1$. Thus H, G can be factorized into the form of (4.3). $\square$

4.2. Filter banks with different symmetric centers. Suppose $H(z) = \sum_{k=0}^{2\gamma+1} h_k z^{-k}, G(z) = \sum_{k=0}^{2\gamma+1} g_k z^{-k}$ satisfy (1.3), (1.1), and

$$\begin{aligned} &z^{-(2\gamma+1)} \mathrm{diag}(S_0 z^2, s_0) H(z^{-1}) \mathrm{diag}(S_0, s_0 z) = H(z), \\ &z^{-(2\gamma+1)} \mathrm{diag}(S_1 z^2, S_2) G(z^{-1}) \mathrm{diag}(S_0, s_0 z) = G(z), \end{aligned} \tag{4.8}$$

where $s_0 = \pm 1$, S_0, S_1, S_2 are diagonal matrices with diagonal entries 1 or -1.

PROPOSITION 4.3. *Suppose a causal multifilter bank $\{H, G\}$ is orthogonal and satisfies (4.8). Then*

$$Tr(S_0) + Tr(S_1) = s_0, \quad Tr(S_2) = 0.$$

Proof. By (4.8),

$$\begin{aligned} &\mathrm{diag}(S_0, s_0, S_1, S_2) \begin{bmatrix} H(1) & H(-1) \\ G(1) & G(-1) \end{bmatrix} \mathrm{diag}(S_0, s_0, S_0, -s_0) \\ &= \begin{bmatrix} H(1) & H(-1) \\ G(1) & G(-1) \end{bmatrix}, \end{aligned} \tag{4.9}$$

and

$$(4.10)\quad (-1)^{\gamma} i \mathrm{diag}(S_0, -s_0, S_1, -S_2) \begin{bmatrix} H(-i) & H(i) \\ G(-i) & G(i) \end{bmatrix} \mathrm{diag}(S_0, s_0 i, S_1, -s_0 i)$$
$$= \begin{bmatrix} H(-i) & H(i) \\ G(-i) & G(i) \end{bmatrix} \begin{bmatrix} & I_r \\ I_r & \end{bmatrix}.$$

By (4.9), Tr(S_0)+Tr(S_1)+Tr$(S_2) = s_0$, and by (4.10), Tr(S_0)+Tr(S_1)-Tr$(S_2) = s_0$. Thus Tr(S_0)+Tr$(S_1) = s_0$ and Tr$(S_2) = 0$. □

In the following we assume that $s_0 = 1$ and suppose

$$S_0 = \mathrm{diag}(I_s, -I_{r-s-1}), \quad S_1 = \mathrm{diag}(I_{r-p-s}, -I_{s-p}), \quad S_2 = \mathrm{diag}(I_p, -I_p)$$

for some $s, p \in N_0$ with $s \leq p, 2p \leq r$. Let E be the polyphase matrix of H, G. Then E satisfies

$$z^{-\gamma} \mathrm{diag}(S_0 z, 1, S_1 z, S_2) E(z^{-1}) \mathrm{diag}(J_0, z) = E(z),$$

where J_0 is the matrix defined by (3.2).

Let M_2 be such a permutation matrix that

$$M_2 \mathrm{diag}(S_0 z, 1, S_1 z, S_2) M_2 = \mathrm{diag}(D_{r-p} z, D_{p+1}).$$

Let R_1 be the matrix defined by (3.4) and M_0 be such a permutation matrix that (3.5) holds. Denote

$$\mathcal{E}(z) := M_2 E(z) \mathrm{diag}(R_1, 1) M_0.$$

Then $\mathcal{E}$ is causal, paraunitary, and satisfies

$$(4.11)\qquad z^{-\gamma} \mathrm{diag}(D_{r-p} z, D_{p+1}) \mathcal{E}(z^{-1}) \mathrm{diag}(z, D_r) = \mathcal{E}(z).$$

In the following we discuss the factorization of $\mathcal{E}$. First we consider the case $\gamma = 1$. For this we introduce a paraunitary matrix $\mathcal{W}(z)$ defined as follows. For $u =: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in O(r), v =: \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in O(r-1)$, where u_1, u_2, v_1 and v_2 are $(r-p) \times r$, $p \times r$, $(r-p-1) \times (r-1)$ and $p \times (r-1)$ matrices respectively, define

$$(4.12)\qquad \mathcal{W}(z) := \frac{1}{2} \begin{bmatrix} 0 & 2u_1 & 0 \\ 0 & 0 & 2v_1 \\ 2 & 0 & 0 \\ 0 & u_2 & v_2 \\ 0 & u_2 & v_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u_2 & -v_2 \\ 0 & -u_2 & v_2 \end{bmatrix} z^{-1},$$

One can show that $\mathcal{W}$ is paraunitary and satisfies (4.11) for $\gamma = 1$.

PROPOSITION 4.4. *A causal paraunitary $\mathcal{E}$ satisfies (4.11) for $\gamma = 1$ if and only it can be written as*

$$(4.13)\qquad \mathcal{E}(z) = diag(I_{2r-2p-1}, c, I_p) \mathcal{W}(z), \quad c \in O(p+1).$$

Proof. It is clear if $\mathcal{E}$ is given by (4.13), then it is paraunitary and satisfies (4.11) for $\gamma = 1$. Conversely, condition (4.11) implies that $\mathcal{E}$ has the form of

$$\begin{bmatrix} 0 & L_1 & 0 \\ 0 & 0 & L_2 \\ c_0 & & d_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & D_{p+1} d_0 D_r & \end{bmatrix} z^{-1},$$

where L_1, L_2 are $(r-p) \times r$ and $(r-p-1) \times (r-1)$ matrices respectively, c_0 is $(2p+1) \times 1$ column vector satisfying $D_{p+1} c_0 = c_0$, and d_0 is a $(2p+1) \times (2r-1)$ matrix. The paraunitaryness of $\mathcal{E}$ implies that

$$(4.14) \quad L_1 L_1^T = I_{r-p}, \quad L_2 L_2^T = I_{r-p-1}, \quad \operatorname{diag}(L_1, L_2) d_0^T = 0, \quad d_0 D_r d_0^T = 0.$$

Again let $\{x_1, x_2, \cdots, x_n\}$ be an orthonormal basis for the columns of d_0^T, where $n \le r-1$ is the rank of d_0. Write

$$[x_1, \cdots, x_n] = \begin{bmatrix} Y_1 \\ Z_1 \end{bmatrix}, \quad Y_1, Z_1 \text{ are } r \times n \text{ and } (r-1) \times n \text{ matrices.}$$

Then $\sqrt{2}Y_1, \sqrt{2}Z_1$ are $r \times n$ and $(r-1) \times n$ matrices of orthonormal columns, respectively. By (4.14), we know L_1, L_2 are $(r-p) \times r$ and $(r-p-1) \times (r-1)$ matrices of orthonormal columns, and $L_1 Y_1 = 0, L_2 Z_1 = 0$. Thus $[\sqrt{2}Y_1, L_1^T], [\sqrt{2}Z_1, L_2^T]$ are $r \times (n+r-p)$ and $(r-1) \times (n+r-p-1)$ matrices of orthonormal columns, respectively. Thus $n \le p$. Let Y_2, Z_2 be $r \times (p-n)$ and $(r-1) \times (p-n)$ matrices such that

$$[\sqrt{2}Y_1, L_1^T, \sqrt{2}Y_2] \in O(r), \quad [\sqrt{2}Z_1, L_2^T, \sqrt{2}Z_2] \in O(r-1).$$

Thus

$$\begin{bmatrix} Y_1^T & -Z_1^T \\ Y_2^T & -Z_2^T \end{bmatrix} x_j = 0.$$

Therefore

$$d_0 \begin{bmatrix} Y_1 & Y_2 \\ -Z_1 & -Z_2 \end{bmatrix} = 0, \quad \operatorname{diag}(L_1, L_2) \begin{bmatrix} Y_1 & Y_2 \\ -Z_1 & -Z_2 \end{bmatrix} = 0.$$

Let $\mathcal{W}(z)$ be the matrix defined by (4.12) with $u_1 = L_1, v_1 = L_2$, and $u_2 = \sqrt{2}[Y_1, Y_2]^T, v_2 = \sqrt{2}[Z_1, Z_2]^T$. Then $\mathcal{E}_0(z) := \mathcal{E}(z)\mathcal{W}(z^{-1})^T$ is a causal paraunitary matrix satisfying

$$\operatorname{diag}(D_{r-p} z, D_{p+1}) \mathcal{E}_0(z^{-1}) \operatorname{diag}(D_{r-p} z^{-1}, D_{p+1}) = \mathcal{E}_0(z),$$

which implies that $\mathcal{E}_0(z)$ is $\operatorname{diag}(a, b, c, d)$ for $a \in O(r-p), b \in O(r-p-1), c \in O(p+1), d \in O(p)$. One can check some parameters in a, b, c, d are redundant, and we can choose $a = I_{r-p}, b = I_{r-p-1}, d = I_p$. Hence $\mathcal{E}$ can be written in the form of (4.13). ◻

Now let us consider the case $\gamma \ge 2$.

LEMMA 4.5. *If a causal, paraunitary $\mathcal{E}(z) = e_0 + e_1 z^{-1} + \cdots e_\gamma z^{-\gamma}$ satisfies (4.11) for $\gamma \ge 2$, then there exists $w_\gamma \in O(r)$ such that*

$$e_0 [0, \tilde{w}_\gamma, -I_{r-1}]^T = 0,$$

where $\tilde{w}_\gamma$ is the matrix consisting of the first $r-1$ rows of w_γ.

Proof. By (4.11), $\mathcal{E}$ can be written as

$$\mathcal{E} = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \cdots +$$
$$\begin{bmatrix} 0 & D_{r-p} b_0 D_r \\ D_{p+1} c_0 & D_{p+1} d_1 D_r \end{bmatrix} z^{-(\gamma-1)} + \begin{bmatrix} 0 & 0 \\ 0 & D_{p+1} d_0 D_r \end{bmatrix} z^{-\gamma},$$

where a_j, b_j, c_j and d_j, $(2r-2p-1)\times 1$, $(2r-2p-1)\times(2r-1)$, $(2p+1)\times 1$ and $(2p+1)\times(2r-1)$ matrices. The paraunitaryness of $\mathcal{E}$ implies that

$$\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & D_{p+1} d_0 D_r \end{bmatrix}^T = 0,$$

and

$$\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \begin{bmatrix} 0 & D_{r-p} b_0 D_r \\ D_{p+1} c_0 & D_{p+1} d_1 D_r \end{bmatrix}^T + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & D_{p+1} d_0 D_r \end{bmatrix}^T = 0.$$

Thus

$$\begin{bmatrix} b_0 \\ d_0 \end{bmatrix} D_r [b_0^T, d_0^T] = 0.$$

Therefore by Lemma 2.1, there exists $w_\gamma \in O(r)$ such that

$$\begin{bmatrix} b_0 \\ d_0 \end{bmatrix} \begin{bmatrix} w_\gamma^T \\ (I_{r-1}, 0) \end{bmatrix} = 0.$$

The proof of Lemma 4.5 is complete. □

By Lemma 4.5, for a causal, paraunitary $\mathcal{E}$ satisfying (4.11) for $\gamma \geq 2$, there exists $w_\gamma \in O(r)$ such that $\mathcal{E}(z) W_\gamma(z^{-1})^T$ is causal, paraunitary and satisfies (4.11) for $\gamma-1$, where $W_\gamma(z)$ is the matrix defined by (3.9) with $w = w_\gamma$. In this way, we can find $w_{\gamma-1}, \cdots, w_2 \in O(r)$ such that $\mathcal{E}(z) W_\gamma(z^{-1})^T \cdots W_2(z^{-1})^T$ is causal, paraunitary and satisfies (4.11) for $\gamma = 1$. This together with Proposition 4.4 leads to the following theorems.

THEOREM 4.6. *A causal paraunitary FIR $\mathcal{E}$ satisfies (4.11) if and only it can be factorized in the form of*

$$\mathcal{E}(z) = diag(I_{2r-2p-1}, c, I_p) \mathcal{W}(z) W_2(z) \cdots W_\gamma(z),$$

where $c \in O(p+1)$, $\mathcal{W}$ is the matrix defined (4.12) for $u \in O(r), v \in O(r-1)$, and $W_j(z)$ are the matrices defined by (3.9) with $w_j \in O(r)$.

THEOREM 4.7. *A causal FIR multifilter bank $\{H, G\}$ is orthogonal and satisfies (4.8) if and only H, G can be factorized in the form*

$$\begin{bmatrix} H(z) \\ G(z) \end{bmatrix} = \frac{\sqrt{2}}{2} M_2 diag(I_{2r-2p-1}, c, I_p) \mathcal{W}(z^2) W_2(z^2) \cdots W_\gamma(z^2) M_0 diag(R_1^T, 1) \begin{bmatrix} I_r \\ I_r z^{-1} \end{bmatrix},$$

where M_2, R_1, M_0 are the matrices defined above, $c \in O(p+1)$, $\mathcal{W}$ is the matrix defined (4.12) for $u \in O(r), v \in O(r-1)$, and $W_j(z)$ are the matrices defined by (3.9) with $w_j \in O(r)$.

For the special case $r = 2, s = p = 1$, another form of the complete factorization of orthogonal $\{H, G\}$ satisfying (4.8) was obtained in [8]. By the parametric expression of symmetric multifilter banks, one can construct multiwavelets with various properties. We will carry out such work elsewhere.

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