On existence and weak stability of matrix refinable functions

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Abstract: We consider the existence of distributional (or L_2) solutions of the matrix refinement equation

$$\widehat{\Phi} = \mathbf{P}(\cdot/2)\widehat{\Phi}(\cdot/2),$$

where $\mathbf{P}$ is an $r \times r$ matrix with trigonometric polynomial entries.

One of the main results of this paper is that the above matrix refinement equation has a compactly supported distributional solution if and only if the matrix $\mathbf{P}(0)$ has an eigenvalue of the form 2^n, $n \in \mathbb{Z}_+$. A characterization of the existence of L_2-solutions of the above matrix refinement equation in terms of the mask is also given.

A concept of L_2-weak stability of a (finite) sequence of function vectors is introduced. In the case when the function vectors are solutions of a matrix refinement equation, we characterize this weak stability in terms of the mask.

AMS Subject Classification: Primary 42C15, 42B05, 41A30

Secondary 39B62, 42B10

Keywords: Refinable function vectors, stable basis

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1. Introduction

The equation

$$\Phi = \sum_{\alpha \in \mathbb{Z}^s} \mathbf{P}_\alpha \Phi(2 \cdot - \alpha), \tag{1.1}$$

where $\mathbf{P}_\alpha$, $\alpha \in \mathbb{Z}^s$ are $r \times r$ real matrices, is considered in wavelet theory in the context of multiwavelet construction and in the area of subdivision in the context of Hermite interpolation. The solution Φ in (1.1) is an r-vector.

Assume that $\mathbf{P}_\alpha = \mathbf{0}$, for all $\alpha \notin [0, N]^s$ for some positive integer N. Define

$$\mathbf{P} := 2^{-s} \sum_{\alpha \in \mathbb{Z}^s} \mathbf{P}_\alpha \exp(-i\alpha \cdot).$$

Then, $\mathbf{P}$ is an $r \times r$ matrix with trigonometric polynomial entries whose Fourier coefficients are real and supported in $[0, N]^s$. Equation (1.1) is then equivalent to

$$\widehat{\Phi} := \mathbf{P}(\cdot/2)\widehat{\Phi}(\cdot/2). \tag{1.2}$$

Either equation (1.1) or equation (1.2) is called a **matrix refinement equation**; $\mathbf{P}$ ($\{\mathbf{P}_\alpha\}$) is called the **(matrix) refinement mask** and a nonzero solution Φ of (1.1) is called a (**P**-)**refinable (function) vector**.

One of the objectives of the present paper is to characterize the existence of distributional solutions of (1.1). We will show in §2 that (1.1) has a compactly supported distributional solution if and only if $\mathbf{P}(0)$ has an eigenvalue of the form 2^n, $n \in \mathbb{Z}_+$. We will further give a characterization in terms of the refinement mask of the existence of L_2-solutions of (1.1) under a mild assumption on $\mathbf{P}(0)$. (Since only nonzero solutions are considered in this paper, a statement that (1.1) has a solution should be understood as a statement that (1.1) has a nonzero solution.)

The existence of compactly supported distributional solutions of (1.1) was previously considered in [HC] for the case $s = 1$. It was shown in [HC] that if $\rho(\mathbf{P}(0)) < 2$ and the largest eigenvalue of $\mathbf{P}(0)$ is simple, (1.1) has a compactly supported distributional solution Φ with $\widehat{\Phi}(0) = \mathbf{r}$, where $\mathbf{r}$ is a right eigenvector of eigenvalue 1 of $\mathbf{P}(0)$. (Here and hereafter the spectral radius of a matrix M is denoted by $\rho(M)$.) This result was improved in [CDP] by removing the condition that the largest eigenvalue of $\mathbf{P}(0)$ should be simple.

We will show at the beginning of §3 that if (1.1) has a solution $\Phi \subset L_2(\mathbb{R}^s)$ with $\det G_\Phi(0) \neq 0$, where $G_\Phi(\omega)$ is the Gramian matrix of Φ, the matrix $\mathbf{P}(0)$ satisfies Condition E(1) (i.e. $\rho(\mathbf{P}(0)) \leq 1$, 1 is a simple eigenvalue of $\mathbf{P}(0)$ and is the only eigenvalue on the unit circle). When $\mathbf{P}(0)$ does not satisfy Condition E(1), we introduce in §3 a concept of the *weak stability* of a sequence of function vectors, and give a characterization of the weak stability of the solutions of (1.1) in terms of the mask.

We conclude this section with some notations. We say that a matrix $M \in L_2(\mathbb{R}^s)$ (or in $\ell_2(\mathbb{Z}^s)$), when each entry of M is in $L_2(\mathbb{R}^s)$ (or in $\ell_2(\mathbb{Z}^s)$). For a given matrix $M = (m_{i,j})$,

$$|M| := \sum_{i,j} |m_{i,j}|.$$

For $1 \le j \le r$, use $\mathbf{i}_j$ to denote the vector $(\delta_j(k))_{k=1}^r$. Also, $\mathbf{l}$ stands for a unit left (row) eigenvector of $\mathbf{P}(0)$ and $\mathbf{r}$ denotes a unit right (column) eigenvector of $\mathbf{P}(0)$. Finally, $\Pi_{j=1}^k M_j := M_1 M_2 \cdots M_k$.

2. Matrix refinement equations

A column vector

$$\Phi = (\phi_i)_{i=1}^r \ne \mathbf{0},$$

where each component ϕ_i is in the space $\mathcal{S}'(\mathbb{R}^s)$ of tempered distributions, is called a solution of (1.1), if Φ satisfies the refinement equation

$$\widehat{\Phi} = \mathbf{P}(\frac{\cdot}{2})\widehat{\Phi}(\frac{\cdot}{2}). \tag{2.1}$$

The **support** of Φ is the set

$$\operatorname{supp}(\Phi) := \cup_{i=1}^r \operatorname{supp}(\phi_i).$$

The solutions of (2.1) are related to the infinite matrix product

$$\Pi_{j=1}^\infty \mathbf{P}(\cdot/2^j) := \lim_{k\to\infty} \Pi_{j=1}^k \mathbf{P}(\cdot/2^j). \tag{2.2}$$

If the infinite product (2.2) converges on every compact subset of $\mathbb{R}^s$ and each entry of the limit matrix has at most polynomial growth at infinity, then (2.1) hence (1.1) has distributional solutions.

We say that an operator M defined on a finite dimensional linear space (such as a matrix M) satisfies **Condition** $E(m)$, when the following conditions are satisfied:

(i) $\rho(M) \le 1$,

(ii) 1 is the only eigenvalue of M on the unit circle,

(iii) 1 is a nondegenerate eigenvalue of M and m is its **multiplicity**, in the sense that both the geometric and the algebraic multiplicity of 1 are m.

We also use the notation $E(m)$ for the set of all such operators.

Suppose $\mathbf{P}(0) \in E(m)$. Then, (2.2) converges on any compact subset of $\mathbb{R}^s$. It is further known that there are then m linearly independent distributional solutions Φ_k, $1 \le k \le m$ with $\widehat{\Phi}_k(0) = \mathbf{r}_k$, where $\mathbf{r}_k$ is a right eigenvector of $\mathbf{P}(0)$ corresponding to eigenvalue 1 (see [HC] and [LCY]). However equation (1.1) may have solutions even in the case (2.2) does not converge. For instance, it was shown in [CDP] that if 1 is an eigenvalue of $\mathbf{P}(0)$ and $\rho(\mathbf{P}(0)) < 2$, then (1.1) has a compactly supported distributional solution.

In this section, we will show that (1.1) has a compactly supported distributional solution if and only if $\mathbf{P}(0)$ has an eigenvalue of the form 2^n for some $n \in \mathbb{Z}_+$. The following lemma establishes the 'only if' part of that result, and was proved previously in [HC] for the case that $s = 1$.

Lemma 2.3. *Suppose (1.1) has a compactly supported distributional solution. Then, the matrix $\mathbf{P}(0)$ has an eigenvalue 2^n for some $n \in \mathbb{Z}_+$.*

Proof. Since Φ is compactly supported, $\widehat{\Phi}$ is analytic. Since $\widehat{\Phi} \neq 0$, there exist $\beta \in \mathbb{Z}_+^s$ such that

$$D^\beta \widehat{\Phi}(0) \neq 0, \; D^\alpha \widehat{\Phi}(0) = 0, \text{ for all } \alpha \in \mathbb{Z}_+^s, \quad (\beta - \alpha) \in \mathbb{Z}_+^s \backslash \{0\}.$$

Therefore,

$$(D^\beta \widehat{\Phi})(0) = \sum_{0 \le \alpha \le \beta} \frac{1}{2^{|\beta|}} \binom{\beta}{\alpha} (D^{\beta-\alpha}\mathbf{P})(0)(D^\alpha \widehat{\Phi})(0) = \frac{1}{2^{|\beta|}} \mathbf{P}(0)(D^\beta \widehat{\Phi})(0).$$

Thus, $2^{|\beta|}$ is an eigenvalue of $\mathbf{P}(0)$ and $D^\beta \widehat{\Phi}(0)$ is the corresponding eigenvector. $\square$

It follows from the next theorem, whose proof will be postponed to the end of this section, that the above necessary condition is also a sufficient condition for the existence of a compactly supported distributional solution of (1.1).

Theorem 2.4. *Suppose that 1 is the only eigenvalue of $\mathbf{P}(0)$ of the form 2^n, $n \in \mathbb{Z}_+$. Then, for each right eigenvector $\mathbf{r}$ of $\mathbf{P}(0)$ corresponding to eigenvalue 1, there is a unique compactly supported distributional solution Φ of (1.1) with $\widehat{\Phi}(0) = \mathbf{r}$.*

For the special case when $r = 2$, a result similar to Theorem 2.4 was independently obtained in [Z].

The following theorem is the main result of this section.

Theorem 2.5. *The matrix refinement equation (1.1) has a compactly supported distributional solution if and only if $\mathbf{P}(0)$ has an eigenvalue of the form 2^n, $n \in \mathbb{Z}_+$.*

Proof. The 'only if' part was proved in Lemma 2.3.

Suppose that $\mathbf{P}(0)$ has some eigenvalues of the form 2^n, $n \in \mathbb{Z}_+$. Let 2^{n_0} be the largest eigenvalue of that form. Define $\mathbf{P}_1(\omega) := 2^{-n_0}\mathbf{P}(\omega)$, then $\mathbf{P}_1(0)$ has 1 as the only eigenvalue of the form 2^n, $n \in \mathbb{Z}_+$. By Theorem 2.4, there exists a compactly supported distribution Φ_1 satisfying

$$\widehat{\Phi}_1 = \mathbf{P}_1(\frac{\cdot}{2})\widehat{\Phi}_1(\frac{\cdot}{2}).$$

Choose $\beta \in \mathbb{Z}_+^s$ such that $|\beta| = n_0$, and define $\Phi := D^\beta \Phi_1$. Then Φ is a compactly supported distribution and $\widehat{\Phi}(\omega) = i^{|\beta|}\omega^\beta \widehat{\Phi}_1(\omega) \neq 0$. Further, Φ satisties

$$\widehat{\Phi}(\omega) = 2^{|\beta|}\mathbf{P}_1(\frac{\omega}{2})\widehat{\Phi}(\frac{\omega}{2}) = \mathbf{P}(\frac{\omega}{2})\widehat{\Phi}(\frac{\omega}{2}).$$

Hence, Φ is a compactly supported distributional solution of (1.1). □

Theorem 2.4, together with the proof of Theorem 2.5, lead to the following corollary:

Corollary 2.6 . *Suppose* 1 *is an eigenvalue of* $\mathbf{P}(0)$ *and* $\mathbf{r}$ *is a corresponding right eigenvector. Then the matrix refinement equation (1.1) has a unique solution* Φ *satisfying* $\widehat{\Phi}(0) = \mathbf{r}$ *if and only if 1 is the only eigenvalue of* $\mathbf{P}(0)$ *of the form* 2^n, $n \in \mathbb{Z}_+$.

This corollary implies the following known fact (see [HC] and [LCY]).

Corollary 2.7. *Suppose that* $\mathbf{P}(0) \in E(m)$. *Let* $\mathbf{r}_1, \ldots, \mathbf{r}_m$ *be a set of linearly independent right eigenvectors of eigenvalue 1 of* $\mathbf{P}(0)$. *Then, there are* m *linearly independent solutions* $\Phi_1, \ldots, \Phi_m$, *with* $\widehat{\Phi}_j(0) = \mathbf{r}_j$, $j = 1, \ldots, m$ *of (1.1). Further, any other solution of (1.1) is a linear combination of* Φ_j, $j = 1, \ldots, m$.

Example 1. Let $\mathbf{P}_\alpha = \mathbf{0}$, $\alpha \neq 0, 1, 2, 3$,

$$\mathbf{P}_0 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad \mathbf{P}_1 = \frac{1}{4}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix},$$

and $\mathbf{P}_2 = S_0\mathbf{P}_1 S_0$, $\mathbf{P}_3 = S_0 \mathbf{P}_0 S_0$, where $S_0 := \operatorname{diag}(1, -1)$. Consider the matrix refinement equation

$$\widehat{\Phi} = \mathbf{P}(\cdot/2)\widehat{\Phi}(\cdot/2), \tag{2.8}$$

where $\mathbf{P}(\omega) = 1/2\sum_{\alpha=0}^{3}\mathbf{P}_\alpha \exp(-i\omega\alpha)$. Since $\mathbf{P}(0) = I_2$, where I_2 is the 2×2 identity matrix, (2.8) has 2 linearly independent compactly supported distributional solutions Φ_1 and Φ_2 with $\widehat{\Phi}_j(0) = \mathbf{i}_j, j = 1, 2$ by Theorem 2.5 and Corollary 2.7. In fact, the first (second) component of Φ_2 is the second (first) component of Φ_1. □

To prove Theorem 2.4, we start with a compactly supported distribution Φ_0. Then, we define Φ_k iteratively by

$$\widehat{\Phi}_k = \mathbf{P}(\cdot/2)\widehat{\Phi}_{k-1}(\cdot/2). \tag{2.9}$$

If $\widehat{\Phi}_k$ converges uniformly on any compact subset K of $\mathbb{R}^s$ to a limit vector denoted by $\widehat{\Phi}$, and if each entry of $\widehat{\Phi}$ is an entire function with at most polynomial growth at infinity, then the well defined compactly supported vector Φ is a distributional solution of (1.1). In the next lemma, we construct a special Φ_0 such that the sequence $\{\widehat{\Phi}_k\}_k$ obtained from Φ_0 converges indeed uniformly on every compact subset K of $\mathbb{R}^s$. Further, the corresponding entries of the limit vector will be shown to have at most polynomial growth at infinity. This will lead to the proof of the existence of a solution of equation (1.1).

Lemma 2.10. *Suppose that 1 is the only eigenvalue of $\mathbf{P}(0)$ of the form 2^n, $n \in \mathbb{Z}_+$. Then, for any $m \in \mathbb{Z}_+$, there exists a compactly supported distribution Φ_0 such that for an arbitrary compact subset K of $\mathbb{R}^s$,*

$$\left|\mathbf{P}(\omega/2)\widehat{\Phi}_0(\omega/2) - \widehat{\Phi}_0(\omega)\right| \le C_K|\omega|^{m+1}, \quad \omega \in K, \tag{2.11}$$

for some ω-independent constant C_K.

Proof. Let $\mathbf{r}$ be a right eigenvector of eigenvalue 1 of $\mathbf{P}(0)$. Define the compactly supported distribution Φ_0 by

$$\widehat{\Phi}_0(\omega) := \sum_{|\beta|\le m} \frac{\omega^\beta}{\beta!}\mathbf{v}_\beta, \tag{2.12}$$

where $\mathbf{v}_\beta$, $|\beta| \le m$ are r-vectors defined inductively as follows:

(i) $\mathbf{v}_0 := \mathbf{r}$;

(ii) $\mathbf{v}_\beta$ is the solution of the system of equations

$$(2^{|\beta|}I - \mathbf{P}(0))\mathbf{v}_\beta = \sum_{0<\alpha\le\beta} \binom{\beta}{\alpha} D^\alpha\mathbf{P}(0)\mathbf{v}_{\beta-\alpha}, \quad 1 < |\beta| \le m. \tag{2.13}$$

The assumption that the matrix $\mathbf{P}(0)$ has no eigenvalues of the form 2^n, $n \in \mathbb{Z}_+\backslash\{0\}$ is used here to show that (2.13) has a unique solution.

Let

$$A(\omega) := \sum_{|\beta|\le m} \frac{D^\beta\mathbf{P}(0)}{\beta!}\omega^\beta.$$

Then, we have that

$$\begin{aligned}
&|\mathbf{P}(\omega/2)\widehat{\Phi}_0(\omega/2) - \widehat{\Phi}_0(\omega)| \\
&\le |(\mathbf{P}(\omega/2) - A(\omega/2))\widehat{\Phi}_0(\omega/2)| + |A(\omega/2)\widehat{\Phi}_0(\omega/2) - \widehat{\Phi}_0(\omega)| \\
&\le C_K|\omega|^{m+1} + |A(\omega/2)\widehat{\Phi}_0(\omega/2) - \widehat{\Phi}_0(\omega)|.
\end{aligned}$$

To prove (2.11), it remains to show that

$$|A(\omega)\widehat{\Phi}_0(\omega) - \widehat{\Phi}_0(2\omega)| \le C_K|\omega|^{m+1}, \quad \omega \in K,$$

for some constant C_K.

First, we calculate $A(\omega)\widehat{\Phi}_0(\omega)$. We write

$$A(\omega)\widehat{\Phi}_0(\omega) = \sum_{\beta \ge 0} c(\beta)\omega^\beta.$$

For $|\alpha| \le m$, we have that

$$c(\beta) = \sum_{0 \le \alpha \le \beta} \frac{D^\alpha \mathbf{P}(0)\mathbf{v}_{\beta-\alpha}}{\alpha!(\beta-\alpha)!} = \frac{2^{|\beta|}\mathbf{v}_\beta}{\beta!}$$

by (2.13). This means in view of the definition of Φ_0 that all the terms of order $\le m$ in $A(\omega)\widehat{\Phi}_0(\omega)$ match with those of $\widehat{\Phi}_0(2\omega)$. Thus, $A(\omega)\widehat{\Phi}_0(\omega) - \widehat{\Phi}_0(2\omega)$ consists of all terms of order $> m$ in the power expansion of $A(\omega)\widehat{\Phi}_0(\omega)$. Since the power expansion of $A(\omega)\widehat{\Phi}_0(\omega) - \widehat{\Phi}_0(2\omega)$ only has finitely many terms, and since K is a compact set, the inequalities (2.11) holds for some ω-independent constant C_K. $\square$

Proof of Theorem 2.4. Since each entry of $\mathbf{P}$ is a trigonometric polynomial, the function matrix $\mathbf{P}$ is bounded for any operator matrix norm $\|\cdot\|$. Let $c_0 := \max_{\omega \in \mathbb{T}^s}\|\mathbf{P}(\omega)\|$ and let m be a nonnegative integer with $c_0 < 2^{m+1}$.

Let K be given compact subset of $\mathbb{R}^s$. Since

$$\left|\widehat{\Phi}_{k+1}(\omega) - \widehat{\Phi}_k(\omega)\right| = \left|\left(\Pi_{j=1}^k \mathbf{P}(2^{-j}\omega)\right)\left(\mathbf{P}(\frac{\omega}{2^{k+1}})\widehat{\Phi}_0(\frac{\omega}{2^{k+1}}) - \widehat{\Phi}_0(\frac{\omega}{2^k})\right)\right|,$$

we have

$$\left|\widehat{\Phi}_{k+1}(\omega) - \widehat{\Phi}_k(\omega)\right| \le C_K\|\Pi_{j=1}^k \mathbf{P}(2^{-j}\omega)\|\left(\frac{|\omega|}{2^k}\right)^{m+1} \le C_K|\omega|^{m+1}\left(\frac{c_0}{2^{m+1}}\right)^k$$

by Lemma 2.10. Therefore, $\{\widehat{\Phi}_k(\omega)\}_k$ is a Cauchy sequence in $L_\infty(K)$. Hence $\{\widehat{\Phi}_k(\omega)\}_k$ converges uniformly on K to a continuous function vector $\widehat{\Phi}(\omega)$.

Consider any ω with $2^j \le |\omega| < 2^{j+1}$ for some $j \in \mathbb{Z}_+$. Since $\widehat{\Phi}(\omega) = \mathbf{P}(\frac{\omega}{2})\Phi(\frac{\omega}{2}) = \mathbf{P}(\frac{\omega}{2})\cdots\mathbf{P}(\frac{\omega}{2^j})\widehat{\Phi}(\frac{\omega}{2^j})$,

$$|\widehat{\Phi}(\omega)| \le c_1c_0^j \le c_1c_0^{\log_2|\omega|} = c_1|\omega|^{\log_2 c_0},$$

where $c_1 := \sup_{|\omega|\le 2} |\widehat{\Phi}(\omega)| < \infty$. Therefore, $\widehat{\Phi}(\omega)$ has at most polynomial growth at infinity, hence, Φ is a distributional solution of (1.1) with $\widehat{\Phi}(0) = \mathbf{r}$. Further, since $\widehat{\Phi}_0$ is a polynomial, Φ_0 is a compactly supported distribution. Since the mask is finitely supported, Φ_k is a compactly supported distribution for all k, and the supports of the vectors Φ_k, $k = 0, 1, \ldots$ are uniformly bounded. Since the sequence Φ_k converges to Φ in the distributional sense, Φ is compactly supported.

Finally, we show that Φ is the unique solution of (1.1) with $\widehat{\Phi}(0) = \mathbf{r}$. Suppose Ψ is a solution of (1.1) with $\widehat{\Psi}(0) = \mathbf{r}$ which is not equal to Φ. Then, $\Phi - \Psi$ is a nontrivial solution of (1.1) with $(\widehat{\Phi} - \widehat{\Psi})(0) = 0$. Applying the proof of Lemma 2.3 to the function $\widehat{\Phi} - \widehat{\Psi}$, one can obtain an eigenvalue of $\mathbf{P}(0)$ with the form 2^n for some $n \ge 1$. This is a contradiction. $\square$

3. Weak stability

In this section, we consider the weak stability of a compactly supported solution $\Phi \in L_2(\mathbb{R}^s)$ of (1.1).

It is very convenient to discuss the weak stability in the Fourier domain by using Gramian analysis. For a given sequence of functions Φ, the **pre-Gramian** matrix at $\omega \in \mathbb{T}^s$ is defined as the following $\Phi \times \mathbb{Z}^s$ matrix

$$J(\omega) := J_\Phi(\omega) := (\widehat{\phi}(\omega + 2\pi\alpha))_{\phi,\alpha},$$

where $\widehat{\phi}$ is the Fourier transform of the function ϕ. Its adjoint matrix

$$J^*(\omega) := J^*_\Phi(\omega) := (\overline{\widehat{\phi}(\omega + 2\pi\alpha)})_{\alpha,\phi}$$

is a $\mathbb{Z}^s \times \Phi$ matrix.

The **Gramian matrix** of functions Φ denoted by G_Φ is the $\Phi \times \Phi$ matrix defined as the product of J_Φ and J^*_Φ, i.e. $G(\omega) := G_\Phi(\omega) := J_\Phi(\omega) J^*_\Phi(\omega)$. The pre-Gramian matrix was first introduced in [RS]; the basic properties of the pre-Gramian and its roles in the Gramian analysis for shift invariant spaces can be found in [RS].

Let $\mathbf{P}$ be the given refinement matrix mask with $\mathbf{P}_\alpha = \mathbf{0}$, $\alpha \notin [0, N]^s$ for some positive integer N. Let $\mathbb{H}$ be the space of all $r \times r$ matrices whose entries are trigonometric polynomials with their Fourier coefficients supported in $[-N, N]^s$. The **transition operator** $\mathbf{T}$ associated to $\mathbf{P}$ is defined on $\mathbb{H}$ by

$$\mathbf{T}H := \sum_{\nu \in \{0,1\}^s} \mathbf{P}(\cdot/2 + \pi\nu) H(\cdot/2 + \pi\nu) \mathbf{P}^*(\cdot/2 + \pi\nu), \quad H \in \mathbb{H}. \tag{3.1}$$

If $\Phi \in L_2(\mathbb{R}^s)$ is a compactly supported solution of (1.1), then G_Φ is an eigenvector of $\mathbf{T}$ corresponding to eigenvalue 1.

At the first part of this section, we will show that if there is a compactly supported solution $\Phi \in L_2(\mathbb{R}^s)$ of (1.1) with $\det G_\Phi(0) \neq 0$, then $\mathbf{P}(0) \in E(1)$. For this, we observe the following facts:

Proposition 3.2. *Let Φ be a compactly supported solution of (1.1). Let λ be an eigenvalue of $\mathbf{P}(0)$ with $|\lambda| \geq 1$ and let $\mathbf{l}$ be an arbitrary left row eigenvector corresponding to λ. Suppose that $\Phi \in L_2(\mathbb{R}^s)$. Then,*

(i)

$$\mathbf{l}\widehat{\Phi}(2\pi\beta) = 0, \quad \beta \in \mathbb{Z}^s \backslash \{0\}. \tag{3.3}$$

Further, if $\lambda \neq 1$, then $\mathbf{l}\widehat{\Phi}(0) = 0$.

(ii) If $\det G_\Phi(\pi\nu) \neq 0$, $\nu \in \{0,1\}^s \backslash \{0\}$, then $\mathbf{l}\mathbf{P}(\pi\nu) = 0$, $\nu \in \{0,1\}^s \backslash \{0\}$.

Proof. Since

$$\begin{aligned} \mathbf{l}G_\Phi(0)\mathbf{l}^* = \mathbf{l}\mathbf{T}G_\Phi(0)\mathbf{l}^* &= |\lambda|^2 \mathbf{l}G_\Phi(0)\mathbf{l}^* + \sum_{\nu \in \{0,1\}^s \backslash \{0\}} \mathbf{l}\mathbf{P}(\pi\nu)G_\Phi(\pi\nu)\mathbf{P}^*(\pi\nu)\mathbf{l}^*, \\ &\geq \mathbf{l}G_\Phi(0)\mathbf{l}^* + \sum_{\nu \in \{0,1\}^s \backslash \{0\}} \mathbf{l}\mathbf{P}(\pi\nu)G_\Phi(\pi\nu)\mathbf{P}^*(\pi\nu)\mathbf{l}^*, \end{aligned}$$

we have

$$\mathbf{l}\mathbf{P}(\pi\nu)G_\Phi(\pi\nu)\mathbf{P}^*(\pi\nu)\mathbf{l}^* = 0, \nu \in \{0,1\}^s \backslash \{0\}. \tag{3.4}$$

Part (ii) follows immediately from (3.4).

For (i), using the fact that $G_\Phi(\pi\nu) = J_\Phi(\pi\nu)J_\Phi^*(\pi\nu)$, we have

$$\sum_{\alpha \in \mathbb{Z}^s} \mathbf{l}\mathbf{P}(\pi\nu)\widehat{\Phi}(\pi\nu + 2\pi\alpha)\widehat{\Phi}^*(\pi\nu + 2\pi\alpha)\mathbf{P}^*(\pi\nu)\mathbf{l}^* = 0, \nu \in \{0,1\}^s \backslash \{0\}.$$

Therefore,

$$\mathbf{l}\widehat{\Phi}(2\pi\nu + 4\pi\alpha) = 0, \ \nu \in \{0,1\}^s \backslash \{0\}, \alpha \in \mathbb{Z}^s.$$

Given $\beta \in \mathbb{Z}^s \backslash \{0\}$, we write it in the form of $\beta = 2^\ell \nu + 2^{\ell+1}\alpha$, for some $\ell \in \mathbb{Z}_+, \nu \in \{0,1\}^s \backslash \{0\}, \alpha \in \mathbb{Z}^s$. Then,

$$\begin{aligned} \mathbf{l}\widehat{\Phi}(2\pi\beta) &= \mathbf{l}\mathbf{P}(\frac{2\pi\beta}{2}) \cdots \mathbf{P}(\frac{2\pi\beta}{2^\ell})\widehat{\Phi}(\frac{2\pi\beta}{2^\ell}) \\ &= \mathbf{l}\mathbf{P}(0)^\ell \widehat{\Phi}(2\pi\nu + 4\pi\alpha) = \lambda^\ell \mathbf{l}\widehat{\Phi}(2\pi\nu + 4\pi\alpha) = 0. \end{aligned}$$

This shows (3.3).

Finally, since $\mathbf{l}\widehat{\Phi}(0) = \mathbf{l}\mathbf{P}(0)\widehat{\Phi}(0) = \lambda\mathbf{l}\widehat{\Phi}(0)$, $\mathbf{l}\widehat{\Phi}(0) = 0$, whenever $\lambda \neq 1$. $\square$

Proposition 3.5. *Let $\Phi \in L_2(\mathbb{R}^s)$ be a compactly supported solution of (1.1). Suppose that* $\det G_\Phi(0) \neq 0$, *then* $\mathbf{P}(0)$ *satisfies condition E(1).*

Proof. It is clear that the vector $\widehat{\Phi}(0) \neq 0$ is a right eigenvector of $\mathbf{P}(0)$ of eigenvalue 1. Let m be the sum of the algebraic multiplicities of all eigenvalues of $\mathbf{P}(0)$ outside the open unit disc. If $m > 1$, there exists a left eigenvector $\mathbf{l}$ of $\mathbf{P}(0)$ such that $\mathbf{l}\widehat{\Phi}(0) = 0$. However, Proposition 3.2 implies that $\mathbf{l}G_\Phi(0)\mathbf{l}^* = |\mathbf{l}\widehat{\Phi}(0)|^2 = 0$, hence $G_\Phi(0)$ must be singular. $\square$

Remark. In the proof, we used the fact that if λ is a non simple eigenvalue of a matrix M, then for a given right column eigenvector $\mathbf{r}$, there exist a left row eigenvector $\mathbf{l}$ such that $\mathbf{lr} = 0$. This can be shown easily, when M has a Jordan canonical form and it is not difficult to extend this observation to the general case. $\square$

Let $\Phi \in L_2(\mathbb{R}^s)$. We say that that Φ is **stable**, if there are constants $0 < c \leq C < \infty$, such that for arbitrary sequences $a_\phi \in \ell_2(\mathbb{Z}^s)$, $\phi \in \Phi$,

$$c \sum_{\phi\in\Phi} \|a_\phi\|^2 \leq \| \sum_{\phi\in\Phi} \sum_{\alpha\in\mathbb{Z}^s} a_\phi(\alpha)\phi(\cdot - \alpha)\|^2 \leq C \sum_{\phi\in\Phi} \|a_\phi\|^2.$$

A compactly supported vector $\Phi \in L_2(\mathbb{R}^s)$ is stable if and only if its Gramian G_Φ is positive definite everywhere (see e.g. [BDR] and [RS]).

A consequence of Proposition 3.2 and Proposition 3.5 is:

Corollary 3.6. *Suppose that (1.1) has a compactly supported stable solution* $\Phi \in L_2(\mathbb{R}^s)$. *Then,* $\mathbf{P}(0)$ *satisfies condition E(1) and* $\mathbf{l}\mathbf{P}(\pi\nu) = 0$, $\nu \in \{0,1\}^s\backslash\{0\}$, *for any left eigenvector* $\mathbf{l}$ *of eigenvalue* 1 *of* $\mathbf{P}(0)$.

Remark. A result similar to Corollary 3.6 was obtained by [DM]. Theorem 3.8 of [S] characterizes the stability of a solution Φ of (1.1) in terms of the mask. The characterization was given under the assumption that $\mathbf{P}(0)$ satisfies condition E(1) and $\mathbf{l}\mathbf{P}(\pi\nu) = 0$ for $\nu \in \{0,1\}^s\backslash\{0\}$, where $\mathbf{l}$ is the unit left (row) eigenvector of eigenvalue 1 of $\mathbf{P}(0)$. Corollary 3.6 shows that this assumption is also necessary for the stability of Φ. Therefore, Theorem 3.8 of [S] holds without the assumption. $\square$

When $\mathbf{P}(0)$ does not satisfy condition E(1), the corresponding solutions of equation (1.1) cannot be stable and, moreover, equation (1.1) can have more than one linearly independent solutions. The solutions can still be stable in the following weak sense.

A finite sequence of r-vectors $\Phi_1, \ldots, \Phi_n$ is **weakly stable**, when the following conditions are satisfied:

(i) $\Phi_j \in L_2(\mathbb{R}^s)$, for $1 \leq j \leq n$, and

(ii) there exists constants $0 < c \le C < \infty$, such that for an arbitrary r-vector sequences $\{\mathbf{v}_\alpha\}_\alpha \in \ell_2(\mathbb{Z}^s)$ the inequality

$$c \sum_{\alpha\in\mathbb{Z}^s} |\mathbf{v}_\alpha|^2 \le \sum_{j=1}^{n} \| \sum_{\alpha\in\mathbb{Z}^s} \mathbf{v}_\alpha^T \Phi_j(x-\alpha)\|_2^2 \le C \sum_{\alpha\in\mathbb{Z}^s} |\mathbf{v}_\alpha|^2$$

holds.

The weak stability in L_∞ norm was introduced in [CDL]. Clearly, if one of the sequence of vectors $\Phi_1,\dots,\Phi_n$ is stable, then the sequence $\Phi_1,\dots,\Phi_n$ is weakly stable.

Let $\Phi \in L_2(\mathbb{R}^s)$ be given. For an arbitrary vector sequence $\mathbf{v} \in \ell_2(\mathbb{Z}^s)$, we have that

$$\| \sum_{\alpha\in\mathbb{Z}^s} \mathbf{v}_\alpha^T \Phi(x-\alpha)\|_2^2 = \int_{\mathbb{T}^s} \widehat{\mathbf{v}}^*(\omega) G_\Phi(\omega)\widehat{\mathbf{v}}(\omega)d\omega,$$

where $\widehat{\mathbf{v}}$ is the Fourier series of the sequence $\mathbf{v} = \{\mathbf{v}_\alpha\}$. This observation leads to the following result:

Proposition 3.7. *Let $\Phi_j \in L_2(\mathbb{R}^s), 1 \le j \le n$, be compactly supported function vectors. The following statements are equivalent:*

(i) the sequence of function vectors $\Phi_1,\dots,\Phi_n$ is weakly stable;

(ii) there exists a positive constant c such that

$$\sum_{j=1}^{n} G_{\Phi_j}(\omega) \ge c\mathbf{I}, \quad \textit{for all } \omega \in \mathbb{T}^s;$$

(iii) the matrix

$$\left((\widehat{\Phi}_1(\omega+2\pi\alpha))_{\alpha\in\mathbb{Z}^s} (\widehat{\Phi}_2(\omega+2\pi\alpha))_{\alpha\in\mathbb{Z}^s} \cdots (\widehat{\Phi}_n(\omega+2\pi\alpha))_{\alpha\in\mathbb{Z}^s}\right)$$

of order $r \times \infty$ has rank r for all $\omega \in \mathbb{T}^s$;

(iv) the matrix

$$\left(G_{\Phi_1}(\omega) \cdots G_{\Phi_n}(\omega)\right)$$

of order $r \times rn$ has rank r for all $\omega \in \mathbb{T}^s$.

If equation (1.1) has a finite sequence of compactly supported solutions $\Phi_1,\dots\Phi_n$ which is weakly stable, we say that refinement equation (1.1) has a sequence of **weakly stable solutions**.

Proposition 3.8. *Suppose that refinement equation (1.1) has a sequence of weakly stable solutions. Then,*

$$\mathbf{P}(\pi\nu) = 0, \nu \in \{0,1\}^s\backslash\{0\}, \tag{3.9}$$

where $\mathbf{l}$ *is any left row eigenvector of eigenvalue* 1 *of* $\mathbf{P}(0)$.

Proof. Let $\Phi_1, \ldots, \Phi_n$ be a sequence of weakly stable solutions of (1.1) and let $\mathbf{l}$ be one of the left row eigenvectors of eigenvalue 1 of $\mathbf{P}(0)$. Then, since each G_{Φ_j} is an eigenvector of eigenvalue 1 of the transition operator,

$$\mathbf{l}\sum_{j=1}^{n} G_{\Phi_j}(0)\mathbf{l}^* = \mathbf{l}\sum_{j=1}^{n} G_{\Phi_j}(0)\mathbf{l}^* + \sum_{\nu\in\{0,1\}^s\setminus\{0\}} \mathbf{l}\mathbf{P}(\pi\nu)(\sum_{j=1}^{n} G_{\Phi_j}(\pi\nu))\mathbf{P}^*(\pi\nu)\mathbf{l}^*.$$

Since the matrix $\sum_{j=1}^{n} G_{\Phi_j}(\pi\nu)$ is a (strictly) positive definite matrix, $\mathbf{l}\mathbf{P}(\pi\nu) = 0$, $\nu \in \{0,1\}^s\setminus\{0\}$. $\square$

Replacing G_Φ in the proof of Proposition 3.5 by $\sum_j G_{\Phi_j}$, we obtain the following result:

Proposition 3.10. *Suppose that refinement equation (1.1) has a sequence of weakly stable solutions. Then,* $\mathbf{P}(0)$ *satisfies Condition E(m) for some positive integer* m.

Proposition 3.10 was also obtained independently by [H].

Later, we will use a special similarity transform of $\mathbf{P}(0)$ that brings $\mathbf{P}(0)$ to a particular Jordan canonical form defined as follows. Let $\mathbf{P}(0) \in E(m)$ and let U be an invertible matrix such that

Similarity transform 3.11.

(i) the matrix $U^{-1}\mathbf{P}(0)U$ *has a Jordan canonical form,*

(ii) the $m \times m$ *leading principal submatrix of* $U^{-1}\mathbf{P}(0)U$ *is the identity matrix of order* m.

We keep the matrix U fixed for the given $\mathbf{P}(0)$ throughout the rest of the paper. We denote by $\mathbf{r}_j$, $j = 1, \ldots, m$ the first m columns of the matrix U and note that they consist of a basis of the right eigenspace of $\mathbf{P}(0) \in E(m)$ corresponding to eigenvalue 1. There are m linearly independent solutions $\Phi_1, \ldots, \Phi_m$ with $\widehat{\Phi}_j(0) = \mathbf{r}_j$. Further, any solution of (1.1) should be a linear combination of $\Phi_1, \ldots, \Phi_m$ (see Corollary 2.7). The vectors $\Phi_1, \ldots, \Phi_m$ with $\widehat{\Phi}_j(0) = \mathbf{r}_j$, $1 \le j \le m$, are called **the basic solutions** of (1.1).

Let $\Psi_1, \ldots, \Psi_n$ be a sequence of weakly stable solutions of (1.1). If $n < m$, then there exists a left row eigenvector $\mathbf{l}$ of eigenvalue 1 of $\mathbf{P}(0)$, so that $\mathbf{l}\widehat{\Psi}_j(0) = 0$, $j = 1, \ldots, n$. Hence,

$$\mathbf{l}(\sum_{j=1}^{n} G_{\Psi_j}(0))\mathbf{l}^* = 0$$

by Proposition 3.2. Thus, $n \ge m$. Since an arbitrary solution of (1.1) is a linear combination of the basic solutions $\Phi_1, \ldots, \Phi_m$ with $\widehat{\Phi}_j(0) = \mathbf{r}_j$, checking whether there exist a sequence of weakly stable solutions of (1.1) is equivalent to checking whether the basic solutions $\Phi_1, \ldots, \Phi_m$ of (1.1) are weakly stable.

Define

$$\Pi_n := \chi_{[0,2\pi]^s}(\cdot/2^n)\Pi_{j=1}^n \mathbf{P}(\cdot/2^j), \quad \text{and} \quad \Pi := \Pi_{j=1}^\infty \mathbf{P}(\cdot/2^j).$$

If $\mathbf{P}(0) \in E(m)$, the sequence Π_n, $n \in \mathbb{Z}_+$ converges to Π pointwise and

$$\Pi(\omega)U = (\widehat{\Phi}_1(\omega), \cdots, \widehat{\Phi}_m(\omega), \mathbf{0}, \cdots, \mathbf{0}),$$

where U gives similarity transform 3.11. Furthermore, for $r \times r$ matrices $H_1(\omega)$ and $H_2(\omega) \in \mathbb{H}$, we have

$$\int_{\mathbb{T}^s} (\mathbf{T}^n H_1)(\omega) H_2(\omega) d\omega = \int_{\mathbb{R}^s} \Pi_n(\omega) H_1(2^{-n}\omega) \Pi_n(\omega)^* H_2(\omega) d\omega \tag{3.12}$$

(see [LCY]).

The next theorem gives a characterization of the existence of L_2-basic solutions of (1.1) in terms of the mask.

Theorem 3.13. *Suppose that $\mathbf{P}(0) \in E(m)$. Then, the basic solutions $\Phi_1, \cdots, \Phi_m$ of equation (1.1) are in $L_2(\mathbb{R}^s)$ if and only if there exists a positive semidefinite matrix $H \in \mathbb{H}$ such that*

(i) $\mathbf{T}H = H$,

(ii) the $m \times m$ leading principal submatrix of the matrix $U^{-1}H(0)(U^{-1})^*$ is positive definite, where U gives similarity transform 3.11.

Proof. Let $\mathbf{l}_j := \mathbf{i}_j^T U^{-1}$. Then $\mathbf{l}_j \mathbf{P}(0) = \mathbf{l}_j$ and $\mathbf{l}_j \mathbf{r}_k = \delta_k(j)$.

Assume that $\Phi_1, \cdots, \Phi_m \in L_2(\mathbb{R}^s)$. Then, the matrix $H(\omega) := \sum_{j=1}^m G_{\Phi_j}(\omega) \geq \mathbf{0}$ and $\mathbf{T}H = H$. Further,

$$\mathbf{i}_j^T U^{-1} H(0)(U^{-1})^* \mathbf{i}_k = \mathbf{l}_j H(0) \mathbf{l}_k^* = \sum_{\ell=1}^m \mathbf{l}_j \mathbf{r}_\ell \mathbf{r}_\ell^* \mathbf{l}_k^* = \delta_k(j),$$

by Proposition 3.2 (i). Hence, the matrix $U^{-1}H(0)(U^{-1})^*$ has I_m as its leading principal submatrix.

Conversely, assume that (i) and (ii) hold. Let M be the $m \times m$ leading principal submatrix of $U^{-1}H(0)(U^{-1})^*$. Since the matrix $\Pi_n(\omega)H(2^{-n}\omega)\Pi_n(\omega)^*$ converges pointwise to the matrix

$$\begin{aligned}\Pi(\omega)H(0)\Pi(\omega)^* &= \Pi(\omega)UU^{-1}H(0)(U^{-1})^*U^*\Pi(\omega)^* \\ &= (\widehat{\Phi}_1(\omega), \cdots, \widehat{\Phi}_m(\omega))M(\widehat{\Phi}_1(\omega), \cdots, \widehat{\Phi}_m(\omega))^*,\end{aligned}$$

we have

$$\begin{aligned}\int_{\mathbb{R}^s} |\Phi_j(\omega)|^2 d\omega &\le c\int_{\mathbb{R}^s} \liminf_{n\to\infty} \mathbf{i}_j^T \Pi_n(\omega)H(2^{-n}\omega)\Pi_n(\omega)^*\mathbf{i}_j d\omega \\ &\le c\liminf_{n\to\infty}\int_{\mathbb{R}^s} \mathbf{i}_j^T \Pi_n(\omega)H(2^{-n}\omega)\Pi_n(\omega)^*\mathbf{i}_j d\omega \\ &< \infty.\end{aligned}$$

The last inequality follows from the fact

$$\int_{\mathbb{R}^s} \Pi_n(\omega)H(2^{-n}\omega)\Pi_n(\omega)^* d\omega = \int_{\mathbb{T}^s} (\mathbf{T}^n H)(\omega)d\omega = \int_{\mathbb{T}^s} H(\omega)d\omega. \qquad \square$$

When $r = 1$, the corresponding refinement mask becomes a finitely supported sequence, hence, $\mathbf{P}$ is a trigonometric polynomial. Theorem 3.13 can be then stated as follows: Suppose $\mathbf{P}(0) = 1$. Equation (1.1) has a L_2-solution if and only if the transition operator $\mathbf{T}$ has an eigenvector H with $H(0) > 0$ corresponding to eigenvalue 1. When $r = 1$ and $s = 1$, this result was given in [V].

Example 2. Let $\{\mathbf{P}_k\}_{k=0}^3$ be the mask given in Example 1. The matrix

$$\mathbf{H}(\omega) = \sum_{k=-2}^{2} H_k e^{-ik\omega}$$

is an eigenvector of eigenvalue 1 of the transition operator $\mathbf{T}$, where

$$H_{-2} = \begin{pmatrix} -13\sqrt{3}/18 & -1.25 \\ 1.25 & 13\sqrt{3}/18 \end{pmatrix}, \quad H_{-1} = \begin{pmatrix} -11\sqrt{3}/18 & -1 \\ 1 & 11\sqrt{3}/18 \end{pmatrix},$$

$$H_0 = \begin{pmatrix} 20.95313601221041 & -13.65950144900945 \\ -13.65950144900945 & 19.79843547383120 \end{pmatrix},$$

and $H_1 = H_{-1}^T, H_2 = H_{-2}^T$. The matrix $\mathbf{H}(\omega)$ is positive definite for all $\omega \in \mathbb{T}$. Therefore, the basic solutions $\Phi_j, j = 1, 2$, are in $L^2(\mathbb{R})$ by Theorem 3.13. Since $\mathbf{P}(0) \in E(2)$, Φ_j is not stable by Proposition 3.5. $\square$

In order to characterize the weak stability in terms of the refinement mask, we introduce the generalized transition operator $\mathcal{T}$. Let $\mathbf{l}_j, 1 \le j \le m$ be the j-th row of U^{-1}, i.e. $\mathbf{l}_j = \mathbf{i}_j^T U^{-1}$, where U gives similarity transform 3.11. Define for each j, $1 \le j \le m$,

$$\mathbb{H}_j := \{H \in \mathbb{H} : \mathbf{l}_i H(0)\mathbf{l}_k^* = 0, (i,k) \ne (j,j), 1 \le i,k \le m\}. \tag{3.14}$$

Suppose that $\mathbf{P}(0) \in E(m)$ and that $\mathbf{P}$ satisfies (3.9). Then, $\mathbb{H}_j$ is an invariant subspace of $\mathbb{H}$ under the transition operator $\mathbf{T}$. Denote by $\mathbf{T}_j$ the restriction of $\mathbf{T}$ to $\mathbb{H}_j$. The **generalized transition operator** $\mathcal{T}$ is defined by

$$\mathcal{T} : \oplus_{j=1}^m \mathbb{H}_j \to \oplus_{j=1}^m \mathbb{H}_j; \quad \text{for} \quad \mathcal{H} := (H_j)_{1\le j\le m}, \quad \mathcal{T}\mathcal{H} := (\mathbf{T}_j H_j)_{1\le j\le m}.$$

If $\Phi_j \in L_2(\mathbb{R}^s)$, $1 \le j \le m$, then 1 is an eigenvalue of $\mathbf{T}_j$ and G_{Φ_j} is a corresponding eigenvector. Hence, $\mathcal{T}$ has at least m independent eigenvectors corresponding to eigenvalue 1. Therefore the operator $\mathcal{T}$ satisfies condition $E(m)$ if and only if $\mathbf{T}_j$ satisfies condition $E(1)$.

Lemma 3.15. *Suppose that the refinement equation (1.1) has a set of weakly stable solutions. Then, for an arbitrary $H \in \mathbb{H}$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^s} \Pi_n(\omega) H(\omega/2^n)) \Pi_n(\omega)^* d\omega = \int_{\mathbb{R}^s} \Pi(\omega) H(0) \Pi(\omega)^* d\omega. \tag{3.16}$$

Proof. Since

$$\int_{\mathbb{R}^s} \Pi_n(\omega) \sum_{j=1}^m G_{\Phi_j}(2^{-n}\omega) \Pi_n(\omega)^* d\omega = \int_{\mathbb{T}^s} \sum_{j=1}^m G_{\Phi_j}(\omega) d\omega,$$

we have that

$$\begin{aligned} \int_{\mathbb{R}^s} \lim_{n\to\infty} \Pi_n(\omega) \sum_{j=1}^m G_{\Phi_j}(2^{-n}\omega) \Pi_n(\omega)^* d\omega &= \int_{\mathbb{T}^s} \sum_{j=1}^m G_{\Phi_j}(\omega) d\omega \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^s} \Pi_n(\omega) \sum_{j=1}^m G_{\Phi_j}(2^{-n}\omega) \Pi_n(\omega)^* d\omega. \end{aligned} \tag{3.17}$$

Equality (3.16) follows from (3.17) and the fact that there exists a constant C such that for ℓ, k, $1 \le \ell, k \le r$,

$$|\mathbf{i}_\ell^T \Pi_n(\omega) H(\omega/2^n)) \Pi_n(\omega)^* \mathbf{i}_k| \le C \sum_{k=1}^r \mathbf{i}_k^T (\Pi_n(\omega) \sum_{j=1}^m G_{\Phi_j}(2^{-n}\omega) \Pi_n(\omega)^*) \mathbf{i}_k.$$

□

Next, we give a characterization of the weak stability in terms of the refinement mask $\mathbf{P}$.

Theorem 3.18. *Suppose that $\mathbf{P}(0) \in E(m)$ and $\mathbf{P}$ satisfies (3.9). Then the refinement equation (1.1) has a sequence of weakly stable solutions if and only if the following conditions hold:*

(i) the generalized transition operator $\mathcal{T}$ satisfies Condition $E(m)$, and

(ii) eigenvalue 1 of the transition operator $\mathcal{T}$ has an eigenvector $(H_1, \cdots, H_m)$ of rank r with each $H_j \ge 0$.

Proof. Suppose that (i) and (ii) hold. Let $\mathcal{H}(\omega) = (H_1(\omega), \cdots, H_m(\omega)) \in \oplus_{j=1}^m \mathbb{H}_j$ be the full rank eigenvector of eigenvalue 1 of $\mathcal{T}$ with $\mathbf{l}_i H_i(0) \mathbf{l}_i^* = 1$. Define $H(\omega) = \sum_{j=1}^m H_j(\omega)$, then $\mathbf{T}H = H, H \ge 0$ and $U^{-1}H(0)(U^{-1})^*$ has $\mathbf{I}_m$ as its leading principal submatrix. By Theorem 3.13, the basic solutions $\Phi_1, \cdots, \Phi_m$ of (1.1) are in $L_2(\mathbb{R}^s)$.

Since $\mathcal{T}$ satisfies Condition $E(m)$, $\mathbf{T}_j$ satisfies condition $E(1)$. This implies that $H_j = G_{\Phi_j}$. Therefore, $\Phi_1, \ldots \Phi_m$ are weakly stable.

Conversely, assume that the basic solutions of (1.1) are weakly stable. To show (i) and (ii), one only needs to show that $\mathbf{T}_j$ as a linear operator on $\mathbb{H}_j$, $1 \le j \le m$, satisfies Condition E(1). The proof can be followed line by line from the corresponding proof of Proposition 3.5 and Lemma 3.6 in [S] by applying Lemma 3.15 and by replacing $\mathbb{H}$ there by $\mathbb{H}_j$ here. □

The weak stability can also be characterized by the action of the transition operator $\mathbf{T}$ on $\mathbb{H}$.

For given $\Phi, \Psi \in L_2(\mathbb{R}^s)$, the mixed Gramian $G_{\Phi,\Psi}$ is defined by

$$G_{\Phi,\Psi}(\omega) := J_\Phi(\omega) J_\Psi^*(\omega).$$

Proposition 3.19. *Suppose that $\mathbf{P}(0) \in E(m)$. Suppose that the basic solutions $\Phi_1, \ldots, \Phi_m$ of (1.1) with $\widehat{\Phi}_i(0) = \mathbf{r}_i$ are in $L_2(\mathbb{R}^s)$. Then, the matrices G_{Φ_i,Φ_j}, $1 \le i, j \le m$ are eigenvectors of eigenvalue 1 of $\mathbf{T}$. Further, matrices G_{Φ_i,Φ_j}, $1 \le i, j \le m$ are linearly independent.*

Proof. Let $\mathbf{l}_j = \mathbf{i}_j^T U^{-1}$, where U gives similarity transform 3.11. Then, $\mathbf{l}_j$ is a left row eigenvector of $\mathbf{P}(0)$. Since $\mathbf{l}_\ell G_{\Phi_i,\Phi_j}(0)\mathbf{l}_k^* = \mathbf{l}_\ell \widehat{\Phi}_i(0)\widehat{\Phi}_j^T(0)\mathbf{l}_k^* = \delta_i(\ell)\delta_j(k)$, $G_{\Phi_i,\Phi_j}, 1 \le i, j \le m$, are linearly independent.

Let Φ be a solution of refinement equation (1.1). Write the matrix $J_\Phi(\omega)$ as a column block matrix

$$J_\Phi(\omega) = (\widehat{\phi}(\omega + 2\pi\nu + 4\pi\alpha))_{\phi\times(\nu,\alpha)\in\Phi\times(\{0,1\}^s\times\mathbb{Z}^s)},$$

then, we have

$$J_\Phi(\omega) = (\mathbf{P}(\omega/2 + \pi\nu) J_\Phi(\omega/2 + \pi\nu))_{\nu\in\{0,1\}^s}.$$

Thus,

$$G_{\Phi_i,\Phi_j}(\omega) = J_{\Phi_i}(\omega) J_{\Phi_j}^*(\omega) = \sum_{\nu\in\{0,1\}^s} \mathbf{P}(\omega/2+\pi\nu) G_{\Phi_i,\Phi_j}(\omega/2+\pi\nu) \mathbf{P}^*(\omega/2+\pi\nu), \tag{3.20}$$

hence the mixed Gramian matrix $G_{\Phi_i,\Phi_j} \in \mathbb{H}$ is an eigenvector of eigenvalue 1 of the transition operator $\mathbf{T}$. $\square$

Theorem 3.21 . *Suppose that $\mathbf{P}(0) \in E(m)$. The refinement equation (1.1) has a sequence of weakly stable solutions if and only if the following conditions hold:*
(i) the transition operator $\mathbf{T}$ defined on $\mathbb{H}$ satisfies Condition $E(m^2)$, and
(ii) there is a positive definite eigenvector of eigenvalue 1 of $\mathbf{T}$.

Proof. Assume that (i) and (ii) hold. Let H_0 be an eigenvector of eigenvalue 1 of $\mathbf{T}$ with $H_0 > 0$. The basic solutions $\Phi_1, \cdots, \Phi_m$ of (1.1) are in $L_2(\mathbb{R}^s)$ by Theorem 3.13. Since $\mathbf{T}$ satisfies Condition $E(m^2)$ and since G_{Φ_i,Φ_j} $1 \le i, j \le m$ are linearly independent eigenvectors of eigenvalue 1 of $\mathbf{T}$, we have that

$$H_0 = \sum_{1\le i,j\le m} c_{ij} G_{\Phi_i,\Phi_j}.$$

Hence,

$$0 < H_0 \le C \sum_{j=1}^{m} G_{\Phi_j},$$

and $\Phi_1, \ldots \Phi_m$ are weakly stable.

Conversely, assume that $\Phi_1, \ldots \Phi_m$ are weakly stable. Let $H_0(\omega) := \sum_{j=1}^m G_{\Phi_j}(\omega)$. It is clear that $\mathbf{T}H_0 = H_0$ with $H_0(\omega) > 0$. This gives (ii).

For (i), let H be an eigenvector of $\mathbf{T}$ corresponding to the eigenvalue λ. Since

$$\lambda^n \int_{\mathbb{T}^s} H(\omega)H(\omega)^* d\omega = \int_{\mathbb{T}^s} \mathbf{T}^n H(\omega)H(\omega)^* d\omega = \int_{\mathbb{R}^s} \Pi_n(\omega)H(2^{-n}\omega)\Pi_n(\omega)^* H(\omega)^* d\omega,$$

$\lim_{n\to\infty} \lambda^n$ exists. Hence, $|\lambda| \le 1$ and 1 is the only eigenvalue of $\mathbf{T}$ on the unit circle.

Next, we show that the geometric multiplicity of the eigenvalue 1 of $\mathbf{T}$ is m^2. Since $\mathbf{P}(0) \in E(m)$, the vectors $\mathbf{l}_i = \mathbf{i}_i^T U^{-1}$, $1 \le i \le m$, where U gives similarity transform 3.11, form a basis of the left eigenspace of $\mathbf{P}(0)$ corresponding to eigenvalue 1. Let H be an eigenvector of eigenvalue 1 of $\mathbf{T}$, and let $c_{ij} := \mathbf{l}_i H(0)\mathbf{l}_j^*$, $1 \le i, j \le m$. Then,

$$\mathbf{i}_\ell^T U^{-1}(H(0) - \sum_{1\le i,j\le m} c_{ij} G_{\Phi_i,\Phi_j}(0))U\mathbf{i}_k = \mathbf{0},$$

for $1 \le \ell, k \le m$. Therefore,

$$\Pi(\omega)(H(0) - \sum_{1\le i,j\le m} c_{ij} G_{\Phi_i,\Phi_j}(0))\Pi^*(\omega) = \mathbf{0}.$$

Applying Lemma 3.15, we have that

$$\begin{aligned}
&\int_{\mathbb{T}^s} (H - \sum_{1\le i,j\le m} c_{ij} G_{\Phi_i,\Phi_j})(H - \sum_{1\le i,j\le m} c_{ij} G_{\Phi_i,\Phi_j})^* d\omega \\
&= \int_{\mathbb{R}^s} \Pi_n(\omega)(H(2^{-n}\omega) - \sum_{i,j} c_{ij} G_{\Phi_i,\Phi_j}(2^{-n}\omega))\Pi_n(\omega)^*(H(\omega) - \sum_{i,j} c_{ij} G_{\Phi_i,\Phi_j}(\omega))^* d\omega \\
&\to \int_{\mathbb{R}^s} \Pi(\omega)(H(0) - \sum_{i,j} c_{ij} G_{\Phi_i,\Phi_j}(0))\Pi(\omega)^*(H(\omega) - \sum_{i,j} c_{ij} G_{\Phi_i,\Phi_j}(\omega))^* d\omega \\
&= \mathbf{0}.
\end{aligned}$$

Therefore,

$$H(\omega) = \sum_{1\le i,j\le m} c_{ij} G_{\Phi_i,\Phi_j}(\omega).$$

Hence, the geometric multiplicity of eigenvalue 1 of $\mathbf{T}$ is m^2.

Finally, we need to show that 1 is a nondegenerate eigenvalue of $\mathbf{T}$. Suppose that 1 is a degenerate eigenvalue of $\mathbf{T}$. Then, there exist matrices $H, G \in \mathbb{H}$ such that $\mathbf{T}G = G$ and $\mathbf{T}H = G + H$. Let

$$H_1 := H - \sum_{1 \le i,j \le m} c_{ij} G_{\Phi_i, \Phi_j},$$

where $c_{ij} = \mathbf{l}_i H(0) \mathbf{l}_j^*$. Then, on the one hand, we have

$$\begin{aligned}\int_{\mathbb{T}^s} \mathbf{T}^n H_1(\omega) G(\omega)^* d\omega &= \int_{\mathbb{R}^s} \Pi_n(\omega) H_1(2^{-n}\omega) \Pi_n(\omega)^* G(\omega)^* d\omega \\ \to \int_{\mathbb{R}^s} \Pi(\omega)(H(0) - &\sum_{1 \le i,j \le m} c_{ij} G_{\Phi_i, \Phi_j}(0)) \Pi(\omega)^* G(\omega)^* d\omega = 0.\end{aligned}$$

On the other hand, we have

$$\mathbf{T}^n H_1 = \mathbf{T}^n H - \sum_{1 \le i,j \le m} c_{ij} G_{\Phi_i, \Phi_j} = nG + H - \sum_{1 \le i,j \le m} c_{ij} G_{\Phi_i, \Phi_j},$$

which gives $\| \int_{\mathbb{T}^s} \mathbf{T}^n H_1(\omega) G(\omega)^* d\omega \| \to \infty$ as $n \to \infty$. This leads to a contradiction. $\square$

Example 3. Let $\{\mathbf{P}_k\}_{k=0}^3$ be the mask given in Example 1. It can be checked easily that the corresponding transition operator $\mathbf{T}$ is in $E(4)$. Example 2 shows that $\mathbf{T}$ has a positive definite eigenvector $\mathbf{H}(\omega)$. Therefore, the sequence of the basic solutions Φ_1, Φ_2 of (2.8) is weakly stable by Theorem 3.21. $\square$

Acknowledgments. The authors would like to thank the anonymous referee who provided us a simple proof of Lemma 2.10. The authors would like to thank Amos Ron for his suggestions and comments, which further improve the proof of Lemma 2.10 and the presentation of this paper. Qingtang Jiang is supported by an NSTB Post-Doctoral Research Fellowship at the National University of Singapore.

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