Correspondence between Frame Shrinkage and High-order Nonlinear Diffusion

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Abstract

Nonlinear diffusion filtering and wavelet/frame shrinkage are two popular methods for signal and image denoising. The relationship between these two methods has been studied recently. In this paper we investigate the correspondence between frame shrinkage and nonlinear diffusion.

We show that the frame shrinkage of Ron-Shen's continuous-linear-spline-based tight frame is associated with a fourth-order nonlinear diffusion equation. We derive high-order nonlinear diffusion equations associated with general tight frame shrinkages. These high-order nonlinear diffusion equations are different from the high-order diffusion equations studied in the literature. We also construct two sets of tight frame filter banks which result in the sixth- and eighth-order nonlinear diffusion equations.

The correspondence between frame shrinkage and diffusion filtering is useful to design diffusion-inspired shrinkage functions with competitive performance. On the other hand, the study of such a correspondence leads to a new type of diffusion equations and helps to design frame-inspired diffusivity functions. The denoising results with diffusion-inspired shrinkages provided in this paper are promising.

Key words and phrases: Nonlinear diffusion filtering, high-order nonlinear diffusion, signal denoising, undecimated frame filter banks, frame shrinkage, connection between nonlinear diffusion and frame shrinkage

1 Introduction

Nonlinear diffusion filtering [26] and wavelet shrinkage (see e.g. [15, 16, 22]) are two powerful methods for signal and image denoising. Correspondence between these two methods has been studied in [23, 31]. In this paper we investigate the correspondence between frame shrinkage and nonlinear diffusion.

For a given 1-D signal f with a noise, nonlinear diffusion filtering is to obtain $u = u(x,t)$ satisfying the nonlinear diffusion equation

$$u_t = \frac{\partial}{\partial x}\left(g(u_x^2)u_x\right), \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x),$$

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and certain boundary conditions, where g is the diffusivity and u_x denotes the first-order partial derivative of $u(x,t)$ with respect to x. The diffusivity g is a nonnegative decreasing function controlling the diffusion. The solution $u(x,t)$ of the above nonlinear equation is a denoised version of $f(x)$.

Since the nonlinear diffusion was introduced by Perona and Malik in 1990, a variety of nonlinear diffusion filters have been proposed, see e.g. [6, 36, 13] and the references therein. The fourth-order nonlinear diffusion was proposed in [38, 39] to solve the problem that (the second-order) Perona and Malik diffusion and its variants tend to produce blocky effects in image denoising. The fourth-order nonlinear diffusion has also been studied in [21], and high-order diffusion with an edge enhancing functional was proposed in [35]. The theoretical properties of high-order diffusion have been studied in [14]. A 1-D high-order diffusion equation is an equation like

$$u_t = (-1)^{n+1} \frac{\partial^n}{\partial x^n} \left(g\left((\frac{\partial^n u}{\partial x^n})^2 \right) \frac{\partial^n u}{\partial x^n} \right), \tag{1.2}$$

for an integer $n \geq 2$.

The discretization of (1.1) could be given as follows. Let h denote the spatial step size and let τ be the time step size. Denote

$$u_k^0 = f(kh), \quad k \in \mathbb{Z}.$$

We use $u_k^j, j \geq 1$ to denote the (approximation) value of the solution $u(x,t)$ at $(kh, j\tau)$. Thus u^j is the approximation solution at time $j\tau$. With the facts that $(u_k^{j+1} - u_k^j)/\tau$ approximates u_t at $(kh, j\tau)$ and $(u_{k+1}^j - u_k^j)/h$ approximates u_x at $(kh, j\tau)$, equation (1.1) can be discretized as

$$u_k^{j+1} = u_k^j + \frac{\tau}{h^2} g\left((\frac{u_{k+1}^j - u_k^j}{h})^2 \right)(u_{k+1}^j - u_k^j) - \frac{\tau}{h^2} g\left((\frac{u_k^j - u_{k-1}^j}{h})^2 \right)(u_k^j - u_{k-1}^j), \tag{1.3}$$

for $j = 0, 1, \cdots$.

Wavelets have been successfully used in signal and image processing [15, 16, 22, 33]. In particular, the undecimated wavelet transform (UWT) (also called the shift-invariant wavelet transform) based denoising [11] has been used widely for signal and image denoising. Let $\{p, q\}$ be a wavelet filter bank. For a given signal $\{c_k\}_k$, the UWT-based denoising consists of the analysis step:

$$L_n = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} p_k c_{k+n}, \; H_n = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} q_k c_{k+n}, \tag{1.4}$$

and the synthesis step:

$$u_k = \frac{\sqrt{2}}{4} \sum_{n \in \mathbb{Z}} p_n L_{k-n} + \frac{\sqrt{2}}{4} \sum_{n \in \mathbb{Z}} q_n S_\theta(H_{k-n}), \tag{1.5}$$

where S_θ is the shrinkage function, depending a parameter θ (or several parameters). With a suitable shrinkage function (for example, the hard or soft shrinkage function), $\{u_k\}_k$ is the denoised signal of the original signal $\{c_k\}_k$ with noise.

It was shown in [23] that when p, q are the Haar filter pair, namely, $p_0 = p_1 = 1$, $q_0 = 1, q_1 = -1$, $p_k = 0, q_k = 0, k \neq 0, 1$, then u_k in (1.5) is u_k^1 in (1.3) provided that shrinkage function S_θ and the diffusivity g satisfy

$$S_\theta(x) = x\left(1 - \frac{4\tau}{h^2} g(\frac{2x^2}{h^2})\right), \tag{1.6}$$

where θ is the parameter with the diffusivity g. Namely, iterated Haar wavelet shrinking and the 2nd-order diffusion filtering result in the same signal. This relationship reveals the connection between nonlinear diffusion filtering and wavelet shrinkage and hence, it opens the gate of exchanging ideas between these two fields. In particular, the connection helps to choose shrinkage functions from diffusivity functions, and vice versa. Refer to [23, 31, 24] for the detailed discussion on the importance of the relationship. The reader is referred to [1, 10] for the relationship between PDE diffusion and the bilateral filter, another popular method for image denoising.

Recently wavelet frames have been successfully used in noise removal [30], image recovery [7, 8], image inpainting/restoration [3, 4, 5], signal classification [9] and medical image analysis [18, 25]. Compared with wavelet systems, the elements in a frame system may be linearly dependent; namely, frames can be redundant. The property of redundancy not only provides a flexibility for the construction of framelets with desirable properties, but also provides high sparsity of frame transform coefficients. Such sparsity is a key property for many applications. In addition, frames work better in a noisy environment [9]. It is very natural to ask whether there is a correspondence between frame shrinkage functions and the nonlinear diffusivity functions of some diffusion equations. In this paper we show that the undecimated frame shrinking corresponds to a high-order nonlinear diffusion such as

$$u_t = \frac{\partial}{\partial x}\left(g_1(u_x^2)u_x\right) - \frac{\partial^2}{\partial x^2}\left(g_2(u_{xx}^2)u_{xx}\right), \tag{1.7}$$

with f as initial condition:

$$u(x,0) = f(x),$$

where u_{xx} denotes the second-order partial derivative of $u(x,t)$ with respect to x. Observe that the high-order diffusion equation corresponding to a frame shrinkage is different from the high-order diffusion equations like (1.2) considered in [38, 39, 21, 14].

The rest of the paper is organized as follows. In Section 2, we show how Ron-Shen's continuous-linear-spline-based tight frame shrinkage corresponds to the diffusion equation given in (1.7). In Section 3, we consider the general case. We show how the vanishing moment of a highpass filter $q^{(\ell)}$ is related to the order of a nonlinear diffusion equation and derive high-order nonlinear diffusion equations associated with general tight frame shrinkages. In Section 4, we construct two sets of tight frame filter banks which result in the 6th-order and 8th-order diffusion equations. In Section 5, we provide some experiment results. We draw the conclusion in Section 6.

2 Fourth-order diffusion and tight frame shrinkage correspondence

In this section we show how Ron-Shen's continuous-linear-spline-based tight frame shrinkage corresponds to a 4th-order nonlinear diffusion equation.

2.1 Ron-Shen's tight frame shrinkage

For a sequence $\{p_k\}_{k\in\mathbb{Z}}$ of real numbers, we use $p(\omega)$ to denote its symbol (also called filter here):

$$p(\omega) = \frac{1}{2}\sum_{k\in\mathbb{Z}} p_k e^{-ik\omega}.$$

Let $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$ be a pair of FIR frame filter banks. Assume that they are biorthogonal, namely,

$$\overline{p(\omega)}\widetilde{p}(\omega) + \sum_{\ell=1}^{L} \overline{q^{(\ell)}(\omega)}\widetilde{q}^{(\ell)}(\omega) = 1, \tag{2.1}$$

$$\overline{p(\omega)}\widetilde{p}(\omega+\pi) + \sum_{\ell=1}^{L} \overline{q^{(\ell)}(\omega)}\widetilde{q}^{(\ell)}(\omega+\pi) = 0. \tag{2.2}$$

If a filter bank $\{p, q^{(1)}, \cdots, q^{(L)}\}$ satisfies (2.1) and (2.2) with $\widetilde{p} = p, \widetilde{q}^{(\ell)} = q^{(\ell)}, 1 \le \ell \le L$, then it is called a tight frame filter bank. It was shown in [28] that if compactly supported scaling functions $\phi, \widetilde{\phi}$ corresponding to lowpass filters $p, \widetilde{p}$ are in $L^2(\mathbb{R})$ with $\int_{\mathbb{R}} \phi(x)dx \neq 0$, $\int_{\mathbb{R}} \widetilde{\phi}(x)dx \neq 0$, and $p(0) = \widetilde{p}(0) = 1, p(\pi) = \widetilde{p}(\pi) = q^{(\ell)}(0) = \widetilde{q}^{(\ell)}(0) = 0$, $1 \le \ell \le L$, then biorthogonal frame filter banks generate wavelet bi-frames (also called dual wavelet frames) of $L^2(\mathbb{R})$.

Let $\{c_k\}_k$ be the initial data. The undecimated frame transform (UFT) based denoising consists of the analysis step:

$$L_n = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} p_k c_{k+n}, \; H_n^{(\ell)} = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} q_k^{(\ell)} c_{k+n}, \quad n \in \mathbb{Z}, \ell = 1, \cdots, L, \tag{2.3}$$

and the synthesis step:

$$u_k = \frac{\sqrt{2}}{4} \sum_{n\in\mathbb{Z}} \widetilde{p}_n L_{k-n} + \frac{\sqrt{2}}{4} \sum_{\ell=1}^{L} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} S_{\theta_\ell}^{\ell}(H_{k-n}^{(\ell)}), \tag{2.4}$$

where $S_{\theta_\ell}^{\ell}, 1 \le \ell \le L$ are the shrinkage functions, depending on parameters θ_ℓ. One can easily verify that when $S_{\theta_\ell}^{\ell}(x) = x, 1 \le \ell \le L$, u_k is c_k provided that $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$ satisfy (2.1). Namely, in this case the synthesis step recovers the original signal.

In this section we consider a particular tight frame filter bank from [27]. The corresponding scaling function is the continuous linear spline function (hat function) supported on $[-1, 1]$. In this paper we call this filter bank Ron-Shen's tight frame filter bank. The nonzero coefficients of the filters are

$$p_0 = 1, \; p_1 = p_{-1} = \frac{1}{2}, \; q_0^{(1)} = 0, \; q_{-1}^{(1)} = \frac{\sqrt{2}}{2}, \; q_1^{(1)} = -\frac{\sqrt{2}}{2}, \; q_0^{(2)} = 1, \; q_{-1}^{(2)} = q_1^{(2)} = -\frac{1}{2}. \tag{2.5}$$

With this tight frame filter bank, $L_n, H_n^{(1)}, H_n^{(2)}$ defined by (2.3) are

$$L_n = \frac{\sqrt{2}}{4}(c_{n-1} + 2c_n + c_{n+1}), \quad H_n^{(1)} = \frac{1}{2}(c_{n-1} - c_{n+1}), \quad H_n^{(2)} = \frac{\sqrt{2}}{4}(2c_n - c_{n-1} - c_{n+1}). \tag{2.6}$$

Let S_θ^1 and S_σ^2 denote the frame shrinkage operators applied to the first and second highpass outputs $\{H_n^{(1)}\}_n$ and $\{H_n^{(2)}\}_n$ respectively. Then the denoised signal u_k after the synthesis step (2.4) is

$$u_k = \frac{\sqrt{2}}{4}(L_k + \frac{1}{2}L_{k-1} + \frac{1}{2}L_{k+1}) + \frac{\sqrt{2}}{4}\frac{\sqrt{2}}{2}\left(S_\theta^1(H_{k+1}^{(1)}) - S_\theta^1(H_{k-1}^{(1)})\right)$$
$$+\frac{\sqrt{2}}{4}\left(S_\sigma^2(H_k^{(2)}) - \frac{1}{2}S_\sigma^2(H_{k+1}^{(2)}) - \frac{1}{2}S_\sigma^2(H_{k-1}^{(2)})\right).$$

With $L_k, H_k^{(1)}$ and $H_k^{(2)}$ given by (2.6), u_k can be written as

$$u_k = \frac{1}{16}(c_{k-2} + 4c_{k-1} + 6c_k + 4c_{k+1} + c_{k+2}) + \frac{1}{4}S_\theta^1(\frac{c_k - c_{k+2}}{2}) - \frac{1}{4}S_\theta^1(\frac{c_{k-2} - c_k}{2}) \quad (2.7)$$
$$+\frac{\sqrt{2}}{4}S_\sigma^2(\frac{\sqrt{2}}{4}\{2c_k - c_{k-1} - c_{k+1}\}) - \frac{\sqrt{2}}{8}S_\sigma^2(\frac{\sqrt{2}}{4}\{2c_{k-1} - c_{k-2} - c_k\})$$
$$-\frac{\sqrt{2}}{8}S_\sigma^2(\frac{\sqrt{2}}{4}\{2c_{k+1} - c_k - c_{k+2}\}).$$

With suitable shrinkage functions S_θ^1 and S_σ^2, u_k is the denoised signal after one step of frame denoising process of the original c_k with noise. We can apply the above denoising process to u_k to get further denoised signal. In fact we can apply the frame shrinkage process repeatedly to the denoised signal to get further denoised signal. We call this process the iterated frame denoising process. In the next subsection we show that the output after iterated denoising process with Ron-Shen's tight frame filter bank is the same signal resulted by the nonlinear diffusion of a 4th-order diffusion equation.

2.2 Fourth-order nonlinear diffusion equation

We consider nonlinear diffusion equation (1.7) for $u = (x,t)$ with $u(x,0) = f(x)$. To discretize the diffusion equation (1.7), we recall two formulas to approximate derivatives of a function. For a function $L(x)$ on $\mathbb{R}$ and $\varepsilon > 0$, we have that (see e.g. [2])

$$L'(x_0) = \frac{1}{2\varepsilon}(L(x_0+\varepsilon) - L(x_0-\varepsilon)) - \frac{\varepsilon^2}{6}L^{(3)}(\xi_1), \quad (2.8)$$

provided that $L \in C^3[x_0-\varepsilon, x_0+\varepsilon]$, where $\xi_1 \in [x_0-\varepsilon, x_0+\varepsilon]$; and that

$$L''(x_0) = \frac{1}{\varepsilon^2}(L(x_0-\varepsilon) - 2L(x_0) + L(x_0+\varepsilon)) - \frac{\varepsilon^2}{12}L^{(4)}(\xi_2), \quad (2.9)$$

provided that $L \in C^4[x_0-\varepsilon, x_0+\varepsilon]$, where $\xi_2 \in [x_0-\varepsilon, x_0+\varepsilon]$.

Next we discretize (1.7) by using (2.8) and (2.9) to approximate the first- and second-order partial derivatives with respect to the variable x in (1.7). Recall that h and τ denote the spatial step size and the time step size respectively. As in Section 1, we use u^j to denote the approximation solution of (1.7) at time $j\tau$, and u_k^j to denote the approximation value of the solution at $(kh, j\tau)$. Thus $(u_k^{j+1} - u_k^j)/\tau$ is the approximation of u_t at $(kh, j\tau)$. We use (2.8) with $\varepsilon = h$ to approximate the first-order partial derivative u_x of $u(x,t)$ and the first-order partial derivative of $g_1(u_x^2)u_x$ in the first term on the right-hand side of (1.7), while we use (2.9) with $\varepsilon = h$ to approximate u_{xx} and the second-order partial derivative with respect to x of $g_2(u_{xx}^2)u_{xx}$ in the second term on the right-hand side of (1.7). Then we have

$$\frac{u_k^{j+1}-u_k^j}{\tau} = \frac{1}{2h}\left\{g_1\Big((\frac{u_{k+2}^j-u_k^j}{2h})^2\Big)\frac{u_{k+2}^j-u_k^j}{2h} - g_1\Big((\frac{u_k^j-u_{k-2}^j}{2h})^2\Big)\frac{u_k^j-u_{k-2}^j}{2h}\right\}$$
$$-\frac{1}{h^2}\left\{g_2\Big((\frac{u_{k-2}^j-2u_{k-1}^j+u_k^j}{h^2})^2\Big)\frac{u_{k-2}^j-2u_{k-1}^j+u_k^j}{h^2}\right.$$
$$-2g_2\Big((\frac{u_{k-1}^j-2u_k^j+u_{k+1}^j}{h^2})^2\Big)\frac{u_{k-1}^j-2u_k^j+u_{k+1}^j}{h^2}$$
$$\left.+g_2\Big((\frac{u_k^j-2u_{k+1}^j+u_{k+2}^j}{h^2})^2\Big)\frac{u_k^j-2u_{k+1}^j+u_{k+2}^j}{h^2}\right\}.$$

Thus, we get

$$\begin{aligned} u_k^{j+1} &= u_k^j + \frac{\tau}{4h^2}g_1\left((u_{k+2}^j-u_k^j)^2/(4h^2)\right)(u_{k+2}^j-u_k^j) \\ &-\frac{\tau}{4h^2}g_1\left((u_k^j-u_{k-2}^j)^2/(4h^2)\right)(u_k^j-u_{k-2}^j) \\ &-\frac{\tau}{h^4}g_2\left((u_{k-2}^j-2u_{k-1}^j+u_k^j)^2/h^4\right)(u_{k-2}^j-2u_{k-1}^j+u_k^j) \\ &+\frac{2\tau}{h^4}g_2\left((u_{k-1}^j-2u_k^j+u_{k+1}^j)^2/h^4\right)(u_{k-1}^j-2u_k^j+u_{k+1}^j) \\ &-\frac{\tau}{h^4}g_2\left((u_k^j-2u_{k+1}^j+u_{k+2}^j)^2/h^4\right)(u_k^j-2u_{k+1}^j+u_{k+2}^j). \end{aligned} \tag{2.10}$$

Next, we obtain that u_k in (2.7) after 1-step frame shrinkage is u_k^1 after 1-step diffusing if S_θ^1 and S_σ^2 are related to $g_1(x)$ and $g_2(x)$ respectively as given in the next theorem.

Theorem 1. *Let u_k in (2.7) be the resulting signal after 1-step frame shrinking with input $c_k = f(kh), k \in \mathbb{Z}$ and u_k^1 in (2.10) be the signal after 1-step diffusing with the initial input $u_k^0 = f(kh), k \in \mathbb{Z}$. If*

$$S_\theta^1(x) = x\left(1-\frac{2\tau}{h^2}g_1(\frac{x^2}{h^2})\right), \quad S_\sigma^2(x) = x\left(1-\frac{16\tau}{h^4}g_2(\frac{8x^2}{h^4})\right), \tag{2.11}$$

then $u_k = u_k^1$ for all k.

Proof. With $u_k^0 = c_k$, u_k^1 in (2.10) after 1 step diffusion is

$$\begin{aligned} u_k^1 &= c_k + \frac{\tau}{4h^2}g_1\left((c_{k+2}-c_k)^2/(4h^2)\right)(c_{k+2}-c_k) \\ &-\frac{\tau}{4h^2}g_1\left((c_k-c_{k-2})^2/(4h^2)\right)(c_k-c_{k-2}) \\ &-\frac{\tau}{h^4}g_2\left((c_{k-2}-2c_{k-1}+c_k)^2/h^4\right)(c_{k-2}-2c_{k-1}+c_k) \\ &+\frac{2\tau}{h^4}g_2\left((c_{k-1}-2c_k+c_{k+1})^2/h^4\right)(c_{k-1}-2c_k+c_{k+1}) \\ &-\frac{\tau}{h^4}g_2\left((c_k-2c_{k+1}+c_{k+2})^2/h^4\right)(c_k-2c_{k+1}+c_{k+2}). \end{aligned}$$

Write c_k as

$$\begin{aligned}c_k = &\tfrac{1}{16}(c_{k-2}+4c_{k-1}+6c_k+4c_{k+1}+c_{k+2}) - \tfrac{1}{8}(c_{k+2}-c_k) + \tfrac{1}{8}(c_k - c_{k-2}) \\ &+\tfrac{1}{16}(c_{k-2}-2c_{k-1}+c_k) - \tfrac{1}{8}(c_{k-1}-2c_k+c_{k+1}) + \tfrac{1}{16}(c_k-2c_{k+1}+c_{k+2}).\end{aligned}$$

Then we have that

$$\begin{aligned}u_k^1 = &\frac{1}{16}(c_{k-2}+4c_{k-1}+6c_k+4c_{k+1}+c_{k+2}) \\ &+\left(\frac{\tau}{4h^2}g_1((c_{k+2}-c_k)^2/(4h^2)) - \frac{1}{8}\right)(c_{k+2}-c_k) \\ &-\left(\frac{\tau}{4h^2}g_1((c_k-c_{k-2})^2/(4h^2)) - \frac{1}{8}\right)(c_k-c_{k-2}) \\ &+\left(\frac{1}{16} - \frac{\tau}{h^4}g_2\left((c_{k-2}-2c_{k-1}+c_k)^2/h^4\right)\right)(c_{k-2}-2c_{k-1}+c_k) \\ &-2\left(\frac{1}{16} - \frac{\tau}{h^4}g_2\left((c_{k-1}-2c_k+c_{k+1})^2/h^4\right)\right)(c_{k-1}-2c_k+c_{k+1}) \\ &+\left(\frac{1}{16} - \frac{\tau}{h^4}g_2\left((c_k-2c_{k+1}+c_{k+2})^2/h^4\right)\right)(c_k-2c_{k+1}+c_{k+2}).\end{aligned} \tag{2.12}$$

Comparing (2.7) with (2.12), we obtain that u_k in (2.7) after 1-step frame shrinking is u_k^1 after 1-step diffusing if S_θ^1, S_σ^2, $g_1(x)$ and $g_2(x)$ satisfy (2.11). □

From Theorem 1, we immediately have the following corollary.

Corollary 1. *With diffusivity functions $g_1(x)$, $g_2(x)$ and shrinkage functions S_θ^1, S_σ^2 satisfying* (2.11)*, iterated frame shrinking with Ron-Shen's tight frame filer bank and nonlinear diffusing with* (1.7) *result in the same signal.*

The correspondence (2.11) between frame shrinkage and diffusion filtering is useful to design diffusion-inspired shrinkage functions for frame signal denoising. On the other hand, this correspondence is useful to design frame-inspired diffusivity functions. In the following we give the corresponding shrinkage functions S_θ^1, S_σ^2 when diffusivity functions g_1, g_2 are the Perona-Malik diffusivity and Weickert diffusivity functions, and provide the associated diffusivity functions g_1, g_2 when S_θ^1, S_σ^2 are the hard shrinkage and soft shrinkage functions. The reader is referred to [23, 31, 24] for more diffusivity and shrinkage functions.

Assume the spatial step size $h=1$. Corresponding to the Perona-Malik diffusivity [26]

$$g(x^2) = \frac{c}{1+(x/\lambda)^2},$$

where c is a constant, shrinkage functions S_θ^1, S_σ^2 are

$$S_\theta^1(x) = x\Big(1-\frac{2\tau c_1}{1+(x/\theta)^2}\Big),\ S_\sigma^2(x) = x\Big(1-\frac{16\tau c_2}{1+(2\sqrt{2}x/\sigma)^2}\Big); \tag{2.13}$$

while corresponding to the TV diffusivity [29], shrinkage functions S_θ^1, S_σ^2 are

$$S_\theta^1(x) = x - 2\tau\,\mathrm{sgn}(x),\ S_\sigma^2(x) = x - 4\sqrt{2}\,\tau\,\mathrm{sgn}(x). \tag{2.14}$$

If g_1, g_2 are the Weickert diffusivity [36] given by

$$g(x^2) = \begin{cases} 1, & \text{if } x = 0, \\ 1 - \exp(-3.31488\lambda^8/x^8), & \text{if } x \neq 0, \end{cases}$$

then the corresponding shrinkage functions S_θ^1, S_σ^2 are

$$S_\theta^1(x) = \begin{cases} 0, & \text{if } x = 0, \\ x(1 - 2\tau + 2\tau\exp(-3.31488\theta^8/x^8)), & \text{if } x \neq 0, \end{cases} \tag{2.15}$$

$$S_\sigma^2(x) = \begin{cases} 0, & \text{if } x = 0, \\ x(1 - 16\tau + 16\tau\exp(-3.31488\sigma^8/(2\sqrt{2}x)^8)), & \text{if } x \neq 0. \end{cases} \tag{2.16}$$

If S_θ^1, S_σ^2 are the hard shrinkage functions [15, 22]:

$$S_\theta^1(x) = \begin{cases} 0, & \text{if } |x| \leq \theta, \\ x, & \text{if } |x| > \theta, \end{cases} \qquad S_\sigma^2(x) = \begin{cases} 0, & \text{if } |x| \leq \sigma, \\ x, & \text{if } |x| > \sigma, \end{cases}$$

then the corresponding diffusivity functions g_1, g_2 are

$$g_1(x^2) = \begin{cases} \frac{1}{2\tau}, & \text{if } |x| \leq \theta, \\ 0, & \text{if } |x| > \theta, \end{cases} \qquad g_2(x^2) = \begin{cases} \frac{1}{16\tau}, & \text{if } |x| \leq 2\sqrt{2}\sigma, \\ 0, & \text{if } |x| > 2\sqrt{2}\sigma. \end{cases} \tag{2.17}$$

When S_θ^1, S_σ^2 are the soft shrinkage functions [16]:

$$S_\theta^1(x) = \begin{cases} 0, & \text{if } |x| \leq \theta, \\ x - \theta\mathrm{sgn}(x), & \text{if } |x| > \theta, \end{cases} \qquad S_\sigma^2(x) = \begin{cases} 0, & \text{if } |x| \leq \sigma, \\ x - \theta\mathrm{sgn}(x), & \text{if } |x| > \sigma, \end{cases}$$

the corresponding diffusivity functions g_1, g_2 are

$$g_1(x^2) = \begin{cases} \frac{1}{2\tau}, & \text{if } |x| \leq \theta, \\ \frac{\theta}{2\tau|x|}, & \text{if } |x| > \theta, \end{cases} \qquad g_2(x^2) = \begin{cases} \frac{1}{16\tau}, & \text{if } |x| \leq 2\sqrt{2}\sigma, \\ \frac{\sigma}{4\sqrt{2}\tau|x|}, & \text{if } |x| > 2\sqrt{2}\sigma. \end{cases} \tag{2.18}$$

Here we should point out that the diffusivity functions g_1, g_2 in either (2.17) or (2.18) are not differentiable. Thus their derivatives in the original equation (1.7) should be understood as the differences in (2.10), a discretized version of (1.7).

2.3 Minimizer of energy functional and Euler-Lagrange equations

In this subsection we show that the nonlinear diffusion equation (1.7) is related to the Euler-Lagrange equation of a variational functional.

Let

$$F(u, u', u'') = (u - f)^2 + \alpha\Psi((u')^2) + \alpha\Phi((u'')^2),$$

where u', u'' denote the first- and second-order derivatives of $u(x)$. Consider the energy functional

$$E(u, u', u'') = \int_a^b F(u, u', u'')dx,$$

for $\alpha > 0$. A necessary condition for E to gain the minimum is that u satisfies the Euler-Lagrange equation (see [32] at p. 245)

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) = 0. \tag{2.19}$$

With

$$\frac{\partial F}{\partial x} = 2(u-f),\ \frac{\partial F}{\partial u'} = 2\alpha\Psi'((u')^2)u',\ \frac{\partial F}{\partial u''} = 2\alpha\Phi'((u'')^2)u'',$$

we know (2.19) is

$$2(u-f) - 2\alpha\frac{d}{dx}\left(\Psi'((u')^2)u'\right) + 2\alpha\frac{d^2}{dx^2}\left(\Phi'((u'')^2)u''\right) = 0.$$

Denote $g_1(x) = \Psi'(x),\ g_2(x) = \Phi'(x)$. Then the above equation can be written as

$$\frac{u-f}{\alpha} = \frac{d}{dx}\left(g_1((u')^2)u'\right) - \frac{d^2}{dx^2}\left(g_2((u'')^2)u''\right). \tag{2.20}$$

By introducing an artificial time variable t to $u(x)$ and letting $u(x,0) = f(x)$, the left-hand side of equation (2.20) can be understood as the discretization to the time variable t of $\frac{\partial}{\partial t}u(x,t)$ with step size α, and equation (2.20) is a time discretization of the nonlinear diffusion equation (1.7). The reader is referred to [14] for more detailed discussions on the relationship between Euler-Lagrange equations and high-order diffusion equations.

3 High-order diffusion and undecimated frame shrinkage correspondence

In this section we consider general frame filter banks and derive the nonlinear diffusion equations associated with them. Recall that for a pair of frame filter banks $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$, L_n and $H_n^{(\ell)}$ are the outputs of initial data $\{c_k\}_k$ after analysis algorithm (2.3) at p.4, and u_k is the shrunk data given by (2.4) with shrinkage functions $S^\ell_{\theta_\ell}, 1 \le \ell \le L$. Observe that if the shrinking operators $S^\ell_{\theta_\ell}$ in (2.4) are the identity operator (namely, no shrinking process is applied), then $u_k = c_k$ if and only if this pair of frame filter banks satisfy (2.1). We call $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$ a pair **undecimated bi-frame filter banks** if they satisfy (2.1). $\{p, q^{(1)}, \cdots, q^{(L)}\}$ is called an **undecimated tight frame filter bank** if it satisfies (2.1) with $\widetilde{p} = p, \widetilde{q}^{(1)} = q^{(1)}, \cdots, \widetilde{q}^{(L)} = q^{(L)}$. In this section we derive high-order nonlinear diffusion equations associated with undecimated frame filter banks. In §3.1, we obtain a proposition which rewrites u_k in a formula which is closely related to a discretized version of some high-order diffusion equations. In §3.2, we derive the correspondence between nonlinear diffusion equations and undecimated bi-frame filter banks.

3.1 Undecimated bi-frame shrinkage

First we have the following lemma.

Lemma 1. *Let $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$ be a pair of undecimated bi-frame filter banks, namely they satisfy* (2.1)*. Then*

$$\sum_{m\in\mathbb{Z}} \widetilde{p}_n p_{n+j} = 4\delta(j) - \sum_{\ell=1}^{L}\sum_{n\in\mathbb{Z}} \widetilde{q}^{(\ell)}_n q^{(\ell)}_{n+j},\ j \in \mathbb{Z}, \tag{3.1}$$

where $\delta(j)$ denotes the Kronecker delta sequence with $\delta(j) = 1$ if $j = 0$, and $\delta(j) = 0$ if $j \neq 0$.

Proof. Denote $\widetilde{q}^{(0)}(\omega) = \widetilde{p}(\omega)$, $q^{(0)}(\omega) = p(\omega)$. From (2.1), we have

$$(\frac{1}{2})^2 \sum_{\ell=0}^{L} \sum_{m\in\mathbb{Z}} q_m^{(\ell)} e^{im\omega} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} e^{-in\omega} = 1.$$

Using the substitution $m = n + j$, we have

$$\frac{1}{4} \sum_{\ell=0}^{L} \sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} q_{n+j}^{(\ell)} e^{ij\omega} = 1,$$

which is equivalent to

$$\sum_{\ell=0}^{L} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} q_{n+j}^{(\ell)} = 4\delta(j), \ j \in \mathbb{Z}.$$

Thus (3.1) holds. □

Proposition 1. *Suppose* $\{p, q^{(1)}, \cdots, q^{(L)}\}$ *and* $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$ *are a pair of undecimated bi-frame filter banks, namely they satisfy* (2.1)*. Let* u_k *be the resulting signal given by* (2.4) *after 1-step frame shrinking of* c_k *with these filter banks. Then*

$$u_k = c_k + \frac{\sqrt{2}}{4} \sum_{\ell=1}^{L} \sum_{m\in\mathbb{Z}} \widetilde{q}_m^{(\ell)} \left(S_{\theta_\ell}^{\ell}(x) - x \right) \Big|_{x=H_{k-m}^{(\ell)}}, \ k \in \mathbb{Z}, \tag{3.2}$$

where $H_m^{(\ell)}$ *is defined by* (2.3)*.*

Proof. By (3.1), we know the first summation in the right hand side of equation (2.4) for u_k is

$$\begin{aligned}
&\frac{1}{2\sqrt{2}} \sum_{n\in\mathbb{Z}} \widetilde{p}_n L_{k-n} = \frac{1}{4} \sum_{n\in\mathbb{Z}} \widetilde{p}_n \sum_{m\in\mathbb{Z}} p_m c_{m+k-n} = \frac{1}{4} \sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widetilde{p}_n p_{n+j} \ c_{k+j} \\
&= \sum_{j\in\mathbb{Z}} \left(\delta(j) - \frac{1}{4} \sum_{\ell=1}^{L} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} q_{n+j}^{(\ell)} \right) c_{k+j} = c_k - \frac{1}{4} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} q_{n+j}^{(\ell)} c_{k+j} \\
&= c_k - \frac{\sqrt{2}}{4} \sum_{\ell=1}^{L} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} \frac{1}{\sqrt{2}} \sum_{j\in\mathbb{Z}} q_{n+j}^{(\ell)} c_{k+j} = c_k - \frac{\sqrt{2}}{4} \sum_{\ell=1}^{L} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} H_{k-n}^{(\ell)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
u_k &= c_k - \frac{\sqrt{2}}{4} \sum_{\ell=1}^{L} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} H_{k-n}^{(\ell)} + \frac{1}{\sqrt{2}} \sum_{\ell=1}^{L} \sum_{n\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} S_{\theta_\ell}^{\ell}(H_{k-n}^{(\ell)}) \\
&= c_k + \frac{\sqrt{2}}{4} \sum_{\ell=1}^{L} \sum_{m\in\mathbb{Z}} \widetilde{q}_n^{(\ell)} \left(S_{\theta_\ell}^{\ell}(x) - x \right) \Big|_{x=H_{k-n}^{(\ell)}},
\end{aligned}$$

as desired. □

3.2 High-order nonlinear diffusion equation

For a (highpass) filter $q(\omega) = \frac{1}{2}\sum_{k\in\mathbb{Z}} q_k e^{-ik\omega}$, we say that it has vanishing moment order J if

$$\sum_{k\in\mathbb{Z}} k^j q_k = 0, \ \forall j \text{ with } 0 \le j < J.$$

The vanishing moments of analysis highpass filters imply the annihilation of discrete polynomials in the analysis step or decomposition algorithm, which results in sparse representations of input data.

Denote

$$C_J = \frac{1}{J!}\sum_{k\in\mathbb{Z}} k^J q_k. \tag{3.3}$$

Clearly, if $q(\omega)$ does not have vanishing moment order $J+1$, then $C_J \neq 0$. Next we have a result which can be found in [37] about using a highpass filter for the approximation of the derivative of a function.

Lemma 2. *If an FIR filter $q(\omega)$ has vanishing moment order J (not $J+1$), then for a function $F(x)$ smooth enough,*

$$\frac{1}{C_J}\frac{1}{\varepsilon^J}\sum_{k\in\mathbb{Z}} q_k F(x+k\varepsilon) = F^{(J)}(x) + o(1), \tag{3.4}$$

$$\frac{1}{C_J}\frac{(-1)^J}{\varepsilon^J}\sum_{k\in\mathbb{Z}} q_k F(x-k\varepsilon) = F^{(J)}(x) + o(1), \tag{3.5}$$

where C_J is defined by (3.3).

Proof. Using L'Hospital's Rule repeatedly, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^J}\sum_{k\in\mathbb{Z}} q_k F(x+k\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{\frac{d}{d\varepsilon}(\varepsilon^J)}\frac{d}{d\varepsilon}\Big(\sum_{k\in\mathbb{Z}} q_k F(x+k\varepsilon)\Big)$$
$$= \lim_{\varepsilon\to 0}\frac{1}{J\varepsilon^{J-1}}\sum_{k\in\mathbb{Z}} kq_k F'(x+k\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{J\frac{d}{d\varepsilon}(\varepsilon^{J-1})}\frac{d}{d\varepsilon}\Big(\sum_{k\in\mathbb{Z}} kq_k F'(x+k\varepsilon)\Big)$$
$$= \cdots = \lim_{\varepsilon\to 0}\frac{1}{J!}\sum_{k\in\mathbb{Z}} k^J q_k F^{(J)}(x+k\varepsilon) = \frac{1}{J!}(\sum_{k\in\mathbb{Z}} k^J q_k)F^{(J)}(x).$$

Thus we have (3.4). (3.5) follows from (3.4) with ε replaced by $-\varepsilon$. $\Box$

Let $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$ be a pair of frame filter banks satisfying (2.1). Assume that $\widetilde{q}^{(\ell)}$ and $q^{(\ell)}$ have vanishing moment orders α_ℓ (not $\alpha_\ell+1$) and β_ℓ (not $\beta_\ell+1$) respectively. Consider the following nonlinear diffusion equation for $u = (x,t)$:

$$u_t = \sum_{\ell=1}^{L}(-1)^{1+\alpha_\ell}\frac{\partial^{\alpha_\ell}}{\partial x^{\alpha_\ell}}\left(g_\ell\Big((\frac{\partial^{\beta_\ell}u}{\partial x^{\beta_\ell}})^2\Big)\frac{\partial^{\beta_\ell}u}{\partial x^{\beta_\ell}}\right), \tag{3.6}$$

with f as initial condition:

$$u(x,0) = f(x).$$

Again, denote $u_k^0 = f(kh)$, and let u_k^j denote the approximation to the value $u(kh, j\tau)$ of $u(x,t)$ at $(kh, j\tau)$, where h and τ are the spatial step size and the time step size. For the ℓ-th term in (3.6), we use following formulas to approximate partial derivatives $\frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}}$ and $\frac{\partial^{\alpha_\ell}}{\partial x^{\alpha_\ell}} G(x,t)$, where $G(x,t) := g_\ell\Big((\frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}})^2 \Big) \frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}}$:

$$\frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}}(kh, j\tau) \approx \frac{1}{C_{\beta_\ell}} \frac{1}{h^{\beta_\ell}} \sum_{n\in\mathbb{Z}} q_n^{(\ell)} u(kh+nh, j\tau) \approx \frac{1}{C_{\beta_\ell}} \frac{1}{h^{\beta_\ell}} \sum_{n\in\mathbb{Z}} q_n^{(\ell)} u_{n+k}^j, \tag{3.7}$$

$$\frac{\partial^{\alpha_\ell}}{\partial x^{\alpha_\ell}} G(kh, j\tau) \approx \frac{(-1)^{\alpha_\ell}}{\widetilde{C}_{\alpha_\ell}} \frac{1}{h^{\alpha_\ell}} \sum_{m\in\mathbb{Z}} \widetilde{q}_m^{(\ell)} G(kh-mh, j\tau), \tag{3.8}$$

where C_{β_ℓ} and $\widetilde{C}_{\alpha_\ell}$ are the constants defined by (3.3) with $q^{(\ell)}$ and $\widetilde{q}^{(\ell)}$ respectively. Observe that (3.7) and (3.8) follow from (3.4) and (3.5) respectively with $\varepsilon = h$.

With (3.7) and (3.8), (3.6) can be discretized as

$$u_k^{j+1} = u_k^j + \tau \sum_{\ell=1}^{L} (-1)^{1+\alpha_\ell} \frac{(-1)^{\alpha_\ell}}{\widetilde{C}_{\alpha_\ell}} \frac{1}{h^{\alpha_\ell}} \sum_{m\in\mathbb{Z}} \widetilde{q}_m^{(\ell)} g_\ell\Big((\frac{1}{C_{\beta_\ell}} \frac{1}{h^{\beta_\ell}} \sum_{n\in\mathbb{Z}} q_n^{(\ell)} u_{n+(k-m)}^j)^2 \Big) (\frac{1}{C_{\beta_\ell}} \frac{1}{h^{\beta_\ell}} \sum_{n\in\mathbb{Z}} q_n^{(\ell)} u_{n+(k-m)}^j).$$

In particular, with $c_k = u_k^0$, the above equation for $j = 0$ is

$$u_k^1 = c_k - \tau \sum_{\ell=1}^{L} \frac{1}{\widetilde{C}_{\alpha_\ell} h^{\alpha_\ell}} \sum_{m\in\mathbb{Z}} \widetilde{q}_m^{(\ell)} g_\ell\Big((\frac{\sqrt{2}}{C_{\beta_\ell} h^{\beta_\ell}} H_{k-m}^{(\ell)})^2 \Big) (\frac{\sqrt{2}}{C_{\beta_\ell} h^{\beta_\ell}} H_{k-m}^{(\ell)}), \tag{3.9}$$

where $H_m^{(\ell)}$ is defined by (2.3).

Comparing (3.2) with (3.9), we have the following result.

Theorem 2. *Let u_k be the resulting signal in (2.4) after 1-step of frame shrinking of $c_k = f(kh), k \in \mathbb{Z}$ with undecimated bi-frame filter banks $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and $\{\widetilde{p}, \widetilde{q}^{(1)}, \cdots, \widetilde{q}^{(L)}\}$ and shrinkage functions $S_{\theta_\ell}^{\ell}$. Let u_k^1 in (3.9) be the signal after 1-step diffusing defined above for diffusion equation (3.6) with $u_k^0 = f(kh), k \in \mathbb{Z}$ as the initial input. If*

$$S_{\theta_\ell}^{\ell}(x) = x \left(1 - \frac{4\tau}{\widetilde{C}_{\alpha_\ell} C_{\beta_\ell} h^{\alpha_\ell+\beta_\ell}} g_\ell\Big(\frac{2x^2}{(C_{\beta_\ell})^2 h^{2\beta_\ell}} \Big) \right), \ 1 \le \ell \le L, \tag{3.10}$$

then $u_k = u_k^1$ for all k.

For the tight frame filter bank, $\widetilde{p} = p$ and $\widetilde{q}^{(\ell)} = q^{(\ell)}$. In this case the nonlinear diffusion equation corresponding to the tight frame shrinking is

$$u_t = \sum_{\ell=1}^{L} (-1)^{1+\beta_\ell} \frac{\partial^{\beta_\ell}}{\partial x^{\beta_\ell}} \left(g_\ell\Big((\frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}})^2 \Big) \frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}} \right), \tag{3.11}$$

with f as initial condition:

$$u(x,0) = f(x).$$

For undecimated tight frame filter banks, the formulas used to discretize partial derivatives $\frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}}$ and $\frac{\partial^{\beta_\ell}}{\partial x^{\beta_\ell}} G(x,t)$, for $G(x,t) := g_\ell\Big((\frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}})^2 \Big) \frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}}$ are

$$\begin{aligned}
\frac{\partial^{\beta_\ell} u}{\partial x^{\beta_\ell}}(kh, j\tau) &\approx \frac{1}{C_{\beta_\ell} h^{\beta_\ell}} \sum_{n\in\mathbb{Z}} q_n^{(\ell)} u_{n+k}^j, \\
\frac{\partial^{\beta_\ell}}{\partial x^{\beta_\ell}} G(kh, j\tau) &\approx \frac{(-1)^{\beta_\ell}}{C_{\beta_\ell} h^{\beta_\ell}} \sum_{m\in\mathbb{Z}} q_m^{(\ell)} G(kh - mh, j\tau).
\end{aligned}$$

Then u_k^1 after 1-step diffusing is

$$u_k^1 = c_k - \tau \sum_{\ell=1}^{L} \frac{1}{C_{\beta_\ell} h^{\beta_\ell}} \sum_{m\in\mathbb{Z}} q_m^{(\ell)} g_\ell \Big((\frac{\sqrt{2}}{C_{\beta_\ell} h^{\beta_\ell}} H_{k-m}^{(\ell)})^2 \Big) (\frac{\sqrt{2}}{C_{\beta_\ell} h^{\beta_\ell}} H_{k-m}^{(\ell)}), \tag{3.12}$$

where $H_m^{(\ell)}$ is defined by (2.3). Comparing (3.2) with (3.12), we have the following result, which is a special case of Theorem 2.

Theorem 3. *Let u_k be the resulting signal in* (2.4) *after 1-step of frame shrinking of $c_k = f(kh), k \in \mathbb{Z}$ with an undecimated tight frame filter bank $\{p, q^{(1)}, \cdots, q^{(L)}\}$ and shrinkage functions $S_{\theta_\ell}^\ell$. Let u_k^1 in* (3.12) *be the signal after 1-step diffusing defined above for diffusion equation* (3.11) *with $u_k^0 = f(kh), k \in \mathbb{Z}$. If*

$$S_{\theta_\ell}^\ell(x) = x \left(1 - \frac{4\tau}{(C_{\beta_\ell})^2 h^{2\beta_\ell}} g_\ell \Big(\frac{2x^2}{(C_{\beta_\ell})^2 h^{2\beta_\ell}} \Big) \right),\ 1 \le \ell \le L, \tag{3.13}$$

then $u_k = u_k^1$ for all k.

Theorem 3 reveals the connection between nonlinear diffusion equations and general undecimated tight frame shrinkages. As in Section 2.3, one can show that the nonlinear diffusion equation (3.11) is related to the Euler-Lagrange equation of a variational functional.

Since bi-frame (tight frame) filter banks are undecimated bi-frame (tight frame) filter banks, all results above hold true for bi-frame (tight frame) filter banks. Next, let us look at Ron-Shen's tight frame filter bank again to illustrate the general theorem.

Example 1. Let $\{p, q^{(1)}, q^{(2)}\}$ be Ron-Shen's tight frame filter bank defined by (2.5). Then $\beta_1 = 1, C_{\beta_1} = -\sqrt{2}$; $\beta_2 = 2, C_{\beta_2} = -\frac{1}{2}$. Thus $S_{\theta_1}^1(x)$ and $S_{\theta_2}^2(x)$ in (3.13) are

$$\begin{aligned}
S_{\theta_1}^1(x) &= x - \frac{4\tau}{(C_{\beta_1})^2 h^2} g_1(\frac{2x^2}{(C_{\beta_1})^2 h^2}) x = x - \frac{4\tau}{(-\sqrt{2})^2 h^2} g_1(\frac{2x^2}{(-\sqrt{2})^2 h^2}) x = x - \frac{2\tau}{h^2} g_1(\frac{x^2}{h^2}) x, \\
S_{\theta_2}^2(x) &= x - \frac{4\tau(-1)^{\beta_2}}{(C_{\beta_2})^2 h^4} g_2(\frac{2x^2}{(C_{\beta_2})^2 h^4}) x = x - \frac{4\tau(-1)^2}{(-\frac{1}{2})^2 h^4} g_2(\frac{2x^2}{(-\frac{1}{2})^2 h^4}) x = x - \frac{16\tau}{h^4} g_2(\frac{8x^2}{h^4}) x.
\end{aligned}$$

Therefore relationship (3.13) of the diffusivity and shrinkage functions for this tight frame bank coincides with that in (2.11) with $\theta_1 = \theta, \theta_2 = \sigma$.

4 More high-order nonlinear diffusion equations

In this section we construct two sets of tight frame filter banks which result in the 6th- and 8th-order nonlinear diffusion equations. In the following, denote

$$z = e^{-i\omega}.$$

4.1 Sixth-order nonlinear diffusion equation

In this subsection we construct a tight frame filter bank with three highpass filters $q^{(\ell)}, 1 \le \ell \le 3$. $q^{(\ell)}$ has vanishing moment order ℓ, and it is symmetric or antisymmetric around the origin, namely, $q^{(\ell)}(-\omega) = q^{(\ell)}(\omega)$ or $q^{(\ell)}(-\omega) = -q^{(\ell)}(\omega)$. We consider the filters that are supported on $[-2, 2]$. That is the coefficients $q_k^{(\ell)} = 0$ if $|k| > 2$.

First let us look at $q^{(3)}$. If it has vanishing moment order 3, and it is antisymmetric around the origin and supported on $[-2, 2]$, then it can be written as

$$q^{(3)}(\omega) = \frac{1}{2} e_0 (-\frac{1}{2} z^{-2} + z^{-1} - z + \frac{1}{2} z^2), \tag{4.1}$$

for some $e_0 \in \mathbb{R}$. The formula in Lemma 2 for the 3rd derivative $L^{(3)}(x)$ of a function $L(x)$ related to such a $q^{(3)}(\omega)$ is

$$L^{(3)}(x_0) = \frac{1}{\varepsilon^3} \Big(- L(x_0 - 2\varepsilon) + 2L(x_0 - \varepsilon) - 2L(x_0 + \varepsilon) + L(x_0 + 2\varepsilon) \Big) + O(\varepsilon^2). \tag{4.2}$$

Next we consider $q^{(1)}(\omega)$ and $q^{(2)}(\omega)$. We choose $q^{(1)}(\omega)$ to be the filter given by

$$q^{(1)}(\omega) = \frac{1}{2} \frac{c_0}{12} (z^{-2} - 8z^{-1} + 8z - z^2), \tag{4.3}$$

where $c_0 \in \mathbb{R}$. The reason for such a choice of $q^{(1)}$ is that the corresponding formula for the derivative $L'(x)$ of a function $L(x)$ is the so-called five-point formula (see e.g. [2]):

$$L'(x_0) = \frac{1}{12\varepsilon} \Big(L(x_0 - 2\varepsilon) - 8L(x_0 - \varepsilon) + 8L(x_0 + \varepsilon) - L(x_0 + 2\varepsilon) \Big) + O(\varepsilon^4). \tag{4.4}$$

For $q^{(2)}(\omega)$, we hope that the corresponding formula for the derivative $L''(x)$ is similar to the five-point formula for $L'(x)$:

$$L''(x_0) = \frac{1}{12\varepsilon^2} \Big(- L(x_0 - 2\varepsilon) + 16L(x_0 - \varepsilon) - 30L(x_0) + 16L(x_0 + \varepsilon) - L(x_0 + 2\varepsilon) \Big) + O(\varepsilon^4). \tag{4.5}$$

$q^{(2)}(\omega)$ corresponding to the formula (4.5) is given by

$$q^{(2)}(\omega) = \frac{1}{2} \frac{d_0}{12} (-z^{-2} + 16z^{-1} - 30 + 16z - z^2), \tag{4.6}$$

where $d_0 \in \mathbb{R}$. Let $p(\omega)$ be the lowpass filter given

$$p(\omega) = \frac{1}{2} (b_0 z^{-2} + \frac{1}{2} z^{-1} + 1 - 2b_0 + \frac{1}{2} z + b_0 z^2), \tag{4.7}$$

where $b_0 \in \mathbb{R}$.

For $p, q^{(1)}, q^{(2)}$ and $q^{(3)}$ given by (4.7), (4.3), (4.6) and (4.1), we find that we are unable to choose b_0, c_0, d_0, e_0 such that $p, q^{(1)}, q^{(2)}$ and $q^{(3)}$ form a tight frame filter bank with the resulting scaling function being in $L^2(\mathbb{R})$. Because of this we consider $q^{(2)}(\omega)$ given by

$$q^{(2)}(\omega) = \frac{1}{2}\frac{d_0}{a_0-2}(-z^{-2} + (a_0+2)z^{-1} - 2a_0 - 2 + (a_0+2)z - z^2), \tag{4.8}$$

where $d_0, a_0 \in \mathbb{R}, a_0 \neq 2$. The corresponding formula for the 2nd derivative $L''(x)$ of a function $L(x)$ is

$$L''(x_0) = \frac{1}{(a_0-2)\varepsilon^2}\Big(- L(x_0 - 2\varepsilon) + (a_0+2)L(x_0-\varepsilon) - (2a_0+2)L(x_0) \tag{4.9}$$
$$+ (a_0+2)L(x_0+\varepsilon) - L(x_0+2\varepsilon)\Big) + O(\varepsilon^2).$$

Then we can choose a_0, b_0, c_0, d_0, e_0 such that the resulting scaling function is in $L^2(\mathbb{R})$. More precisely, if

$$b_0 = \frac{3\sqrt{2}-14}{178},\ c_0 = \sqrt{2+16b_0},\ d_0 = \frac{7+104b_0}{6},\ e_0 = \frac{\sqrt{-14-256b_0}}{6},\ a_0 = \frac{233+2314b_0}{21},$$

then ϕ is $W^{1.20414}(\mathbb{R})$ and $p, q^{(1)}, q^{(2)}, q^{(3)}$ form a tight frame filter bank. The corresponding nonlinear diffusion equation is

$$u_t = \frac{\partial}{\partial x}\left(g_1\left((\frac{\partial u}{\partial x})^2\right)\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial x^2}\left(g_2\left((\frac{\partial^2 u}{\partial x^2})^2\right)\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^3}{\partial x^3}\left(g_3\left((\frac{\partial^3 u}{\partial x^3})^2\right)\frac{\partial^3 u}{\partial x^3}\right). \tag{4.10}$$

In the above paragraph and also in the next subsection, for $s > 0$, $W^s(\mathbb{R})$ denotes the Sobolev space which consists of all functions f on $\mathbb{R}$ satisfying $\int_{\mathbb{R}}(1+|\omega|^2)^s|\widehat{f}(\omega)|^2 d\omega < \infty$, where $\widehat{f}(\omega)$ denotes the Fourier transform of $f(x)$. The Sobolev exponent of a compactly supported scaling function ϕ can be characterized by the eigenvalues of the transition operator associated with the refinement mask of ϕ. The reader is referred to [17, 34, 19] for the characterization, and to [20] for the Matlab routines about calculating the Sobolev smoothness of ϕ.

For $q^{(1)}, q^{(2)}$ and $q^{(3)}$ given by (4.3), (4.8) and (4.1) respectively, one can calculate directly that their vanishing moment orders β_j and the corresponding C_{β_j} defined by (3.3) are

$$\beta_1 = 1,\ C_{\beta_1} = c_0;\ \beta_2 = 2,\ C_{\beta_2} = d_0;\ \beta_3 = 3, C_{\beta_3} = e_0.$$

Thus, the relationship between $S^\ell_{\theta_\ell}$ and g_ℓ in Theorem 3 is given by

$$S^1_{\theta_1}(x) = x\left(1 - \frac{4\tau}{c_0^2h^2}g_1(\frac{2x^2}{c_0^2h^2})\right), S^2_{\theta_2}(x) = x\left(1 - \frac{4\tau}{d_0^2h^4}g_2(\frac{2x^2}{d_0^2h^4})\right), S^3_{\theta_3}(x) = x\left(1 - \frac{4\tau}{e_0^2h^6}g_3(\frac{2x^2}{e_0^2h^6})\right), \tag{4.11}$$

where h and τ are the spatial step size and the time step size respectively.

In conclusion, with the relationship in (4.11) for the diffusivity and shrinkage functions, the signal resulted from iterated denoising with p, $q^{(1)}, q^{(2)}$ and $q^{(3)}$ of the tight frame filter bank given by (4.7), (4.3), (4.8) and (4.1) respectively and that resulted from diffusion governed by equation (4.10) with the discretization of the 1st, 2nd and 3rd partial derivatives given by (4.4), (4.9) and (4.2) respectively are the same.

4.2 Eighth-order nonlinear diffusion equation

In this subsection, we construct a tight frame filter bank with four symmetric/antisymmetric highpass filters $q^{(\ell)}, 1 \le \ell \le 4$. $q^{(\ell)}$ has vanishing moment order ℓ, and all the filters constructed are supported on $[-2, 2]$. This filter bank results in a 8th-order nonlinear diffusion equation.

Let $p, q^{(1)}, q^{(2)}$ and $q^{(3)}$ be the filters given by (4.7), (4.3), (4.6) and (4.1) respectively. We construct the 4th highpass $q^{(4)}$ to have vanishing moment order 4 and to be symmetric around the origin. Then $q^{(4)}$ is given by

$$q^{(4)}(\omega) = \frac{1}{2} f_0 (z^{-2} - 4z^{-1} + 6 - 4z + z^2), \tag{4.12}$$

where $f_0 \in \mathbb{R}$. The corresponding formula for the 4th-order derivative $L^{(4)}$ of a function $L(x)$ is

$$L^{(4)}(x_0) = \frac{1}{\varepsilon^4} \Big(L(x_0 - 2\varepsilon) - 4L(x_0 - \varepsilon) + 6L(x_0) - 4L(x_0 + \varepsilon) + L(x_0 + 2\varepsilon) \Big) + O(\varepsilon^2). \tag{4.13}$$

If we choose

$$b_0 = \frac{\sqrt{7057} - 95}{192}, \ c_0 = \sqrt{2 + 16 b_0}, \ d_0 = \frac{\sqrt{7 + 104 b_0}}{4}, \ e_0 = \frac{\sqrt{-14 - 256 b_0}}{6}, \ f_0 = \frac{589 + 624 b_0}{54},$$

then ϕ is in the Sobolev space $W^{1.19195}(\mathbb{R})$, and $\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$ is a tight frame filter bank. The corresponding nonlinear diffusion equation is

$$\begin{aligned} u_t = & \frac{\partial}{\partial x} \left(g_1 \Big((\frac{\partial u}{\partial x})^2 \Big) \frac{\partial u}{\partial x} \right) - \frac{\partial^2}{\partial x^2} \left(g_2 \Big((\frac{\partial^2 u}{\partial x^2})^2 \Big) \frac{\partial^2 u}{\partial x^2} \right) \\ & + \frac{\partial^3}{\partial x^3} \left(g_3 \Big((\frac{\partial^3 u}{\partial x^3})^2 \Big) \frac{\partial^3 u}{\partial x^3} \right) - \frac{\partial^4}{\partial x^4} \left(g_4 \Big((\frac{\partial^4 u}{\partial x^4})^2 \Big) \frac{\partial^4 u}{\partial x^4} \right). \end{aligned} \tag{4.14}$$

For $q^{(1)}, q^{(2)}, q^{(3)}$ and $q^{(4)}$ given by (4.3), (4.6), (4.1) and (4.12), one can calculate directly that their vanishing moment orders β_j and the corresponding C_{β_j} defined by (3.3) are

$$\beta_1 = 1, \ C_{\beta_1} = c_0; \ \beta_2 = 2, \ C_{\beta_2} = d_0; \ \beta_3 = 3, C_{\beta_3} = e_0; \ \beta_4 = 4, C_{\beta_4} = f_0.$$

Thus the relationship between $S^{\ell}_{\theta_\ell}$ and g_ℓ in Theorem 3 is given by

$$\begin{aligned} S^1_{\theta_1}(x) &= x \left(1 - \frac{4\tau}{c_0^2 h^2} g_1 (\frac{2x^2}{c_0^2 h^2}) \right), \ S^2_{\theta_2}(x) = x \left(1 - \frac{4\tau}{d_0^2 h^4} g_2 (\frac{2x^2}{d_0^2 h^4}) \right), \\ S^3_{\theta_3}(x) &= x \left(1 - \frac{4\tau}{e_0^2 h^6} g_3 (\frac{2x^2}{e_0^2 h^6}) \right), \ S^4_{\theta_4}(x) = x \left(1 - \frac{4\tau}{f_0^2 h^8} g_4 (\frac{2x^2}{f_0^2 h^8}) \right). \end{aligned}$$

With such a relation among the diffusivity and shrinkage functions, the signal resulted from iterated denoising with the tight frame filter bank given by (4.7), (4.3), (4.6), (4.1) and (4.12) and that resulted from diffusion governed by equation (4.14) with the discretization of the 1st to 4th partial derivatives given by (4.4), (4.5), (4.2), (4.13) are the same.

The reader is referred to [12] for a tight frame filter bank consisting of 4 highpass filters $q^{(\ell)}, 1 \le \ell \le 4$ with $q^{(\ell)}$ having vanishing moment order ℓ. Its corresponding scaling function is the C^2 cubic spline supported on $[-2, 2]$; and its first two highpass filters $q^{(1)}, q^{(2)}$ are different from these given above (they cannot result in five-point formulas (4.4)(4.5) for the 1st- and 2nd-order derivatives), while up to constants, $q^{(3)}$ and $q^{(4)}$ are the filters given in (4.1) and (4.12).

Remark 1. In this section we construct two sets of tight frame filter banks which result in the 6th- and 8th-order nonlinear diffusion equations. If we consider undecimated tight frame filter banks, then we will have more flexibility for the construction which will result in smoother scaling functions. The details related to such construction are omitted here.

5 Experimental results

We carried out various experiments of signal denoising based on Ron-Shen's tight frame filter bank with different shrinkage functions. The overall performances of the diffusion-inspired shrinking (except for the one from TV diffusivity) are comparable with hard and soft threshold denoising. Actually, they perform slightly better. Here we provide experimental results with two "toy" signals, denoted as S_1 and S_2.

For S_1, which is shown on the top-left of Fig. 1, five noised signals are generated by adding zero-mean Gaussian noise five times to the original signal S_1. Each noised signal has signal-to-noise ratio (SNR)=6. SNR is defined as

$$\mathrm{SNR} = 20(\log_{10}|s-\overline{s}|_2 - \log_{10}|n|_2),$$

where s is the ideal signal and $\overline{s}$ is the mean of s, and n is the noise. We apply 1-level Ron-Shen's frame shrinking iteratively 50 times to each noised signal. We provide in Table 1 the SNRs of the denoised signals with different shrinkage functions. The SNR for each case in Table 1 is the average of the SNRs of the denoised signals of the five noised signals mentioned above. When we apply the Perona-Malik (denoted as P_M) diffusivity function, we choose $h = 1, \tau = \frac{1}{4}$. We choose $c_1 = 1$ when Perona-Malik diffusivity-based S^1_θ given in (2.13) is applied to the first highpass output $H^{(1)}_n$ while we set $c_2 = \frac{1}{8}$ when Perona-Malik diffusivity-based S^2_σ in (2.13) is applied to the second highpass output $H^{(2)}_n$. We set $h = 1, \tau = \frac{1}{4}$ when Weickert diffusivity-based S^1_θ defined by (2.15) is applied to the first highpass output $H^{(1)}_n$, while we use $h = 1, \tau = \frac{1}{16}$ when Weickert diffusivity-based S^2_σ in (2.16) is applied to the second highpass output $H^{(2)}_n$. The parameters θ and σ are selected such that SNRs of the denoised signals are as big as possible. The TV diffusivity-based S^1_θ, S^2_σ defined by (2.14) are independent of the parameters. Here we choose a smaller τ with $\tau = \frac{1}{32}$. From Table 1, we know for S_1, Perona-Malik diffusivity-inspired and Weickert diffusivity-inspired shrinkages perform slightly better than hard and soft shrinkages.

Shrinkage Method	S^1_θ=P_M S^2_σ=P_M	S^1_θ=Weickert S^2_σ=Weickert	S^1_θ=Hard S^2_σ=Hard	S^1_θ=Soft S^2_σ=Soft	S^1_θ=P_M S^2_σ=Weickert	S^1_θ=Weickert S^2_σ=P_M	S^1_θ=TV S^2_σ=TV
SNR (for S_1)	18.0560	18.0643	17.4083	17.8367	18.0589	18.0539	16.2321
SNR (for S_2)	26.1805	25.6973	24.4256	24.4167	26.3698	25.6202	16.2660

Table 1: Signal denoising results with different shrinkage functions

The second signal S_2 we consider is shown on the top-left of Fig. 2. Again five noised signals are generated by adding zero-mean Gaussian noise five times to S_2. In this case each noised signal has SNR=16 and we apply Ron-Shen's frame shrinking iteratively 100 times to each noised signal

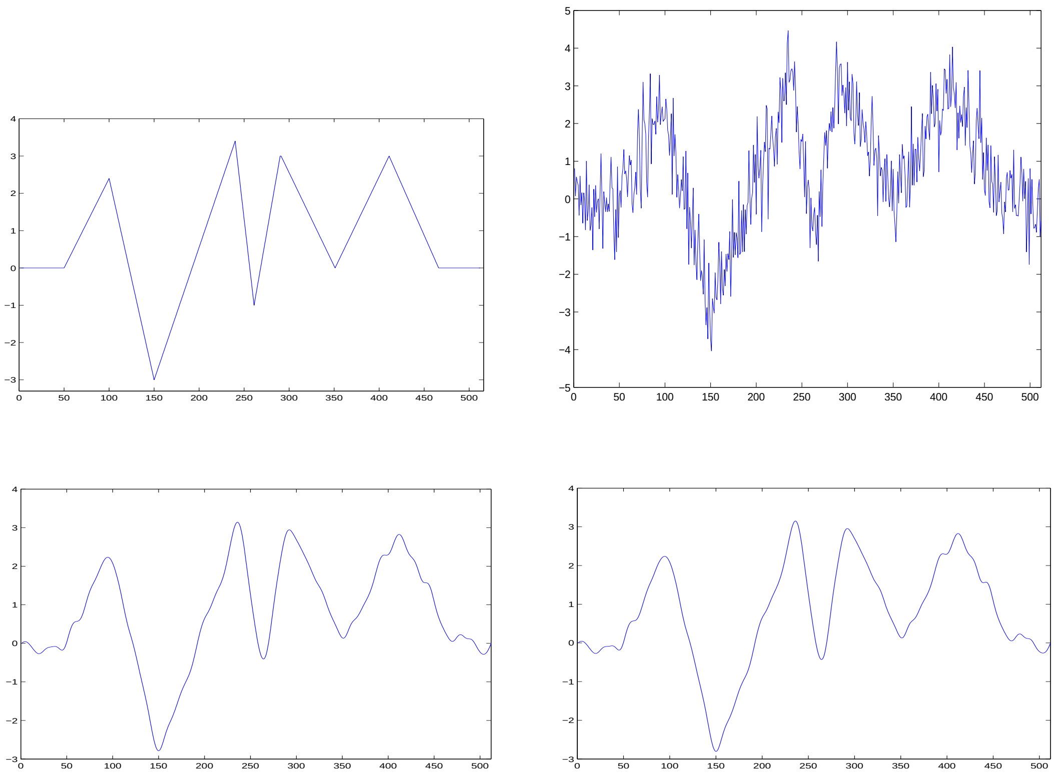

Figure 1: *Top-left: Original signal* S_1*; Top-right: Noised signal with SNR=6; Bottom-left: Denoised signal with Perona-Malik shrinkage; Bottom-right: Denoised signal with Weickert shrinkage*

(still 50 times for TV diffusivity-inspired shrinking). The constant c_1, c_2 and τ are chosen as above. The SNRs of the denoised signals with different shrinkage functions are provided also in Table 1. Again, for each case, the SNR for S_2 in Table 1 is the average of the SNRs of the denoised signals of five noised signals. This example also shows that Perona-Malik diffusivity-inspired and Weickert diffusivity-inspired shrinkages perform better than hard and soft shrinkages.

6 Conclusion and future work

In this paper we establish the correspondence between frame shrinkage functions and the diffusivity functions of certain high-order nonlinear diffusion equations. We start with the frame shrinkage based on a Ron-Shen's continuous-linear-spline-based tight frame filter bank and obtain a 4th-order nonlinear diffusion equation associated with this filter bank. After that we derive high-order nonlinear diffusion equations associated with general tight frame filter banks. These high-order nonlinear diffusion equations are different from the high-order diffusion equations studied in the literature. In addition, we construct two sets of tight frame filter banks which result in the 6th- and 8th-order nonlinear diffusion equations. We also present signal denoising experiments with various shrinkage functions including diffusivity-inspired shrinkage functions.

The study of relationship between the frame shrinkage and diffusion filtering leads to a new type of diffusion equations. The derived relationship is useful to design diffusion-inspired shrinkage functions with competitive performance. On the other hand, the relationship is helpful to design

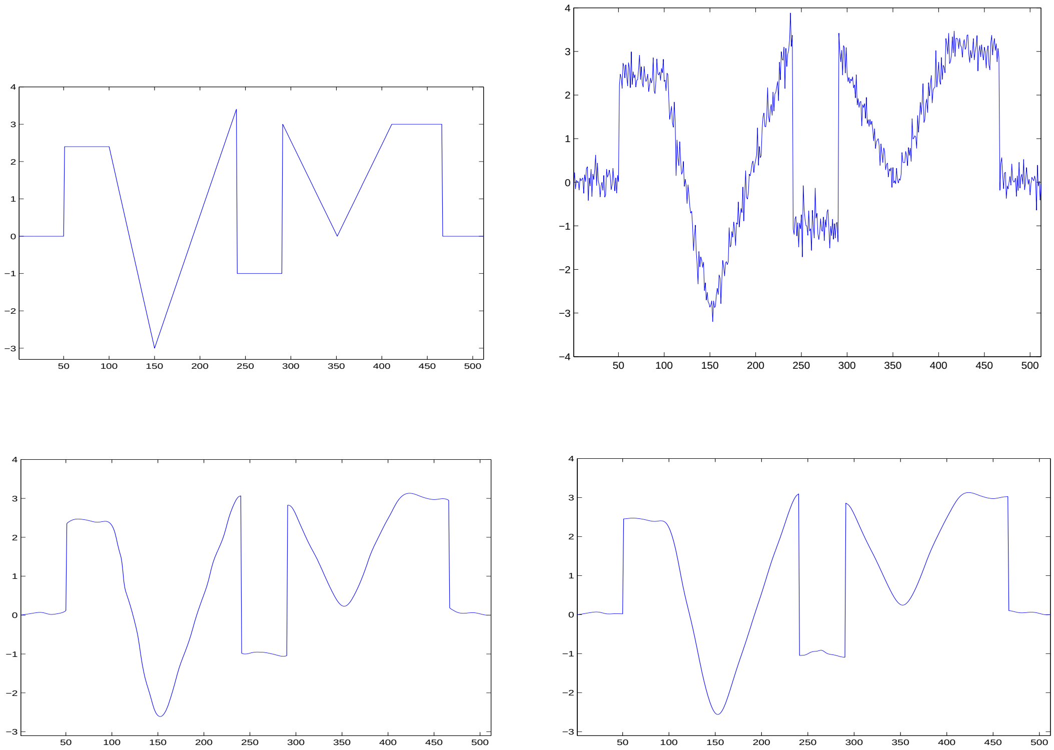

Figure 2: *Top-left: Original signal S_2; Top-right: Noised signal with SNR=16; Bottom-left: Denoised signal with Perona-Malik shrinkage; Bottom-right: Denoised signal with Weickert shrinkage*

frame-inspired diffusivity functions.

In this paper we consider frame shrinkage and diffusion filtering correspondence in the 1-D case. Our future work will include the study of the correspondence between 2-D frame shrinkage and diffusion filtering, the design of competitive diffusion-inspired shrinkage functions for image denoising and the construction of tight frame filter banks which result in nonlinear diffusion equations with good performances in image noise removal.

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References

[1] D. Barash, Fundamental relationship between bilateral filtering, adaptive smoothing, and the nonlinear diffusion equation, IEEE Trans. Pattern Analysis and Machine Intelligence 24 (2002) 844–847.

[2] R.L. Burden, J.D. Faires, Numerical Analysis (8th Ed.), Thomson Books/Cole, Belmont, CA, 2005.

[3] J.F. Cai, R.H. Chan, L.X. Shen, Z.W. Shen, Restoration of chopped and nodded images by framelets, SIAM J. Sci. Comput. 24 (2008) 1205–1227.

[4] J.F. Cai, R.H. Chan, Z.W. Shen, A framelet-based image inpainting algorithm, Appl. Comput. Harmonic Anal. 24 (2008) 131–149.

[5] J.F. Cai, S. Osher, Z.W. Shen, Split Bregman methods and frame based image restoration, Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal 8 (2009) 337–369.

[6] F. Catté, P.-L. Lions, J.-M. Morel, T. Coll, Image selective smoothing and edge detection by nonlinear diffusion, SIAM J. Numer. Anal. 29 (1992) 182–193.

[7] R.H. Chan, T.F. Chan, L.X. Shen, Z.W. Shen, Wavelet algorithms for high-resolution image reconstruction, SIAM Journal Sci. Comput. 24 (2003) 1408–1432.

[8] R.H. Chan, S.D. Riemenschneider, L.X. Shen, Z.W. Shen, Tight frame: An efficient way for high-resolution image reconstruction, Appl. Comput. Harmonic Anal. 17 (2004) 91–115.

[9] A. Chebira, Adaptive Multiresolution Frame Classification of Biomedical Images, Ph.D. Dissertation, Carnegie Mellon University, 2008.

[10] C.K. Chui, J.Z. Wang, PDE models associated with the bilateral filter, Adv. Comput. Math. 31 (2009) 131–156.

[11] R.R. Coifman, D.L. Donoho, Translation-invariant de-noising, in: Wavelets and Statistics, Springer Lecture Notes in Statistics 103, Springer-Verlag, New York, 1994, pp. 125–150.

[12] I. Daubechies, B. Han, A. Ron, Z.W. Shen, Framelets: MRA-based construction of wavelet frames, Appl. Comput. Harmonic Anal. 14 (2003) 1-46.

[13] S. Didas, Denoising and enhancement of digital images–variational methods, integrodifferential equations, and wavelets, Ph.D. Dissertation, Saarland University (2008). http://www.itwm.de/bv/employees/didas/dissertation.pdf.

[14] S. Didas, J. Weickert, B. Burgeth, Properties of higher order nonlinear diffusion filtering, Journal of Mathematical Imaging and Vision 35 (2009) 208–226.

[15] D.L. Donoho, De-noising by soft-thresholding, IEEE Trans. Inform. Theory 41 (1995) 613–627.

[16] D.L. Donoho, I.M. Johnstone, Ideal spatial adaptation via wavelet shrinkage, Biometrika 81 (1994) 425–455.

[17] T. Eirola, Sobolev characterization of solutions of dilation equations, SIAM J. Math. Anal. 23 (1992) 1015–1030.

[18] S.D. Gertz, B.G. Bodmann, D. Vela, M. Papadakis, *et al*, Three-dimensional isotropic wavelets for post-acquisitional extraction of latent images of atherosclerotic plaque components from micro-computed tomography of human coronary arteries, Academic Radiology 17 (2007) 1509–1519.

[19] R.Q. Jia, Q.T. Jiang, Spectral analysis of transition operators and its applications to smoothness analysis of wavelets, SIAM J. Matrix Anal. Appl. 24 (2003) 1071–1109.

[20] Q.T. Jiang, P. Oswald, Triangular $\sqrt{3}$-subdivision schemes: the regular case, J. Comput. Appl. Math. 156 (2003) 47–75.

[21] M. Lysaker, A. Lundervold, X.-C. Tai, Noise removal using fourth-order partial differential equation with applications to medical magnetic resonance images in space and time, IEEE Trans. Image Process. 12 (2003) 1579–1590.

[22] S. Mallat, A Wavelet Tour of Signal Processing (2nd Ed.), Academic Press, San Diego, 1999.

[23] P. Mrázek, J. Weickert, G. Steidl, Correspondences between wavelet shrinkage and nonlinear diffusion", in: L.D. Griffin and M. Lillholm (Eds.), Scale-Space Methods in Computer Vision, Lecture Notes in Computer Science vol. 2695, Springer, Berlin, 2003, pp. 101–116.

[24] P. Mrázek, J. Weickert, G. Steidl, Diffusion-inspired shrinkage functions and stability results for wavelet denoising, International Journal of Computer Vision 64 (2005) 171–186.

[25] M. Papadakis, B.G. Bodmann, S.K. Alexander, D. Vela, *et al*, Texture based tissue characterization for high-resolution CT-scans of coronary arteries, Comm. in Numer. Methods in Engineering 25 (2009) 597–613.

[26] P. Perona, J. Malik, Scale space and edge detection using anisotropic diffusion, IEEE Trans. Pattern Analysis and Machine Intelligence 12 (1990) 629-639.

[27] A. Ron, Z.W. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operators, J. Funct. Anal. 148 (1997) 408–447.

[28] A. Ron, Z.W. Shen, Affine systems in $L_2(\mathbb{R}^d)$ II: Dual systems, J. Fourier Anal. Appl. 3 (1997) 617–637.

[29] L. Rudin, S. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D 60 (1992) 259–268.

[30] L.X. Shen, M. Papadakis, I.A. Kakadiaris, I.Konstantinidis, I. Kouri, D. Hoffman, Image denoising using a tight frame, IEEE Trans. Image Proc. 15 (2006) 1254–1263.

[31] G. Steidl, J. Weickert, T. Brox, P. Mrázek, M. Welk, On the equivalence of soft wavelet shrinkage, total variation diffusion, total variation regularization, and SIDEs, SIAM J. Numer. Anal. 42 (2004) 686-713.

[32] G. Strang, Introduction to Applied Math, Wellesley–Cambridge Press, Wellesley, MA, 1986.

[33] G. Strang, T. Nguyen, Wavelets and Filter Banks, Wellesley Cambridge Press, Wellesley, 1996.

[34] L. Villemoes, Energy moments in time and frequency for two-scale difference equation solutions and wavelets, SIAM J. Math. Anal. 23 (1992) 1519–1543.

[35] G.W. Wei, Generalized Perona-Malik equation for image restoration, IEEE Signal Process. Lett. 6 (1999) 165–167.

[36] J. Weickert, Anisotropic Diffusion in Image Processing, B.G. Teubner, Stuttgart, 1998.

[37] J. Weickert, G. Steidl, P. Mrázek, M. Welk, T. Brox, Diffusion filters and wavelets: What can they learn from each other?, in: N. Paragios, Y. Chen, O. Faugeras (Eds.), Handbook of Mathematical Models in Computer Vision, Springer, New York, 2006, pp. 3–16.

[38] Y.-L. You, M. Kaveh, Image enhancement using fourth order partial differential equations, in: 32nd Asilomar Conf. Signals, Systems, Computers, vol. 2, 1998, pp. 1677–1681.

[39] Y.-L. You, M. Kaveh, Fourth-order partial differential equations for noise removal, IEEE Trans. Image Process. 9 (2000) 1723–1730.