On the design of multifilter banks and orthonormal multiwavelet bases

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Abstract— **Several forms of parametric expressions for orthogonal multifilter banks are presented. The explicit expressions for a group of orthogonal multifilter banks which generate symmetric/antisymmetric scaling functions and orthonormal multiwavelets are obtained. Based on these parametric expressions for orthogonal multifilter banks, orthonormal multiwavelet pairs with good time–frequency localization are constructed and examples of optimal multifilter banks are provided.**

Keywords— **Multifilter bank, parametrization, symmetry, time–frequency resolution, scaling function, orthonormal multiwavelet, balanced orthonormal multiwavelet, orthonormal multiwavelet pair.**

I. Introduction

Recently, the construction of multiwavelets is an area of active research (see [7], [26], [17], [2], [5], [4], [18], [13], and [15]). A set of functions $\psi_1, \cdots, \psi_r \in L^2(R)$ are called *orthonormal multiwavelets of multiplicity* r if $\psi_1(2^j x - k), \cdots, \psi_r(2^j x - k), j, k \in Z$, form an orthonormal basis of $L^2(R)$. Multiwavelet construction is associated with multiresolution analysis (MRA) of multiplicity r (see [8]). More precisely, an *MRA of multiplicity* r is a nested sequence of closed subspaces (V_j) in $L^2(R)$ satisfying the following conditions: (1°) $V_j \subset V_{j+1}, j \in Z$; (2°) $\bigcap_{j\in Z} V_j = \{0\}$, $\bigcup_{j\in Z} V_j$ is dense in $L^2(R)$; (3°) $f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$; (4°) There exist r functions $\phi_1, \cdots, \phi_r$ such that $\{\phi_j(\cdot - k) : 1 \le j \le r, k \in Z\}$ is a Riesz basis of V_0. Such functions $\phi_1, \cdots, \phi_r$ are called *scaling functions* and they are said to generate the MRA (V_j). If there is a set of compactly supported scaling functions whose integer translates form an orthonormal basis of V_0, then (V_j) is called an orthonormal MRA. For an orthonormal MRA (V_j), let $W_j := V_{j+1} \ominus V_j$, the orthogonal complement of V_j in V_{j+1}. Then Conditions (1°) and (2°) imply that $W_j \perp W_k$ for $j \ne k$ and $\sum_{j\in Z} \oplus W_j = L^2(R)$. If there exist functions $\psi_1, \cdots, \psi_r$ such that the integer translates of them form an orthonormal basis of W_0, then $\psi_1, \cdots, \psi_r$ are a set of orthonormal multiwavelets.

For a set of functions $f_1, \cdots, f_r$ in $L^2(R)$, write $\mathbf{F} = (f_1, \cdots, f_r)^T$, where throughout the paper, $\mathbf{B}^T$ denotes the transpose of the matrix $\mathbf{B}$. We shall say that $\mathbf{F}$ is orthonormal if the integer translates of $f_1, \cdots, f_r$ form an orthonormal basis of their closed linear span in $L^2(R)$, and that $\mathbf{F}$ is a scaling function (an orthonormal multiwavelet) if $f_1, \cdots, f_r$ are a set of scaling functions (a set of orthonormal multiwavelets).

If $\Phi = (\phi_1, \cdots, \phi_r)^T$ is a scaling function, then Conditions (1°), (3°) and (4°) imply that there exist $r \times r$ matrices $\mathbf{H}_k$ such that

$$\Phi(x) = 2 \sum_{k\in Z} \mathbf{H}_k \Phi(2x - k). \tag{1}$$

Let $\mathbf{H}$ denote the matrix frequency response for $\{\mathbf{H}_k\}$ defined by $\mathbf{H}(\omega) := \sum_{k\in Z} \mathbf{H}_k e^{-ik\omega}$. Then in the frequency domain, the functional equation (1) can be written as $\widehat{\Phi}(\omega) = \mathbf{H}(\omega/2)\widehat{\Phi}(\omega/2)$.

Assume that $\{\mathbf{H}_k\}$ is a finitely supported $r \times r$ matrix sequence, and let $\mathbf{H}$ denote the corresponding matrix frequency response. If there exists a compactly supported solution Φ of the equation (1) such that Φ generates an orthonormal MRA (V_j), we say that $\mathbf{H}$ generates the orthonormal scaling function Φ. In this case, $\Phi(\omega) = \lim_{n\to\infty} \Phi_n(\omega)$, where Φ_n is defined by

$$\widehat{\Phi}_n(\omega) := \mathbf{H}(\frac{\omega}{2}) \cdots \mathbf{H}(\frac{\omega}{2^n}) \mathbf{v}_0 \frac{\sin(\omega/2^{n+1})}{\omega/2^{n+1}} e^{-i\omega/2^{n+1}} \tag{2}$$

and $\mathbf{v}_0$ is the normalized right 1–eigenvector of $\mathbf{H}(0)$ (see e.g., [16]). Let $\{\mathbf{G}_k\}$ be another finitely supported $r \times r$ matrix sequence, and let $\Psi = (\psi_1, \cdots, \psi_r)^T$ be the vector–valued function defined by $\Psi(x) := 2\sum_{k\in Z} \mathbf{G}_k \Phi(2x - k)$, or equivalently, $\widehat{\Psi}(\omega) = \mathbf{G}(\omega/2)\widehat{\Phi}(\omega/2)$, where $\mathbf{G}(\omega) := \sum_{k\in Z} \mathbf{G}_k e^{-ik\omega}$. If Ψ is a compactly supported orthonormal multiwavelet, we say that $\{\mathbf{H}, \mathbf{G}\}$ generates the orthonormal multiwavelet Ψ (or $\{\mathbf{H}, \mathbf{G}\}$ generates an orthonormal multiwavelet basis). The pair $\{\mathbf{H}, \mathbf{G}\}$ is called a multiwavelet filter bank (often abbreviated *multifilter bank*), and $\mathbf{H}$ ($\mathbf{G}$ respectively) is called a *matrix low–pass filter* (*matrix high–pass filter* respectively). For a multifilter bank $\{\mathbf{H}, \mathbf{G}\}$, it is said to be *causal* if $\mathbf{H}_k = \mathbf{0}, \mathbf{G}_k = \mathbf{0}, k < 0$. It is said to be a *finite impulse response (FIR)* multifilter bank if there exists an integer N such that $\mathbf{H}_k = \mathbf{0}, \mathbf{G}_k = \mathbf{0}, |k| \le N$. Let $\mathbf{H}_m(\omega)$ denote the modulation matrix of an FIR multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ defined by

$$\mathbf{H}_m(\omega) := \begin{bmatrix} \mathbf{H}(\omega) & \mathbf{H}(\omega + \pi) \\ \mathbf{G}(\omega) & \mathbf{G}(\omega + \pi) \end{bmatrix}. \tag{3}$$

It was shown in [8] (see also [7]) that if $\{\mathbf{H}, \mathbf{G}\}$ generates an orthogonal multiwavelet, then $\mathbf{H}_m(\omega)$ is lossless (or paraunitary), i.e., $\mathbf{H}_m(\omega)$ is unitary for all ω:

$$\begin{cases} \mathbf{H}(\omega)\mathbf{H}^*(\omega) + \mathbf{H}(\omega+\pi)\mathbf{H}^*(\omega+\pi) = \mathbf{I}_r, \\ \mathbf{H}(\omega)\mathbf{G}^*(\omega) + \mathbf{H}(\omega+\pi)\mathbf{G}^*(\omega+\pi) = \mathbf{0}_r, \\ \mathbf{G}(\omega)\mathbf{G}^*(\omega) + \mathbf{G}(\omega+\pi)\mathbf{G}^*(\omega+\pi) = \mathbf{I}_r. \end{cases} \tag{4}$$

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Throughout this paper, $\mathbf{B}^*$ denotes the Hermitian adjoint of the matrix $\mathbf{B}$, and $\mathbf{I}_r$ and $\mathbf{0}_r$ denote the $r \times r$ identity matrix and zero matrix respectively. We also let $\mathbf{0}_{l\times k}$ denote the l by k zero matrix, and we would drop the subscript $l \times k$ when it does not cause any confusion. If the multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ satisfies (4), we say that $\{\mathbf{H}, \mathbf{G}\}$ is orthogonal. Condition (4) is necessary for $\{\mathbf{H}, \mathbf{G}\}$ to generate an orthonormal multiwavelet. Indeed, assume that $\text{supp}\{\mathbf{H}_k\} \subset [0, N]$ for a positive integer N, i.e., $\mathbf{H}_k = 0$ if $k < 0$ or $k > N$. Then $\{\mathbf{H}, \mathbf{G}\}$ generates an orthonormal multiwavelet basis if and only if $\{\mathbf{H}, \mathbf{G}\}$ satisfies (4) and the matrix $\mathcal{T}_\mathbf{H}$ defined by

$$\mathcal{T}_\mathbf{H} := (2\mathcal{A}_{2i-j})_{1-N\le i,j\le N-1}, \tag{5}$$

where $\mathcal{A}_j$ is the $r^2 \times r^2$ matrix given by $\mathcal{A}_j := \sum_{\kappa=0}^{N} \mathbf{H}_{\kappa-j} \otimes \mathbf{H}_\kappa$, satisfies Conditions E (see [23] and [13]). For two matrices $\mathbf{B} = (b_{ij})$ and $\mathbf{C} = (c_{ij})$, $\mathbf{B} \otimes \mathbf{C} := (b_{ij}\mathbf{C})$ denotes the Kronecker product of $\mathbf{B}$ and $\mathbf{C}$; and for a matrix $\mathbf{D}$, we say that $\mathbf{D}$ satisfies *Condition E* if the spectral radius of $\mathbf{D}$ is 1, 1 is the unique eigenvalue of $\mathbf{D}$ on the unit circle and it is a simple eigenvalue.

The first example of orthonormal multiwavelets was constructed in [7] and [5], and more examples were provided in [26], [2], [4]. In [13], a group of orthonormal scaling functions and orthonormal multiwavelets with good regularity and better time–frequency localization were constructed. However, up to now, there is still no systematic method to construct orthonormal multiwavelets. In this paper, we provide a method to construct orthonormal multiwavelets based on the lattice structures for $M \times M$ causal FIR lossless systems. The elegant factorization theory for $M \times M$ FIR lossless systems was begun by Belevitch and completed by Vaidyanathan *et al*, and it plays a fundamental role in developing filter banks with desirable properties (see [29], [28] and references therein). (As for the relation between scalar wavelets and filter banks, see e.g., [1], [9], [24], [25], and [30].) By the factorization theory, we will give parametric expressions for orthogonal causal FIR multifilter banks. With such explicit expressions at hand, the scaling functions and orthonormal multiwavelets with various properties can be constructed. Symmetry and time–frequency localization of scaling functions and multiwavelets will also be discussed in this paper.

This paper is organized as follows. In Section II, we give several forms of the parametric expressions for orthogonal causal FIR multifilter banks based on the lattice structures for $M \times M$ causal FIR lossless systems. In Section III, we derive the parametric expressions for a group of orthogonal causal FIR multifilter banks which generate symmetric/antisymmetric scaling functions and orthonormal multiwavelets. In Section IV, we discuss the time–frequency resolution of the scaling functions and multiwavelets, and construct the orthonormal multiwavelet pairs with optimum time–frequency resolution. The conclusions are given in Section V. The proofs of some lemmas and propositions are presented in Appendix A, and some optimal filters are provided in Appendix B.

Scaling functions, orthonormal multiwavelets, and the filter coefficients of the multifilter banks discussed in this paper are real. In this paper, we let R^n denote the n–dimensional Euclidean space (let R denote R^1, the set of all real numbers), let S^{n-1} denote the unit sphere in R^n, i.e., $S^{n-1} := \{\mathbf{x} = (x_1, \cdots, x_n)^T \in R^n : \mathbf{x}^T\mathbf{x} = 1\}$, and let T denote S^1, the unit circle. In the following, we also let T denote the interval $[0, 2\pi)$. We use $O(n)$ to denote the set of all $n \times n$ real matrices $\mathbf{B}$ with $\mathbf{B}\mathbf{B}^T = \mathbf{I}_n$.

II. Parametrization of orthogonal causal FIR multifilter banks

Let $\{\mathbf{H}, \mathbf{G}\}$ be a causal FIR multifilter bank. Write

$$\begin{aligned}
\mathbf{H}(\omega) &= \sum \mathbf{H}_{2k} e^{-i2k\omega} + (\sum \mathbf{H}_{2k+1} e^{-i2k\omega}) e^{-i\omega} \\
&=: \mathbf{E}_{00}(2\omega) + \mathbf{E}_{01}(2\omega) e^{-i\omega}, \\
\mathbf{G}(\omega) &= \sum \mathbf{G}_{2k} e^{-i2k\omega} + (\sum \mathbf{G}_{2k+1} e^{-i2k\omega}) e^{-i\omega} \\
&=: \mathbf{E}_{10}(2\omega) + \mathbf{E}_{11}(2\omega) e^{-i\omega}.
\end{aligned}$$

Then the polyphase matrix $\mathbf{E}_p(\omega)$ of the multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ is given by

$$\mathbf{E}_p(\omega) := \begin{bmatrix} \mathbf{E}_{00}(\omega) & \mathbf{E}_{01}(\omega) \\ \mathbf{E}_{10}(\omega) & \mathbf{E}_{11}(\omega) \end{bmatrix}.$$

Let $\mathbf{H}_m(\omega)$ denote the modulation matrix of $\mathbf{H}, \mathbf{G}$ defined by (3) and $\mathbf{U}_{2r}(\omega)$ denote the unitary matrix defined by

$$\mathbf{U}_{2r}(\omega) := \frac{\sqrt{2}}{2} \begin{bmatrix} \mathbf{I}_r & \mathbf{I}_r \\ e^{-i\omega}\mathbf{I}_r & -e^{-i\omega} I_r \end{bmatrix}.$$

Then the polyphase matrix and modulation matrix of $\mathbf{H}, \mathbf{G}$ have the following relation

$$\mathbf{H}_m(\omega) = \sqrt{2}\mathbf{E}_p(2\omega)\mathbf{U}_{2r}(\omega).$$

Since $\mathbf{U}_{2r}(\omega)$ is unitary for any $\omega \in T$, $\mathbf{H}_m(\omega)^*\mathbf{H}_m(\omega) = \mathbf{I}_{2r}$ if and only if

$$(\sqrt{2}\mathbf{E}_p(2\omega))^* \sqrt{2}\mathbf{E}_p(2\omega) = \mathbf{I}_{2r}, \quad \omega \in T.$$

Thus the causal FIR multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ is orthogonal if and only if $\sqrt{2}\mathbf{E}_p(\omega)$ is causal and lossless. If $\sqrt{2}\mathbf{E}_p(\omega)$ is causal and lossless, then it can be factorized as (see e.g., [9], [25], [28], and [29])

$$\sqrt{2}\mathbf{E}_p(\omega) = \mathbf{V}_\gamma(z)\mathbf{V}_{\gamma-1}(z)\cdots\mathbf{V}_1(z)\mathbf{U}_0, \quad z = e^{i\omega},$$

where γ is the (McMillan) degree of $\sqrt{2}\mathbf{E}_p(\omega)$, $\mathbf{U}_0 \in O(2r)$ and

$$\mathbf{V}_k(z) := \mathbf{I}_{2r} + (z^{-1} - 1)\mathbf{v}_k\mathbf{v}_k^T, \quad \mathbf{v}_k \in S^{2r-1}. \tag{6}$$

Thus, by its definition, $\{\mathbf{H}, \mathbf{G}\}$ is factorized as

$$\begin{aligned}
\begin{bmatrix} \mathbf{H}(\omega) \\ \mathbf{G}(\omega) \end{bmatrix} &= \mathbf{E}_p(2\omega) \begin{bmatrix} \mathbf{I}_r \\ z^{-1}\mathbf{I}_r \end{bmatrix} \\
&= \frac{\sqrt{2}}{2}\mathbf{V}_\gamma(z^2)\mathbf{V}_{\gamma-1}(z^2)\cdots\mathbf{V}_1(z^2)\mathbf{U}_0 \begin{bmatrix} \mathbf{I}_r \\ z^{-1}\mathbf{I}_r \end{bmatrix}, \quad (7)
\end{aligned}$$

where $z = e^{i\omega}$.

Theorem 1: If $\{\mathbf{H}, \mathbf{G}\}$ is a causal FIR multifilter bank which generates an orthonormal multiwavelet basis, then there exist vectors $\mathbf{v}_1, \cdots, \mathbf{v}_\gamma \in S^{2r-1}$ and $\mathbf{U}_0 \in O(2r)$ such that $\{\mathbf{H}, \mathbf{G}\}$ is given by (7), and the matrix $\mathcal{T}_\mathbf{H}$ associated to $\mathbf{H}$ satisfies Condition E. Conversely, for any positive integer γ, vectors $\mathbf{v}_1, \cdots, \mathbf{v}_\gamma \in S^{2r-1}$ and $\mathbf{U}_0 \in O(2r)$, if $\{\mathbf{H}, \mathbf{G}\}$ is the causal FIR multifilter bank defined by (7), then $\{\mathbf{H}, \mathbf{G}\}$ is orthogonal. Furthermore, if $\mathcal{T}_\mathbf{H}$ satisfies Condition E, then the multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ generates an orthonormal multiwavelet basis.

Recall that a matrix $\mathbf{A}$ is called a projection matrix if $\mathbf{A}$ is real and satisfies $\mathbf{A}^T = \mathbf{A}, \mathbf{A}^2 = \mathbf{A}$. For $2r \times 2r$ projection matrices $\mathbf{A}_k$, define

$$\mathbf{V}_k(z) := \mathbf{I}_{2r} + (z^{-1} - 1)\mathbf{A}_k. \tag{8}$$

Then $\mathbf{V}_k(z)\mathbf{V}_k(z^{-1})^T = \mathbf{I}_{2r}$, and we have the following proposition.

Proposition 1: For any positive integer γ, projection matrices $\mathbf{A}_1, \cdots, \mathbf{A}_\gamma$ and $\mathbf{U}_0 \in O(2r)$, if $\{\mathbf{H}, \mathbf{G}\}$ is the FIR multifilter bank given by (7) with $\mathbf{V}_k(z)$ defined by (8), then $\{\mathbf{H}, \mathbf{G}\}$ is orthogonal. Furthermore, if $\mathcal{T}_\mathbf{H}$ satisfies Condition E, then $\{\mathbf{H}, \mathbf{G}\}$ generates an orthonormal multiwavelet basis.

For a projection matrix $\mathbf{A}_k$, if $\mathrm{rank}(\mathbf{A}_k) = 1$, then $\mathbf{A}_k = \mathbf{v}_k\mathbf{v}_k^T$, $\mathbf{v}_k \in S^{2r-1}$ and $\mathbf{V}_k(z)$ defined by (8) is exactly the one defined by (6). In the construction of orthonormal multiwavelets, we shall also consider the case $\mathrm{rank}(\mathbf{A}_k) > 1$. This is because there will be more free parameters available to construct scaling functions and orthonormal multiwavelets with small supports.

Assume that the matrix filter $\mathbf{H}$ generates an orthonormal scaling function Φ. Then $\mathbf{H}(0)$ satisfies Condition E and $\mathbf{H}$ satisfies the vanishing moment conditions of order at least one, i.e., there exists a vector $\mathbf{v} \in S^{r-1}$ such that (see [3] and [16])

$$\mathbf{v}^T\mathbf{H}(0) = \mathbf{v}^T, \quad \mathbf{v}^T\mathbf{H}(\pi) = 0. \tag{9}$$

By $\mathbf{H}(\omega)\widehat{\Phi}(\omega) = \widehat{\Phi}(2\omega)$ and $\widehat{\Phi}(0) \neq 0$ (otherwise Φ is a zero function), $\widehat{\Phi}(0)$ is a right 1–eigenvector of $\mathbf{H}(0)$. In fact, we have the following proposition.

Proposition 2: If $\mathbf{H}$ generates an orthonormal scaling function Φ, then $\widehat{\Phi}(0)$ is a normalized right and left 1–eigenvector of $\mathbf{H}(0)$, and $\widehat{\Phi}(0)^T\mathbf{H}(\pi) = \mathbf{0}$.

In Proposition 2 and throughout this paper, a vector $\mathbf{v}$ is called a right and left eigenvector of a matrix $\mathbf{B}$ if $\mathbf{v}$ ($\mathbf{v}^T$ respectively) is a right (left respectively) eigenvector of $\mathbf{B}$. The proof of Proposition 2 is provided in Appendix A.

In the next theorem, we will give another form of factorization for an orthogonal causal FIR multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ with $\mathbf{H}$ satisfying (9) taken into account. For this, we recall in the following that any vector $\mathbf{x} \in S^{r-1}$ can be given by the operations of rotations on $\mathbf{e}_1 := (1, 0, \cdots, 0)^T$. Indeed, for $\mathbf{x} = (x_1, x_2, \cdots, x_r)^T \in S^{r-1} \subset R^r$, $\mathbf{x}$ can be written as

$$\begin{cases} x_1 = \cos\alpha_1, \quad x_2 = \sin\alpha_1\cos\alpha_2, \quad \cdots, \\ x_{r-1} = \sin\alpha_1 \cdots \sin\alpha_{r-2}\cos\alpha_{r-1}, \\ x_r = \sin\alpha_1 \cdots \sin\alpha_{r-2}\sin\alpha_{r-1}, \end{cases}$$

where $\alpha_{r-1} \in T, 0 \le \alpha_k < \pi, 0 \le k \le r-2$. Let $\mathbf{R}(\alpha) \in O(2)$ denote the rotation by an angle α in the (x_2, x_1)–plane:

$$\mathbf{R}(\alpha) := \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \tag{10}$$

For $1 \le k \le r$, let $\mathbf{R}_k(\alpha) \in O(r)$ denote the rotation by an angle α in the (x_{k+1}, x_k)–plane, i.e.,

$$\mathbf{R}_k(\alpha) := \mathrm{diag}(\mathbf{I}_{k-1}, \mathbf{R}(\alpha), \mathbf{I}_{r-k-1}).$$

Denote

$$\mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1}) := \mathbf{R}_{r-1}(\alpha_{r-1}) \cdots \mathbf{R}_1(\alpha_1).$$

Then $\mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1}) \in O(r)$ since $\mathbf{R}_k(\alpha_k) \in O(r)$. One can check that $\mathbf{x} = \mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})\mathbf{e}_1$.

For any matrices $\mathbf{u}_1, \mathbf{u}_2 \in O(r)$ and multifilter bank $\{\mathbf{H}, \mathbf{G}\}$, define

$$\mathbf{H}_1 = \mathbf{u}_1\mathbf{H}\mathbf{u}_1^T, \quad \mathbf{G}_1 = \mathbf{u}_2\mathbf{G}\mathbf{u}_1^T.$$

Then $\{\mathbf{H}, \mathbf{G}\}$ is orthogonal if and only if $\{\mathbf{H}_1, \mathbf{G}_1\}$ is orthogonal; and $\{\mathbf{H}, \mathbf{G}\}$ generates a scaling function Φ and an orthonormal multiwavelet Ψ if and only if $\{\mathbf{H}_1, \mathbf{G}_1\}$ generates the scaling function Φ_1 and the orthonormal multiwavelet Ψ_1, where $\Phi_1 = \mathbf{u}_1\Phi, \Psi_1 = \mathbf{u}_2\Psi$.

Assume that $\{\mathbf{H}_1, \mathbf{G}_1\}$ is an orthogonal FIR multifilter bank generating a scaling function Φ_1 and an orthonormal multiwavelet Ψ_1 with $\widehat{\Phi}_1(0) = \mathbf{e}_1$. For any $\mathbf{v} \in S^{r-1}$, there exist $\alpha_{r-1} \in T, 0 \le \alpha_k < \pi, 0 \le k \le r-2$, such that $\mathbf{v} = \mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})\mathbf{e}_1$. Define for $\mathbf{U} \in O(r)$,

$$\begin{cases} \mathbf{H}(\omega) = \mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})\mathbf{H}_1(\omega)\mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})^T, \\ \mathbf{G}(\omega) = \mathbf{U}\mathbf{G}_1(\omega)\mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})^T. \end{cases} \tag{11}$$

Then $\{\mathbf{H}, \mathbf{G}\}$ is an orthogonal FIR multifilter bank generating a scaling function Φ and an orthonormal multiwavelet Ψ with $\widehat{\Phi}(0) = \mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})\widehat{\Phi}_1(0) = \mathbf{v}$. Conversely, suppose that $\{\mathbf{H}, \mathbf{G}\}$ is an orthogonal FIR multifilter bank generating a scaling function Φ and an orthonormal multiwavelet Ψ. Then by Proposition 2, $\widehat{\Phi}(0) \in S^{r-1}$, and therefore there exist $\alpha_{r-1} \in T, 0 \le \alpha_k < \pi, 0 \le k \le r-2$ such that $\widehat{\Phi}(0) = \mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})\mathbf{e}_1$. Define $\{\mathbf{H}_1, \mathbf{G}_1\}$ by

$$\begin{cases} \mathbf{H}_1(\omega) = \mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1})^T\mathbf{H}(\omega)\mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1}), \\ \mathbf{G}_1(\omega) = \mathbf{U}^T\mathbf{G}(\omega)\mathbf{R}^{(r)}(\alpha_1, \cdots, \alpha_{r-1}), \end{cases}$$

where $\mathbf{U} \in O(r)$. Then $\{\mathbf{H}_1, \mathbf{G}_1\}$ is also orthogonal and generates a scaling function Φ_1 and an orthonormal multiwavelet Ψ_1 with $\widehat{\Phi}_1(0) = \mathbf{e}_1$. Thus to factorize the orthogonal causal FIR multifilter bank $\{\mathbf{H}, \mathbf{G}\}$, we need only to factorize its corresponding orthogonal causal FIR multifilter bank $\{\mathbf{H}_1, \mathbf{G}_1\}$, where $\mathbf{H}_1$ generates an orthonormal scaling function Φ_1 with $\widehat{\Phi}_1(0) = \mathbf{e}_1$. In this case, $\mathbf{e}_1$ is a right and left 1–eigenvector of $\mathbf{H}_1(0)$ with $\mathbf{e}_1^T\mathbf{H}_1(\pi) = \mathbf{0}$. By $\mathbf{e}_1^T\mathbf{H}_1(0)\mathbf{G}_1(0)^* + \mathbf{e}_1^T\mathbf{H}_1(\pi)\mathbf{G}_1(\pi)^* = \mathbf{0}$, $\mathbf{e}_1^T\mathbf{G}_1(0)^T = \mathbf{0}$, i.e., $\mathbf{G}_1(0)\mathbf{e}_1 = \mathbf{0}$. Let $\mathbf{H}_m(\omega)$ denote the modulation matrix of $\mathbf{H}_1, \mathbf{G}_1$. Then

$$\mathbf{H}_m(0) = \begin{bmatrix} \mathbf{H}_1(0) & \mathbf{H}_1(\pi) \\ \mathbf{G}_1(0) & \mathbf{G}_1(\pi) \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \tag{12}$$

for some $2r-1$ by $2r-1$ real matrix $\mathbf{U}_1$. Thus the polyphase matrix $\mathbf{E}_p(\omega)$ of $\mathbf{H}_1, \mathbf{G}_1$ satisfies

$$\mathbf{E}_p(0) = \mathbf{H}_m(0) \begin{bmatrix} \mathbf{I}_r & \mathbf{I}_r \\ \mathbf{I}_r & -\mathbf{I}_r \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_r & \mathbf{I}_r \\ \mathbf{I}_r & -\mathbf{I}_r \end{bmatrix}.$$

Since the causal FIR multifilter bank $\{\mathbf{H}_1, \mathbf{G}_1\}$ is orthogonal, $\sqrt{2}\mathbf{E}_p(\omega)$ is causal and lossless. Hence $\mathbf{U}_1$ is also orthogonal, i.e., $\mathbf{U}_1 \in O(2r-1)$. Let $\sqrt{2}\mathbf{E}_p(\omega) = \mathbf{V}_\gamma(z)\mathbf{V}_{\gamma-1}(z)\cdots\mathbf{V}_1(z)\mathbf{U}_0, z = e^{i\omega}$, be the lattice structure for $\sqrt{2}\mathbf{E}_p(\omega)$, where $\mathbf{V}_k(z)$ are defined by (8) and $\mathbf{U}_0 \in O(2r)$. Note that $\mathbf{V}_k(1) = \mathbf{I}_{2r}$. Thus

$$\mathbf{U}_0 = \sqrt{2}\mathbf{E}_p(0) = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_r & \mathbf{I}_r \\ \mathbf{I}_r & -\mathbf{I}_r \end{bmatrix}.$$

Hence we have

$$\begin{bmatrix} \mathbf{H}_1(\omega) \\ \mathbf{G}_1(\omega) \end{bmatrix} = \frac{1}{2}\mathbf{V}_\gamma(z^2)\mathbf{V}_{\gamma-1}(z^2)\cdots\mathbf{V}_1(z^2) \cdot \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \left(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix} + \begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix} e^{-i\omega} \right), \quad (13)$$

where $z = e^{i\omega}, \mathbf{V}_k(z)$ are defined by (8) with projection matrices $\mathbf{A}_k$, and $\mathbf{U}_1 \in O(2r-1)$. The preceding discussions lead to the following theorem.

Theorem 2: If $\{\mathbf{H}, \mathbf{G}\}$ is a causal FIR multifilter bank generating an orthonormal multiwavelet basis, then there exist $\alpha_1, \cdots, \alpha_{r-1}$, $0 \le \alpha_1, \cdots, \alpha_{r-2} < \pi$, $\alpha_{r-1} \in T$, projection matrices $\mathbf{A}_1, \cdots, \mathbf{A}_\gamma$, $\mathbf{U}_1 \in O(2r-1)$ and $\mathbf{U} \in O(r)$ such that $\{\mathbf{H}, \mathbf{G}\}$ is given by (11) with $\{\mathbf{H}_1, \mathbf{G}_1\}$ given by (13), and the matrix $\mathcal{T}_{\mathbf{H}_1}$ associated to $\mathbf{H}_1$ satisfies Condition E. Conversely, for any $\alpha_1, \cdots, \alpha_{r-1} \in R$, projection matrices $\mathbf{A}_1, \cdots, \mathbf{A}_\gamma$ and $\mathbf{U}_1 \in O(2r-1)$, if $\{\mathbf{H}_1, \mathbf{G}_1\}$ is the multifilter bank defined by (13) and $\{\mathbf{H}, \mathbf{G}\}$ is defined by (11) for some $\mathbf{U} \in O(r)$, then the FIR multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ is orthogonal. Furthermore, if $\mathcal{T}_{\mathbf{H}_1}$ satisfies Condition E, then $\{\mathbf{H}, \mathbf{G}\}$ generates an orthonormal multiwavelet basis.

Regarding the matrices $\mathbf{U}_0 \in O(2r)$ in Theorem 1, Proposition 1, and $\mathbf{U}_1 \in O(2r-1)$ in Theorem 2, they can be given in terms of parameters such as Euler angles (see [31]) or Givens rotations (see [28]). Therefore Theorem 1, Proposition 1, and Theorem 2 give parametric expressions for orthogonal causal FIR multifilter banks. Comparing to Proposition 1, the expressions in Theorem 2 have some advantages. Firstly if we want to design the scaling function Φ such that $\widehat{\Phi}(0)$ is a desirable value, we need only choose appropriate parameters α_k in (11). Secondly, the scaling function Φ and orthonormal multiwavelet Ψ generated by $\{\mathbf{H}, \mathbf{G}\}$ and Φ_1, Ψ_1 generated by $\{\mathbf{H}_1, \mathbf{G}_1\}$ have the same approximation and smoothness properties, but $\dim(O(2r)) - -\dim(O(2r-1)) = 2r-1$. Thus by Theorem 2, we use less free parameters to construct Φ_1, Ψ_1 that have the same approximation and smoothness properties as Φ, Ψ constructed by Proposition 1.

Usually the support lengths of the scaling functions and orthonormal multiwavelets constructed with (11) and (13) are odd integers. If we want to construct scaling functions and orthonormal multiwavelets with support lengths being even integers using (11) and (13), we would choose a projection matrix $\mathbf{A}_1$ in (8) that satisfies

$$\mathbf{A}_1 \mathrm{diag}(1, \mathbf{U}_1) \begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix} = \mathbf{0}_{2r\times r}. \quad (14)$$

The next lemma characterizes the matrix $\mathbf{A}_1$ that satisfies (14).

Lemma 1: Assume that $\mathbf{A}_1$ is a $2r \times 2r$ matrix with $\mathrm{rank}(\mathbf{A}_1) = s$. Then $\mathbf{A}_1$ is a projection matrix and satisfies (14) for some $\mathbf{U}_1 \in O(2r-1)$ if and only if $s \le r$ and

$$\mathbf{A}_1 = \frac{1}{2}\mathrm{diag}(1, \mathbf{U}_1) \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{B}^T & \mathbf{B}^T \end{bmatrix} \mathrm{diag}(1, \mathbf{U}_1^T) \quad (15)$$

for some $r \times s$ real matrix $\mathbf{B}$ with $\mathbf{B}^T\mathbf{B} = \mathbf{I}_s$.

The proof of Lemma 1 is given in Appendix A.

If $\mathbf{A}_1$ is given by (15), then

$$\begin{aligned} \begin{bmatrix} {}_2\mathbf{H}_1(\omega) \\ {}_2\mathbf{G}_1(\omega) \end{bmatrix} &:= \frac{1}{2}(\mathbf{I}_{2r} + (z^{-2}-1)\mathbf{A}_1) \cdot \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \left(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix} + \begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix} e^{-i\omega} \right) \\ &= \frac{1}{2}\mathrm{diag}(1, \mathbf{U}_1) \left(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix} (\mathbf{I}_r - \mathbf{B}\mathbf{B}^T) + \begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix} e^{-i\omega} + \begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix} \mathbf{B}\mathbf{B}^T e^{-i2\omega} \right). \quad (16) \end{aligned}$$

Thus to construct scaling functions and orthonormal multiwavelets with even integer support lengths, we will use the following parametric expressions for orthogonal causal FIR multifilter banks:

$$\begin{bmatrix} \mathbf{H}_1(\omega) \\ \mathbf{G}_1(\omega) \end{bmatrix} = \mathbf{V}_\gamma(z^2)\mathbf{V}_{\gamma-2}(z^2)\cdots\mathbf{V}_2(z^2) \begin{bmatrix} {}_2\mathbf{H}_1(\omega) \\ {}_2\mathbf{G}_1(\omega) \end{bmatrix}, \quad (17)$$

where $z = e^{i\omega}, \mathbf{V}_k(z)$ are given by (8) with projection matrices $\mathbf{A}_k$, $\{{}_2\mathbf{H}_1, {}_2\mathbf{G}_1\}$ is defined by (16) with $\mathbf{U}_1 \in O(2r-1)$ and $r \times s$ matrix $\mathbf{B}$ satisfying $\mathbf{B}\mathbf{B}^T = \mathbf{I}_s$, $s < r$.

For the rest of this section, we will consider the case $r = 2$ in greater details. In this case, $\mathbf{R}^{(2)}(\alpha_1) = \mathbf{R}(\alpha_1)$, $\{\mathbf{H}, \mathbf{G}\}$ and $\{\mathbf{H}_1, \mathbf{G}_1\}$ are related by

$$\mathbf{H}(\omega) = \mathbf{R}(\alpha_1)\mathbf{H}_1(\omega)\mathbf{R}(\alpha_1)^T, \mathbf{G}(\omega) = \mathbf{R}(\alpha_2)\mathbf{H}_1(\omega)\mathbf{R}(\alpha_1)^T, \quad (18)$$

where $\alpha_1, \alpha_2 \in T$, and $\mathbf{U}_1$ in (13) and (16) is a 3×3 orthogonal matrix. The matrix $\mathbf{U}_1 \in O(3)$ can be given by Euler angles or Given rotations (see [28] and [31]), but here we will not write down its parametric expression. For the projection matrix $\mathbf{A}_k$ in (8), if $\mathrm{rank}(\mathbf{A}_k) = 1$, then $\mathbf{A}_k = \mathbf{v}_k\mathbf{v}_k^T, \mathbf{v}_k \in S^3$; if $\mathrm{rank}(\mathbf{A}_k) = 3$, $\mathbf{A}_k = \mathbf{I}_4 - \mathbf{v}_k\mathbf{v}_k^T, \mathbf{v}_k \in S^3$; and if $\mathrm{rank}(\mathbf{A}_k) = 2$, $\mathbf{A}_k$ can be written as

$$\mathbf{A}_k = \mathbf{R}^{(4)}(\alpha_1, \alpha_2, \alpha_3)\mathrm{diag}(1, \mathbf{w}_k\mathbf{w}_k^T)\mathbf{R}^{(4)}(\alpha_1, \alpha_2, \alpha_3)^T, \quad (19)$$

where $\mathbf{w}_k = (\cos\alpha_4, \sin\alpha_4\cos\alpha_5, \sin\alpha_4\sin\alpha_5)^T \in S^2$ with $\alpha_1, \alpha_2, \alpha_4 \in T, 0 \le \alpha_3, \alpha_5 < \pi$.

In the case $r = 2$, $\mathbf{B}$ in (16) is $\begin{bmatrix} \cos\alpha_0 \\ \sin\alpha_0 \end{bmatrix} [\cos\alpha_0, \sin\alpha_0]$ with $\alpha_0 \in T$. Thus to construct scaling functions and orthonormal wavelets of multiplicity 2, we will use the explicit expressions (13) and (17) with $z = e^{i\omega}, \mathbf{U}_1 \in O(3)$, $\mathbf{V}_k(z) = \mathbf{I}_4 + (z^{-1} - 1)\mathbf{A}_k$, $\mathbf{A}_k = \mathbf{v}_k\mathbf{v}_k^T, \mathbf{v}_k \in S^3$, or $\mathbf{A}_k$ given by (19), and $\mathbf{B} = \begin{bmatrix} \cos\alpha_0 \\ \sin\alpha_0 \end{bmatrix} [\cos\alpha_0, \sin\alpha_0]$ in (16).

III. Symmetric/antisymmetric orthonormal multiwavelets

In this section, we will consider causal FIR multifilter banks which generate orthonormal scaling functions Φ and orthonormal multiwavelets Ψ with symmetry. Here we discuss the case $r = 2$, and consider the situation where the first components of Φ and Ψ are symmetric, while the second components are antisymmetric. For a function $\mathbf{f} = (f_1, f_2)^T$, we say that $\mathbf{f}$ is symmetric/antisymmetric about a center of symmetry $c_0 \in R$ if f_1 (f_2 respectively) is symmetric (antisymmetric respectively) about the center c_0. If Φ is symmetric/antisymmetric, then $\widehat{\Phi}(0) = (1, 0)^T$, and thus we choose $\mathbf{R}^{(2)}(\alpha_1), \mathbf{R}^{(2)}(\alpha_2)$ in (18) to be $\mathbf{I}_2$. Hence $\mathbf{H} = \mathbf{H}_1, \mathbf{G} = \mathbf{G}_1$ and they are given by (13) or (17) with $r = 2$.

Denote

$$\mathbf{S}_0 := \operatorname{diag}(1, -1), \quad \mathbf{S}_1 := \operatorname{diag}(\mathbf{S}_0, \mathbf{S}_0).$$

Let $\{{}_N\mathbf{H}, {}_N\mathbf{G}\}$ with ${}_N\mathbf{H}(\omega) = \sum_{k=0}^N \mathbf{H}_k e^{-ik\omega}$, ${}_N\mathbf{G}(\omega) = \sum_{k=0}^N \mathbf{G}_k e^{-ik\omega}$ be an orthogonal FIR multifilter bank. In order that the corresponding scaling function ${}_N\Phi$ and orthonormal multiwavelet ${}_N\Psi$ are symmetric/antisymmetric about the symmetry center $\frac{N}{2}$ for some positive integer N, we shall construct matrix filters $\{\mathbf{H}_k\}_{k=0}^N, \{\mathbf{G}_k\}_{k=0}^N$ satisfying (see e.g., [2]):

$$\mathbf{S}_0\mathbf{H}_{N-k}\mathbf{S}_0 = \mathbf{H}_k, \mathbf{S}_0\mathbf{G}_{N-k}\mathbf{S}_0 = \mathbf{G}_k, k = 0, \cdots, N, \quad (20)$$

or equivalently

$$z^N\mathbf{S}_1 \begin{bmatrix} {}_N\mathbf{H}(\omega) \\ {}_N\mathbf{G}(\omega) \end{bmatrix} \mathbf{S}_0 = \begin{bmatrix} {}_N\mathbf{H}(-\omega) \\ {}_N\mathbf{G}(-\omega) \end{bmatrix}, z = e^{i\omega}, \omega \in T. \quad (21)$$

Thus to construct symmetric/antisymmetric scaling functions ${}_N\Phi$ and orthonormal multiwavelets ${}_N\Psi$ based on (13) and (17), we shall determine $\mathbf{U}_1$, $\mathbf{A}_k$ in (13) and (17) such that ${}_N\mathbf{H}, {}_N\mathbf{G}$ satisfy (21). However here we would like to determine $\mathbf{U}_1$, $\mathbf{A}_k$ such that if ${}_k\mathbf{H}, {}_k\mathbf{G}$ satisfy (21), then

$$\begin{bmatrix} {}_{k+2}\mathbf{H}(\omega) \\ {}_{k+2}\mathbf{G}(\omega) \end{bmatrix} := \mathbf{V}_k(z^2) \begin{bmatrix} {}_k\mathbf{H}(\omega) \\ {}_k\mathbf{G}(\omega) \end{bmatrix},$$

also satisfies (21), which implies that $z\mathbf{S}_1\mathbf{V}_k(z)\mathbf{S}_1 = \mathbf{V}_k(z^{-1})$. We need the following two lemmas, whose proofs are presented in Appendix A.

Lemma 2: Assume that $\mathbf{V}_k(z) = \mathbf{I}_4 + (z^{-1} - 1)\mathbf{A}_k$ for some projection matrix $\mathbf{A}_k$. Then $z\mathbf{S}_1\mathbf{V}_k(z)\mathbf{S}_1 = \mathbf{V}_k(z^{-1})$ if and only if

$$\mathbf{A}_k = \frac{1}{2} \begin{bmatrix} 1 & \cos\theta_k & 0 & \mp\sin\theta_k \\ \cos\theta_k & 1 & \sin\theta_k & 0 \\ 0 & \sin\theta_k & 1 & \pm\cos\theta_k \\ \mp\sin\theta_k & 0 & \pm\cos\theta_k & 1 \end{bmatrix} \quad (22)$$

with $\theta_k \in T$.

Lemma 3: If ${}_N\mathbf{H}, {}_N\mathbf{G}$ satisfy (21), then $\mathbf{U}_1$ in (12) for the modulation matrix of ${}_N\mathbf{H}, {}_N\mathbf{G}$ is

$$\mathbf{U}_1 = \begin{bmatrix} \mp\sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & \pm 1 \\ \pm\cos\theta_0 & \sin\theta_0 & 0 \end{bmatrix} (\text{ for } N = 2N_1 + 1), \quad (23)$$

or

$$\mathbf{U}_1 = \begin{bmatrix} \mp\sin\theta_0 & 0 & \cos\theta_0 \\ 0 & \pm 1 & 0 \\ \pm\cos\theta_0 & 0 & \sin\theta_0 \end{bmatrix} (\text{ for } N = 2N_1), \quad (24)$$

where $\theta_0 \in T$.

By Lemma 3, if $\mathbf{H}_1(\omega), \mathbf{G}_1(\omega)$, defined by (13) or (17) for $r = 2$, satisfy (21), then $\mathbf{U}_1$ in (13) and (16) is determined by (23) and (24), respectively.

Let $\mathbf{U}_1$ be the matrix defined by (23) for the case $N = 2N_1 + 1$. Then denote

$$\begin{aligned} \begin{bmatrix} {}_1\mathbf{H}(\omega) \\ {}_1\mathbf{G}(\omega) \end{bmatrix} &:= \frac{1}{2} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \left(\begin{bmatrix} \mathbf{I}_2 \\ \mathbf{I}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{I}_2 \\ -\mathbf{I}_2 \end{bmatrix} e^{-i\omega} \right) \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ \cos\theta_0 & \mp\sin\theta_0 \\ 0 & \pm 1 \\ \sin\theta_0 & \pm\cos\theta_0 \end{bmatrix} + \\ &\quad \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -\cos\theta_0 & \mp\sin\theta_0 \\ 0 & \mp 1 \\ -\sin\theta_0 & \pm\cos\theta_0 \end{bmatrix} e^{-i\omega}. \end{aligned} \quad (25)$$

Let $\{{}_2\mathbf{H}, {}_2\mathbf{G}\}$ be the FIR multifilter bank defined by (16) with $\mathbf{B} = \begin{bmatrix} \cos\alpha_0 \\ \sin\alpha_0 \end{bmatrix} [\cos\alpha_0, \sin\alpha_0]$, $\alpha_0 \in T$, and $\mathbf{U}_1$ is given by (24) for the case $N = 2N_1$. If $\{{}_2\mathbf{H}, {}_2\mathbf{G}\}$ satisfies (21), then $\cos^2\alpha_0 = \sin^2\alpha_0$. Here we choose $\alpha_0 = \pi/4$. By the facts that

$$\begin{cases} \cos\theta_0 \mp \sin\theta_0 = \sqrt{2}\cos(\theta_0 \pm \frac{\pi}{4}), \\ \sin\theta_0 \pm \cos\theta_0 = \sqrt{2}\sin(\theta_0 \pm \frac{\pi}{4}), \end{cases}$$

and by the change of variable $\theta_0 \pm \pi/4 \to \theta_0$, we have

$$\begin{aligned} \begin{bmatrix} {}_2\mathbf{H}(\omega) \\ {}_2\mathbf{G}(\omega) \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -\sqrt{2}\cos\theta_0 & \sqrt{2}\cos\theta_0 \\ \pm 1 & \mp 1 \\ -\sqrt{2}\sin\theta_0 & \sqrt{2}\sin\theta_0 \end{bmatrix} + \\ &\quad \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & \mp\sqrt{2}\sin\theta_0 \\ \mp 1 & 0 \\ 0 & \pm\sqrt{2}\cos\theta_0 \end{bmatrix} e^{-i\omega} + \\ &\quad \frac{1}{4} \begin{bmatrix} 1 & 1 \\ \sqrt{2}\cos\theta_0 & \sqrt{2}\cos\theta_0 \\ \pm 1 & \pm 1 \\ \sqrt{2}\sin\theta_0 & \sqrt{2}\sin\theta_0 \end{bmatrix} e^{-i2\omega}, \end{aligned} \quad (26)$$

with $\theta_0 \in T$. Therefore to construct symmetric/antisymmetric scaling functions and orthonormal multiwavelets ${}_N\Phi, {}_N\Psi$, we use

$$\begin{bmatrix} {}_{2N_1}\mathbf{H}(\omega) \\ {}_{2N_1}\mathbf{G}(\omega) \end{bmatrix} = \mathbf{V}_{N_1-1}(z^2)\cdots\mathbf{V}_1(z^2)\begin{bmatrix} {}_2\mathbf{H}(\omega) \\ {}_2\mathbf{G}(\omega) \end{bmatrix}, \quad (27)$$

and

$$\begin{bmatrix} {}_{2N_1+1}\mathbf{H}(\omega) \\ {}_{2N_1+1}\mathbf{G}(\omega) \end{bmatrix} = \mathbf{V}_{N_1}(z^2)\cdots\mathbf{V}_1(z^2)\begin{bmatrix} {}_1\mathbf{H}(\omega) \\ {}_1\mathbf{G}(\omega) \end{bmatrix}, \quad (28)$$

where $z = e^{i\omega}, \mathbf{V}_k(z) = \mathbf{I}_4 + (z^{-1} - 1)\mathbf{A}_k$, $\mathbf{A}_k$ are given by (22), ${}_2\mathbf{H}(\omega), {}_2\mathbf{G}(\omega)$ and ${}_1\mathbf{H}(\omega), {}_1\mathbf{G}(\omega)$ are defined by (26) and (25) respectively. Lemma 2, Lemma 3 and the fact that ${}_1\mathbf{H}(\omega), {}_1\mathbf{G}(\omega)$ satisfy (21) for $N = 1$ lead to the following theorem.

Theorem 3: Let $\{{}_N\mathbf{H}, {}_N\mathbf{G}\}$ be an orthogonal FIR multifilter bank defined by (27) or (28) with $\mathbf{V}_k(z) = \mathbf{I}_4 + (z^{-1} - 1)\mathbf{A}_k$ and $\mathbf{A}_k$ given by (22). Then ${}_N\mathbf{H}(\omega), {}_N\mathbf{G}(\omega)$ satisfy (21). Furthermore, if the matrix $\mathcal{T}_{{}_N\mathbf{H}}$ satisfies Condition E, then $\{{}_N\mathbf{H}, {}_N\mathbf{G}\}$ generates symmetric/antisymmetric orthonormal scaling functions and orthonormal multiwavelets.

Theorem 3 gives parametric expressions for a group of orthogonal causal FIR multifilter banks that generate symmetric/antisymmetric scaling functions and orthonormal multiwavelets. Note that the number of free parameters is N_1 for ${}_{2N_1}\mathbf{H}, {}_{2N_1}\mathbf{G}$ and $N_1 + 1$ for ${}_{2N_1+1}\mathbf{H}, {}_{2N_1+1}\mathbf{G}$. For $2 \le N \le 6$, the explicit expressions of the orthogonal FIR multifilter banks for symmetric/antisymmetric ${}_N\Phi$ and ${}_N\Psi$ supported in $[0, N]$ were also provided in [13]. In [15], the complete factorization of orthogonal causal FIR multifilter banks for symmetric/antisymmetric scaling functions and orthonormal multiwavelets of dilation factor M is discussed.

Using the parametric expressions for orthogonal causal FIR multifilter banks provided in (13), (17), (27) and (28), we can construct lots of scaling functions and orthonormal multiwavelets with good approximation and smoothness properties. Assume that $\mathbf{H}$ is an FIR matrix filter and $\mathbf{H}$ generates an orthonormal scaling function Φ. Then Φ provides an approximation of order m for some positive integer m if and only if there exist real vectors $\mathbf{y}_k \in R^r$ with $\mathbf{y}_0 \ne 0$, $0 \le k < m$, such that (see e.g., [11], [21], and [14])

$$\begin{cases} \sum_{\ell=0}^{k} \binom{k}{\ell}(2i)^{-\ell}\mathbf{y}_{k-\ell}^T D^\ell\mathbf{H}(0) = 2^{-k}\mathbf{y}_k^T, \\ \sum_{\ell=0}^{k} \binom{k}{\ell}(2i)^{-\ell}\mathbf{y}_{k-\ell}^T D^\ell\mathbf{H}(\pi) = \mathbf{0}, \end{cases} \quad (29)$$

where $D^\ell\mathbf{H}$ denotes the matrix formed by the ℓth derivatives of the entries of $\mathbf{H}$. Thus to construct orthonormal scaling functions Φ with good approximation, we need only to find the parameters for $\mathbf{H}$ and the real vectors $\mathbf{y}_k$ which satisfy (29). We refer the reader to [13] for examples of scaling functions and orthonormal multiwavelets with good regularity, where we used the smoothness estimate of scaling functions provided in [12].

IV. Multiwavelets with optimum time–frequency resolution

In many still–image and video processing applications, the time–frequency localization of the decomposition technique is an important consideration (see [10] and [20]), and in these applications, the time–frequency resolution property of the scaling functions and wavelets is important. In the scalar case, the design of optimum time–frequency resolution (*OPTFR*) wavelets was considered in [6]. Further studies were carried out and more optimal filters were designed in [33] and [20]. As for the vector case, OPTFR–multiwavelets were studied in [13]. In this section, we will design OPTFR–multiwavelets which are more suitable for image processing.

For a window function f, the *time–duration* Δ_f of f is defined by

$$\Delta_f := (\int_R (t - \overline{t})^2 |f(t)|^2 dt/E)^{\frac{1}{2}},$$

where $\overline{t}$ is the center in the time domain defined by $\overline{t} := \int_R t|f(t)|^2dt/E$, $E := \int_R |f(t)|^2dt$. The *frequency–bandwidth* of f denoted by $\Delta_{\widehat{f}}$ is defined in the same way with f replaced by $\widehat{f}$. Then it is well known that $\Delta_f\Delta_{\widehat{f}} \ge 1/2$. This inequality is called the uncertainty principle, and the product $\Delta_f\Delta_{\widehat{f}}$ is called the *resolution cell.*

Since every component ψ_j of the orthonormal multiwavelet Ψ is a bandpass function (see [13]), as in the scalar case, we shall also consider the frequency–bandwidth $\widetilde{\Delta}_{\widehat{\psi}_j}$ of ψ_j defined by ([10], [6])

$$\widetilde{\Delta}_{\widehat{\psi}_j} := (\int_0^{+\infty} (\omega - \overline{\overline{\omega}})^2 |\widehat{\psi}_j(\omega)|^2 d\omega \Big/ \int_0^{+\infty} |\widehat{\psi}_j(\omega)|^2 d\omega)^{\frac{1}{2}},$$

where

$$\overline{\overline{\omega}} := \int_0^{+\infty} \omega|\widehat{\psi}_j(\omega)|^2 d\omega \Big/ \int_0^{+\infty} |\widehat{\psi}_j(\omega)|^2 d\omega.$$

One can check that for a real function ψ_j, $\widetilde{\Delta}^2_{\widehat{\psi}_j} = \Delta^2_{\widehat{\psi}_j} - (\overline{\overline{\omega}})^2$. If $\widehat{\psi}_j(0) = 0$, then $\Delta_{\psi_j}\widetilde{\Delta}_{\widehat{\psi}_j} > 1/2$ holds (see [6] and [10]).

In [13], formulas to compute the energy moments of scaling functions and multiwavelets in the time–frequency plane were provided. Let Φ and Ψ be the scaling function and orthonormal multiwavelet supported in $[0, N]$ that correspond to a multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ given by $\mathbf{H}(\omega) = \sum_{k=0}^N \mathbf{H}_k e^{-ik\omega}, \mathbf{G}(\omega) = \sum_{k=0}^N \mathbf{G}_k e^{-ik\omega}$. It was shown in [13] that the time–durations of the components of Φ, Ψ are given in terms of the eigenvectors of the matrices:

$$\mathcal{T}^\beta_{\mathbf{H}} := (2\mathcal{A}^\beta_{2\ell-j})_{1-N\le\ell,j\le N-1}, \quad 0 \le \beta \le 2,$$

and

$$\mathcal{T}^\beta_{\mathbf{G}} := (2\mathcal{B}^\beta_{2\ell-j})_{1-N\le\ell,j\le N-1}, \quad 0 \le \beta \le 2,$$

where $\mathcal{A}_j^\beta := \sum_{k=0}^N k^\beta \mathbf{H}_{k-j} \otimes \mathbf{H}_k$ and $\mathcal{B}_j^\beta := \sum_{k=0}^N k^\beta \mathbf{G}_{k-j} \otimes \mathbf{G}_k$. If Φ provides approximation order 2 and $1/4$ is a simple eigenvalue of the matrix $\mathcal{T}_\mathbf{H}$ defined by (5), then the frequency–bandwidths $\Delta_{\widehat{\phi}_j}$ and $\Delta_{\widehat{\psi}_j}$ are also given in terms of $\mathcal{T}_\mathbf{G}^0$ and the 1/4–eigenvector of $\mathcal{T}_\mathbf{H}$ (see [13]). When we consider the resolution cell $\Delta_{\psi_j}\widetilde{\Delta}_{\widehat{\psi}_j}$, by the fact that $\widetilde{\Delta}^2_{\widehat{\psi}_j} = \Delta^2_{\widehat{\psi}_j} - (\overline{\widetilde{\omega}})^2$, what we need to compute is the center $\overline{\widetilde{\omega}}$ of ψ_j in the frequency domain since Δ_{ψ_j} and $\Delta_{\widehat{\psi}_j}$ can be obtained by the formulas provided in [13]. In this case, we will use the cascade algorithm to approximate the multiwavelets, i.e., we will compute the centers $\overline{\widetilde{\omega}}$ of the components of Ψ_n, where $\Psi_n(x) = 2\sum_k \mathbf{G}_k \Phi_n(2x-k)$ and Φ_n is defined by (2) for some n (e.g., $n = 8$). In the case when $1/4$ is a non–simple eigenvalue of $\mathcal{T}_\mathbf{H}$, or we are not constructing scaling functions which provide approximation order 2, we can also use the cascade algorithm to compute approximately the frequency–bandwidths of the scaling functions and multiwavelets. As for the convergence of the cascade algorithm, see [23] and [19].

Assume that $\Phi = (\phi_1, \phi_2)^T$ and $\Psi = (\psi_1, \psi_2)^T$ are the scaling function and orthonormal multiwavelet supported in $[0, N]$ that correspond to a multifilter bank $\{\mathbf{H}, \mathbf{G}\}$ given by $\mathbf{H}(\omega) = \sum_{k=0}^N \mathbf{H}_k e^{-ik\omega}, \mathbf{G}(\omega) = \sum_{k=0}^N \mathbf{G}_k e^{-ik\omega}$. Denote

$$\mathbf{L} := \begin{bmatrix} \cdots & & & & & & \\ & \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{0} & \mathbf{0} & \\ & \mathbf{H}_5 & \mathbf{H}_4 & \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 & \\ & & & & & & & \cdots \end{bmatrix}.$$

Assume that (V_j) is the orthogonal multiresolution analysis generated by Φ and $W_j := V_{j+1} \ominus V_j, j \in Z$. Any continuous–time function $f(t)$ in V_0 can be expanded as

$$f(t) = \sum_n v_{1,n}^{(0)} \phi_1(t-n) + v_{2,n}^{(0)} \phi_2(t-n).$$

The function f is completely determined by the sequences $\{v_{1,n}^{(0)}\}, \{v_{2,n}^{(0)}\}$. The coarse approximation (component in V_{-1}) is computed with the low–pass part of the orthogonal multifilter bank (see [27] and [32]):

$$\begin{aligned} &[\cdots, [v_{1,n}^{(-1)}, v_{2,n}^{(-1)}], [v_{1,n-1}^{(-1)}, v_{2,n-1}^{(-1)}], \cdots]^T \\ &= \mathbf{L}[\cdots, [v_{1,n}^{(0)}, v_{2,n}^{(0)}], [v_{1,n-1}^{(0)}, v_{2,n-1}^{(0)}], \cdots]^T. \end{aligned} \quad (30)$$

Similarly, the details $w_{1,n}^{(-1)}, w_{2,n}^{(-1)}$ in W_{-1} are computed with the high–pass part $\{\mathbf{G}_k\}$ of the orthogonal multifilter bank.

As in [18], an orthonormal multiwavelet Ψ is said to be *balanced* if its corresponding scaling function Φ satisfies $\widehat{\Phi}(0) = (1,1)^T/\sqrt{2}$. From (30), we note that a multifilter bank differs from the usual scalar filter bank in the sense that it requires two input streams (a vector stream). If Φ is unbalanced, simple methods for the vectorization, like splitting the input signal into blocks of size two, may lead to mixing of coarse resolution and details which creates strong oscillations in the reconstructed signal after compression (see [18]). In [27] and [32], prefiltering is used when the orthonormal multiwavelet is not balanced. However, in this section, we will construct directly balanced scaling functions and orthonormal multiwavelets which do not need any prefiltering.

From (30), the low–pass frequency responses for this system are ([22])

$$h_\alpha(\omega) := \sum_{k=0}^N \mathbf{H}_k(\alpha, 1) e^{-2ik\omega} + \mathbf{H}_k(\alpha, 2) e^{-i(2k+1)\omega}, \quad (31)$$

and similarly the high–pass frequency responses are

$$q_\alpha(\omega) := \sum_{k=0}^N \mathbf{G}_k(\alpha, 1) e^{-2ik\omega} + \mathbf{G}_k(\alpha, 2) e^{-i(2k+1)\omega}, \quad (32)$$

where $\alpha = 1, 2$, and for a matrix $\mathbf{B}$, $\mathbf{B}(\ell, j)$ denotes the (ℓ, j)–entry of $\mathbf{B}$. The filters h_1, h_2 act as low–pass filters, while q_1, q_2 act as high–pass filters. Thus it is required that $|h_\alpha(0)| = |q_\alpha(\pi)| = 1, h_\alpha(\pi) = q_\alpha(0) = 0, \alpha = 1, 2$.

Proposition 3: If an orthonormal multiwavelet Ψ is balanced, then $h_\alpha(0) = 1, q_\alpha(0) = 0, \alpha = 1, 2$.

Proof: Note that

$$\begin{bmatrix} h_1(0) \\ h_2(0) \end{bmatrix} = \mathbf{H}(0) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix} = \mathbf{G}(0) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If Ψ is balanced, then $(1,1)^T$ is a right 1–eigenvector of $\mathbf{H}(0)$ and $\mathbf{G}(0)(1,1)^T = (0,0)^T$. Thus $h_\alpha(0) = 1, q_\alpha(0) = 0, \alpha = 1, 2$. ■

From Proposition 3, what we need in our design are the requirements: $h_1(\pi) = h_2(\pi) = 0$ and $q_1(\pi) = q_2(\pi) = \pm 1$. In the following, we will design the balanced optimum time–frequency resolution scaling functions and orthonormal multiwavelets with the corresponding multifilter banks satisfying

$$h_1(\pi) \approx 0, h_2(\pi) \approx 0, q_1(\pi) \approx \pm 1, q_2(\pi) \approx \pm 1. \quad (33)$$

We use (27) and (28) to construct the balanced OPTFR–multiwavelets. Assume that $\{{}_N\mathbf{H}, {}_N\mathbf{G}\}$ is an orthogonal multifilter banks given by (27) or (28) and $\{{}_N\mathbf{H}, {}_N\mathbf{G}\}$ generates the scaling function ${}_N\Phi = ({}_N\phi_1, {}_N\phi_2)^T$ and orthonormal multiwavelet ${}_N\Psi = ({}_N\psi_1, {}_N\psi_2)^T$ supported in $[0, N]$. Then $(1,0)^T$ is a right 1–eigenvector of ${}_N\mathbf{H}(0)$. By the rotation of an angle $\pi/4$, we get the balanced scaling function ${}_N\Phi^b = ({}_N\phi_1^b, {}_N\phi_2^b)^T$ and orthonormal multiwavelet ${}_N\Psi^b = ({}_N\psi_1^b, {}_N\psi_2^b)^T$ by

$${}_N\Phi^b = \mathbf{R}_0 {}_N\Phi, {}_N\Psi^b = \mathbf{R}_0 {}_N\Psi, \quad (34)$$

where i.e.,

$$\mathbf{R}_0 := \mathbf{R}^{(2)}(\pi/4) = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \quad (35)$$

In this case the multifilter bank corresponding to ${}_N\Phi^b, {}_N\Psi^b$ is

$${}_N\mathbf{H}^b(\omega) = \mathbf{R}_0 {}_N\mathbf{H}(\omega) \mathbf{R}_0^T, {}_N\mathbf{G}^b(\omega) = \mathbf{R}_0 {}_N\mathbf{G}(\omega) \mathbf{R}_0^T. \quad (36)$$

A scaling function $\Phi = (\phi_1, \phi_2)^T$ (an orthonormal multiwavelet $\Psi = (\psi_1, \psi_2)^T$ respectively) is called a *scaling function pair* (*an orthonormal multiwavelet pair* respectively) about a center $c_0/2$ if there exists $c_0 \in R$ such that $\phi_2(c_0 - x) = \phi_1(x)$ ($\psi_2(c_0 - x) = \psi_1(x)$ respectively). In fact in these definitions, Φ and Ψ denote the sets of $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ respectively. By a direct calculation, one has the following proposition.

Proposition 4: Assume that vector–valued functions $\mathbf{F}, \mathbf{F}^b$ satisfy $\mathbf{F}^b = \mathbf{R}_0\mathbf{F}$. Then $\mathbf{F}$ is symmetric/antisymmetric about a center $N/2$ if and only if $\mathbf{F}^b = (f_1^b, f_2^b)^T$ satisfies $f_2^b(N-x) = f_1^b(x)$.

By Proposition 4, ${}_N\Phi^b$ (${}_N\Psi^b$ respectively) defined by (34) is an orthonormal scaling function pair (multiwavelet pair respectively). Clearly if ${}_N\Phi^b$ is a scaling function pair, then ${}_N\Phi^b$ is balanced. In [13], OPTFR–multiwavelet pairs were constructed by minimizing the sums of the areas of the resolution cells of the scaling functions and orthonormal multiwavelets. In this section, we will construct OPTFR–multiwavelet pairs under the constrained conditions (33).

Proposition 5: Suppose that $\{{}_N\mathbf{H}^b, {}_N\mathbf{G}^b\}$ is the multifilter bank defined by (36) with ${}_N\mathbf{H}$, ${}_N\mathbf{G}$ satisfying (21). Let ${}_Nh_\alpha^b, {}_Nq_\alpha^b, \alpha = 1, 2$, be the corresponding frequency responses defined by (31) and (32) respectively. Then

$$\begin{aligned} {}_Nh_2^b(\omega) &= e^{-i(2N+1)\omega}\,{}_Nh_1^b(-\omega), \\ {}_Nq_2^b(\omega) &= e^{-i(2N+1)\omega}\,{}_Nq_1^b(-\omega), \end{aligned}$$

and ${}_Nh_1^b(\pi) = (2,2)$–entry of ${}_N\mathbf{H}(0)$, ${}_Nq_1^b(\pi) = (2,2)$–entry of ${}_N\mathbf{G}(0)$.

The proof of Proposition 5 is presented in Appendix A.

In the following, ${}_N\mathbf{H}^b$ and ${}_N\mathbf{G}^b$ are given by (36) with ${}_N\mathbf{H}$ and ${}_N\mathbf{G}$ given by (27) and (28). Here we will choose $\pm$, $\mp$ in $\mathbf{A}_k$ of (22), $\mathbf{U}_1$ of (23) (in (25)) and (24) (in (26)) to be $+$ and $-$ respectively. By Proposition 3.2, ${}_N\mathbf{H}(0) = \mathrm{diag}(1, -\sin\theta_0)$ and ${}_N\mathbf{G}(0) = \mathrm{diag}(0, \cos\theta_0)$ with $\theta_0 \in T$. Proposition 5 implies that ${}_Nh_1^b(\pi) = -\sin\theta_0$, ${}_Nq_1^b(\pi) = \cos\theta_0$. Therefore ${}_Nh_1^b(\pi) = 0$ if and only if ${}_Nq_1^b(\pi) = \pm 1$. Thus the constrained conditions (33) reduce to $\sin\theta_0 \approx 0$. Since ${}_N\phi_2^b(N-x) = {}_N\phi_1^b(x), {}_N\psi_2^b(N-x) = {}_N\psi_1^b(x)$, we construct OPTFR–multiwavelet pairs by minimizing the sums ${}_NS^b := \Delta_{{}_N\phi_1^b}\Delta_{\widehat{{}_N\phi_1^b}} + \Delta_{{}_N\psi_1^b}\Delta_{\widehat{{}_N\psi_1^b}}$ and ${}_N\widetilde{S}^b := \Delta_{{}_N\phi_1^b}\Delta_{\widehat{{}_N\phi_1^b}} + \Delta_{{}_N\psi_1^b}\widetilde{\Delta}_{\widehat{{}_N\psi_1^b}}$ under the condition that $\sin\theta_0 \approx 0$. Let ${}_N\Phi^{bo}$ and ${}_N\Psi^{bo}$ (${}_N\widetilde{\Phi}^{bo}$ and ${}_N\widetilde{\Psi}^{bo}$ respectively) denote the corresponding OPTFR-scaling function and orthonormal multiwavelet pairs constructed by minimizing ${}_NS^b$ (${}_N\widetilde{S}^b$ respectively). In the following, we will construct the OPTFR-scaling function and orthonormal multiwavelet pairs for $3 \le N \le 7$. The constrained condition $\sin\theta_0 \approx 0$ used is $|\sin\theta_0| \le 10^{-4}$.

For $N = 3, 4$, there are two free parameters for ${}_N\mathbf{H}^b$ and ${}_N\mathbf{G}^b$, and we will minimize ${}_NS^b$ and ${}_N\widetilde{S}^b$ under the constrained condition $|\sin\theta_0| \le 10^{-4}$ and the resulting ${}_N\Phi^b$ provides an approximation of order 1. For $N = 5, 6, 7$, there are more free parameters, and in these cases, we minimize ${}_NS^b$ under the constrained condition $|\sin\theta_0| \le 10^{-4}$ and the resulting ${}_N\Phi^b$ provides an approximation of order 2. The parameters for the optimal multifilter banks generating ${}_N\Phi^{bo}$, ${}_N\Psi^{bo}$ and ${}_N\widetilde{\Phi}^{bo}$, ${}_N\widetilde{\Psi}^{bo}$ are provided in Table 1 and Table 2 respectively. One can check that for such choices of parameters, the matrices $\mathcal{T}_{{}_N\mathbf{H}^b}$ associated to the low-pass parts ${}_N\mathbf{H}^b$ of the optimal multifilter banks satisfy Condition E. Thus ${}_N\Phi^{bo}$, ${}_N\Psi^{bo}$ and ${}_N\widetilde{\Phi}^{bo}$, ${}_N\widetilde{\Psi}^{bo}$ are orthonormal scaling functions and orthonormal multiwavelets. The areas of the resolution cells of OPTFR-scaling function and orthonormal multiwavelet pairs are given in Table 3. Based on the results in Table 3, we conclude that ${}_NS^b < {}_{N+1}S^b, {}_N\widetilde{S}^b < {}_{N+1}\widetilde{S}^b, N = 3, 5$. We show the graphs of ${}_3\Phi^{bo}$, ${}_3\Psi^{bo}$ and ${}_5\Phi^{bo}$, ${}_5\Psi^{bo}$ in Fig.1 and Fig.3 respectively. The magnitude responses $|{}_3h_1(\omega)|$, $|{}_3q_1(\omega)|$ corresponding to the optimal multifilter bank generating ${}_3\Phi^{bo}$, ${}_3\Psi^{bo}$ are shown in Fig.2, while in Fig.4 the figures shown are the magnitude responses $|{}_5h_1(\omega)|$, $|{}_5q_1(\omega)|$ corresponding to the optimal multifilter bank generating ${}_5\Phi^{bo}(5-x)$, ${}_5\Psi^{bo}(5-x)$, i.e., corresponding to the multifilter bank $\{e^{-i5\omega}{}_5\mathbf{H}^b(-\omega), e^{-i5\omega}{}_5\mathbf{G}^b(-\omega)\}$, where $\{{}_5\mathbf{H}^b(\omega), \mathbf{G}^b(\omega)\}$ is the optimal multifilter bank provided in Appendix B. The optimal multifilter banks for $N = 3, 5, 7$ are provided in Appendix B.

Remark 1: In the design of OPTFR-multiwavelets, we obtain multifilters $\mathbf{H}_0^b, \mathbf{H}_1^b, \cdots, \mathbf{H}_N^b$ and $\mathbf{G}_0^b, \mathbf{G}_1^b, \cdots, \mathbf{G}_N^b$ by minimizing the sums of the areas of the the resolution cells ${}_N\mathbf{S}^b$, ${}_N\widetilde{\mathbf{S}}^b$. However sometimes, the optimal multifilters required in practice may be $\mathbf{H}_N^b, \mathbf{H}_{N-1}^b, \cdots, \mathbf{H}_0^b$ and $\mathbf{G}_N^b, \mathbf{G}_{N-1}^b, \cdots, \mathbf{G}_0^b$ instead. Indeed, if $\{\{\mathbf{H}_k^b\}_{k=0}^N, \{\mathbf{G}_k^b\}_{k=0}^N\}$ generates a scaling function $\Phi(x)$ and an orthonormal multiwavelet $\Psi(x)$, then $\{\{\mathbf{H}_k^b\}_{k=N}^0, \{\mathbf{G}_k^b\}_{k=N}^0\}$ generates the scaling function $\Phi(N-x)$ and the orthonormal multiwavelet $\Psi(N-x)$. Furthermore, for a window function f, $f(x)$ and $f(c-x)$ have the same time–frequency resolution cell for any $c \in R$. Choosing $\{\mathbf{H}_k^b\}_{k=0}^N, \{\mathbf{G}_k^b\}_{k=0}^N$ or $\{\mathbf{H}_k^b\}_{k=N}^0, \{\mathbf{G}_k^b\}_{k=N}^0$ depends on their frequence responses defined by (31) and (32). For example, for the optimal filters generating ${}_5\Phi^{bo}(x)$, ${}_5\Psi^{bo}(x)$ provided in Appendix B, we shall use $\{\mathbf{H}_5^b, \cdots, \mathbf{H}_0^b\}$ and $\{\mathbf{G}_5^b, \cdots, \mathbf{G}_0^b\}$, but not $\{\mathbf{H}_0^b, \cdots, \mathbf{H}_5^b\}$ and $\{\mathbf{G}_0^b, \cdots, \mathbf{G}_5^b\}$, in image processing applications.

V. Conclusions

In this paper, several forms of parametric expressions for orthogonal causal FIR multifilter banks are obtained based on the lattice structures for $M \times M$ causal FIR lossless systems. The explicit expressions for a group of orthogonal causal FIR multifilter banks which generate symmetric/antisymmetric scaling functions and orthonormal multiwavelets are presented. Based on the parametric expressions for orthogonal multifilter banks, orthonormal multiwavelet pairs with good time–frequency resolution are constructed, and examples of optimal multifilter banks are provided. Future research problems include: (i) to design optimal multifilter banks with even better time–frequency resolution; (ii) to use the optimal multifilter banks in image processing applications.

Appendix A

A. Proof of Proposition 2

Proof: Since $\mathbf{H}$ generates the scaling function Φ, $\mathbf{H}(0)$ satisfies Condition E, and there exists a vector $\mathbf{v} \in S^{r-1}$ such that $\mathbf{v}^T\mathbf{H}(\pi) = \mathbf{v}^T, \mathbf{v}^T\mathbf{H}(\pi) = 0$. Thus $\mathbf{v}^T\widehat{\Phi}(2k\pi) = 0, k \in Z\backslash\{0\}$ (see [16]). By the first equation in (4), $\mathbf{v}^T\mathbf{H}(0)^T = \mathbf{v}^T$, i.e., $\mathbf{H}(0)\mathbf{v} = \mathbf{v}$. By our assumption that Φ is real, $\widehat{\Phi}(0)$ is also real. Since 1 is a simple eigenvalue of $\mathbf{H}(0)$ and $\widehat{\Phi}(0)$ is a right 1-eigenvector of $\mathbf{H}(0)$, $\mathbf{v} = c_0\widehat{\Phi}(0)$ for some constant $c_0 \neq 0$. Thus $\widehat{\Phi}(0)$ is a right and left 1-eigenvector of $\mathbf{H}(0)$. To complete the proof of Proposition 2, we need to show that $\widehat{\Phi}(0) \in S^{r-1}$. By the orthonormality of Φ, $G_\Phi(\omega) = \mathbf{I}_r$, $\omega \in T$ (see [8]). Here

$$G_\Phi(\omega) := \sum_{k\in Z}\widehat{\Phi}(\omega+2\pi k)\widehat{\Phi}^*(\omega+2\pi k)$$

is the Gram matrix of Φ. Therefore

$$\begin{aligned}&\widehat{\Phi}(0)^T\widehat{\Phi}(0) = \widehat{\Phi}(0)^T G_\Phi(0)\widehat{\Phi}(0)\\ &= \widehat{\Phi}(0)^T(\sum_{k\in Z}\widehat{\Phi}(2\pi k)\widehat{\Phi}(2\pi k)^*)\widehat{\Phi}(0) = (\widehat{\Phi}(0)^T\widehat{\Phi}(0))^2,\end{aligned}$$

which implies that $\widehat{\Phi}(0)^T\widehat{\Phi}(0) = 1$, i.e., $\widehat{\Phi}(0) \in S^{r-1}$. ∎

B. Proof of Lemma 1

Proof: Clearly if $\mathbf{A}_1$ is given by (15), then $\mathbf{A}_1$ is a projection matrix and satisfies (14). Conversely, if $\mathbf{A}_1$ is a projection matrix , then there exists a matrix $\mathbf{P} \in O(2r)$ such that $\mathbf{A}_1 = \mathbf{P}^T\mathrm{diag}(\mathbf{I}_s, \mathbf{0})\mathbf{P} = \mathbf{P}_1^T\mathbf{P}_1$, where $\mathbf{P}_1$ is the $s \times 2r$ matrix consisting of the first s rows of $\mathbf{P}$. By (14), $\mathrm{diag}(\mathbf{I}_s, \mathbf{0})\mathbf{P}\ \mathrm{diag}(1, \mathbf{U}_1)\begin{bmatrix}\mathbf{I}_r\\ -\mathbf{I}_r\end{bmatrix} = \mathbf{0}_{2r\times r}$. Thus

$$\begin{aligned}&\begin{bmatrix}\mathbf{P}_1\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}1 & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_1\end{bmatrix}\begin{bmatrix}\mathbf{I}_r & \mathbf{I}_r\\ -\mathbf{I}_r & \mathbf{I}_r\end{bmatrix}\\ &= \sqrt{2}\left[\mathbf{0}_{2r\times r}\quad \begin{bmatrix}\mathbf{B}^T\\ \mathbf{0}_{(2r-s)\times r}\end{bmatrix}\right],\end{aligned}$$

where $\mathbf{B}$ is an $r \times s$ matrix. Therefore

$$\begin{aligned}&\begin{bmatrix}\mathbf{P}_1\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}1 & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_1\end{bmatrix}\\ &= \frac{\sqrt{2}}{2}\left[\mathbf{0}_{2r\times r}\quad \begin{bmatrix}\mathbf{B}^T\\ \mathbf{0}_{(2r-s)\times r}\end{bmatrix}\right]\begin{bmatrix}\mathbf{I}_r & -\mathbf{I}_r\\ \mathbf{I}_r & \mathbf{I}_r\end{bmatrix}\\ &= \frac{\sqrt{2}}{2}\begin{bmatrix}\mathbf{B}^T & \mathbf{B}^T\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\end{aligned}$$

which leads to $\mathbf{P}_1 = \left[\mathbf{B}^T\quad \mathbf{B}^T\right]\mathrm{diag}(1, \mathbf{U}_1^T)/\sqrt{2}$. By $\mathbf{P}_1\mathbf{P}_1^T = \mathbf{I}_s$, we conclude that $\mathbf{B}\mathbf{B}^T = \mathbf{I}_s$. Thus $s \leq r$ and $\mathbf{A}_1 = \mathbf{P}_1^T\mathbf{P}_1$ can be written in the form (15). ∎

C. Proof of Lemma 2

Proof: If $\mathbf{A}_k$ is given by (22), then one can check that $z\mathbf{S}_1\mathbf{V}_k(z)\mathbf{S}_1 = \mathbf{V}_k(z^{-1})$. Conversely, if $z\mathbf{S}_1\mathbf{V}_k(z)\mathbf{S}_1 = \mathbf{V}_k(z^{-1})$, then $\mathbf{S}_1\mathbf{A}_k\mathbf{S}_1 = \mathbf{I}_4 - \mathbf{A}_k$. This fact, together with the relations $\mathbf{A}_k^T = \mathbf{A}_k, \mathbf{A}_k^2 = \mathbf{A}_k$, imply that $\mathbf{A}_k$ can be written in the form (22). ∎

D. Proof of Lemma 3

Proof: If ${}_N\mathbf{H}, {}_N\mathbf{G}$ satisfy (21), then by (20), ${}_N\mathbf{H}(0) = \mathrm{diag}(1, a)$, ${}_N\mathbf{H}(\pi) = \mathrm{diag}(0, b)$, ${}_N\mathbf{G}(0) = \mathrm{diag}(0, c)$ and

$${}_N\mathbf{G}(\pi) = \begin{bmatrix}0 & d\\ e & 0\end{bmatrix}\ (\text{for } N = 2N_1 + 1)$$

or

$${}_N\mathbf{G}(\pi) = \begin{bmatrix}d & 0\\ 0 & e\end{bmatrix}\ (\text{for } N = 2N_1)$$

for some $a, b, c, d, e \in R$. Consequently, (4) implies that $d = \pm 1$, $a^2 + b^2 = 1$, $c^2 + e^2 = 1$ and $ac + be = 0$. Thus $\mathbf{U}_1$ in (12) is given by (24) or (23). ∎

E. Proof of Proposition 5

Proof: Recall that $\mathbf{S}_0 = \mathrm{diag}(1, -1)$, $\mathbf{R}_0 = \mathbf{R}^{(2)}(\pi/4)$. By the assumption of the proposition, $\mathbf{S}_0\mathbf{H}_{N-k}\mathbf{S}_0 = \mathbf{H}_k$. Then we have

$$\begin{aligned}{}_Nh_1^b(\omega) &= \sum_{k=0}^N (1,0)\mathbf{R}_0\mathbf{H}_k\mathbf{R}_0^T(1, e^{-i\omega})^T e^{-i2k\omega}\\ &= \frac{1}{2}\sum_{k=0}^N (1,-1)\mathbf{H}_k\begin{bmatrix}1+e^{i\omega}\\ -1+e^{i\omega}\end{bmatrix}e^{-i2k\omega},\end{aligned}$$

which gives

$$\begin{aligned}{}_Nh_2^b(\omega) &= \sum_{k=0}^N (0,1)\mathbf{R}_0\mathbf{H}_k\mathbf{R}_0^T(1, e^{-i\omega})^T e^{-i2k\omega}\\ &= \frac{1}{2}\sum_{k=0}^N (1,1)\mathbf{H}_k(1+e^{-i\omega}, -1+e^{-i\omega})^T e^{-i2k\omega}\\ &= \frac{1}{2}\sum_{k=0}^N (1,-1)\mathbf{S}_0\mathbf{H}_k\mathbf{S}_0(1+e^{-i\omega}, 1-e^{-i\omega})^T e^{-i2k\omega}\\ &= \frac{1}{2}\sum_{k=0}^N (1,-1)\mathbf{H}_{N-k}\begin{bmatrix}1+e^{i\omega}\\ -1+e^{i\omega}\end{bmatrix}e^{i2(N-k)\omega}e^{-i(2N+1)\omega}\\ &= e^{-i(2N+1)\omega}\,{}_Nh_1^b(-\omega).\end{aligned}$$

The relationship between $q_1^b(\omega)$ and $q_2^b(\omega)$ can be established in a similar way. By Proposition 3.2, ${}_N\mathbf{H}(0) = \mathrm{diag}(1, \mp\sin\theta_0)$ and ${}_N\mathbf{G}(0) = \mathrm{diag}(0, \pm\cos\theta_0)$ for some $\theta_0 \in T$. Thus

$$\begin{aligned}{}_Nh_1^b(\pi) &= \sum_{k=0}^N (1,-1)\mathbf{H}_k(0,-2)^T/2\\ &= (-1,1)\,{}_N\mathbf{H}(0)(0,1)^T = \mp\sin\theta_0,\end{aligned}$$

and ${}_Nq_1^b(\pi) = (1,-1)\,{}_N\mathbf{G}(0)(0,-1)^T = \pm\cos\theta_0$. ∎

Appendix B

Denote $\mathbf{J}_2 := \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$. The optimal multi-filter banks $\{{}_N\mathbf{H}^b, {}_N\mathbf{G}^b\}$ are given by ${}_N\mathbf{H}^b(\omega) = \mathbf{R}_0\,{}_N\mathbf{H}(\omega)\mathbf{R}_0^T, {}_N\mathbf{G}^b(\omega) = \mathbf{R}_0\,{}_N\mathbf{G}(\omega)\mathbf{R}_0^T$, where $\mathbf{R}_0$ is defined by (35) and ${}_N\mathbf{H}, {}_N\mathbf{G}$ are as follows.

${}_3\mathbf{H}, {}_3\mathbf{G}$ for ${}_3\Phi^{bo}$ and ${}_3\Psi^{bo}$:

$$\mathbf{H}_0 = \begin{bmatrix} .00790248504499 & .06236018964540 \\ .00789624898652 & -.06236097958210 \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} .49209751495501 & .06236018964540 \\ -.49210374851348 & .06231097958210 \end{bmatrix},$$

and $\mathbf{H}_j = \mathbf{S}_0\mathbf{H}_{3-j}\mathbf{S}_0$, $j = 2, 3$, $\mathbf{G}_k = (-1)^{k+1}\mathbf{H}_k\mathbf{J}_2, 0 \le k \le 3$.

${}_3\mathbf{H}, {}_3\mathbf{G}$ for ${}_3\widetilde{\Phi}^{bo}$ and ${}_3\widetilde{\Psi}^{bo}$:

$$\mathbf{H}_0 = \begin{bmatrix} .00840489892798 & .06427913454613 \\ .00839847097250 & -.06427997471463 \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} .49159510107202 & .06427913454613 \\ -.49160152652750 & .06422997471463 \end{bmatrix},$$

and $\mathbf{H}_j = \mathbf{S}_0\mathbf{H}_{3-j}\mathbf{S}_0$, $j = 2, 3$, $\mathbf{G}_k = (-1)^{k+1}\mathbf{H}_k\mathbf{J}_2, 0 \le k \le 3$.

${}_5\mathbf{H}, {}_5\mathbf{G}$ for ${}_5\Phi^{bo}$ and ${}_5\Psi^{bo}$:

$$\mathbf{H}_0 = \begin{bmatrix} -.00880400349405 & .00249794465312 \\ -.00880425324450 & -.00249706424028 \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} .01505927978451 & -.05307641691552 \\ -.01505397206753 & -.05307792257812 \end{bmatrix},$$

$$\mathbf{H}_2 = \begin{bmatrix} .49374472370954 & -.05557436156864 \\ .49375027867697 & .05552498681839 \end{bmatrix},$$

and $\mathbf{H}_j = \mathbf{S}_0\mathbf{H}_{5-j}\mathbf{S}_0$, $3 \le j \le 5$, $\mathbf{G}_k = (-1)^{k+1}\mathbf{H}_k\mathbf{J}_2, 0 \le k \le 5$.

${}_5\mathbf{H}, {}_5\mathbf{G}$ for ${}_5\widetilde{\Phi}^{bo}$ and ${}_5\widetilde{\Psi}^{bo}$:

$$\mathbf{H}_0 = \begin{bmatrix} -.01374612215754 & .00537394370532 \\ -.01374665948318 & -.00537256906623 \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} .02055687876217 & -.05258286691076 \\ -.02055162037270 & -.05258492233572 \end{bmatrix},$$

$$\mathbf{H}_2 = \begin{bmatrix} .49318924339537 & -.05795681061608 \\ .49319503661048 & .05790749140195 \end{bmatrix},$$

and $\mathbf{H}_j = \mathbf{S}_0\mathbf{H}_{5-j}\mathbf{S}_0$, $3 \le j \le 5$, $\mathbf{G}_k = (-1)^{k+1}\mathbf{H}_k\mathbf{J}_2, 0 \le k \le 5$.

${}_7\mathbf{H}, {}_7\mathbf{G}$ for ${}_7\Phi^{bo}$ and ${}_7\Psi^{bo}$:

$$\mathbf{H}_0 = \begin{bmatrix} -.00021301558643 & -.00186927059418 \\ -.00021282865831 & .00186929188640 \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} -.00209035969785 & -.00023821013302 \\ .00209038350841 & -.00023800109586 \end{bmatrix},$$

$$\mathbf{H}_2 = \begin{bmatrix} .00966240240680 & .06082475346760 \\ .00965631988315 & -.06082571940372 \end{bmatrix},$$

$$\mathbf{H}_3 = \begin{bmatrix} .49264097287748 & .05919369300644 \\ -.49264688978357 & .05914442861318 \end{bmatrix},$$

and $\mathbf{H}_j = \mathbf{S}_0\mathbf{H}_{7-j}\mathbf{S}_0$, $4 \le j \le 7$, $\mathbf{G}_k = (-1)^{k+1}\mathbf{H}_k\mathbf{J}_2, 0 \le k \le 7$.

${}_7\mathbf{H}, {}_7\mathbf{G}$ for ${}_7\widetilde{\Phi}^{bo}$ and ${}_7\widetilde{\Psi}^{bo}$:

$$\mathbf{H}_0 = \begin{bmatrix} -.00107171534355 & -.00050451477624 \\ .00107176578967 & -.00050440760218 \end{bmatrix},$$

$$\mathbf{H}_1 = \begin{bmatrix} .00368435270613 & .00782648499544 \\ .00368357003920 & -.00782685339158 \end{bmatrix},$$

$$\mathbf{H}_2 = \begin{bmatrix} .00812402691514 & -.06809947882033 \\ -.00811721692664 & -.06810029088252 \end{bmatrix},$$

$$\mathbf{H}_3 = \begin{bmatrix} .48926333572228 & -.07643047859201 \\ .48927097632382 & .07638155187628 \end{bmatrix},$$

and $\mathbf{H}_j = \mathbf{S}_0\mathbf{H}_{7-j}\mathbf{S}_0$, $4 \le j \le 7$, $\mathbf{G}_k = (-1)^{k+1}\mathbf{H}_k\mathbf{J}_2, 0 \le k \le 7$.

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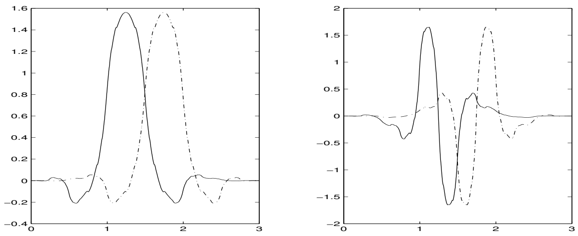

Fig. 1. OPTFR-scaling function pair ${}_3\Phi^{bo}$ (on the left) and multiwavelet pair ${}_3\Psi^{bo}$ (on the right).

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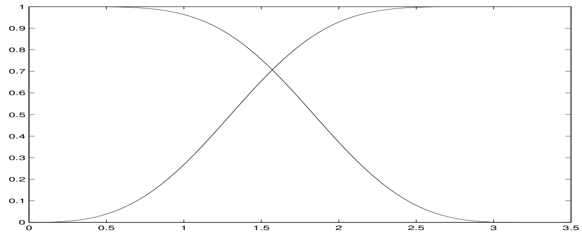

Fig. 2. The magnitude responses $|{}_3h_1^b|$ and $|{}_3q_1^b|$ corresponding to the optimal multifilter bank generating ${}_3\Phi^{bo}$ and ${}_3\Psi^{bo}$.

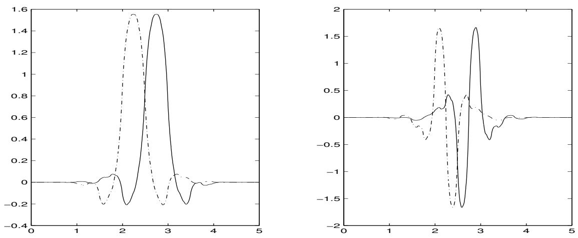

Fig. 3. OPTFR-scaling function pair ${}_5\Phi^{bo}$ (on the left) and multiwavelet pair ${}_5\Psi^{bo}$ (on the right).

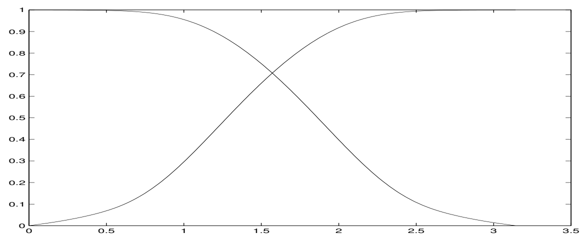

Fig. 4. The magnitude responses $|{}_5h_1^b|$ and $|{}_5q_1^b|$ corresponding to the optimal multifilter bank generating ${}_5\Phi^{bo}(5-\cdot)$ and ${}_5\Psi^{bo}(5-\cdot)$.

N	θ_0	θ_1	θ_2	θ_3
3	.00010000000017	.25200271611776		
4	.78549816339761	2.85341425815471		
5	.00010000000017	.32865488725439	-2.58876752016828	
6	-2.35629449019251	-.38893951271608	2.98074180633618	
7	.00010000000017	1.45914057145477	-1.70226608079784	.22683410549091

TABLE 1. The parameters for the optimal multifilter bank generating ${}_N\Phi^{bo}, {}_N\Psi^{bo}$.

N	θ_0	θ_1	θ_2	θ_3
3	.00010000000017	.25993723804186		
4	.78549816339761	2.84191811629411		
5	.00010000000017	.51129213165796	-2.39634484202025	
6	-2.35629449019251	-.63531529405017	2.74303786280756	
7	3.14149265358963	2.86951654391665	2.29813695533660	-.87985732107116

TABLE 2. The parameters for the optimal multifilter bank generating ${}_N\widetilde{\Phi}^{bo}, {}_N\widetilde{\Psi}^{bo}$.

N	$\Delta_{{}_N\phi_1^{bo}}\Delta_{\widehat{{}_N\phi_1^{bo}}}$	$\Delta_{{}_N\psi_1^{bo}}\Delta_{\widehat{{}_N\psi_1^{bo}}}$	$\Delta_{{}_N\widetilde{\phi}_1^{bo}}\Delta_{\widehat{{}_N\widetilde{\phi}_1^{bo}}}$	$\Delta_{{}_N\widetilde{\psi}_1^{bo}}\widetilde{\Delta}_{\widehat{{}_N\widetilde{\psi}_1^{bo}}}$
3	.67464	2.11477	.67464	.81788
4	.68200	2.12525	.68380	.84363
5	.67665	2.12357	.68764	.72582
6	.68533	2.14327	.69285	.77665
7	.67719	2.11694	.66186	.68065

TABLE 3. The areas of the resolution cells of OPTFR-scaling function pairs and multiwavelet pairs.