Tangents and curvatures of matrix-valued subdivision curves and their applications to curve design

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Abstract

Subdivision provides an efficient method to generate smooth curves and surfaces. Recently matrix-valued subdivision schemes were introduced to provide more flexibility and smaller subdivision templates for curve and surface design. For matrix-valued subdivision, the input is a set of vectors with the first components being the vertices of the control polygon (or the control net for surface subdivision) and the other components being the so-called control (or shape) parameters. It was observed that the control parameters can change the shape of limiting curve/surfaces significantly. However, how to choose these parameters has not been fully discussed in the literature. In this paper we address this issue for matrix-valued curve subdivision by providing easy-to-implement formulas for normals and curvature of subdivision curves and a method for defining shape parameters. We also do some analysis using data from a sample planar curve.

1. Introduction

Subdivision is an efficient method to generate smooth curves and surfaces [21, 23, 26]. To construct a smooth curve/surface, the subdivision process is carried out iteratively, starting from an initial polygon/polyhedron, to generate a sequence of finer and finer polygons/polyhedra that eventually converges to the desired limiting curve/surface, called the subdivision (limiting) curve/surface. During each iteration step of a (dyadic) subdivision, to form the finer polygon/polyhedron, new vertices (also called "odd" vertices) are inserted among old vertices (also called "even" vertices) on the coarser polygon/polyhedron and the positions of "even" vertices may be updated. If all the vertices (nodes) of each coarser polygon/polyhedron are among the vertices of the finer polygon/polyhedron, then the subdivision scheme is called an *interpolatory subdivision scheme*. Otherwise, it is called an *approximating subdivision scheme*. The exact positions of the new vertices and old vertices in the finer polygon/polyhedron are given by the local averaging rule.

A curve subdivision scheme is in general related to a refinement equation

$$\phi(x) = \sum_{\ell\in\mathbb{Z}} p_\ell \phi(2x-\ell), \quad x \in \mathbb{R},$$

where $\phi(x)$ is called the refinable function, and $\{p_\ell\}_{\ell\in\mathbb{Z}}$ is called the refinement mask. The refinement of ϕ results in the subdivision scheme:

$$v_k^m = \sum_j v_j^{m-1} p_{k-2j}, \quad m = 1, 2, \cdots, \tag{1.1}$$

where $v_k^0, k \in \mathbb{Z}$ are given real numbers. In practice, when (1.1) is applied to curve subdivisions, v_k^0 (where k belongs to a finite set in $\mathbb{Z}$) are vertices of the given control polygon. Hence these

v_k^0 are 2×1 vectors in $\mathbb{R}^2$ (for planar curve subdivision) or 3×1 vectors in $\mathbb{R}^3$ (for 3-D curve subdivision). In this case, (1.1) is applied to each component of v_k^0, and the resulting $v_k^1, v_k^2, \cdots$ are also column vectors. Then the finer and finer polygons with vertices $\{v_k^m\}_k$ provide an accurate discrete approximation to the limiting subdivision curve.

Recently matrix-valued subdivision schemes were introduced in [2, 3, 4, 13, 14] to provide more flexibility and smaller subdivision templates for curve/surface design. For matrix-valued curve subdivision, the refinement mask is a sequence $\{P_\ell\}_{\ell\in\mathbb{Z}}$ of matrices P_ℓ with only finitely many of them being nonzero matrices. In this paper, for simplicity of presentation, we assume P_ℓ are 2×2 matrices although they could be matrices of larger sizes. The associated refinement equation is

$$\Phi(x) = \sum_{\ell\in\mathbb{Z}} P_\ell \Phi(2x - \ell), \quad x \in \mathbb{R}, \tag{1.2}$$

where $\Phi(x) = \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \end{bmatrix}$ is called the refinable function vector.

For given initial vectors $\mathbf{v}_k^0 = [v_k^0, s_k^0], k \in \mathbb{Z}$, where $v_k^0, s_k^0 \in \mathbb{R}$, (1.2) leads to the subdivision scheme:

$$\mathbf{v}_k^m = \sum_j \mathbf{v}_j^{m-1} P_{k-2j}, \quad m = 1, 2, \cdots, \tag{1.3}$$

where $\mathbf{v}_k^m$ are 1×2 vectors, and we use v_k^m and s_k^m to denote their first and second components: $\mathbf{v}_k^m =: [v_k^m, s_k^m]$. Like the conventional (or scalar) curve subdivision, when matrix-valued subdivision (1.3) is applied to curve subdivision, v_k^0, s_k^0 are column vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and (1.3) is applied to each row of $\mathbf{v}_k^0 = [v_k^0, s_k^0]$, then to each row of $\mathbf{v}_k^1, \mathbf{v}_k^2, \cdots$ iteratively. In the following we assume v_k^0, s_k^0 and hence $v_k^m, s_k^m, m \geq 1$ to be real numbers unless it is specifically indicated.

Observe that for matrix-valued subdivision, the input is a (finite or infinite) sequence of vectors $\mathbf{v}_k^0 = [v_k^0, s_k^0]$. We can use the first components v_k^0 of $\mathbf{v}_k^0$ as the vertices of the initial control polygon, and the first components v_k^m of $\mathbf{v}_\mathbf{k}^m$ as the vertices of the subdivision polygons generated after m-step subdivision iterations. The other component, s_k^0 of $\mathbf{v}_k^0$, can be used to control the geometric shape of the limiting curve. Then one can show as in [3], that the vertices v_k^m provide an accurate discrete approximation of the target subdivision (limiting) curve. The limiting curve depends on not only the initial vertices v_k^0, but also on s_k^0 which are called *shape control parameters.*

In this paper we discuss how to choose shape control parameters s_k^0. We will investigate the relationship between shape control parameters and some geometric property such as tangents, normals and curvatures of the limiting curves. In particular we obtain formulas for the normals and curvatures of the limiting curves in terms of the shape control parameters and the initial control polygon. With these formulas, we can design our shape parameters such that the resulting subdivision curve has specific normals and curvatures.

The rest of this paper is organized as follows. In § 2, we recall some C^2 matrix-valued curve subdivision schemes introduced in [4], and discuss the difference between matrix-valued **interpolatory** schemes and the Hermite type schemes studied by other researchers. In § 3 we derive the first and second derivatives of the limiting curve. In § 4 we provide the tangents and curvatures of the subdivision curves generated by 3-point and 4-point schemes. Finally, in § 5, we define our shape parameters and apply these formulas to curve design.

2. C^2 matrix-valued approximating scheme and C^2 matrix-valued interpolatory scheme

In this section, we recall some matrix-valued curve subdivision schemes obtained in [4]. Tangents and curvatures of the subdivision curves generated by these schemes will be analyzed in later sections. In this section we also have a look at the difference between these schemes and the Hermite type schemes investigated in the literature.

The subdivision scheme (also called the local averaging rule) (1.3) can be described and represented in a straight line along with a set of subdivision matrix-valued templates. For example, Fig. 1 shows the templates for a 4-point matrix-valued scheme of a mask $\{P_\ell\}_{\ell=-3,\cdots,3}$, where $P_0 = W, P_1 = P_{-1} = X, P_2 = P_{-2} = Z, P_3 = P_{-3} = Y$, with the local averaging rule:

$$\begin{aligned} \mathbf{v}_{2k}^m &= \mathbf{v}_{k-1}^{m-1} Z + \mathbf{v}_k^{m-1} W + \mathbf{v}_{k+1}^{m-1} Z, \\ \mathbf{v}_{2k+1}^m &= \mathbf{v}_{k-1}^{m-1} Y + \mathbf{v}_k^{m-1} X + \mathbf{v}_{k+1}^{m-1} X + \mathbf{v}_{k+2}^{m-1} Y, \quad m \geq 1. \end{aligned} \tag{2.1}$$

Figure 1: *Matrix-valued 4-point scheme*

An example of a matrix-valued 3-point approximating scheme in [4] with corresponding scaling functions ϕ_0, ϕ_1 in C^2 is:

$$W = \frac{1}{20}\begin{bmatrix} 18 & -2 \\ 2 & -3 \end{bmatrix}, \; X = \frac{1}{8}\begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}, \; Z = \frac{1}{20}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \; Y = \mathbf{0}. \tag{2.2}$$

Two more examples from [4] are a 3-point matrix-valued scheme with corresponding ϕ_0, ϕ_1 in C^2:

$$W = \frac{1}{32}\begin{bmatrix} 32 & -21 \\ 0 & -6 \end{bmatrix}, \; X = \frac{1}{8}\begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}, \; Z = \frac{1}{64}\begin{bmatrix} 0 & 21 \\ 0 & -7 \end{bmatrix}, \; Y = \mathbf{0}; \tag{2.3}$$

and a 4-point scheme with the corresponding scaling functions ϕ_0, ϕ_1 being compactly supported C^2 cubic splines:

$$W = \frac{1}{4}\begin{bmatrix} 4 & 1 \\ 0 & -1 \end{bmatrix}, \; X = \frac{1}{48}\begin{bmatrix} 25 & -1 \\ 13 & 11 \end{bmatrix}, \; Z = \frac{1}{8}\begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}, \; Y = \frac{1}{48}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{2.4}$$

Observe that for the schemes in both (2.3) and (2.4), the $(1,1)$-entry and $(2,1)$-entry of W are 1 and 0 respectively, and the first column of Z is the zero vector, thus with the formula in (2.1), we know that the first component v_{2k}^m of $\mathbf{v}_{2k}^m$ is

$$v_{2k}^m = \mathbf{v}_{k-1}^{m-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mathbf{v}_k^{m-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{v}_{k+1}^{m-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = v_k^{m-1}, \; m = 1, 2, \cdots.$$

This means that the positions of vertices v_k^0 are not changed during the matrix-valued subdivision scheme (recall that we use the first components v_k^m of $\mathbf{v}_\mathbf{k}^m$ as the vertices of the subdivision polygons generated after m-step matrix-valued subdivision iterations). Thus these two schemes are "interpolatory". This is the definition for matrix-valued interpolatory scheme introduced in [2, 4]. It was shown in [4] that the refinable function vector $\Phi = [\phi_0, \phi_1]^T$ corresponding a matrix-valued interpolatory scheme satisfies

$$\phi_0(k) = \delta(k), \quad \phi_1(k) = 0, \quad k \in \mathbb{Z}, \tag{2.5}$$

where $\delta(k)$ is the Dirac-delta sequence.

As we know, conventional interpolatory subdivision schemes cannot achieve such smoothness and small template size simultaneously as the schemes in (2.3) and (2.4). For example the 4-point (scalar) interpolatory scheme in [9] is C^1 only, and to have C^2 interpolatory schemes, one needs to use a 6-point scheme or a ternary scheme [24, 15]. In addition, except for the hat function, the compactly supported refinable function associated with an interpolatory scheme cannot be a spline function. In contrast, for matrix-valued schemes, it is possible to have 3-point C^2 interpolatory schemes and even a 4-point C^2 interpolatory scheme with spline functions as the refinable functions.

The curve subdivision templates considered in this paper, such as the one in Fig. 1, are symmetric (with mask elements $P_\ell = P_{-\ell}$ for $\ell \in \mathbb{Z}$), and therefore they can be readily adopted as boundary subdivision templates for matrix-valued surface subdivision.

The minimum-supported Hermite interpolatory C^1 cubic splines ϕ_0, ϕ_1 produce a matrix-valued curve subdivision using the following mask (see [5]):

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \; P_1 = \frac{1}{8}\begin{bmatrix} 4 & 6 \\ -1 & -1 \end{bmatrix}, \; P_{-1} = \frac{1}{8}\begin{bmatrix} 4 & -6 \\ 1 & -1 \end{bmatrix}, \; P_\ell = \mathbf{0}, |\ell| \geq 2.$$

The refinable functions (C^1 piecewise cubic polynomials supported on $[-1, 1]$) ϕ_0, ϕ_1 satisfy not only (2.5) but also $\phi_0'(k) = 0, \phi_1'(k) = \delta(k), k \in \mathbb{Z}$. However, the subdivision mask **does not have the symmetry of $P_\ell = P_{-\ell}$**. Hence it is hard to use this scheme as a boundary subdivision scheme for matrix-valued surface subdivision. Hermite-type interpolatory curve subdivision schemes have been studied in [10, 6, 7]. In these papers the constructed ϕ_0, ϕ_1 satisfy $\phi_1(x) = \phi_0'(x)$. Again, the corresponding masks do not meet the symmetry specification. In addition, since Hermite-type interpolatory schemes require the restriction $\phi_1(x) = \phi_0'(x)$, the constructed schemes have larger templates than the matrix-valued interpolatory schemes. [18] and [19] discuss classes of 4-point Hermite subdivision schemes that generate C^2 functions and the masks consist of eight 2×2 matrices.

As mentioned in § 1, we will study the normals and curvatures of subdivision curves. Since the curvature involves the second derivative, in the following we assume that the refinable function vector $\Phi = [\phi_0, \phi_1]^T$ is in $C^2(\mathbb{R})$, namely $\phi_0, \phi_1 \in C^2(\mathbb{R})$. We assume that the associated subdivision mask $P = \{P_\ell\}_{\ell \in \mathbb{Z}}$ is supported in $[-N, N]$ for some $N > 0$, meaning $P_\ell = \mathbf{0}$ for $|\ell| > N$, and that it has sum rule of order 3, namely, there exist constant vectors $\mathbf{y}_0 = [1, 0]$, $\mathbf{y}_1 = [c_1, c_2], \mathbf{y}_2 = [d_1, d_2]$ such that

$$\begin{aligned}
&\mathbf{y}_0 \sum_k P_{2k} = \mathbf{y}_0 \sum_k P_{2k+1} = \mathbf{y}_0, \\
&\sum_k \big(2\mathbf{y}_1 + (-2k)\mathbf{y}_0\big) P_{2k} = \sum_k \big(2\mathbf{y}_1 + (-2k-1)\mathbf{y}_0\big) P_{2k+1} = \mathbf{y}_1 \\
&\sum_k \big(4\mathbf{y}_2 + 4(-2k)\mathbf{y}_1 + (-2k)^2\mathbf{y}_0\big) P_{2k} \\
&\qquad = \sum_k \big(4\mathbf{y}_2 + 4(-2k-1)\mathbf{y}_1 + (-2k-1)^2\mathbf{y}_0\big) P_{2k+1} = \mathbf{y}_2.
\end{aligned}$$

Sum rule order of P implies the accuracy order of the associated Φ. **More precisely, with**

$$\mathbf{y}_0(j) = \mathbf{y}_0,\ \mathbf{y}_1(j) = \mathbf{y}_1 + j\mathbf{y}_0,\ \mathbf{y}_2(j) = \mathbf{y}_2 + 2j\mathbf{y}_1 + j^2\mathbf{y}_0,\ j \in \mathbb{Z},$$

we have (see [16, 17] and the references therein) for $n = 0, 1, 2$**,**

$$x^n = \sum_{j\in\mathbb{Z}} \mathbf{y}_n(j)\Phi(x-j),\ x \in \mathbb{R}. \tag{2.6}$$

That is, the polynomials of degree ≤ 2 **can be reproduced by** $\phi_0(x-j), \phi_1(x-j), j \in \mathbb{Z}$ **when** P **has sum rule order** 3**.**

Let $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2$ be $1 \times 2(2N+1)$ vectors defined by

$$\begin{aligned} \mathbf{l}_0 &= [\mathbf{y}_0(j)]_{-N\le j\le N} = [\mathbf{y}_0, \cdots, \mathbf{y}_0, \cdots, \mathbf{y}_0], \\ \mathbf{l}_1 &= [\mathbf{y}_1(j)]_{-N\le j\le N} = [\mathbf{y}_1 - N\mathbf{y}_0, \cdots, \mathbf{y}_1 - \mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_1 + \mathbf{y}_0, \cdots, \mathbf{y}_1 + N\mathbf{y}_0], \\ \mathbf{l}_2 &= [\mathbf{y}_2(j)]_{-N\le j\le N} = [\mathbf{y}_2 + 2j\mathbf{y}_1 + j^2\mathbf{y}_0]_{-N\le j\le N}. \end{aligned} \tag{2.7}$$

Then $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2$ are left (row) eigenvectors of eigenvalues $1, \frac{1}{2}, \frac{1}{4}$ resp. for the subdivision matrix:

$$\mathbf{S} = \Big[P_{k-2j}\Big]_{-N\le j\le N, -N\le k\le N}. \tag{2.8}$$

In this paper we assume $1, \frac{1}{2}, \frac{1}{4}$ are simple eigenvalues of $\mathbf{S}$ and other eigenvalues λ of $\mathbf{S}$ satisfy $|\lambda| < \frac{1}{4}$. The reader refers to [8] on the relation between the convergence of Hermite subdivision schemes and the eigenvalues of $\mathbf{S}$.

For an initial sequence of control vectors $\mathbf{v}^0 = \{\mathbf{v}_k^0\}$, let $\mathbf{v}_k^m = [v_k^m, s_k^m]$ be the vectors derived by the local averaging rule (1.3) after m steps of iterations with a subdivision mask $P = \{P_\ell\}_\ell$. We say the scheme with mask P is convergent if for each $\mathbf{v}^0$, there exists a continuous function f (f is a function vector if v_k^0 are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$) such that $f \not\equiv 0$ for at least one $\mathbf{v}^0$ and

$$\lim_{m\to\infty} \sup_k \left|v_k^m - f(\frac{k}{2^m})\right| = 0.$$

3. Derivatives

In this section we derive the first and second derivatives of the limiting curve f at an initial point, k_0. Here we derive the formulas for 3-D curve subdivision. Thus $\mathbf{v}_k^0 = [v_k^0, s_k^0]$ are given 3×2 matrices with v_k^0 being the vertices of the control polygon in 3-D, and f is a function vector with 3 components. We will follow the approach used in [22] where the formula was obtained for the normals of limiting surfaces generated by the butterfly scheme [11].

For simplicity, let $\mathbf{u}_j^m, -N \le j \le N$, denote the vectors of $\mathbf{v}_j^m$ after m iterations around $\mathbf{v}_{k_0}^0$:

$$\mathbf{u}_j^m := \mathbf{v}^m_{2^m k_0 + j},\ -N \le j \le N.$$

Then, with $U_0^m = [\mathbf{u}_{-N}^m, \cdots, \mathbf{u}_{-1}^m, \mathbf{u}_0^m, \mathbf{u}_1^m, \cdots, \mathbf{u}_N^n]$, we have

$$U_0^m = U_0^{m-1}\mathbf{S},\ m = 1, 2, \cdots.$$

We consider interpolatory and approximation schemes in the following two subsections.

3.1 Interpolatory schemes

For an interpolatory scheme, the first components of $\mathbf{u}_j^m$, namely, $v_{2^m k_0+j}^m$ lie on the limiting curve, denoted by $f(x)$. Thus $v_{2^m k_0+j}^m = f(k_0 + \frac{j}{2^m})$. Therefore, we have

$$\begin{aligned}
&\lim_{m\to\infty} 2^m (v_{2^m k_0+1}^m - v_{2^m k_0}^m) = \lim_{m\to\infty} 2^m (f(k_0 + \frac{1}{2^m}) - f(k_0)) = f'(k_0), \\
&\lim_{m\to\infty} 2^{2m} (v_{2^m k_0+1}^m + v_{2^m k_0-1}^m - 2v_{2^m k_0}^m) = \\
&\qquad \lim_{m\to\infty} 2^{2m} \Big(f(k_0 + \frac{1}{2^m}) + f(k_0 - \frac{1}{2^m}) - 2f(k_0) \Big) = f''(k_0).
\end{aligned} \tag{3.1}$$

Let $\{\mathbf{l}_j : 0 \le j \le 4N+1\}$ be a set of linearly independent generalized eigenvectors of $\mathbf{S}$ with $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2$ the eigenvectors of $1, 1/2, 1/4$ given in (2.7). So U_0^0 can be written as

$$U_0^0 = \alpha_0 \mathbf{l}_0 + \alpha_1 \mathbf{l}_1 + \alpha_2 \mathbf{l}_2 + \sum_{j=3}^{4N+1} \alpha_j \mathbf{l}_j, \tag{3.2}$$

where $\alpha_j \in \mathbb{R}^3$, $0 \le j \le 4N+1$.

Theorem 1 *Suppose an interpolatory scheme is convergent with limiting curve in C^2. Let f be the limiting curve of an initial control vector "polygon" $\{\mathbf{v}_k^0\}_k$. Let $\alpha_1, \alpha_2 \in \mathbb{R}^3$ be the column vectors in (3.2). Then*

$$f'(k_0) = \alpha_1, \; f''(k_0) = 2\alpha_2. \tag{3.3}$$

Proof. By the assumption that other eigenvalues excluding $1, \frac{1}{2}, \frac{1}{4}$ of $\mathbf{S}$ are smaller in modulus than $\frac{1}{4}$, we have

$$U_0^m = U_0^0 \mathbf{S}^m = \alpha_0 \mathbf{l}_0 + \alpha_1 2^{-m} \mathbf{l}_1 + \alpha_2 2^{-2m} \mathbf{l}_2 + o(2^{-2m}). \tag{3.4}$$

Thus

$$\mathbf{u}_1^m - \mathbf{u}_0^m = \alpha_0 (\mathbf{y}_0 - \mathbf{y}_0) + 2^{-m} \alpha_1 (\mathbf{y}_1 + \mathbf{y}_0 - \mathbf{y}_1) + o(2^{-m}) = 2^{-m} \alpha_1 \mathbf{y}_0 + o(2^{-m}).$$

Since the first component of $\mathbf{y}_0$ is 1,

$$2^m (v_{2^m k_0+1}^m - v_{2^m k_0}^m) = \alpha_1 + o(1).$$

Hence,

$$\lim_{m\to\infty} 2^m (v_{2^m k_0+1}^m - v_{2^m k_0}^m) = \alpha_1.$$

This and the first equation in (3.1) imply that $f'(k_0) = \alpha_1$.

Similarly, from (3.4), we have

$$\begin{aligned}
&\mathbf{u}_1^m + \mathbf{u}_{-1}^m - 2\mathbf{u}_0^m \\
&= \alpha_0 (\mathbf{y}_0 + \mathbf{y}_0 - 2\mathbf{y}_0) + \alpha_1 2^{-m} (\mathbf{y}_1 + \mathbf{y}_0 + \mathbf{y}_1 - \mathbf{y}_0 - 2\mathbf{y}_1) + \\
&\qquad \alpha_2 2^{-2m} (\mathbf{y}_2 - 2\mathbf{y}_1 + \mathbf{y}_0 + \mathbf{y}_2 + 2\mathbf{y}_1 + \mathbf{y}_0 - 2\mathbf{y}_2) + o(2^{-2m}) \\
&= 2^{-2m} 2\alpha_2 \mathbf{y}_0 + o(2^{-2m}).
\end{aligned}$$

Thus,

$$v^m_{2^m k_0+1} + v^m_{2^m k_0-1} - 2v^m_{2^m k_0} = 2^{-2m}2\alpha_2 + o(2^{-2m}),$$

and therefore,

$$\lim_{m\to\infty} 2^{2m}(v^m_{2^m k_0+1} + v^m_{2^m k_0-1} - 2v^m_{2^m k_0}) = 2\alpha_2.$$

This, together with the second equation in (3.1), implies that $f''(k_0) = 2\alpha_2$. ■

We can similarly obtain the derivatives of f at point

$$k_0 + \frac{i}{2^n}$$

for $k_0, i, n \in \mathbb{Z}$ with $i \neq 0$ and $n > 0$. In this case, let $\mathbf{u}^0_j, -N \leq j \leq N$, denote the vectors of $\mathbf{v}^n_j$ around the point $k_0 + \frac{i}{2^n}$:

$$\mathbf{u}^0_j = \mathbf{v}^n_{2^n k_0+i+j},\ -N \leq j \leq N.$$

In general, denote $\mathbf{u}^m_j = \mathbf{v}^{n+m}_{2^{n+m}k_0+2^m i+j}$, $-N \leq j \leq N$. Then, with

$$U^m_i = [\mathbf{u}^m_{-N}, \cdots, \mathbf{u}^m_{-1}, \mathbf{u}^m_0, \mathbf{u}^m_1, \cdots, \mathbf{u}^n_N],$$

we have $U^m_i = U^{m-1}_i S, m = 1, 2, \cdots$. Expand U^0_i in the form of (3.2), namely,

$$\left[\mathbf{v}^n_{2^n k_0+i+j}\right]_{-N\leq j\leq N} = \tilde{\alpha}_0\mathbf{l}_0 + \tilde{\alpha}_1\mathbf{l}_1 + \tilde{\alpha}_2\mathbf{l}_2 + \sum_{j=3}^{4N+1}\tilde{\alpha}_j\mathbf{l}_j, \tag{3.5}$$

where $\mathbf{l}_j$ are (generalized) eigenvectors of $\mathbf{S}$ and $\tilde{\alpha}_j \in \mathbb{R}^3$, $0 \leq j \leq 4N+1$. Then as shown above we have

$$\tilde{\alpha}_1 = \lim_{m\to\infty} 2^m(u^m_1 - u^m_0), \quad 2\tilde{\alpha}_2 = \lim_{m\to\infty} 2^{2m}(u^m_1 + u^m_{-1} - 2u^m_0),$$

where each u^m_j denotes the first column of $\mathbf{u}^m_j$. On the other hand,

$$\begin{aligned}
2^m(u^m_1 - u^m_0) &= 2^m(v^{m+n}_{2^{m+n}k_0+2^m i+1} - v^{m+n}_{2^{m+n}k_0+2^m i}) \\
&= 2^m\Big(f(k_0 + \frac{i}{2^n} + \frac{1}{2^{m+n}}) - f(k_0 + \frac{i}{2^n})\Big) \\
&\to \frac{1}{2^n}f'(k_0 + \frac{i}{2^n}),\ \text{as } m \to \infty,
\end{aligned}$$

$$\begin{aligned}
2^{2m}(u^m_1 + u^m_{-1} - 2u^m_0) &= 2^{2m}(v^{m+n}_{2^{m+n}k_0+2^m i+1} + v^{m+n}_{2^{m+n}k_0+2^m i-1} - 2v^{m+n}_{2^{m+n}k_0+2^m i}) \\
&= 2^{2m}\Big(f(k_0 + \frac{i}{2^n} + \frac{1}{2^{m+n}}) + f(k_0 + \frac{i}{2^n} - \frac{1}{2^{m+n}}) - 2f(k_0 + \frac{i}{2^n})\Big) \\
&\to \frac{1}{2^{2n}}f''(k_0 + \frac{i}{2^n}),\ \text{as } m \to \infty.
\end{aligned}$$

Therefore, we have the following corollary.

Corollary 1 *Suppose an interpolatory scheme is convergent with limiting curve in C^2. Let f be the limiting curve of an initial control vector "polygon" $\{\mathbf{v}_k^0\}_k$. Let $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{R}^3$ be the column vectors in (3.5). Then*

$$f'(k_0 + \frac{i}{2^n}) = 2^n \tilde{\alpha}_1, \; f''(k_0 + \frac{i}{2^n}) = 2^{2n+1} \tilde{\alpha}_2.$$

To calculate α_1, α_2 in (3.2) and $\tilde{\alpha}_1, \tilde{\alpha}_2$ in (3.5), we use the right eigenvectors of $\mathbf{S}$. For $k = 0, 1, 2$, let $\mathbf{r}_k$ be the right (column) eigenvectors of eigenvalues $1, \frac{1}{2}, \frac{1}{4}$ resp. of $\mathbf{S}$ such that $\mathbf{l}_j \mathbf{r}_k = \delta_{j,k}, 0 \le k \le 2, 0 \le j \le 4N+1$. Denote $\mathbf{r}_k$ as

$$\mathbf{r}_k = [r_{-N}^k, t_{-N}^k, r_{1-N}^k, t_{1-N}^k, ..., r_0^k, t_0^k, ..., r_N^k, t_N^k]^T.$$

Then

$$\begin{aligned} U_0^m \, \mathbf{r}_k &= U_0^0 S^m \, \mathbf{r}_k \\ &= \alpha_0 \mathbf{l}_0 \, \mathbf{r}_k + \alpha_1 \left(\frac{1}{2}\right)^m \mathbf{l}_1 \, \mathbf{r}_k + \alpha_2 \left(\frac{1}{4}\right)^m \mathbf{l}_2 \, \mathbf{r}_k + \sum_{j=3}^{4N+1} c_{j,m} \mathbf{l}_j \, \mathbf{r}_k \\ &= \alpha_k \left(\frac{1}{2^k}\right)^m, \end{aligned}$$

where $c_{j,m}$ are some points in $\mathbb{R}^3$. Thus, $\alpha_1 = 2^m U_0^m \, \mathbf{r}_1$ and $\alpha_2 = 2^{2m} U_0^m \, \mathbf{r}_2$.

Similarly, for nonzero $i \in \mathbb{Z}$, $\tilde{\alpha}_1 = 2^m U_i^m \, \mathbf{r}_1$ and $\tilde{\alpha}_2 = 2^{2m} U_i^m \, \mathbf{r}_2$. In particular, letting $m = 0$, we have

$$\begin{aligned} \alpha_1 &= U_0^0 \, \mathbf{r}_1, \quad \tilde{\alpha}_1 = U_i^0 \, \mathbf{r}_1, \\ \alpha_2 &= U_0^0 \, \mathbf{r}_2, \quad \tilde{\alpha}_2 = U_i^0 \mathbf{r}_2. \end{aligned}$$

Hence, we have the following result.

Corollary 2 *Suppose an interpolatory scheme is convergent with limiting curve in C^2. Let f be the limiting curve of an initial control vector "polygon" $\{\mathbf{v}_k^0\}_k$. Then for $i \in \mathbb{Z}$*

$$f'(k_0 + \frac{i}{2^n}) = 2^n U_i^0 \, \mathbf{r}_1 = 2^n \Big(\sum_{j=-N}^{N} v_{2^n k_0 + i + j}^n \, r_j^1 + \sum_{j=-N}^{N} s_{2^n k_0 + i + j}^n \, t_j^1 \Big), \tag{3.6}$$

$$f''(k_0 + \frac{i}{2^n}) = 2^{2n+1} U_i^0 \, \mathbf{r}_2 = 2^{2n+1} \Big(\sum_{j=-N}^{N} v_{2^n k_0 + i + j}^n \, r_j^2 + \sum_{j=-N}^{N} s_{2^n k_0 + i + j}^n \, t_j^2 \Big). \tag{3.7}$$

Before we move on to the next subsection, we remark that (3.3) **can be derived in the following way.**

Assume Φ is in C^2. Taking the nth derivative of the both sides of (1.2)**, we have**

$$\Phi^{(n)}(x) = 2^n \sum_{\ell} P_\ell \Phi^{(n)}(2x - \ell), \; n = 0, 1, 2.$$

Thus

$$\Phi^{(n)}(-j) = 2^n \sum_{\ell} P_\ell \Phi^{(n)}(-2j - \ell) = 2^n \sum_{\ell} P_{\ell - 2j} \Phi^{(n)}(-\ell) = 2^n \sum_{\ell=-N}^{N} P_{\ell - 2j} \Phi^{(n)}(-\ell),$$

where the last equality follows from the fact that both ϕ_0 and ϕ_1 are supported in $[-N, N]$. Therefore,

$$[\Phi^{(n)}(-j)]_{j=N}^{-N} = 2^n \left[P_{\ell-2j}\right]_{-N\le j\le N, -N\le \ell\le N} [\Phi^{(n)}(-\ell)]_{\ell=N}^{-N}.$$

That is $[\Phi^{(n)}(-j)]_{j=N}^{-N}$ is a right 2^{-n}-eigenvector of the subdivision matrix S defined by (2.8). Since we assume that $1, \frac{1}{2}, \frac{1}{4}$ are simple eigenvalues of S and other eigenvalues λ of S satisfy $|\lambda| < \frac{1}{4}$, we have that for $n = 0, 1, 2,$

$$\mathbf{l}_k[\Phi^{(n)}(-j)]_{j=N}^{-N} = 0, \quad \textbf{for all } k = 0, 1, \cdots, 4N+1 \textbf{ with } k \neq n. \tag{3.8}$$

Furthermore, taking the nth derivative of the both sides of (2.6), we have

$$n! = \sum_j \mathbf{y}_n(j)\Phi^{(n)}(x-j), \ n = 0, 1, 2.$$

Hence, letting $x = 0$, we have

$$n! = \sum_j \mathbf{y}_n(j)\Phi^{(n)}(-j) = \sum_{j=-N}^{N} \mathbf{y}_n(j)\Phi^{(n)}(-j) = \mathbf{l}_n[\Phi^{(n)}(-j)]_{j=N}^{-N}, \ n = 0, 1, 2. \tag{3.9}$$

The limiting curve f in Theorem 1 is given by (refer to [3])

$$f(x) = \sum_k \mathbf{v}_k^0 \Phi(x-k).$$

Thus, with $U_0^0 = [\mathbf{v}_{k+k_0}^0]_{-N\le k\le N}$ expanded as (3.2), we have

$$\begin{aligned} f^{(n)}(k_0) &= \sum_k \mathbf{v}_k^0 \Phi^{(n)}(k_0-k) = \sum_k \mathbf{v}_{k+k_0}^0 \Phi^{(n)}(-k) = \sum_{k=-N}^{N} \mathbf{v}_{k+k_0}^0 \Phi^{(n)}(-k) \\ &= U_0^0[\Phi^{(n)}(-k)]_{j=N}^{-N} = \Big(\alpha_0\mathbf{l}_0 + \alpha_1\mathbf{l}_1 + \alpha_2\mathbf{l}_2 + \sum_{j=3}^{4N+1} \alpha_j \mathbf{l}_j\Big)[\Phi^{(n)}(-k)]_{j=N}^{-N} \\ &= \alpha_n n!, \end{aligned}$$

where the last equality follows from (3.8) and (3.9). This proves (3.3) .

3.2 Approximation schemes

In this subsection, we consider approximation schemes. To derive formulas for $f'(k_0), f''(k_0)$ similar to those in (3.1), we consider the C^1 and C^2 convergence of the cascade algorithm.

For a subdivision mask $P = \{P_k\}$, let T_P denote the cascade algorithm operator defined by

$$(T_P\Phi_0)(x) = \sum_k P_k \Phi_0(2x-k),$$

where Φ_0 is a compactly supported function vector. Clearly, the refinable function vector Φ associated with P is a fixed-point of T_P, and if $\{T_P^m\Phi_0\}_m$ converges to a nonzero limit function, then the limiting function is Φ. In addition, for any control vector "polygon" $\{\mathbf{v}_k^0\}_k$, we have

$$\sum_k \mathbf{v}_k^0 \ (T_P^m\Phi_0)(x-k) = \sum_k \mathbf{v}_k^m \ \Phi_0(2^m x - k), \tag{3.10}$$

where $\mathbf{v}_k^m, k \in \mathbb{Z}$ are exactly the vectors defined by the subdivision algorithm in (1.3). Therefore, if we choose $\Phi_0(x) = [h(x), 0]^T$, where $h(x)$ is the hat function $h(x) = (1-|x|)\chi_{[-1,1]}(x)$, then $\sum_k \mathbf{v}_k^m\ \Phi_0(2^m x - k) = \sum_k v_k^m\ h(2^m x - k)$ is the refined polygon after m step of subdivision iterations with vertices v_k^m. Hence, the convergence of the subdivision scheme is equivalent to the uniform convergence of the cascade algorithm sequence $\{T_P^m \Phi_0\}_m$.

We say a subdivision scheme with mask P converges in C^k if for any compactly supported $\Phi_0 \in C^k$ the sequences $\{T_P^m\Phi_0(x)\}_m$, $\{(T_P^m\Phi_0)'(x)\}_m$, $\cdots$, $\{(T_P^m\Phi_0)^{(k)}(x)\}_m$ converge uniformly (to $\Phi, \Phi', \cdots, \Phi^{(k)}$). Note that such a subdivision scheme will have sum rule order of (at least) $k+1$ and Φ_0 will have accuracy of order (at least) $k+1$ (see [1]). The characterization of cascade algorithm convergence in the Sobolev norm is provided in some papers, see e.g. [1, 14]. The convergence of cascade algorithm in C^k can be characterized by the spectral radius of $\mathbf{S}|_{V_k}$ and $\widetilde{\mathbf{S}}|_{V_k}$, where $\mathbf{S}$ is the subdivision matrix in (2.8), $\widetilde{\mathbf{S}} = \left[P_{k-2j+1}\right]_{j,k}$, and V_k is a subspace of $\mathbb{R}^{4N+2}$ determined by the vectors $\mathbf{y}_0, \cdots, \mathbf{y}_k$ for the sum rule order $k+1$ of P. Since our basic task in this paper is to study tangents and curvatures of the limiting curves, we will not discuss the characterization for C^k convergence here.

Proposition 1 *Suppose a subdivision scheme with mask P converges in C^1. For an initial control vector "polygon" $\mathbf{v}^0 = \{\mathbf{v}_k^0\}_k$, let f be the limiting curve. Then*

$$\lim_{m\to\infty} 2^m (v^m_{2^m k_0+1} - v^m_{2^m k_0}) = f'(k_0). \tag{3.11}$$

If in addition, the scheme is also C^2 convergent, then

$$\lim_{m\to\infty} 2^{2m} (v^m_{2^m k_0+1} + v^m_{2^m k_0-1} - 2v^m_{2^m k_0}) = f''(k_0), \tag{3.12}$$

Proof. Let $\varphi_0(x) = \chi_{[-1,\ 0]} * h(x)$, the convolution of the characteristic function $\chi_{[-1,\ 0]}(x)$ of $[-1,\ 0]$ and the hat function $h(x)$. Then $\varphi_0(x)$ is the C^1 quadratical B-spline function. Thus $\varphi_0(x)$ has accuracy of order 3. Furthermore, one can verify directly

$$\varphi_0'(x) = h(x+1) - h(x). \tag{3.13}$$

Let $\Phi_0 = [\varphi_0, 0]^T$. Then Φ_0 also has accuracy of order 3. Let $f_m(x) = \sum_k \mathbf{v}_k^m \Phi_0(2^m x - k)$. By (3.10), $f_m(x) = \sum_k \mathbf{v}_k^0 (T_P^m \Phi_0)(x-k)$. Thus by the C^1 convergence assumption, $f_m'(x) = \sum_k \mathbf{v}_k^0 (T_P^m\Phi_0)'(x-k) \to \sum_k \mathbf{v}_k^0 \Phi'(x-k) = f'(x)$, as $m \to \infty$. On the other hand, by (3.13),

$$\begin{aligned} f_m'(x) &= \sum_k \mathbf{v}_k^m 2^m \Phi_0'(2^m x - k) = \sum_k 2^m \mathbf{v}_k^m [\varphi_0'(2^m x - k),\ 0]^T \\ &= \sum_k 2^m v_k^m (h(2^m x - k + 1) - h(2^m x - k)) \\ &= \sum_k 2^m (v_{k+1}^m - v_k^m) h(2^m x - k). \end{aligned}$$

Let $x = k_0$. Then $f_m'(k_0) = 2^m(v^m_{2^m k_0+1} - v^m_{2^m k_0})$. Thus

$$2^m(v^m_{2^m k_0+1} - v^m_{2^m k_0}) \to f'(k_0),\ \text{as}\ m \to \infty$$

To obtain (3.12), we choose $\Phi_0 = [\widetilde{\varphi}_0, 0]^T$, where $\widetilde{\varphi}_0(x) = \chi[0,\ 1] * \chi_{[-1,\ 0]} * h(x)$. $\widetilde{\varphi}_0(x)$ is the C^2 cubic B-spline function, and it has accuracy of order 4. Furthermore,

$$\widetilde{\varphi}_0''(x) = h(x+1) + h(x-1) - 2h(x). \tag{3.14}$$

With this initial function vector Φ_0 and the formula (3.14), one can show similarly as above that

$$f''_m(k_0) = 2^{2m}(v^m_{2^m k_0+1} + v^m_{2^m k_0-1} - 2v^m_{2^m k_0}) \to f''(k_0),$$

as $m \to \infty$. ■

Theorem 2 *Suppose that an (approximation) subdivision scheme is convergent in C^2. Let f be the limiting curve of an initial control vector "polygon" $\{\mathbf{v}^0_k\}_k$. Let $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{R}^3$ be the column vectors in (3.5). Then for integer $i \neq 0$*

$$f'(k_0 + \frac{i}{2^n}) = 2^n \tilde{\alpha}_1,\ f''(k_0 + \frac{i}{2^n}) = 2^{2n+1}\tilde{\alpha}_2.$$

Using (3.11) and (3.12), one can give the proof of Theorem 2 as that for Corollary 1.

Again, as before, we use the right eigenvectors $\mathbf{r}_k$ of $\mathbf{S}$ to get $\tilde{\alpha}_1, \tilde{\alpha}_2$.

Corollary 3 *Suppose an (approximation) scheme is convergent in C^2. Let f be the limiting curve of an initial control vector "polygon" $\{\mathbf{v}^0_k\}_k$. Then for $i \in \mathbb{Z}$ $f'(k_0 + \frac{i}{2^n})$ and $f''(k_0 + \frac{i}{2^n})$ are given by the formulas in (3.6) and (3.7) resp.*

From

$$\Big[\mathbf{v}^m_{2^m k_0+j}\Big]_{-N \le j \le N} = \alpha_0 \mathbf{l}_0 + \alpha_1 2^{-m}\mathbf{l}_1 + 2^{-2m}\alpha_2 \mathbf{l}_2 + o(2^{-2m}), \tag{3.15}$$

we know

$$\Big[\mathbf{v}^m_{2^m k_0+j}\Big]_{-N \le j \le N} \to \alpha_0 \mathbf{l}_0 = \alpha_0[\mathbf{y}_0, \cdots, \mathbf{y}_0] \quad \text{as } m \to \infty.$$

Thus

$$\mathbf{v}^m_{2^m k_0+j} = [v^m_{2^m k_0+j}, s^m_{2^m k_0+j}] \to \alpha_0 \mathbf{y}_0.$$

Since $\mathbf{y}_0 = [1, 0]$, we have

$$\lim_{m\to\infty} v^m_{2^m k_0+j} = \alpha_0, \quad \lim_{m\to\infty} s^m_{2^m k_0+j} = 0. \tag{3.16}$$

By definition, $\lim_{m\to\infty} v^m_{2^m k_0} = f(k_0)$. Thus $\alpha_0 = f(k_0)$ and hence, $\lim_{m\to\infty} v^m_{2^m k_0+j} = f(k_0), |j| \le N$. The second limit in (3.16) implies $s^m_k \to 0$ as $m \to \infty$ for $k \in [2^m k_0 - N, 2^m k_0 + N]$. Since the limit of the second components s^m_k of $\mathbf{v}^m_k$ is zero, it is reasonable to the use the first components v^m_k as the vertices of the refined the polygon.

With $\alpha_0 = f(k_0), \alpha_1 = f'(k_0), \alpha_2 = \frac{1}{2}f''(k_0)$, from (3.15), we have

$$\begin{aligned}
&2^m\Big(\big[\mathbf{v}^m_{2^m k_0+j}\big]_{-N\le j\le N} - f(k_0)\mathbf{l}_0\Big) = f'(k_0)\mathbf{l}_1 + o(1),\\
&2^{2m}\Big(\big[\mathbf{v}^m_{2^m k_0+j}\big]_{-N\le j\le N} - f(k_0)\mathbf{l}_0 - 2^{-m}f'(k_0)\mathbf{l}_1\Big) = \tfrac{1}{2}f''(k_0)\mathbf{l}_2 + o(1).
\end{aligned}$$

These and the structures of $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2$ in (2.7) immediately yield the formulas shown in the following corollary.

Corollary 4 *Suppose an (approximation) scheme is convergent in C^2. Let f be the limiting curve of an initial control vector "polygon" $\{\mathbf{v}^0_k\}_k$. Then for $|j| \le N$,*

$$\begin{aligned}
&\lim_{m\to\infty} 2^m\Big(v^m_{2^m k_0+j} - f(k_0)\Big) = (j + c_1)f'(k_0),\\
&\lim_{m\to\infty} 2^{2m}\Big(v^m_{2^m k_0+j} - f(k_0) - 2^{-m}(j+c_1)f'(k_0)\Big) = \frac{1}{2}(d_1 + 2jc_1 + j^2)f''(k_0),
\end{aligned}$$

where c_1 and d_1 are the first components of $\mathbf{y}_1$ and $\mathbf{y}_2$ resp. for sum rule order 3 of the mask P.

4. Tangents and curvatures of 3-point and 4-point matrix-valued subdivision curves

For an initial control vector "polygon" $\mathbf{v}^0 = \{\mathbf{v}_k^0\}_k$, with derivatives $f'(k_0)$ and $f''(k_0)$ at $v_{k_0}^0$ given by (3.6) and (3.7), we have the unit tangent and normal vectors and curvature of f at this point. Here and below we assume that f is not singular at each x, namely, $\|f'(x)\| \neq 0$. Then, the unit tangent vector at $v_{k_0}^0$, denoted as $T(k_0)$, is

$$T(k_0) = \frac{f'(k_0)}{\|f'(k_0)\|}.$$

The formula for the principal normal (vector) N of a curve $\alpha(t)$ in $\mathbb{R}^3$ is

$$N = \frac{(\alpha' \times \alpha'') \times \alpha'}{\|\alpha' \times \alpha''\| \, \|\alpha'\|}$$

and the curvature τ of $\alpha(t)$ is

$$\tau = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$$

where for $u = [u_1, u_2, u_3]^T$, $w = [w_1, w_2, w_3]^T \in \mathbb{R}^3$, the cross product $u \times w$ of u and w is defined as

$$u \times w = [u_2 w_3 - u_3 w_2, u_3 w_1 - u_1 w_3, u_1 w_2 - u_2 w_1]^T.$$

The reader may refer to some textbooks, e.g. [20], for the formulas of N and τ. With f' and f'' of the subdivision curve at $v_{k_0}^0$, we have the normals and curvature of f at this point. In the following we focus on the tangent and curvature of f generated by the 3-point and 4-point matrix-valued C^2 interpolatory schemes and the 3-point matrix-valued C^2 approximating scheme with templates given by (2.3), (2.4) and (2.2).

4.1 3-point C^2 interpolatory scheme

For this scheme, the nonzero coefficients P_k of the mask are $P_0 = W, P_1 = P_{-1} = X, P_2 = P_{-2} = Z$, where W, X, Z are defined by (2.3). Thus the subdivision matrix $\mathbf{S} = \left[P_{k-2j}\right]_{-2\le j,k\le 2}$ is a 10×10 matrix. This subdivision mask has sum rule order at least 3 (it has sum rule order 4 actually) with $\mathbf{y}_0 = [1, 0], \mathbf{y}_1 = [0, 0], \mathbf{y}_2 = [0, 1]$. Thus the left eigenvectors $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2$ defined in (2.7) are

$$\begin{aligned}
\mathbf{l}_0 &= [1, 0, 1, 0, 1, 0, 1, 0, 1, 0], \\
\mathbf{l}_1 &= [-2, 0, -1, 0, 0, 0, 1, 0, 2, 0], \\
\mathbf{l}_2 &= [4, 1, 1, 1, 0, 1, 1, 1, 4, 1].
\end{aligned}$$

$1, \frac{1}{2}, \frac{1}{4}$ are simple eigenvalues of $\mathbf{S}$ and other eigenvalues of $\mathbf{S}$ are smaller (in modulus) than $\frac{1}{4}$. The appropriately normalized right eigenvectors $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ of eigenvalues $1, \frac{1}{2}, \frac{1}{4}$ resp. of $\mathbf{S}$ are:

$$\begin{aligned}
\mathbf{r}_0 &= [0, 0, 0, 0, 1, 0, 0, 0, 0, 0]^T, \\
\mathbf{r}_1 &= [0, 0, -\frac{1}{2}, \frac{1}{6}, 0, 0, \frac{1}{2}, -\frac{1}{6}, 0, 0]^T, \\
\mathbf{r}_2 &= [0, 0, \frac{7}{4}, -\frac{7}{12}, -\frac{7}{2}, -\frac{4}{3}, \frac{7}{4}, -\frac{7}{12}, 0, 0]^T.
\end{aligned}$$

These $\mathbf{r}_k$ satisfy $\mathbf{l}_j\ \mathbf{r}_k = \delta_{j,k}, 0 \le k, j \le 2$. Thus the first derivative f' at point $v^n_{2^n k_0+i}$ after n subdivision iterations, denoted as $f'(k_0 + \frac{i}{2^n})$, is

$$f'(k_0 + \frac{i}{2^n}) = 2^n\big(-\frac{1}{2}v^n_{2^n k_0+i-1} + \frac{1}{6}s^n_{2^n k_0+i-1} + \frac{1}{2}v^n_{2^n k_0+i+1} - \frac{1}{6}s^n_{2^n k_0+i+1}\big).$$

In particular, we have

$$f'(k_0) = -\frac{1}{2}v^0_{k_0-1} + \frac{1}{6}s^0_{k_0-1} + \frac{1}{2}v^0_{k_0+1} - \frac{1}{6}s^0_{k_0+1}. \tag{4.1}$$

The second derivative f'' at point $v^n_{2^n k_0+i}$, denoted as $f''(k_0 + \frac{i}{2^n})$, is

$$f''(k_0 + \frac{i}{2^n}) =$$
$$2^{2n+1}\big(\frac{7}{4}v^n_{2^n k_0+i-1} - \frac{7}{12}s^n_{2^n k_0+i-1} - \frac{7}{2}v^n_{2^n k_0+i} - \frac{4}{3}s^n_{2^n k_0+i} + \frac{7}{4}v^n_{2^n k_0+i+1} - \frac{7}{12}s^n_{2^n k_0+i+1}\big).$$

With $i, n = 0$, we have

$$f''(k_0) = \frac{7}{2}v^0_{k_0-1} - \frac{7}{6}s^0_{k_0-1} - 7v^0_{k_0} - \frac{8}{3}s^0_{k_0} + \frac{7}{2}v^0_{k_0+1} - \frac{7}{6}s^0_{k_0+1}. \tag{4.2}$$

4.2 4-point C^2 interpolatory scheme

The nonzero coefficients P_k of the mask for this scheme are $P_0 = W, P_1 = P_{-1} = X, P_2 = P_{-2} = Z, P_3 = P_{-3} = Y$, where W, X, Z, Y are defined by (2.4). Thus the subdivision matrix $\mathbf{S} = [P_{k-2j}]_{-3\le j\le 3, -3\le k\le 3}$ is a 14×14 matrix. This subdivision mask has sum rule order at least 3 (it has sum rule order 4 actually) with $\mathbf{y}_0 = [1, 0], \mathbf{y}_1 = [0, 0], \mathbf{y}_2 = [0, -\frac{1}{3}]$. Thus the left eigenvectors $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2$ defined in (2.7) are

$$\begin{aligned}
\mathbf{l}_0 &= [1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0],\\
\mathbf{l}_1 &= [-3, 0, -2, 0, -1, 0, 0, 0, 1, 0, 2, 0, 3, 0],\\
\mathbf{l}_2 &= [9, -\frac{1}{3}, 4, -\frac{1}{3}, 1, -\frac{1}{3}, 0, -\frac{1}{3}, 1, -\frac{1}{3}, 4, -\frac{1}{3}, 9, -\frac{1}{3}].
\end{aligned}$$

$1, \frac{1}{2}, \frac{1}{4}$ are simple eigenvalues of $\mathbf{S}$ and other eigenvalues of $\mathbf{S}$ are smaller (in modulus) than $\frac{1}{4}$. The appropriately normalized right eigenvectors $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ of eigenvalues $1, \frac{1}{2}, \frac{1}{4}$ resp. of $\mathbf{S}$ are:

$$\begin{aligned}
\mathbf{r}_0 &= [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]^T,\\
\mathbf{r}_1 &= [0, 0, 0, 0, -\frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0]^T,\\
\mathbf{r}_2 &= [0, 0, 0, 0, \frac{3}{2}, \frac{3}{2}, -3, 3, \frac{3}{2}, \frac{3}{2}, 0, 0, 0, 0]^T.
\end{aligned}$$

These $\mathbf{r}_k$ satisfy $\mathbf{l}_j\ \mathbf{r}_k = \delta_{j,k}, 0 \le k, j \le 2$. Thus the first derivative f' at point $v^n_{2^n k_0+i}$ after n subdivision iterations, denoted as $f'(k_0 + \frac{i}{2^n})$, is

$$f'(k_0 + \frac{i}{2^n}) = 2^n\big(-\frac{1}{2}v^n_{2^n k_0+i-1} - \frac{1}{2}s^n_{2^n k_0+i-1} + \frac{1}{2}v^n_{2^n k_0+i+1} + \frac{1}{2}s^n_{2^n k_0+i+1}\big).$$

In particular, we have

$$f'(k_0) = -\frac{1}{2}v^0_{k_0-1} - \frac{1}{2}s^0_{k_0-1} + \frac{1}{2}v^0_{k_0+1} + \frac{1}{2}s^0_{k_0+1}. \tag{4.3}$$

The second derivative f'' at point $v^n_{2^nk_0+i}$, denoted as $f''(k_0+\frac{i}{2^n})$, is

$$f''(k_0+\frac{i}{2^n}) = \\ 2^{2n+1}(\frac{3}{2}v^n_{2^nk_0+i-1} + \frac{3}{2}s^n_{2^nk_0+i-1} - 3v^n_{2^nk_0+i} + 3s^n_{2^nk_0+i} + \frac{3}{2}v^n_{2^nk_0+i+1} + \frac{3}{2}s^n_{2^nk_0+i+1}).$$

With $i, n = 0$, we have

$$f''(k_0) = 3v^0_{k_0-1} + 3s^0_{k_0-1} - 6v^0_{k_0} + 6s^0_{k_0} + 3v^0_{k_0+1} + 3s^0_{k_0+1}. \tag{4.4}$$

With f' and f'' given in (4.1) and (4.2) or in (4.3) and (4.4), we then have the normals and curvature of f at a particular point. More precisely, using the 3-point C^2 interpolatory scheme, we obtain (assuming a planar curve):

Normal at $k_0 =$

$$\Big[\frac{1}{2}v^0_{k_0-1,2} - \frac{1}{6}s^0_{k_0-1,2} - \frac{1}{2}v^0_{k_0+1,2} + \frac{1}{6}s^0_{k_0+1,2}, -\frac{1}{2}v^0_{k_0-1,1} + \frac{1}{6}s^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1} - \frac{1}{6}s^0_{k_0+1,1}\Big]^T,$$

Curvature at $k_0 =$

$$\Big|H_3\Big|\Big/\Big[\big(-\frac{1}{2}v^0_{k_0-1,1} + \frac{1}{6}s^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1} - \frac{1}{6}s^0_{k_0+1,1}\big)^2 + \big(-\frac{1}{2}v^0_{k_0-1,2} + \frac{1}{6}s^0_{k_0-1,2} + \frac{1}{2}v^0_{k_0+1,2} - \frac{1}{6}s^0_{k_0+1,2}\big)^2\Big]^{3/2}. \tag{4.5}$$

where

$$H_3 = \Big(\frac{7}{2}v^0_{k_0-1,1} - \frac{7}{6}s^0_{k_0-1,1} - 7v^0_{k_0,1} - \frac{8}{3}s^0_{k_0,1} + \frac{7}{2}v^0_{k_0+1,1} - \frac{7}{6}s^0_{k_0+1,1}\Big)\Big(-\frac{1}{2}v^0_{k_0-1,2} + \frac{1}{6}s^0_{k_0-1,2} + \frac{1}{2}v^0_{k_0+1,2} - \frac{1}{6}s^0_{k_0+1,2}\Big) \\ -\Big(\frac{7}{2}v^0_{k_0-1,2} - \frac{7}{6}s^0_{k_0-1,2} - 7v^0_{k_0,2} - \frac{8}{3}s^0_{k_0,2} + \frac{7}{2}v^0_{k_0+1,2} - \frac{7}{6}s^0_{k_0+1,2}\Big)\Big(-\frac{1}{2}v^0_{k_0-1,1} + \frac{1}{6}s^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1} - \frac{1}{6}s^0_{k_0+1,1}\Big). \tag{4.6}$$

Using the 4-point C^2 interpolatory scheme, we obtain (assuming a planar curve):

Normal at $k_0 =$

$$\Big[\frac{1}{2}v^0_{k_0-1,2} + \frac{1}{2}s^0_{k_0-1,2} - \frac{1}{2}v^0_{k_0+1,2} - \frac{1}{2}s^0_{k_0+1,2}, -\frac{1}{2}v^0_{k_0-1,1} - \frac{1}{2}s^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1} + \frac{1}{2}s^0_{k_0+1,1}\Big]^T,$$

Curvature at $k_0 =$

$$\Big|H_4\Big|\Big/ \Big[\big(-\frac{1}{2}v^0_{k_0-1,1} - \frac{1}{2}s^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1} + \frac{1}{2}s^0_{k_0+1,1}\big)^2 + \big(-\frac{1}{2}v^0_{k_0-1,2} - \frac{1}{2}s^0_{k_0-1,2} + \frac{1}{2}v^0_{k_0+1,2} + \frac{1}{2}s^0_{k_0+1,2}\big)^2\Big]^{3/2}.$$

where

$$H_4 = \Big(3v^0_{k_0-1,1} + 3s^0_{k_0-1,1} - 6v^0_{k_0,1} + 6s^0_{k_0,1} + 3v^0_{k_0+1,1} + 3s^0_{k_0+1,1}\Big)\Big(-\frac{1}{2}v^0_{k_0-1,2} - \frac{1}{2}s^0_{k_0-1,2} + \frac{1}{2}v^0_{k_0+1,2} + \frac{1}{2}s^0_{k_0+1,2}\Big) \\ -\Big(3v^0_{k_0-1,2} + 3s^0_{k_0-1,2} - 6v^0_{k_0,2} + 6s^0_{k_0,2} + 3v^0_{k_0+1,2} + 3s^0_{k_0+1,2}\Big)\Big(-\frac{1}{2}v^0_{k_0-1,1} - \frac{1}{2}s^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1} + \frac{1}{2}s^0_{k_0+1,1}\Big).$$

4.3 3-point C^2 approximating scheme

The nonzero coefficients P_k of the mask for a C^2 approximating scheme in [4] are $P_0 = W, P_1 = P_{-1} = X, P_2 = P_{-2} = Z$, where W, X, Z are defined by (2.2). Thus the subdivision matrix $\mathbf{S} = \left[P_{k-2j}\right]_{-2\le j\le 2,-2\le k\le 2}$ is a 10×10 matrix. This subdivision mask has sum rule order at least 3 (it has sum rule order 4 actually) with $\mathbf{y}_0 = [1,0], \mathbf{y}_1 = [0,0], \mathbf{y}_2 = [-\frac{2}{15}, \frac{1}{5}]$. Thus the left eigenvectors $\mathbf{l}_0, \mathbf{l}_1, \mathbf{l}_2$ defined in (2.7) are

$$
\begin{aligned}
\mathbf{l}_0 &= [1,0,1,0,1,0,1,0,1,0],\\
\mathbf{l}_1 &= [-2,0,-1,0,0,0,1,0,2,0],\\
\mathbf{l}_2 &= \frac{1}{15}[58,3,13,3,-2,3,13,3,58,3].
\end{aligned}
$$

$1, \frac{1}{2}, \frac{1}{4}$ are simple eigenvalues of $\mathbf{S}$ and other eigenvalues of $\mathbf{S}$ are smaller (in modulus) than $\frac{1}{4}$. The appropriately normalized right eigenvectors $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2$ of eigenvalues $1, \frac{1}{2}, \frac{1}{4}$ resp. of $\mathbf{S}$ are:

$$
\begin{aligned}
\mathbf{r}_0 &= \frac{1}{12}[0,0,1,-1,10,0,1,-1,0,0]^T,\\
\mathbf{r}_1 &= [0,0,-\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},-\frac{1}{2},0,0]^T,\\
\mathbf{r}_2 &= [0,0,1,-1,-2,-3,1,-1,0,0]^T.
\end{aligned}
$$

These $\mathbf{r}_k$ satisfy $\mathbf{l}_j\ \mathbf{r}_k = \delta_{j,k}, 0\le k,j\le 2$. Thus using Theorem 2 we get

$$
f'(k_0+\frac{i}{2^n}) = 2^n\big(-\frac{1}{2}v^n_{2^nk_0+i-1}+\frac{1}{2}s^n_{2^nk_0+i-1}+\frac{1}{2}v^n_{2^nk_0+i+1}-\frac{1}{2}s^n_{2^nk_0+i+1}\big).
$$

In particular, we have

$$
f'(k_0) = -\frac{1}{2}v^0_{k_0-1}+\frac{1}{2}s^0_{k_0-1}+\frac{1}{2}v^0_{k_0+1}-\frac{1}{2}s^0_{k_0+1}.
$$

Using Theorem 2 for f'' we obtain

$$
f''(k_0+\frac{i}{2^n}) = 2^{2n+1}\big(v^n_{2^nk_0+i-1}-s^n_{2^nk_0+i-1}-2v^n_{2^nk_0+i}-3s^n_{2^nk_0+i}+v^n_{2^nk_0+i+1}-s^n_{2^nk_0+i+1}\big).
$$

With $i, n = 0$, we have

$$
f''(k_0) = v^0_{k_0-1}-s^0_{k_0-1}-2v^0_{k_0}-3s^0_{k_0}+v^0_{k_0+1}-s^0_{k_0+1}.
$$

As with the interpolating case, we can obtain the normals and curvature of f at a particular point (assuming a planar curve):

Normal at $k_0 =$

$$
\Big[\frac{1}{2}v^0_{k_0-1,2}-\frac{1}{2}s^0_{k_0-1,2}-\frac{1}{2}v^0_{k_0+1,2}+\frac{1}{2}s^0_{k_0+1,2}, -\frac{1}{2}v^0_{k_0-1,1}+\frac{1}{2}s^0_{k_0-1,1}+\frac{1}{2}v^0_{k_0+1,1}-\frac{1}{2}s^0_{k_0+1,1}\Big]^T,
$$

Curvature at $k_0 =$

$$
\big|H_5\big|\Big/\Big[\Big(-\frac{1}{2}v^0_{k_0-1,1}+\frac{1}{2}s^0_{k_0-1,1}+\frac{1}{2}v^0_{k_0+1,1}-\frac{1}{2}s^0_{k_0+1,1}\Big)^2+\Big(-\frac{1}{2}v^0_{k_0-1,2}+\frac{1}{2}s^0_{k_0-1,2}+\frac{1}{2}v^0_{k_0+1,2}-\frac{1}{2}s^0_{k_0+1,2}\Big)^2\Big]^{3/2}. \tag{4.7}
$$

where

$$H_5 = \Big(v^0_{k_0-1,1} - s^0_{k_0-1,1} - 2v^0_{k_0,1} - 3s^0_{k_0,1} + v^0_{k_0+1,1} - s^0_{k_0+1,1}\Big)\Big(-\frac{1}{2}v^0_{k_0-1,2} + \frac{1}{2}s^0_{k_0-1,2} + \frac{1}{2}v^0_{k_0+1,2} - \frac{1}{2}s^0_{k_0+1,2}\Big)$$
$$- \Big(v^0_{k_0-1,2} - s^0_{k_0-1,2} - 2v^0_{k_0,2} - 3s^0_{k_0,2} + v^0_{k_0+1,2} - s^0_{k_0+1,2}\Big)\Big(-\frac{1}{2}v^0_{k_0-1,1} + \frac{1}{2}s^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1} - \frac{1}{2}s^0_{k_0+1,1}\Big). \tag{4.8}$$

5. Applications to curve design

In this section we consider the selection of the shape control parameters s^0_k for curve design. In § 5.1, for a given initial control polygon with vertices v^0_k, we consider a selection of the shape parameters s^0_k such that the normals of the resulting limiting curve at all vertices v^0_k are equal or close to the well-defined discrete normals of the initial control polygon. In § 5.2 and § 5.3, we consider the cases where we are wanting the limiting curve f (generated from the initial control vector "polygon" $\{[v^0_k, s^0_k]\}_k$) to have a certain unit normal and/or curvature at a vertex $v^0_{k_0}$ of the initial points. In the following, we focus on planar curves and we use both a 3-point C^2 interpolatory scheme and a 3-point C^2 approximating scheme for curve design.

5.1 General selection of s^0_k

For planar curve design we will select s^0_k so that there is only one free variable, ω_k. Let $\mathbf{v}^0_k = [v^0_k, s^0_k]$, with $v^0_k = [v^0_{k,1}, v^0_{k,2}]^T, s^0_k = [s^0_{k,1}, s^0_{k,2}]^T \in \mathbb{R}^2$. For a polygon $\{v^0_k\}_k$ define, as in [25], the unit tangent at v^0_k, denoted as T_k, to be

$$T_k = \frac{T^-_k + T^+_k}{\|T^-_k + T^+_k\|},$$

where

$$T^-_k = \frac{v^0_k - v^0_{k-1}}{\|v^0_k - v^0_{k-1}\|}, \quad T^+_k = \frac{v^0_{k+1} - v^0_k}{\|v^0_{k+1} - v^0_k\|}.$$

Assuming that v^0_{k-1}, v^0_k and v^0_{k+1} are not collinear, we define the normal (vector) at v^0_k, denoted as n_k, to be a vector on the plane generated by T^-_k and T^+_k and orthogonal to T_k. Therefore, we may define

$$n_k = \frac{T^-_k - T^+_k}{\|T^-_k - T^+_k\|}. \tag{5.1}$$

If v^0_{k-1}, v^0_k and v^0_{k+1} are collinear, we define

$$n_k = \frac{[v^0_{k,2} - v^0_{k+1,2}, v^0_{k+1,1} - v^0_{k,1}]^T}{\|v^0_{k+1} - v^0_k\|}. \tag{5.2}$$

As in [25], we say that a line segment $\overline{v^0_{k-1}v^0_k}$ is convex if $l_k r_k > 0$, where l_k and r_k are the projections of $\overline{v^0_{k-1}v^0_k}$ onto n_{k-1} and n_k resp.:

$$l_k = (v^0_{k-1} - v^0_k)^T n_{k-1}, \quad r_k = (v^0_k - v^0_{k-1})^T n_k.$$

We say $\overline{v_{k-1}^0 v_k^0}$ is an inflection line segment if $l_k r_k < 0$, and we say $\overline{v_{k-1}^0 v_k^0}$ is a straight line segment if $l_k r_k = 0$. Next, we define s_k as follows, depending on the convexity of $\overline{v_{k-1}^0 v_k^0}$ and $\overline{v_k^0 v_{k+1}^0}$.

(i) If one or both of $\overline{v_{k-1}^0 v_k^0}$ and $\overline{v_k^0 v_{k+1}^0}$ is a straight line segment, or if both of them are inflective, then set $s_k^0 = \mathbf{0}$;

(ii) If one of $\overline{v_{k-1}^0 v_k^0}$ and $\overline{v_k^0 v_{k+1}^0}$ is convex and the other is inflective, then if $\overline{v_{k-1}^0 v_k^0}$ is convex, set $s_k^0 = \omega_k \Big((v_k^0 - v_{k-1}^0)^T n_k \Big) n_k$ and if $\overline{v_k^0 v_{k+1}^0}$ is convex, set $s_k^0 = \omega_k \Big((v_k^0 - v_{k+1}^0)^T n_k \Big) n_k$;

(iii) If both $\overline{v_{k-1}^0 v_k^0}$ and $\overline{v_k^0 v_{k+1}^0}$ are convex, then if $\|v_k^0 - v_{k-1}^0\| \le \|v_{k+1}^0 - v_k^0\|$, set $s_k^0 = \omega_k \Big((v_k^0 - v_{k-1}^0)^T n_k \Big) n_k$ and if $\|v_k^0 - v_{k-1}^0\| > \|v_{k+1}^0 - v_k^0\|$, set $s_k^0 = \omega_k \Big((v_k^0 - v_{k+1}^0)^T n_k \Big) n_k$.

In (ii) and (iii), ω_k is a scalar. Its range of values that produce curves free of waviness varies according to the mask of the scheme.

Testing with values for ω_k ranging from -3 to 1, the limiting curve appears satisfactory for values of ω_k between -0.5 and 0.3 for the scheme in (2.2). Similarly the limiting curve appears nice for values of ω_k between -1.5 and -0.1 for the scheme in (2.3). See Fig. 2 and Fig. 3, where "unit normal" in Fig. 3 is the discrete normal to the initial polygon defined by (5.1). We summarize the matrix-valued curve subdivision with the above described general selection of s_k^0 as Algorithm 1 below.

Algorithm 1

Step 1. For each point v_k^0 on the initial control polygon, define n_k as given in (5.1). If the points are collinear, define n_k using (5.2).

Step 2. Letting $\omega_k = \alpha$ (where one may choose α to be number between -0.5 to -0.1), define s_k^0 using the procedure given in (i) to (iii) immediately above.

Step 3. Apply the subdivision scheme (1.3) to these initial $[v_k^0, s_k^0]$ and determine visually if the limiting curve is suitable for one's needs. If it isn't suitable, then return to **Step 2** and adjust α.

5.2 Selection of s_k^0 for specific unit normal

For a polygon $\{v_k^0\}_k$, if a normal unit vector $n_{k_0} = [n_{k_0,1}, n_{k_0,2}]^T$ is desired for the limiting curve at the point $v_{k_0}^0$ we will consider the necessary relationship between ω_{k_0-1} and ω_{k_0+1}. We will assume that for the adjoining points ($v_{k_0-1}^0$ and $v_{k_0+1}^0$) both the discrete unit normals (n_{k_0-1}, n_{k_0+1}) and the shape parameters (s_{k_0-1}, s_{k_0+1}) will be defined as in §5.1. We will determine the corresponding ω_{k_0-1} and ω_{k_0+1}. We will consider the 3-point interpolatory scheme from §4.1 and the 3-point approximating scheme from §4.3.

In order that $n_{k_0} \cdot f'(k_0) = 0$, from (4.1), we have

$$n_{k_0,1}\Big(\frac{1}{2}\left(v_{k_0+1,1}^0 - v_{k_0-1,1}^0\right) + \frac{1}{6}\left(s_{k_0-1,1}^0 - s_{k_0+1,1}^0\right)\Big)$$
$$+n_{k_0,2}\Big(\frac{1}{2}\left(v_{k_0+1,2}^0 - v_{k_0-1,2}^0\right) + \frac{1}{6}\left(s_{k_0-1,2}^0 - s_{k_0+1,2}^0\right)\Big) = 0.$$

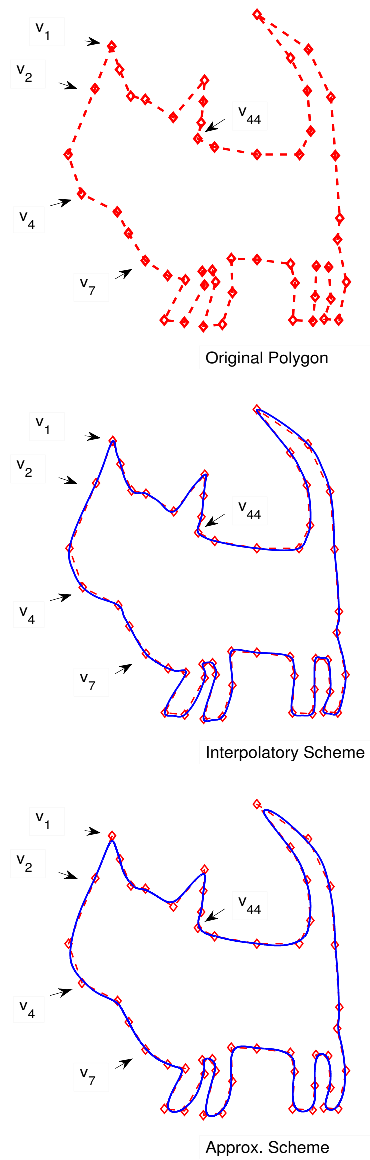

Figure 2: *Top: Initial polygon; Middle: Subdivision curve with 3-point interpolatory scheme where* $\omega_k = -1$*;   Bottom: Subdivision curve with 3-point approximating scheme where* $\omega_k = -0.4$

So

$$\sum_{j=1}^{2} n_{k_0,j} s^0_{k_0-1,j} = \sum_{j=1}^{2} n_{k_0,j} s^0_{k_0+1,j} + 3 \sum_{j=1}^{2} n_{k_0,j} \left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)$$

In a similar fashion for the approximating scheme

$$\sum_{j=1}^{2} n_{k_0,j} s^0_{k_0-1,j} = \sum_{j=1}^{2} n_{k_0,j} s^0_{k_0+1,j} + \sum_{j=1}^{2} n_{k_0,j} \left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)$$

Notice that s_{k_0} does not appear. So we will define s_{k_0} and its corresponding ω_{k_0} as in §5.1.

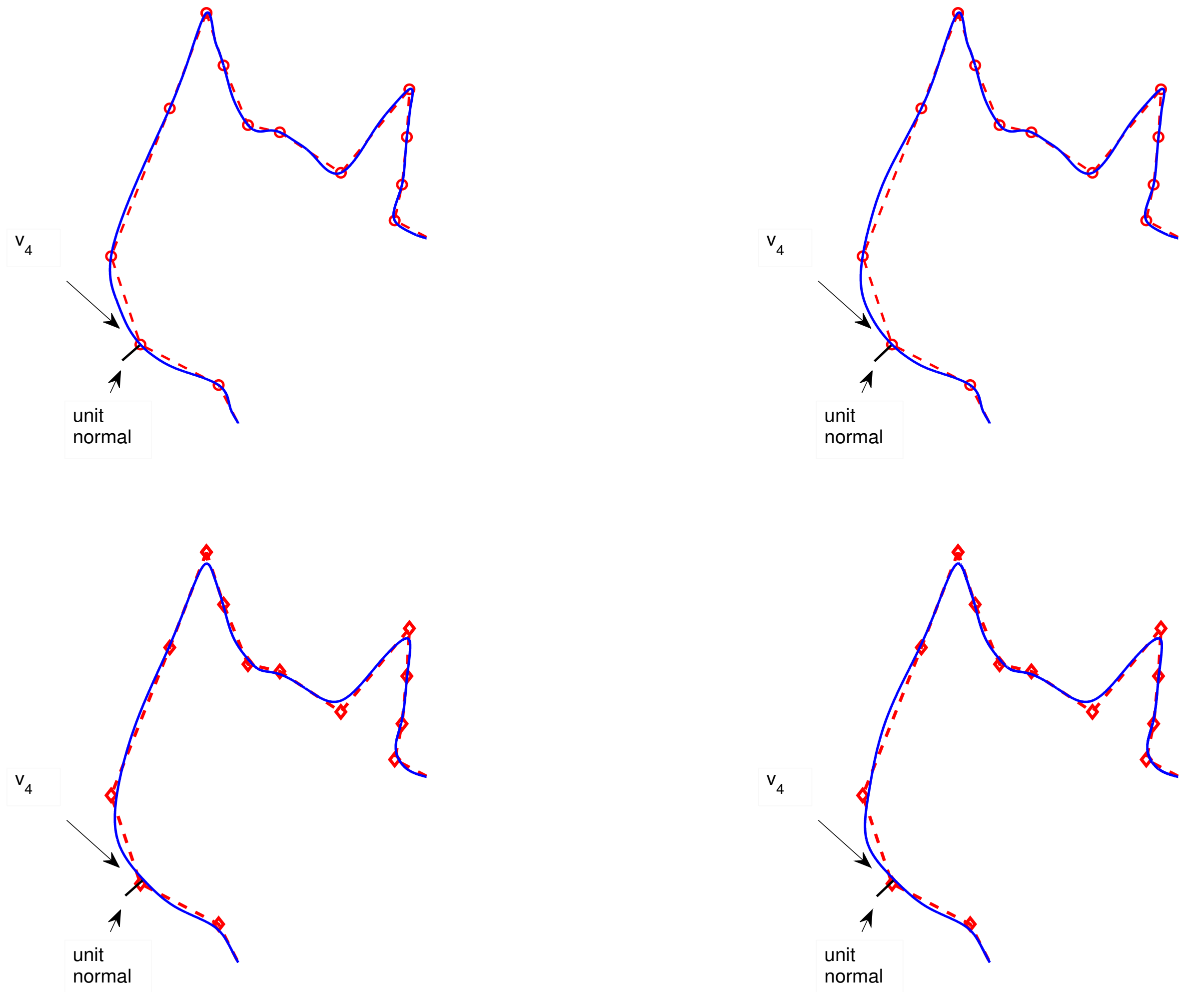

Figure 3: *Upper Left: 3-point interpolating scheme showing segment of subdivision curve where* $\omega_k = -1$*; Upper Right: Same interpolating scheme with a* ***specific desired*** *normal at vertex* v_4*:* $\mathbf{n}_4 = [-.7151, -.6990]^T$ *which varies* $4.35°$ *from the polygon normal Lower Left: 3-point approximating showing same segment with* $\omega_k = -0.4$*; Lower Right: Same approx. scheme with* ***same specific desired*** *normal at vertex* v_4*:* $\mathbf{n}_4 = [-.7151, -.6990]^T$*.*

(However, contrast this with the treatment of ω_{k_0} in §5.3.) Hence assuming that conditions (ii) or (iii) from §5.1 apply for the determination of s_{k_0+1} and s_{k_0-1} we obtain for the interpolatory scheme

$$\omega_{k_0-1} = \frac{\omega_{k_0+1}(n_{k_0,1}\alpha + n_{k_0,2}\beta) + 3\sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\gamma + n_{k_0,2}\tau}, \tag{5.3}$$

or alternatively

$$\omega_{k_0+1} = \frac{\omega_{k_0-1}(n_{k_0,1}\gamma + n_{k_0,2}\tau) - 3\sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\alpha + n_{k_0,2}\beta}, \tag{5.4}$$

where, depending on whether condition (ii) or condition (iii) in § 5.1 applies, $\alpha,\ \beta$ are the respective components of either

$$\left((v^0_{k_0+1} - v^0_{k_0})^T n_{k_0+1}\right) n_{k_0+1} \ \text{ or } \ \left((v^0_{k_0+1} - v^0_{k_0+2})^T n_{k_0+1}\right) n_{k_0+1}$$

and $\gamma,\ \tau$ are the respective components of either

$$\left((v^0_{k_0-1} - v^0_{k_0-2})^T n_{k_0-1}\right) n_{k_0-1} \ \text{ or } \ \left((v^0_{k_0-1} - v^0_{k_0})^T n_{k_0-1}\right) n_{k_0-1}.$$

Similarly for the approximating case

$$\omega_{k_0-1} = \frac{\omega_{k_0+1}(n_{k_0,1}\alpha + n_{k_0,2}\beta) + \sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\gamma + n_{k_0,2}\tau}, \tag{5.5}$$

or alternatively

$$\omega_{k_0+1} = \frac{\omega_{k_0-1}(n_{k_0,1}\gamma + n_{k_0,2}\tau) - \sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\alpha + n_{k_0,2}\beta}, \tag{5.6}$$

Note that in (5.3) and (5.5) we assume that the desired n_{k_0} is not orthogonal to n_{k_0-1} and similarly in (5.4) and (5.6) we assume that n_{k_0} is not orthogonal to n_{k_0+1}. Also note that using (5.3) ((5.5) resp. for approx. scheme) ω_{k_0-1} and ω_{k_0+1} lay on a straight line with slope $a_0 = \frac{n_{k_0,1}\alpha + n_{k_0,2}\beta}{n_{k_0,1}\gamma + n_{k_0,2}\tau}$ and y-intercept

$$c_0 = \frac{3\sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\gamma + n_{k_0,2}\tau}$$

$$\left(c_0 = \frac{\sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\gamma + n_{k_0,2}\tau} \text{ resp. for approx. scheme}\right).$$

We propose the following method to select the values for ω_{k_0-1} and ω_{k_0+1}. Choose the point on the straight line such that its distance from the origin is minimized. Notice that this point will be the origin only if the y-intercept equals 0 (i.e. if n_{k_0} is orthogonal to $v_{k_0-1} - v_{k_0+1}$). We obtain (different c_0 for approximating case as given above)

$$\omega_{k_0-1} = \frac{c_0}{a_0^2+1}, \quad \omega_{k_0+1} = \frac{-a_0 c_0}{a_0^2+1},$$

Using the initial polygon on the left side in Fig. 2 and looking at v_7, we consider eight different desired unit normals and examine the resulting curves and ω values. Our experiments with both the 3-point interpolatory scheme and the 3-point approximating scheme show that when the specifically chosen unit normals are close to the discrete normal to the initial polygon at v_7, the resulting curves look nice. The detailed experimental results can be found in a long version of this paper which can be downloaded from the website of one of the authors. We also observed that as it would be expected the subdivision curves by the approximating scheme in general look better than those by the interpolatory scheme. See Fig. 4 for some of the limiting curves.

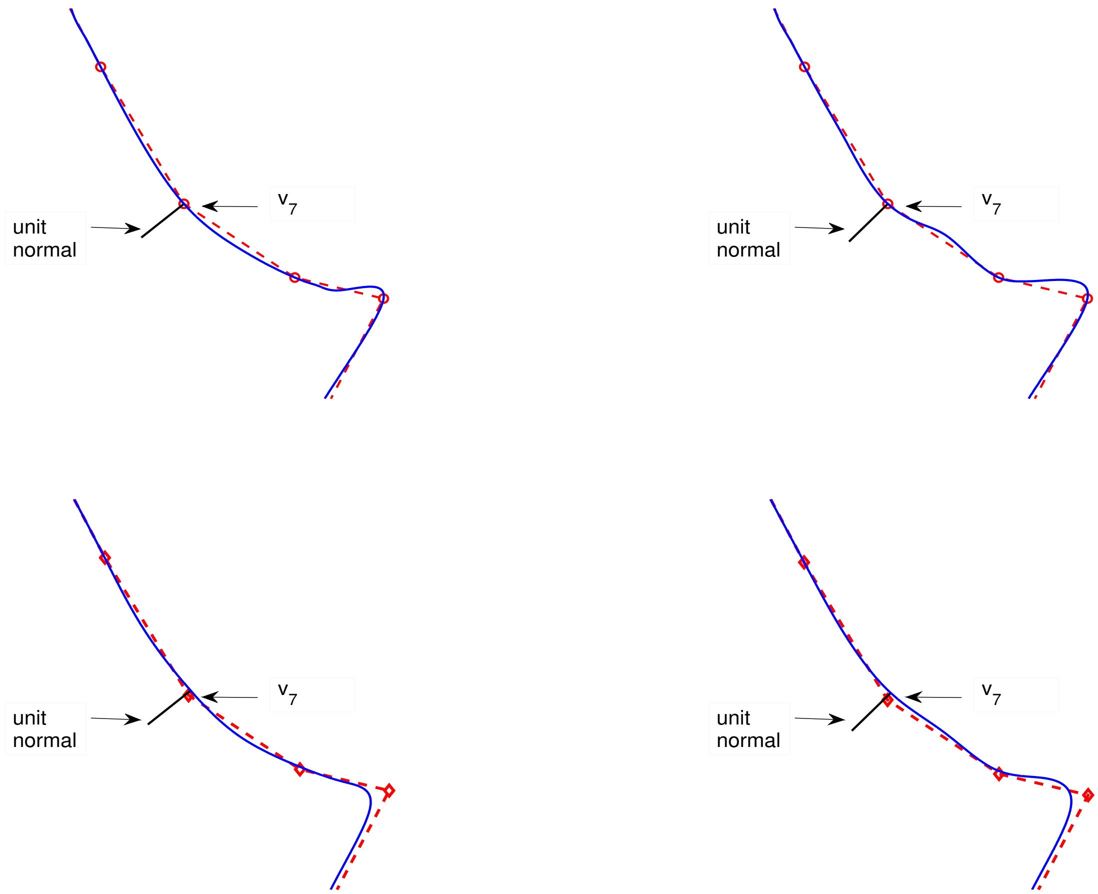

Figure 4: *Top Left: Interpolating scheme using* $n_7 = [-.7677, -.6408]^T$*; Top Right: Interpolating scheme using same segment with* $n_7 = [-.6977, -.7164]$*; Bottom Left: Approximating scheme using* $n_7 = [-.7677, -.6408]^T$*; Bottom Right: Approximating scheme with* $n_7 = [-.6977, -.7164]$*; Blue (solid) line is limit curve. Unit normal in the pictures is* $[-.7373, -.6755]^T$*, which is the calculated discrete normal to the initial polygon at* v_7*.*

If condition (i) of § 5.1 applies to s_{k_0+1} (s_{k_0-1}) but not to s_{k_0-1} (s_{k_0+1}) then ω_{k_0-1} (ω_{k_0+1}) is predetermined. For instance, assume that condition (i) applies to s_{k_0+1} but not to s_{k_0-1}. Then

$s_{k_0+1} = 0$ and we have (approximating case in parentheses)

$$\omega_{k_0-1} = \frac{3\sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\gamma + n_{k_0,2}\tau} \quad \left(\omega_{k_0-1} = \frac{\sum_{j=1}^{2} n_{k_0,j}\left(v^0_{k_0-1,j} - v^0_{k_0+1,j}\right)}{n_{k_0,1}\gamma + n_{k_0,2}\tau}\right).$$

5.3 Selection of s^0_k for specific curvature

We also want to consider how to select ω values so that we can obtain a desired curvature at a particular vertex of our curve. Since we start with a control (planar) polygon, it would be beneficial to have a method to define curvature at a vertex of a planar polygon. In Differential Geometry there is a unique circle that most closely resembles the curve at any point P (the osculating circle with radius R) and the curvature equals $\frac{1}{R}$. So we will define the estimated curvature at vertex v_{k_0} as the reciprocal of the radius of the unique circle that v_{k_0} and its two adjacent vertices (v_{k_0-1} and v_{k_0+1}) lay on:

$$\text{Estimated curvature at } v_{k_0} = \frac{1}{R} \tag{5.7}$$

where R is radius of circle on which $v_{k_0-1}, v_{k_0}, v_{k_0+1}$ lay.

So for a polygon $\left\{v^0_k\right\}_k$, if we want not only a certain normal unit vector, $n_{k_0} = [n_{k_0,1}, n_{k_0,2}]^T$ but also a certain curvature κ_{k_0} for the limiting curve at the point $v^0_{k_0}$ we will consider how to calculate a suitable ω_k. Our basic procedure for calculating the shape parameters will only vary from § 5.1 in the way corresponding ω_k values are decided upon.

We will (initially) assume that conditions (ii) or (iii) as stated in § 5.1 apply for the calculation of the shape parameters s_{k_0-1}, s_{k_0} and s_{k_0+1}. Since we are wanting a specific normal to the limit curve at v_{k_0} we will determine ω_{k_0-1} and ω_{k_0+1} as in § 5.2.

Using the formula for curvature (4.5) and assuming that H_3 in (4.6) is nonnegative we obtain (for the interpolatory scheme)

$$\begin{aligned}\omega_+ = {} & \frac{\kappa_{k_0}\left(\left(d+\frac{1}{6}r\omega_{k_0-1}-\frac{1}{6}t\omega_{k_0+1}\right)^2+\left(b+\frac{1}{6}u\omega_{k_0-1}-\frac{1}{6}m\omega_{k_0+1}\right)^2\right)^{3/2}}{\left(\frac{4}{9}qm-\frac{4}{9}pt\right)\omega_{k_0+1}+\left(-\frac{4}{9}uq+\frac{4}{9}pr\right)\omega_{k_0-1}-\frac{8}{3}qb+\frac{8}{3}pd} \\ & +\frac{-ab+cd-\left(\frac{1}{6}ua-\frac{7}{6}rb-\frac{1}{6}rc+\frac{7}{6}ud\right)\omega_{k_0-1}}{\left(\frac{4}{9}qm-\frac{4}{9}pt\right)\omega_{k_0+1}+\left(-\frac{4}{9}uq+\frac{4}{9}pr\right)\omega_{k_0-1}-\frac{8}{3}qb+\frac{8}{3}pd} \\ & +\frac{-\left(-\frac{1}{6}ma-\frac{7}{6}tb+\frac{1}{6}tc+\frac{7}{6}md\right)\omega_{k_0+1}+\frac{7}{18}(tu-mr)\,\omega_{k_0-1}\omega_{k_0+1}}{\left(\frac{4}{9}qm-\frac{4}{9}pt\right)\omega_{k_0+1}+\left(-\frac{4}{9}uq+\frac{4}{9}pr\right)\omega_{k_0-1}-\frac{8}{3}qb+\frac{8}{3}pd}\end{aligned} \tag{5.8}$$

and using the formula for curvature (4.7) and assuming that H_5 in (4.8) is nonnegative we obtain for the 3-point approximating scheme

$$\begin{aligned}\omega_+ = {} & \frac{\kappa_{k_0}\left(\left(d+\frac{1}{2}r\omega_{k_0-1}-\frac{1}{2}t\omega_{k_0+1}\right)^2+\left(b+\frac{1}{2}u\omega_{k_0-1}-\frac{1}{2}m\omega_{k_0+1}\right)^2\right)^{3/2}}{\left(-\frac{3}{2}qm+\frac{3}{2}pt\right)\omega_{k_0+1}+\left(\frac{3}{2}uq-\frac{3}{2}pr\right)\omega_{k_0-1}+3qb-3pd} \\ & +\frac{\tilde{a}b-\tilde{c}d+\left(\frac{1}{2}u\tilde{a}-rb-\frac{1}{2}r\tilde{c}+ud\right)\omega_{k_0-1}}{\left(-\frac{3}{2}qm+\frac{3}{2}pt\right)\omega_{k_0+1}+\left(\frac{3}{2}uq-\frac{3}{2}pr\right)\omega_{k_0-1}+3qb-3pd} \\ & +\frac{\left(-\frac{1}{2}m\tilde{a}-tb+\frac{1}{2}t\tilde{c}+md\right)\omega_{k_0+1}+(rm-tu)\,\omega_{k_0-1}\omega_{k_0+1}}{\left(-\frac{3}{2}qm+\frac{3}{2}pt\right)\omega_{k_0+1}+\left(\frac{3}{2}uq-\frac{3}{2}pr\right)\omega_{k_0-1}+3qb-3pd}\end{aligned} \tag{5.9}$$

where κ_{k_0}=desired curvature at $v^0_{k_0}$ and

$$
\begin{aligned}
a &= \frac{7}{2}v^0_{k_0-1,1} - 7v^0_{k_0,1} + \frac{7}{2}v^0_{k_0+1,1}, & \tilde{a} &= v^0_{k_0-1,1} - 2v^0_{k_0,1} + v^0_{k_0+1,1},\\
c &= \frac{7}{2}v^0_{k_0-1,2} - 7v^0_{k_0,2} + \frac{7}{2}v^0_{k_0+1,2}, & \tilde{c} &= v^0_{k_0-1,2} - 2v^0_{k_0,2} + v^0_{k_0+1,2},\\
d &= -\frac{1}{2}v^0_{k_0-1,1} + \frac{1}{2}v^0_{k_0+1,1}, & b &= -\frac{1}{2}v^0_{k_0-1,2} + \frac{1}{2}v^0_{k_0+1,2},
\end{aligned}
$$

and where, depending on whether condition (ii) or condition (iii) in § 5.1 applies, $r,\ u$ are the respective components of either

$$\left(\left(v^0_{k_0-1} - v^0_{k_0-2}\right)^T n_{k_0-1}\right) n_{k_0-1} \ \text{ or } \left(\left(v^0_{k_0-1} - v^0_{k_0}\right)^T n_{k_0-1}\right) n_{k_0-1}$$

and $q,\ p$ are the respective components of either

$$\left(\left(v^0_{k_0} - v^0_{k_0-1}\right)^T n_{k_0}\right) n_{k_0} \ \text{ or } \left(\left(v^0_{k_0} - v^0_{k_0+1}\right)^T n_{k_0}\right) n_{k_0}$$

and $t,\ m$ are the respective components of either

$$\left(\left(v^0_{k_0+1} - v^0_{k_0}\right)^T n_{k_0+1}\right) n_{k_0+1} \ \text{ or } \left(\left(v^0_{k_0+1} - v^0_{k_0+2}\right)^T n_{k_0+1}\right) n_{k_0+1}.$$

Alternatively, assuming that H_3 in (4.6) is negative we obtain for the interpolatory scheme:

$$
\begin{aligned}
\omega_- = {} & \frac{\kappa_{k_0}\left(\left(d + \frac{1}{6}r\omega_{k_0-1} - \frac{1}{6}t\omega_{k_0+1}\right)^2 + \left(b + \frac{1}{6}u\omega_{k_0-1} - \frac{1}{6}m\omega_{k_0+1}\right)^2\right)^{3/2}}{-\left(\frac{4}{9}qm - \frac{4}{9}pt\right)\omega_{k_0+1} - \left(-\frac{4}{9}uq + \frac{4}{9}pr\right)\omega_{k_0-1} + \frac{8}{3}qb - \frac{8}{3}pd}\\
& + \frac{+ab - cd + \left(\frac{1}{6}ua - \frac{7}{6}rb - \frac{1}{6}rc + \frac{7}{6}ud\right)\omega_{k_0-1}}{-\left(\frac{4}{9}qm - \frac{4}{9}pt\right)\omega_{k_0+1} - \left(-\frac{4}{9}uq + \frac{4}{9}pr\right)\omega_{k_0-1} + \frac{8}{3}qb - \frac{8}{3}pd}\\
& + \frac{\left(-\frac{1}{6}ma - \frac{7}{6}tb + \frac{1}{6}tc + \frac{7}{6}md\right)\omega_{k_0+1} - \frac{7}{18}\left(tu - mr\right)\omega_{k_0-1}\omega_{k_0+1}}{-\left(\frac{4}{9}qm - \frac{4}{9}pt\right)\omega_{k_0+1} - \left(-\frac{4}{9}uq + \frac{4}{9}pr\right)\omega_{k_0-1} + \frac{8}{3}qb - \frac{8}{3}pd}.
\end{aligned}
\tag{5.10}
$$

If H_5 in (4.8) is negative we obtain for the approximating scheme:

$$
\begin{aligned}
\omega_- = {} & \frac{\kappa_{k_0}\left(\left(d + \frac{1}{2}r\omega_{k_0-1} - \frac{1}{2}t\omega_{k_0+1}\right)^2 + \left(b + \frac{1}{2}u\omega_{k_0-1} - \frac{1}{2}m\omega_{k_0+1}\right)^2\right)^{3/2}}{\left(\frac{3}{2}qm - \frac{3}{2}pt\right)\omega_{k_0+1} - \left(\frac{3}{2}uq - \frac{3}{2}pr\right)\omega_{k_0-1} - 3qb + 3pd}\\
& + \frac{-\tilde{a}b + \tilde{c}d - \left(\frac{1}{2}u\tilde{a} - rb - \frac{1}{2}r\tilde{c} + ud\right)\omega_{k_0-1}}{\left(\frac{3}{2}qm - \frac{3}{2}pt\right)\omega_{k_0+1} - \left(\frac{3}{2}uq - \frac{3}{2}pr\right)\omega_{k_0-1} - 3qb + 3pd}\\
& + \frac{-\left(-\frac{1}{2}m\tilde{a} - tb + \frac{1}{2}t\tilde{c} + md\right)\omega_{k_0+1} + \left(tu - rm\right)\omega_{k_0-1}\omega_{k_0+1}}{\left(\frac{3}{2}qm - \frac{3}{2}pt\right)\omega_{k_0+1} - \left(\frac{3}{2}uq - \frac{3}{2}pr\right)\omega_{k_0-1} - 3qb + 3pd}.
\end{aligned}
\tag{5.11}
$$

Having obtained ω_+ and ω_- we choose ω_{k_0} to be the one with smaller absolute value.

If condition (i) of § 5.1 applies to s_{k_0+1} (s_{k_0-1}) then we need to modify (5.8), (5.10) or for the approximating case (5.9), (5.11) by setting ω_{k_0+1} (ω_{k_0-1}) equal to 0.

Using the initial polygon in Fig. 2, we carried out experiments with various curvatures at two vertices, v_7 that appears to have a small curvature and v_{44} that has a larger curvature

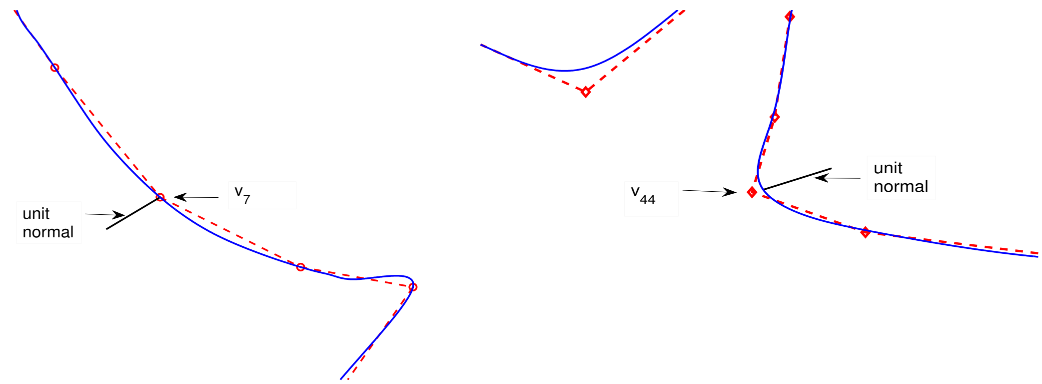

Figure 5: *Left: Subdivision curve by 3-point interpolating scheme with* $n_7 = [-.7677, -.6408]^T$, *curvature*=.1592 (*which is the discrete curvature to the initial polygon at* v_7)*; Right: Subdivision curve by 3-point approximating scheme with* $n_{44} = [.9040, .4275]^T$ *and curvature* =.7382 (*which is the discrete curvature to the initial polygon at* v_{44}). *Unit normal in the pictures is the calculated discrete normal to the initial polygon at vertex* v_7 *or* v_{44}.

with 3-point interpolatory scheme and 3-point approximation scheme. For the approximating scheme, if the desired curvature at v_7 or v_{44} is close to the estimated discrete curvature, the resulting curve looks very nice. For the interpolatory scheme, if the selected curvature at v_7 is close to the estimated discrete curvature, the resulting curve looks good. At v_{44} which has large curvature, the resulting curve with interpolatory scheme has some waviness if the resulted curvature is close to the estimated (discrete) curvature. This is an area where more research is needed. See Fig. 5 for the resulting curves at v_7 with interpolatory scheme and at v_{44} with approximation scheme.

6. Conclusions and future work

We have seen that we could obtain explicit formulas for both the first and second derivatives of the limiting curve for both interpolatory and approximating matrix-valued curve subdivision schemes whose refinement mask and subdivision matrix satisfied certain assumptions. Using these formulas, we then could calculate the normals and curvature of the limiting curve. The first components $\{v_k^0\}_k$ of the initial control vector net $\{\mathbf{v}_k^0\}_k$ represent the initial control polygon and the second components $\{s_k^0\}_k$ are the so-called shape control parameters. We used well-defined discrete normals to the control polygon based on [25] to guide us toward some suitable selection of these shape parameters.

We presented an algorithm to formulate the shape parameters based on projections of the polygon line segments onto these defined discrete normals. Depending on the outcome, $s_k = [0, 0]^T$ or $s_k = \omega_k$ times some calculated multiple of n_k, where n_k is the above-mentioned discrete unit normal to the control polygon at vertex v_k and where ω_k is some scalar. One of the benefits of such an approach is that we now just need to determine one variable (ω_k) rather than two when selecting a suitable shape parameter s_k. Using a sample control polygon we found that limiting curves were nice (i.e. relatively free of waviness) for ω_k values between -1.5 and .3.

We also presented a method for selecting ω_{k-1} and ω_{k+1} (and as a result the respective shape parameters) if we desired to have a specific unit normal to the limit curve at vertex v_k. In some cases the choice of a desired normal that is close to the above-mentioned discrete unit normal can improve the appearance of the final curve. Other choices, however, can increase the waviness. Such results reinforce the work done in [3] that showed how the selection of the shape control parameters change the shapes of the limiting surfaces dramatically.

In addition we presented a method for the selection of ω_k (and thus the shape parameter s_k) if we also wanted a specific desired curvature for the limit curve at v_k. If dealing with a point on the curve with relatively small curvature then selecting a curvature near an estimated (discrete) curvature can produce a nice curve.

Future work will include finding explicit formulas for first and second partial derivatives of a limiting surface for both interpolatory and approximating matrix-valued surface subdivision schemes whose refinement mask and subdivision matrix satisfy certain assumptions. We will then be able to calculate normals and curvature to the limit surface at both regular and extraordinary vertices. We will explore how we can select shape parameters based on defined discrete normals of the initial control polyhedron.

Acknowledgements. The authors would like to thank anonymous referees for their valuable comments and suggestions towards improving the content of the manuscript.

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