Optimal multifilter banks: Design, related symmetric extension transform and application to image compression

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Abstract—The design of optimal multifilter banks and optimum time-frequency resolution multiwavelets with different objective functions is discussed. The symmetric extension transform related to multifilter banks with the symmetric properties is presented. It is shown that such a symmetric extension transform is nonexpansive. More optimal multifilter banks for image compression are constructed and some of them are used in image compression. Experiments show that optimal multifilter banks have better performances in image compression than Daubechies' orthogonal wavelet filters and Daubechies' least asymmetric wavelet filters, and for some images, they even have better performances than the scalar (9,7)-tap biorthogonal wavelet filters. Experiments also show that the symmetric extension transform provided in this paper improves the rate-distortion performance, compared with the periodic extension transform.

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I. INTRODUCTION

Multiwavelets, wavelets generated by a finite set of scaling functions, have several advantages in comparison to scalar wavelets (see [18]). One of the advantages is that a multiwavelet can possess the orthogonality and symmetry simultaneously, while except for the Haar system, a scalar wavelet cannot have these two properties at the same time ([4] and [17]). Thus, as stated in [18], multiwavelets offer the possibility of superior performance for image processing applications, compared with the scalar wavelets. As it was shown in [18], the DGHM multiwavelet, which was constructed by Donovan, Geninno, Hazlim and Massopust in [5] and [7], has a good performance in signal denoising. However for image compression, the DGHM multiwavelet does not have a good performance as one expected [18], and more multiwavelets which are more suitable for image compression are desired. In [11], the construction of optimum time-frequency resolution (OPTFR) multiwavelets was proposed and the optimal multiwavelet filter banks suitable for image compression were designed in [12]. In this paper, we discuss the design of OPTFR multiwavelets with different objective functions and provide more optimal multiwavelet filter banks. The constructed optimal multiwavelet filter banks have the symmetric properties. For symmetric multiwavelet filter banks, a symmetric extension transform of the finite-length input signals is desired. This paper provides such a symmetric extension transform and proves that the symmetric extension transform is nonexpansive. Based on the symmetric extension transform, we use optimal multiwavelet filter banks in image compression. Experiments show that OPTFR multiwavelets have better performances in image compression than Daubechies' orthogonal wavelets and Daubechies' least asymmetric wavelets, and for some images, OPTFR multiwavelets even have better performance than the scalar (9,7)-tap biorthogonal wavelet.

This paper is organized as follows. In Section 2, we review the definition of multiwavelets, the discrete multiwavelet transform, and the procedure to construct OPTFR multiwavelets. In Section 3, we develop the nonexpansive symmetric extension transform for multiwavelet filter banks with the symmetric property. In Section 4, first we discuss the objective functions for the construction of OPTFR multiwavelets for image compression, then we give the implementation of OPTFR multiwavelets for image compression and provide the image compression results with scalar wavelets and OPTFR multiwavelets. The conclusions are given in Sections 5.

II. MULTIFILTER BANKS AND MULTIWAVELETS

In this section, we review the definitions of multiwavelet filter banks and multiwavelets, the discrete multiwavelet transform and the procedure to construct OPTFR multiwavelets.

A. Discrete multiwavelet transform

A vector-valued function \( \Psi := (\psi_1, \ldots, \psi_r)^T \) is called a multiwavelet if the collections of the integer translates and the dilations of factor 2 of \( \psi_1, \ldots, \psi_r \) form an orthonormal basis of \( L^2(\mathbb{R}) \). To construct a compactly supported multiwavelet, we begin with two \( r \times r \) matrices \( H(\omega) = \sum_{k \in \mathbb{Z}} H_k e^{-i k \omega} \) and \( G(\omega) = \sum_{k \in \mathbb{Z}} G_k e^{-i k \omega} \) of
trigonometric polynomials satisfying

\[
\begin{aligned}
H(\omega)H^*(\omega) + H(\omega + \pi)H^*(\omega + \pi) &= I_r, \\
G(\omega)G^*(\omega) + G(\omega + \pi)G^*(\omega + \pi) &= 0_r,
\end{aligned}
\]

where \(-\pi < \omega \leq \pi\). Throughout this paper, \(B^*\) denotes the Hermitian adjoint of the matrix \(B\), and \(I_r\) and \(0_r\) denote the \(r \times r\) identity matrix and zero matrix respectively. If the transition operator \(\ell_1\) associated with \(H\) satisfies Condition E (A matrix \(D\) is said to satisfy Condition E if 1 is a simple eigenvalue of \(D\) and all other eigenvalues lie inside the open unit disk), then there exists a unique compactly supported solution \(\Phi\) with \(\Phi(0) \neq 0\) of the refinement equation

\[
\Phi(x) = 2 \sum_{k \in \mathbb{Z}} H_k \Phi(2x - k).
\]

Furthermore \(\Phi\) is a scaling function, i.e., \(\Phi\) generates a multiresolution analysis of multiplicity \(r\), and \(\Psi\) defined by

\[
\Psi(x) = 2 \sum_{k \in \mathbb{Z}} G_k \Phi(2x - k),
\]

is a multiwavelet, see e.g., [12]. In this case we say that \(\{H, G\}\) generates the scaling function \(\Phi\) and the multiwavelet \(\Psi\). The pair \(\{H, G\}\) is called a multiwavelet filter bank (this is often abbreviated multi filter bank), and \(H\) (\(G\), respectively) is called a matrix lowpass filter (matrix highpass filter, respectively). For a multi filter bank \(\{H, G\}\), it is said to be a finite impulse response (FIR) multi filter bank if there exists an integer \(N\) such that \(H_k = 0, G_k = 0, |k| > N\). A multi filter bank \(\{H, G\}\) is said to be orthogonal if it satisfies (1). In this paper, scaling functions, multiwavelets, and the filter coefficients of the multi filter banks discussed are real.

Assume that \(\{H, G\}\) is an orthogonal FIR multi filter bank and it generates a compactly supported scaling function \(\Phi = (\phi_1, \ldots, \phi_r)^T\) and a multiwavelet \(\Psi = (\psi_1, \ldots, \psi_r)^T\). Let \(V_j\) be the multi resolution analysis generated by \(\Phi\), i.e., \(V_j = \overline{\bigcap_{s \leq j} \Phi(2^j \cdot -k)}\). Let \(V_j \subseteq \mathbb{R}^r\) be the orthogonal complement of \(V_j\) in \(V_{j+1}\). Then \(2^j \Phi(2^j \cdot -k)\) is an orthonormal basis of \(V_j\) and \(2^j \psi(2^j \cdot -k)\) is an orthonormal basis of \(W_j\), respectively. For any continuous function \(f(t)\) in \(V_0\), it can be expanded as

\[
f(t) = \sum_{n} \sum_{\ell=0}^{r} c_{\ell,n}^{(0)} \psi_{\ell}(2^{r}t - n).
\]

The function \(f\) is completely determined by the sequence \(\{c_{\ell,n}^{(0)}\}_{\ell,n} = 0\). Let \(J\) be a negative integer. By the fact \(V_0 = W_{-J} \oplus V_{-J-1} = \cdots = W_{-J} \oplus \cdots \oplus W_J \oplus V_J\), and the orthonormality of \(2^j \Phi(2^j \cdot -k)\) and \(2^j \psi_{\ell}(2^j \cdot -k)\), \(f\) also can be expanded as

\[
f(t) = \sum_{n} \sum_{\ell=1}^{r} d_{\ell,n}^{(J)} \frac{1}{\sqrt{2}} \phi_{\ell}(2^{r}t - n)
\]

where \(c_{\ell,n}^{(J)} = \sum_{r=0}^{\infty} c_{\ell,n}^{(r)} 2^{r} \psi_{\ell}(2^{r}t - n)\).

Denote \(c_{n}^{(j)} := (c_{1,n}^{(j)}, \ldots, c_{r,n}^{(j)})^T\), \(d_{n}^{(j)} := (d_{1,n}^{(j)}, \ldots, d_{r,n}^{(j)})^T\). Then we have

\[
\sum_{n} (c_{n}^{(j)})^T 2^{j} \Phi(2^j t - n)
\]

\[
= \sum_{n} (c_{n}^{(j-1)})^T 2^{j} \Phi(2^j t - n) + \sum_{n} (d_{n}^{(j-1)})^T 2^{j} \Psi(2^j t - n).
\]

Multiplying both sides of (4) with \(2^{-j} \Phi(2^j t - k)\) and taking integral over \(R\), we have

\[
c_{k}^{(j-1)} = \sqrt{2} \sum_{n} H_{n-k} c_{n}^{(j)}.
\]

Similarly by multiplying both sides of (4) with \(2^{-j} \Phi(2^j t - k)\) and taking integral over \(R\), one has

\[
d_{k}^{(j-1)} = \sqrt{2} \sum_{n} G_{n-k} d_{n}^{(j)}.
\]

Finally, multiply both sides of (4) with \(2^{-j} \Phi(2^j t - k)\) and take integral over \(R\), we have

\[
c_{k}^{(j)} = \sqrt{2} \sum_{n} H_{n-k} c_{n}^{(j-1)} + \sqrt{2} \sum_{n} G_{n-k} d_{n}^{(j-1)}.
\]

Equations (5) and (6) are the discrete multiwavelet transform decomposition algorithm, while (7) is the reconstruction algorithm (see also [18] and [22]). Thus to determine \(c_{k}^{(j)}\) and \(d_{k}^{(j-1)}\), \(J \leq j \leq -1\), we need only to determine \(c_{k}^{(0)}\) from \(f(t)\).

For the scalar wavelet transform, since the scaling function \(\phi\) satisfies \(\phi(0) = 1\), \(c_{k}^{(0)}\) is close to \(f(n/N_0)\) and we simply let \(c_{n}^{(0)}\) to be \(f(n/N_0)\), where \(f(n/N_0)\) are samples of \(f(t)\) with sampling rate \(1/N_0\). However for the multiwavelet transform, \(\Phi(0)\) is a normalized right 1-eigenvector of \(H(0)\) (see [12]), and \(\Phi(0)\) needs not to have \(\Phi(0) = \frac{1}{\sqrt{r}} (1, \ldots, 1)^T\). In this case, we can not simply determine \(c_{k}^{(0)}\) in such a way as in the scalar case. There are
two methods to deal with this problem. One method is the pre/postfilter techniques carried out in [18], [21] and [22]. Another method is to use another pair of multiband filter \( \{H^b, G^b\} \) constructed from \( \{H, G\} \). This new multiband filter \( \{H^b, G^b\} \) generates the scaling function \( \Phi^b \) and multivariate \( \Psi^b \) with \( \Phi^b(0) = \frac{1}{\sqrt{r}}(1, \ldots, 1)^T \) (see [15]). In fact if \( \Phi(0) \in R^r \) is a vector with \( \Phi(0)^T \Phi(0) = 1 \), then there exists a \( r \times r \) orthogonal matrix \( U \) such that \( \Phi^b(0) = \frac{1}{\sqrt{r}}(1, \ldots, 1)^T \). Let \( \{H^b, G^b\} \) be the multiband filter defined by

\[
H^b = UH^T, \quad G^b = U^TGU^T,
\]

where \( U \) is an orthogonal matrix. Then \( \{H^b, G^b\} \) is orthogonal and generates the scaling function \( \Phi^b \) and the multivariate \( \Psi^b \) with \( \Phi^b(0) = \frac{1}{\sqrt{r}}(1, \ldots, 1)^T \), where \( \Phi^b = U\Phi, \Psi^b = U\Psi \). As in [15], multivariate \( \Phi^b \) is said to be balanced if its corresponding scaling function \( \Phi^b \) satisfies \( \Phi^b(0) = \frac{1}{\sqrt{r}}(1, \ldots, 1)^T \). In this paper, we will use the second method and we will design the balanced multivariate.

Consider the case \( r = 2 \). From the decomposition algorithm (5) with multiband filter \( \{H^b, G^b\} \), the normalized lowpass frequency responses for this system are (see [19])

\[
h^b_k(\omega) := \sum_{k=0}^N H^b_k(\alpha, 1)e^{-2\pi kw} + H^b_k(\alpha, 2)e^{-\frac{\pi}{2}(2k+1)w}, \tag{8}
\]

where \( \alpha = 1, 2 \); and for a matrix \( B, B(\ell, j) \) denotes the \( (\ell, j) \)-entry of \( B \). The filters \( h_1^b, h_2^b \) act as lowpass filters. Thus it is required that \( h^b_0(0) = 1, h^b_0(\pi) = 0, \alpha = 1, 2 \).

It was shown in [12] that if \( \{H^b, G^b\} \) generates a balanced wavelet of multiplicity 2, then \( h^b_0(\pi) = 1, \alpha = 2 \). Thus in this case what in our design are the requirements:

\[
h^b_1(\pi) \approx 0, \quad h^b_2(\pi) \approx 0. \tag{9}
\]

\section{Optimum time-frequency resolution multiwavelets}

The design of OPTFR multivariate was studied in [11] and [12]. In this subsection, we review the procedure to design OPTFR multivariate.

The time-duration \( \Delta_T \) of a window function \( f \) is defined by

\[
\Delta_T := (\int_R (t - T)^2 |f(t)|^2 dt/E)^{\frac{1}{2}},
\]

where

\[
T := \int_R |f(t)|^2 dt/E, \quad E := \int_R |f(t)|^2 dt.
\]

The frequency-bandwidth of \( f \) denoted by \( \Delta_f \) is defined in the same way with \( f \) replaced by \( \hat{f} \). Then \( \Delta_T \Delta_f \geq \frac{1}{\pi} \). This inequality is called the uncertainty principle, and the product \( \Delta_T \Delta_f \) is called the resolution cell.

Since every component \( \psi_j \) of a multiresolution \( \Psi \) is a bandpass function (see [11]), i.e., \( \hat{\psi}_j(0) = 0 \), as in the scalar case, we also consider the frequency-bandwidth \( \Delta^p \psi_j \) of \( \psi_j \) defined by ([8], [6])

\[
\Delta^p \psi_j := \left( \int_0^\infty (\omega - \bar{\omega})^2 |\hat{\psi}_j(\omega)|^2 d\omega \right)^{\frac{1}{2}},
\]

where

\[
\bar{\omega} := \int_0^\infty |\hat{\psi}_j(\omega)|^2 d\omega / \int_0^\infty |\hat{\psi}_j(\omega)|^2 d\omega.
\]

In [11], formulas to compute the energy moments of scaling functions and multivariate in the time-frequency plane were provided. In the following we use \( \Theta_f \) and \( \Theta_f^* \) to denote \( \Delta_T \Delta_f \) and \( \Delta^p \psi_j \) respectively.

To construct OPTFR multivariate, since the scaling functions and multivariate with good time-frequency localization are constructed simultaneously, the parametric expressions for the matrix coefficients \( H_k, G_k \) of multiband filters are required. In [12], several forms of factorizations for orthogonal causal FIR multiband filters are provided based on the lattice structures for \( M \times M \) FIR lossless systems. (About \( M \times M \) FIR lossless systems, see, e.g., [20] and the references therein.) The factorization for orthogonal causal FIR multiband filters which generate symmetric/antisymmetric scaling functions and multivariate is discussed in [13]. In the following we review the parametric expressions for the multiband filters which generate wavelets of multiplicity 2 with symmetry property.

For a vector-valued function \( f = (f_1, f_2)^T \), we say \( f \) is symmetric/antisymmetric about a symmetry center \( c_0 \in \mathbb{R} \) if \( f_1 \) (\( f_2 \), respectively) is symmetric (antisymmetric, respectively) about the center \( c_0 \). Let \( \{N, H, N, G\} \) with \( N(\omega) = \sum_{k=0}^N H_k e^{-ik\omega}, \quad N(\omega) = \sum_{k=0}^N G_k e^{-ik\omega} \) be an orthogonal FIR multiband filter. If \( \{N, H, N, G\} \) generates symmetric/antisymmetric scaling function \( N \Psi \) and multivariate \( \Psi \) about the symmetry center \( \frac{N}{2} \), then \( \{H_k\}_{k=0}^N \{G_k\}_{k=0}^N \) satisfy (see, e.g., [8]):

\[
D_0 H_{N-k} D_0 = H_k, \quad D_0 G_{N-k} D_0 = G_k, 0 \leq k \leq N, \tag{10}
\]

where

\[
D_0 := \text{diag}(1, -1).
\]

If \( N \) is odd, \( N = 2\gamma + 1, \gamma \in \mathbb{Z} \), to say, then it was shown in [13] that \( \{2\gamma+1 H, 2\gamma+1 G\} \) is orthogonal and satisfies (10) if and only if

\[
[2\gamma+1 H(\omega) \quad 2\gamma+1 G(\omega)] = \begin{bmatrix} M_0 V_\gamma(z^2) \cdots V_1(z^2) M_0 & [H(\omega) \quad G(\omega)] \end{bmatrix},
\]

where

\[
z = e^{i\omega}, \quad V_k(z) := \frac{1}{2} \begin{bmatrix} I_2 & v_k \ v_k^* & I_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} I_2 & -v_k \ -v_k^* & I_2 \end{bmatrix} z^{-1}, \tag{11}
\]

in (11), the matrices \( M_0 \) and \( V_k \) are such that

\[
M_0 := \begin{bmatrix} I_2 & v_k \ v_k^* & I_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} I_2 & -v_k \ -v_k^* & I_2 \end{bmatrix} z^{-1}, \tag{12}
\]
with \( \mathbf{v}_k \in O(2) \),

\[
\begin{bmatrix}
\mathbf{1} \mathbf{H}(\omega) \\
\mathbf{1} \mathbf{G}(\omega)
\end{bmatrix}
:=
\begin{bmatrix}
\frac{1}{2} & 1 \\
\cos \theta_0 & \sin \theta_0 & \pm 1 \\
0 & \sin \theta_0 & \pm 1 \\
\frac{1}{2} & -1 & -\cos \theta_0 & \pm \sin \theta_0 \\
0 & \pm 1 & \sin \theta_0 & \pm \cos \theta_0
\end{bmatrix} 
\] (13)

with \(-\pi < \theta_0 \leq \pi\), and

\[
\mathbf{M}_0 :=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Here \( O(2) \) denotes the set consisting of all \( 2 \times 2 \) orthogonal matrices. For \( \mathbf{v}_k \in O(2) \), \( 0 \leq k \leq \gamma \), \( \mathbf{v}_k \) are given by \( \mathbf{v}_k = \mathbf{R}(\theta_k) \mathbf{diag}(1, \pm 1) \), \(-\pi < \theta_k \leq \pi\), where \( \mathbf{R}(\theta_k) = \begin{bmatrix} \cos \theta_k & -\sin \theta_k \\ \sin \theta_k & \cos \theta_k \end{bmatrix} \).

If \( N = 2\gamma \) for some positive integer \( \gamma \), a family of orthogonal multiframe banks \( \{ \mathbf{2}_1 \mathbf{H}_1, \mathbf{2}_1 \mathbf{G}_1 \} \) which satisfy (10) are provided in [12]:

\[
\begin{bmatrix}
\mathbf{2}_1 \mathbf{H}(\omega) \\
\mathbf{2}_1 \mathbf{G}(\omega)
\end{bmatrix}
=
\mathbf{M}_0 \mathbf{V}_{\gamma-1}(z^2) \mathbf{V}_{\gamma-2}(z^2) \cdots \mathbf{V}_1(z^2) \mathbf{M}_0 \begin{bmatrix}
\mathbf{2}_1 \mathbf{H}(\omega) \\
\mathbf{2}_1 \mathbf{G}(\omega)
\end{bmatrix},
\]

where \( \mathbf{V}_k \) are given by (12) for some \( \mathbf{v}_k \in O(2) \) and

\[
\begin{bmatrix}
\mathbf{2}_1 \mathbf{H}(\omega) \\
\mathbf{2}_1 \mathbf{G}(\omega)
\end{bmatrix}
:=
\begin{bmatrix}
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\
-\sqrt{2}\cos \theta_0 & \sqrt{2}\cos \theta_0 & \pm 1 \\
\pm 1 & \pm 1 & \sqrt{2}\sin \theta_0 & \sqrt{2}\sin \theta_0 \\
0 & 0 & \pm \sqrt{2}\sin \theta_0 & \pm \sqrt{2}\sin \theta_0 \\
0 & 0 & \pm \sqrt{2}\cos \theta_0 & \pm \sqrt{2}\cos \theta_0 \\
0 & 0 & \pm 1 & \pm 1
\end{bmatrix} 
\] (15)

with \(-\pi < \theta_0 \leq \pi, z = e^{i\omega}\).

The scaling functions \( \mathcal{N} \Phi \) and multiwavelets \( \mathcal{N} \Psi \) generated by \( \{ \mathcal{N} \mathbf{H}, \mathcal{N} \mathbf{G} \} \) which are given in (11) and (14) are symmetric/antisymmetric. Thus \( \mathcal{N} \Phi(0) = (1, 0)^T \) and hence \( \mathcal{N} \Psi \) is not balanced. By a rotation of the angle \( \pi/4 \), we get the balanced scaling function \( \mathcal{N} \Phi^b \) and multiwavelet \( \mathcal{N} \Psi^b \) by

\[
\mathcal{N} \Phi^b = \mathbf{R}_0 \mathcal{N} \Phi, \quad \mathcal{N} \Psi^b = \mathbf{R}(\theta) \mathcal{N} \Psi,
\]

where \( \theta \in [-\pi, \pi) \) and

\[
\mathbf{R}_0 := \mathbf{R}(\frac{\pi}{4}) = \sqrt{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

The problem is how to choose the number \( \theta \). In [15] and [19], \( \theta \) is chosen to be zero, while in [12] \( \theta \) is \( \frac{\pi}{4} \). In our experiments about image compression, we find that for most cases we will get a little better results for the choice of \( \theta = 0 \) than for \( \theta = \frac{\pi}{4} \). In the following, we set \( \theta = 0 \). In this case, the scale functions \( \mathcal{N} \Phi = (\mathcal{N} \phi^b_1, \mathcal{N} \phi^b_2)^T \) satisfy \( \mathcal{N} \phi^b_2(N - t) = \mathcal{N} \phi^b_1(t) \) and multiwavelets \( \mathcal{N} \Psi = (\mathcal{N} \psi^b_1, \mathcal{N} \psi^b_2)^T \) are symmetric/antisymmetric. The corresponding multiframe banks, denoted by \( \{ \mathcal{N} \mathbf{H}^b, \mathcal{N} \mathbf{G}^b \} \), are given by

\[
\mathcal{N} \mathbf{H}^b(\omega) = \mathbf{R}_0 \mathcal{N} \mathbf{H}(\omega) \mathbf{R}_0^T, \quad \mathcal{N} \mathbf{G}^b(\omega) = \mathbf{N} \mathbf{G}(\omega) \mathbf{R}_0^T.
\] (18)

It is shown in [12] that for \( N = 2\gamma + 1 \), the constrained conditions (9) for \( \mathcal{N} \mathbf{H}^b \) are

\[
\sin \theta_0 \approx 0.
\] (19)

while for \( N = 2\gamma \), the constrained conditions (9) are

\[
\cos(\theta_0 + \pi/4) \approx 0.
\] (20)

Using the parametric expression of \( \{ \mathcal{N} \mathbf{H}^b, \mathcal{N} \mathbf{G}^b \} \), we construct the extended OPTFR multiwavelets by minimizing the objective function \( \mathcal{N} S(1) := \mathcal{N} \mathbf{S} + \mathcal{N} \mathbf{S}^2 \), or \( \mathcal{N} S(2) := \mathcal{N} \mathbf{S} + \mathcal{N} \mathbf{S}^2 \), or by minimizing the sum \( \mathcal{N} S(3) := 2\mathcal{N} \mathbf{S} + \mathcal{N} \mathbf{S}^2 + \mathcal{N} \mathbf{S}^2 \), or the sum \( \mathcal{N} S(4) := 2\mathcal{N} \mathbf{S} + \mathcal{N} \mathbf{S}^2 + \mathcal{N} \mathbf{S}^2 + \mathcal{N} \mathbf{S}^2 \), where \( \mathcal{N} \Psi = \mathbf{R}_0 \mathcal{N} \Psi \).

III. NONEXPANSIVE SYMMETRIC EXTENSION TRANSFORM

When we apply filter banks to image compression, we encounter the boundary conditions at the ends of the signals since in practice all signals are finite-length. In other words, we should extend the signal to the infinite interval. The extension approaches should satisfy the following conditions:

(1) Easy to implement. Due to the demand for time complexity, we can not design too complicated means to extend signals.

(2) Perfect reconstruction. If no error exists in the coding and quantization stages, the original signals can be perfectly reconstructed.

(3) Smoothness. No discontinuity is introduced by the extension process.

(4) Nonexpansive. That means no extra space overheads are imposed for the perfect reconstruction of the extended signals.

The most popular method for extension is the periodic extension. It works for any perfect reconstruction multirate
filter bank. However, condition (3) cannot be met anymore due to the differences between the first samples and the last samples of image signals.

Another extension approach is the symmetric extension. In comparison with the periodic extension, the symmetric extension improves the rate-distorion performance about 0.1–1.0 dB in image compression. However its drawback is that it works only for the linear phase filters. As mentioned in the introduction, 2-channel orthogonal scalar wavelet filters can not be linear phases except for the trivial case (Haar system) [4]. So for 2-channel scalar wavelet orthogonal filters, periodic extension is the most used extension method for perfect reconstruction. Other method includes linear extrapolation of boundary samples [14].

Brislawn investigated the symmetric extension for the perfect reconstruction filter banks thoroughly [2]. A 2-channel orthogonal multilter bank can be regarded as a 4-channel perfect reconstruction filter bank in some sense. However Brislawn’s approach cannot be applied or generalized directly to our multilter banks which generate symmetric/antisymmetric multilters. Strela et al have considered this issue for multilter banks. But their method can only deal with the DGHM multilter bank [18]. In this section we give our scheme for symmetric extension of the finite-length vector signals for the multilter banks with the symmetric property. The finite–length vector signals v considered in this section are

\[ v = (v_0, v_1, \ldots, v_{\ell-1}) \]

where \( \ell \) is the length of the signal \( v \), and \( v_i = (v_{i,1}, v_{i,2})^T \in R^2, 0 \leq i \leq \ell - 1 \).

In the following we consider the symmetric extension transform for the matrix filters with a more general symmetry. Assume throughout that \( S \) is a \( 2 \times 2 \) real matrix satisfying

\[ S^T S = I_2, \quad \det(S) = -1. \]  

(21)

Clearly \( D_0 \) is a matrix satisfying (21).

\[ \text{Definition 1 (Symmetric multilter).} \quad \text{A matrix filter} \ H \text{ is said to be symmetric under} \ S \text{ with the symmetric center} \ N \text{ for some integer} \ N \text{ if} \]

\[ SH_{N-k}S = H_k, \quad k \in Z. \]  

(22)

If a matrix filter \( H \) is symmetric (under \( S \)) with the symmetric center \( \frac{N}{2} \), then we denote it by \( \text{cen}(H) = \frac{N}{2} \).

For a matrix filter \( H(\omega) = \sum_{k=-N_2}^{N_2} H_k e^{-i\omega k} \) with \( H_0 \neq 0 \) for \( k = N_1, N_2 \), we use \( \text{len}(H) := N_2 - N_1 + 1 \) to denote the filter length of \( H \).

For 2-channel filter banks, the transations of filters with 2, \( k, k \in Z \) do not affect the performance (see [2]). This property still holds for multilter banks. So, without loss of the generality, we classify the symmetric matrix filters \( H \) into 4 types:

- **Type I**: \( \text{len}(H) = 2N_1 + 1, N_1 \in Z, \text{ and cen}(H) = 0, \text{ i.e.} \]

\[ H(\omega) = \sum_{k=-N_1}^{N_1} H_k e^{-i\omega k}, \quad H_k = SH_k S; \]

- **Type II**: \( \text{len}(H) = 2N_1 + 1, N_1 \in Z, \text{ and cen}(H) = 1, \text{ i.e.} \]

\[ H(\omega) = \sum_{k=-N_1}^{N_1+1} H_k e^{-i\omega k}, \quad H_k = SH_{2-k} S; \]

- **Type III**: \( \text{len}(H) = 2N_1, N_1 \in Z, \text{ and cen}(H) = -\frac{1}{2}, \text{ i.e.} \]

\[ H(\omega) = \sum_{k=-N_1}^{N_1-1} H_k e^{-i\omega k}, \quad H_k = SH_{-1-k} S; \]

- **Type IV**: \( \text{len}(H) = 2N_1, N_1 \in Z, \text{ and cen}(H) = \frac{1}{2}, \text{ i.e.} \]

\[ H(\omega) = \sum_{k=-N_1}^{N_1} H_k e^{-i\omega k}, \quad H_k = SH_{1-k} S. \]

**Definition 2 (Symmetric signal).** For an infinite vector signal \( v = (v_0, v_1, \ldots, v_m) \), an infinite signal \( v_{ext} = (v_{ext,0}, v_{ext,1}, v_{ext,2}, \ldots) \), where \( v_k \in R^2, k \in Z \), if there exists an integer \( n \) such that \( Sv_{n-k} = v_k, \forall k \in Z \), then we say that \( v \) is symmetric (under \( S \)) with the symmetric center \( \frac{n}{2} \). If \( v \) has two symmetric centers \( c_1 \) and \( c_2 \) with \( c_1 < c_2 \), we denote it by \( \text{cen}(v) = (c_1, c_2) \).

**Definition 3 (Symmetric extension).** For an infinite vector signal \( v = (v_0, v_1, \ldots, v_m) \), an infinite signal \( v_{ext} = (v_{ext,0}, v_{ext,1}, v_{ext,2}, \ldots) \) is said to be the extension of \( v \), if \( v_{ext,k} = v_{0,k}, 0 \leq k \leq m \). If \( v_{ext} \) is symmetric with the symmetric centers \( c_1, c_2 \), i.e., \( \text{cen}(v_{ext}) = (c_1, c_2) \), then \( v_{ext} \) is said to be the symmetric extension of \( v \) with the left symmetric center \( c_1 \) and the right symmetric center \( c_2 \). Let \( v_{ext}(c_1, c_2) \) denote \( v_{ext} \) to emphasize the symmetric centers of \( v_{ext} \).

As pointed out in [2], for subband coding, symmetric extensions of scalar signals can be reduced to two different types of extension depending on the parities of the lengths of the scalar filters considered. For the vector case, the extension of a vector signal also depends on the parities of the lengths of the matrix filters. However the vector signals are extended symmetrically in a slight different way. It depends also on the matrix \( S \) in (22).

Assume that the input finite-length vector signal is \( (v_0, v_1, \ldots, v_{\ell-1}) \). For the even length matrix filters, the symmetric extension \( v_{ext} \) of \( v \) is given by

\[ v_{ext}(-\ell), \ell = \frac{1}{2}, \frac{\ell - 1}{2} \]

\[ \cdots, Sv_2, Sv_1, Sv_0, v_0, v_1, \cdots, v_{\ell-1}, Sv_{\ell-1}, Sv_{\ell-2}, \cdots, \]

(23)

while for the odd length matrix filters, the symmetric extension \( v_{ext} \) of \( v \) is given by

\[ v_{ext}(0), \ell = \frac{1}{2}, \frac{\ell - 1}{2} \]

\[ \cdots, Sv_2, Sv_1, v_0, v_1, \cdots, v_{\ell-2}, v_{\ell-1}, Sv_{\ell-1}, v_{\ell-2}, \cdots. \]

(24)

For the symmetric extension (24), it is required that the input signal satisfies

\[ Sv_0 = v_0, \quad Sv_{\ell-1} = v_{\ell-1}. \]  

(25)
The map from \( \mathbf{v} \) to its symmetric extension \( \mathbf{v}_{\text{ext}} \) defined by (23) or (24) is called the symmetric extension transform. We have the following two theorems about the symmetric extension transform.

**Theorem 1:** Suppose \( \mathbf{H} \) is a matrix filter of Type I or Type II. Assume that the input signal \( \mathbf{v}^{(0)} = (v_0^{(0)}, \ldots, v_{L-1}^{(0)}) \) satisfies \( \mathbf{Sv}_0^{(0)} = v_0^{(0)} \). Then \( \mathbf{v}_{\text{ext}}(0, \ell - 1) \) be the symmetric extension of \( \mathbf{v}^{(0)} \) defined by (24). Let \( \mathbf{v}^{(-1)} \) be the output of subband filter \( \mathbf{H} \):

\[
\mathbf{v}_{\mathbf{H}}^{(-1)} = \sqrt{2} \sum_i H_i v_{2k+i}^{(0)} = \sqrt{2} \sum_i \mathbf{S}_{\mathbf{H}_i} v_{2k+i}^{(0)} = \mathbf{Sv}_{\mathbf{H}}^{(-1)},
\]

and

\[
\mathbf{v}_{\mathbf{H}}^{(-1)} = \sqrt{2} \sum_i H_i v_{2k+i}^{(0)} = \sqrt{2} \sum_i H_i s_{2k+i}^{(0)} = \mathbf{Sv}_{\mathbf{H}}^{(-1)}.
\]

That is \( \mathbf{v}^{(-1)} \) is symmetric about 0 and \( \frac{L-1}{2} \).

For the case that \( \text{cen} (\mathbf{H}) = 1 \), the symmetry of \( \mathbf{v}^{(-1)} \) can be shown in a similar way, and the details are omitted here.

**Theorem 2:** Suppose \( \mathbf{H} \) is a matrix filter of Type III or Type IV. Let \( \mathbf{v}_{\text{ext}}^{(0)}(-\frac{1}{2}, \ell - \frac{1}{2}) \) be the symmetric extension of \( \mathbf{v}^{(0)} \) defined by (23) and \( \mathbf{v}^{(-1)} \) be the output of subband filter \( \mathbf{H} \):

\[
\mathbf{v}_{\mathbf{H}}^{(-1)} = \sqrt{2} \sum_i H_i v_{2k+i}^{(0)} = \sqrt{2} \sum_i \mathbf{S}_{\mathbf{H}_i} v_{2k+i}^{(0)} = \mathbf{Sv}_{\mathbf{H}}^{(-1)},
\]

and

\[
\mathbf{v}_{\mathbf{H}}^{(-1)} = \sqrt{2} \sum_i H_i v_{2k+i}^{(0)} = \sqrt{2} \sum_i H_i s_{2k+i}^{(0)} = \mathbf{Sv}_{\mathbf{H}}^{(-1)}.
\]

That is \( \mathbf{v}^{(-1)} \) is symmetric about 0 and \( \frac{L-1}{2} \).

For \( c \in \mathbb{R} \), let \( [c] \) denotes the largest integer not greater than \( c \). In the following, we define the 

**Definition 4 (Storage length).** Suppose \( \mathbf{v}(c_1, c_2) \) is an infinite signal with the left symmetric center \( c_1 \) and the right symmetric center \( c_2 \). Define \( \mathbf{v} = (v_{L+1}, v_L, \ldots, v_2, v_1, v_0, v_1, \ldots, v_L, v_{L+1}) \), where \( l_1 = [c_1 + \frac{1}{2}], l_2 = [c_2] \). Define \( \text{stol}(\mathbf{v}) = \frac{1}{2} \) if \( c_1 = \ell_1 \) is an integer and \( \text{stol}(\mathbf{v}) = \frac{1}{2} \) if \( c_1 \) is not an integer, and define \( \text{stol}(\mathbf{v}) \) in a similar way. Define \( \text{stol}(\mathbf{v}^0) = \ell_2 - \ell_1 - 1 + \text{stol}(\mathbf{v}_L) + \text{stol}(\mathbf{v}_{L+1}) \) and \( \text{stol}(\mathbf{v}) = \text{stol}(\mathbf{v}^0) \).

**Remark 1:** In Definition 4, if the left center \( c_1 \) is an integer, i.e., \( \ell_1 = \ell_1 \) for some integer \( \ell_1 \), then \( v_{L+1} \) satisfies \( \mathbf{Sv}_{L+1} = v_{L+1} \). Thus \( v_{L+1} = k \mathbf{v}_L \), where \( k \) is a constant and \( \mathbf{v}_L \) is the normalized real eigenvector of \( \mathbf{S} \) corresponding to eigenvalue 1. Therefore one parameter \( k \) is enough to determine \( \mathbf{v}_{L+1} \). Thus in this case, compared with other vectors \( \mathbf{v}_j, \ell_1 < j < \ell_2 \), half storage is needed to store \( \mathbf{v}_{L+1} \); and we define \( \text{stol}(\mathbf{v}_L) \) to be \( \frac{1}{2} \). For the same reason, if \( \ell_2 = \ell_2 \) for some integer \( \ell_2 \), then we define \( \text{stol}(\mathbf{v}_{L+1}) = \frac{1}{2} \).

The number \( \text{stol}(\mathbf{v}) \) is the actual number we need to store the vector signal \( \mathbf{v} \).

By a direct calculation, we have the following propositions.

**Proposition 1:** Let \( \mathbf{v}(c_1, c_2) \) be an infinite vector signal with the left symmetric center \( c_1 \) and the right symmetric center \( c_2 \). Then \( \text{stol}(\mathbf{v}) = c_2 - c_1 \).

In the symmetric extension of a finite-length vector signal for the odd length matrix filter, (25) imposes an extra constraint on the original vector signal. However, most vector signals do not meet this condition automatically. To overcome this problem, we develop a transform to generate the input vector signal from the original scalar signal, and the vector signal derived can be used as the input signal for a filter matrix no matter the length of the matrix filter is odd or even.

If \( \mathbf{S} \) is a matrix satisfying (21), then 1 and 1 are eigenvalues of \( \mathbf{S} \). Thus there is a non-singular matrix \( \mathbf{U} \) such that

\[
\mathbf{U}^{-1} \mathbf{S} \mathbf{U} = \mathbf{D}_0.
\]

Recall \( \mathbf{D}_0 = \text{diag}(1, -1) \). Let \( \mathbf{R}_0 \) be the matrix defined by (17). In the following we define the transform \( \mathbf{T}_S \) associated with \( \mathbf{S} \).

**Transform \( \mathbf{T}_S \):** Suppose \( f = (f_0, f_1, \ldots, f_{2^{L-1}}) \) is the sample of the original scalar signal.

(1) If the length of the matrix filter is even, define the vector signal \( \mathbf{v} = (v_0, v_1, v_2, \ldots, v_{L-1}) \) of length \( \ell \) by

\[
v_i = \mathbf{U}^{-T} \mathbf{f}_i, 0 \leq i \leq \ell - 1.
\]

(2) If the length of the matrix filter is odd, define the vector signal \( \mathbf{v} = (v_0, v_1, v_2, \ldots, v_{L-1}, v_L) \) of length \( \ell + 1 \) by

\[
\begin{align*}
v_0 &= \mathbf{U}^{-T} \mathbf{f}_0, \quad v_L = \mathbf{U}^{-T} \mathbf{f}_{L-1}, \\
v_i &= \mathbf{U}^{-T} \mathbf{f}_{i-1}, \quad 1 \leq i \leq \ell - 1.
\end{align*}
\]

Let \( \mathbf{T}_S \) denote the transform from the original scalar signal \( f \) to the vector signal \( \mathbf{v} \), i.e., \( \mathbf{T}_S f = \mathbf{v} \).

**Proposition 2:** Suppose \( f = (f_0, f_1, \ldots, f_{2^{L-1}}) \) is the sample of the original signal. Let \( \mathbf{T}_S f = (v_0, v_1, v_2, \ldots, v_{L-1}, v_L) \) be the vector signal defined by (27), then \( \mathbf{T}_S f \) satisfies (25).

**Proof:** Since \( \mathbf{SU} = \mathbf{U} \mathbf{D}_0 \), one has

\[
\mathbf{Sv}_0 = \mathbf{U} \mathbf{D}_0 \mathbf{U}^{-T} \mathbf{f}_0 = \mathbf{U} \mathbf{D}_0 \left( \sqrt{2} \mathbf{f}_0 \right) = \mathbf{UR}_0^{-T} \mathbf{f}_0 = \mathbf{v}_0.
\]

Similarly we have \( \mathbf{Sv}_L = \mathbf{v}_L \).

**Proposition 2** ensures that the vector signal derived by transform \( \mathbf{T}_S \) meets the requirement of the symmetric extension for the odd-length matrix filters. Since \( \mathbf{UR}_0^{-T} \) is a
linear operator, transform $T_S$ preserves the continuity of the input signal.

Theorems 1, 2 and Proposition 1 lead to the following theorem.

**Theorem 3**: Suppose that $\{H, G\}$ is a multfilter bank, $H, G$ are symmetric under $S$ and the lengths of $H, G$ have the same parity. For a scalar input signal $f = (f_0, f_1, \ldots, f_{2\ell-1})$, let $v^{(0)} = T_S f$ be the vector signal defined by (26) if the length of $H$ is even and by (27) if the length of $H$ is odd. Let $v^{(0)}_{ext}$ be the symmetric extension of $v^{(0)}$ defined by (23) or (24) and $v^{(-1)}, u^{(-1)}$ be the subband outputs of the input vector signal $v^{(0)}_{ext}$ with multfilters $H, G$, respectively. Then

$$\text{storl}(v^{(-1)}) + \text{storl}(u^{(-1)}) = \ell. \quad (28)$$

Proof. First we consider the case that $\text{len}(H)$ is even. In this case $v^{(0)}_{ext}$ is the symmetric extension of $v^{(0)}$ by (24). Then $\text{cen}(v^{(-1)}) = (0, \frac{\ell}{2})$ if $H$ is of Type III, while $\text{cen}(v^{(-1)}) = (-\frac{\ell}{2}, \frac{\ell}{2})$ if $H$ is of Type IV. Proposition 1 implies that for both cases,

$$\text{storl}(v^{(-1)}) = \frac{\ell}{2}.$$ 

Since $\text{len}(G)$ is also even, we also have $\text{storl}(u^{(-1)}) = \frac{\ell}{2}$. Thus $\text{storl}(v^{(-1)}) + \text{storl}(u^{(-1)}) = \ell$. That is (28) holds true for the case that $\text{len}(H)$ is even.

Now let us consider the case that $\text{len}(H)$ is odd. Note that in this case, $v^{(0)}_{ext}$ is the symmetric extension of $v^{(0)} = (v_0^{(0)}, v_1^{(0)}, \ldots, v^{(0)}_{\ell})$ given by (23). Then $\text{cen}(v^{(-1)}) = (0, \frac{\ell}{2})$ if $H$ is of Type I, while $\text{cen}(v^{(-1)}) = (-\frac{\ell}{2}, \frac{\ell}{2})$ if $H$ is of Type II. From Proposition 1, for both cases,

$$\text{storl}(v^{(-1)}) = \frac{\ell}{2}.$$ 

In the same way, one has $\text{storl}(u^{(-1)}) = \frac{\ell}{2}$. Thus $\text{storl}(v^{(-1)}) + \text{storl}(u^{(-1)}) = \ell$. That is (28) holds for the case that $\text{len}(H)$ is odd. 

**Theorem 3** implies

**Corollary 1**: The symmetric extension transforms defined by (23) and (24) are nonepisymmetric.

In the following let us consider two special cases. If $\{H, G\}$ is an FIR multfilter bank generating symmetric/antisymmetric scaling functions and multiwavelets, then $H, G$ satisfy (22) with $S = D_0$. In this case, we can choose $U$ to be $I_2$. Thus the corresponding transform is given by (26) and (27) with $U = I_2$. By Corollary 1, for such a multfilter bank $\{H, G\}$, the symmetric extension transforms defined by (23) and (24) with $S = D_0$ are nonepisymmetric.

Assume that $\{H, G\}$ is an FIR multfilter bank. Let $\{H^b, G^b\}$ be the multfilter multfilter banks defined by

$$H^b = R_0 H R_0^T, \quad G^b = R_0 G R_0^T.$$ 

Then one can show that both $H$ and $G$ are symmetric under $D_0$ if and only if $H^b$ and $G^b$ are symmetric under $E$, where

$$E := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the exchange matrix. For $S = E$, choose $U = R_0$. Then $U^{-1} EU = D_0$. Thus in this case for the the original scalar signal $f = (f_0, f_1, f_2, \ldots, f_{2\ell-1})$, the corresponding transform, denoted by $T_{E, f}$, is given by $T_{E, f} := v$, where $v = (v_0, v_1, v_2, \ldots, v_{\ell-1})$ with

$$v_i = \begin{cases} \frac{f_{2i}}{f_{2i-1}}, & 0 \leq i \leq \ell - 1 \end{cases} \quad (29)$$

if the length of the matrix filter is even, and $v = (v_0, v_1, v_2, \ldots, v_{\ell-1}, v_\ell)$ with

$$v_0 = (f_0), v_i = \begin{cases} \frac{f_{2i-1}}{f_{2i-2}}, & 1 \leq i \leq \ell - 1 \end{cases} \quad (30)$$

if the length of the matrix filter is odd. Therefore, when we use $\{H^b, G^b\}$ as the analysis and synthesis multfilter banks, we use the transform $T_{E, f}$ and the symmetric extension transforms given by (23) or (24) with $S = E$. Corollary 1 shows again that the symmetric extension transform with matrix $E$ is nonepisymmetric. The vector input from a scalar signal defined by (29) and (26) with $U = I_2$ were also used in [18] and [19], respectively.

**IV. OPTIMAL MULTFILTER BANKS FOR IMAGE COMPRESSION**

**A. Objective function and optimal multfilter banks**

Let $\{X H^b, X G^b\}$ be the orthogonal FIR multfilter filter banks given by (18) with $X H$ and $X G$ given by (11) and (14), and $\pm, \mp$ in (13) and (15) being $+$ and $-$, respectively, and $v \in O(2)$ in (12) being $R(\theta_k)$. Let $X \Phi, X \Psi$ denote the scaling functions and multiwavelets generated by $\{X H^b, X G^b\}$. To construct OPTFR multiwavelets, we shall determine which objective function $n S(j), 1 \leq j \leq 4$ we shall use. In the following the constrained conditions (19) and (20) used are respectively

$$\left\{ \begin{array}{l} |\sin \theta_0| \leq 10^{-4} \quad \text{for } N = 2 \gamma + 1, \\ |\cos(\theta_0 + \pi/4)| \leq 10^{-4} \quad \text{for } N = 2 \gamma. \end{array} \right. \quad (31)$$

We consider the case $N = 9$. By minimizing different objective functions under (31), we obtain the corresponding parameters and hence the scaling functions $\Phi$ and multiwavelets $\Psi$. In Table 1, we provide the areas of the resolution cells of $\Phi$ and $\Psi$. All the resulting scaling functions $\Phi$ provide approximation order 1. (About the approximation order of functions, see [9].) Let $\text{Ort}_{10}(j)$, denote the corresponding optimal multfilter banks obtained by minimizing $n S(j), 1 \leq j \leq 4$. In Table 2, we list the compression results for $512 \times 512$ standard images Lena and Barbara using $\text{Ort}_{10}(j), 1 \leq j \leq 4$ (see the implementation of $\text{Ort}_{10}(j)$ for image compression in the next subsection).

From Tables 2, the performances in image compression of the optimal multfilter banks obtained by minimizing different objective functions are comparable. The optimal
multifilter bank obtained with $S(4)$ has a little better performance. We also tested optimal multifilter banks of other lengths. We find in some cases and for the Barbara image, the optimal multifilter banks obtained by minimizing $N(S(3))$ also provide a good performance. However in general, the optimal multifilter banks by minimizing $N(S(4))$ provide a good and stable performance for both Lena image and Barbara image. The reasons might be that in the multiwavelet decomposition and reconstruction algorithms, the lowpass filters $\lambda H$ are used as the highpass filters $\lambda G^4$, and it plays the similar role as $\lambda G^4$. Thus the scaling functions shall also have good time–frequency localization as multiwavelets. In the following we choose $N(S(4))$ as the objective function, and let $\text{Ort}_{N+1}$ denote the corresponding optimal multifilter banks.

In Table 3, we provide some parameters for the optimal multifilter banks and the areas of the resolution cells of the corresponding OPTRF scaling functions (denoted by $N\Phi^0 = (N\phi_0^0, N\phi_1^0, N\phi_2^0)$) and multiwavelets (denoted by $N\Psi^0 = (N\psi_1^0, N\psi_2^0)$). From Table 3, we find the scaling functions and multiwavelets with longer lengths have smaller areas of the resolution cells. By the parameters in Table 3 and the expressions of the multifilter banks, one can get the optimal multifilter banks. In Appendix, we provide $\text{Ort}_{N+1}$ for $N = 3, 4, 5$.

B. Image compression

Let $\{\lambda H, \lambda G^4\}$ be the multifilter bank given by (18). Since $\lambda G^4$ does not satisfy the symmetric properties as $\lambda H$ and we cannot use directly the symmetric extension transform provided in the above section, we first discuss the implementation of the symmetric extension transform for $\{\lambda H, \lambda G^4\}$. Define $\lambda G^4_1 := R_0 \lambda G^4$. Then $\lambda H, \lambda G^4_1$ are symmetric under $E$. Thus for scalar signals, we generate the vector signals $v$ by the transform $T_E$ defined by (29) and (30). Then we use the symmetric extension transform of $v$ for $\{\lambda H, \lambda G^4_1\}$ provided in Section III to produce the symmetric vector signals and compute $v^{(-j)}$ and $w^{(-j)}$, $1 \leq j \leq J$ with the discrete multiwavelet decomposition algorithm. Since $\lambda G^4_j = R_0 \lambda G^4_1$, the highpass outputs for $\{\lambda H, \lambda G^4_j\}$ are $R^T w^{(-j)}$. We develop an algorithm based on embedded zero-tree wavelet (EZW) (see [16]) with 5 level multiwavelet decomposition $(J = 5)$ for further quantization and coding of $v^{(-5)}$ and $R^T w^{(-j)}$, $1 \leq j \leq 5$. The compression results for $512 \times 512$ standard images Lena and Barbara using different optimal multifilter banks are listed in Table 4.

In Table 4, $\text{Ort}_4$(vmd3) denotes the multifilter bank of length 4 with the corresponding scaling function $S(4)$ providing approximation order 3. The corresponding parameters are $\theta_0 = -0.25661167176, \theta_1 = 0.256600255142$. The areas of the resolution cells for $\lambda G^4$ and the corresponding multiwavelet $\lambda \Psi$ are: $\Box_{\lambda \phi_1} = 70136$, $\Box_{\lambda \psi_1} = 1.51150$ and $\Box_{\lambda \psi_2} = 1.58041$. $\lambda \Psi$ is the multiwavelet constructed in [3]. In Table 4, $\text{Ort}_0$(smth-v2) denotes the multifilter bank of length 6 with the corresponding scaling function $S(4)$ providing approximation order 2, $S(4)$ as smooth as possible and $\text{Ort}_0$(smth-v2) satisfying (9). The corresponding parameters are: $\theta_0 = 0.0001, \theta_1 = -459212307370, \theta_2 = -2.456942641174$. The scaling function $S(4)$ and the corresponding multiwavelet $5 \Psi$ are in $C^{0.257285}(R)$ (here we use the smoothness estimate provided in [10]). The areas of the resolution cells for $5 \Psi^0$ and $5 \Psi$ are: $\Box_{5 \psi_1} = 0.67903$, $\Box_{5 \psi_1} = 1.15052$ and $\Box_{5 \psi_2} = 1.04253$. From Table 4, we find that good smoothness and approximation do not provide good multifilter banks for image compression.

In Tables 4, we also provide the image compression results with Daubechies’ wavelet filter of length 8, denoted by $D_8$, Daubechies’ least asymmetric wavelet filter of length 8, denoted by $L_{ asym}$, and with the scalar (9,7)–tap biorthogonal wavelet filters, denoted by $S_{biort}_7$. (See [1] and [4] about Daubechies’ wavelet filters and $S_{biort}_7$.) We find that for both Lena and Barbara images, the optimal multifilter banks $\text{Ort}_N$ have better performances than $D_8$ and $L_{ asym}$. For the Barbara image, $\text{Ort}_N$ even have better performances than biorthogonal wavelet $S_{biort}_7$, and for the Lena image, $\text{Ort}_{16}$ has a better performance than $S_{biort}_7$ for compression ratio (CR) $32 : 1$. The original image of Barbara is shown in Fig. 1. The reconstructed images with compression ratio $32 : 1$ by filter $L_{ asym}$, $S_{biort}_7$ and $\text{Ort}_8$ are shown in Fig. 2, Fig. 3 and Fig. 4, respectively. From Fig. 1–4, one can find that there is less distortion in texture parts of the image by the optimal multifilters than other scalar filters. See, for example, the scarf, trousers in Fig. 1–4. See also Fig. 5, Fig. 6, Fig. 7 and Fig. 8, the zooming in images of the right trousers in Fig. 1, Fig. 2, Fig. 3, and Fig. 4, respectively.

In Tables 4, we also provide the compression results using the periodic extension transform for some multifilter banks. From Tables 4, we conclude that the symmetric extension transform presented in Section 3 improves the rate–distortion performance, compared with the periodic extension transform. Finally let us discuss the time complexity for the multiwavelet decomposition and reconstruction algorithms. For a scalar signal $v$ of length $L$ and a symmetric/antisymmetric matrix filter of length $M$, by the symmetric property of the matrix filter one can obtain that the time complexity for one level multiwavelet decomposition algorithm (5) and (6) and for the symmetric extension transform is $LM$ multiplications and $L(2M + 1)$ additions. The time complexity for the same length orthogonal scalar wavelet is about $LM$ multiplications and $L(M – 1)$ additions. In addition, the time complexity of the scalar (9,7)–tap biorthogonal wavelet filter is 4.5L multiplications and 7L additions, higher than those for the symmetric/antisymmetric matrix filter of length 4. Thus the time complexity for the multiwavelet decomposition and reconstruction algorithms and for our symmetric extension transform of the symmetric/antisymmetric matrix filters is very still low.

V. CONCLUSIONS

The design of optimal multifilter banks and optimum time–frequency resolution multiwavelets with different ob-
jective functions is discussed. The symmetric extension transform relates to multifilter banks with a more general symmetric property is presented. We show that the symmetric extension transforms are nonexpansive. The symmetric extension transforms for two kinds of special interesting multifilter banks, multifilter banks generate symmetric/antisymmetric multiwavelets and multifilter banks generate balanced multiwavelets, are discussed. More optimal multifilter banks for image compression are constructed and some of them are used in image compression. The experiments show that optimum time–frequency resolution multiwavelets have better performances in image compression than Daubechies' orthogonal wavelets and Daubechies' least asymmetric wavelets, and for some images, OPTFR multiwavelets even have better performances than the scalar (9,7)-tap biorthogonal wavelets. The experiments also show that the symmetric extension transform presented in this paper improves the rate–distortion performance, compared with the periodic extension transform.

APPENDIX

Denote \( \mathbf{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). The optimal multifilter banks \( \text{Ort}_{N+1} = \{N \mathbf{H}^k \} \) are given by \( N \mathbf{H}^k(\omega) = R_0 N \mathbf{H}(\omega) R_0^T \), \( N \mathbf{G}^k(\omega) = N \mathbf{G}(\omega) R_0^T \), where \( R_0 \) is defined by (17). In the following, we provide \( N \mathbf{H}, N \mathbf{G} \) with \( N = 3, 4, 5 \).

For \( N = 3 \):

\[
\mathbf{H}_0 = \begin{bmatrix} -0.08533247511 & 0.06475961274 \\ -0.085326771507 & -0.064760465743 \end{bmatrix},
\]

\[
\mathbf{H}_1 = \begin{bmatrix} 0.491466752489 & 0.06475961274 \\ -0.491473225993 & 0.064710465743 \end{bmatrix},
\]

and \( \mathbf{H}_j = D_0 \mathbf{H}_{3-j} D_0, 2 \leq j \leq 3, G_k = (-1)^{k+1} \mathbf{H}_j \mathbf{J}, 0 \leq k \leq 3 \).

For \( N = 4 \):

\[
\mathbf{H}_0 = \begin{bmatrix} -0.031578613037 & 0.31578613037 \\ -0.042947475241 & 0.042947475241 \end{bmatrix},
\]

\[
\mathbf{H}_1 = \begin{bmatrix} 0.25 & -1.64111400451 \\ 0.313173635648 & -2.50024998750 \end{bmatrix},
\]

\[
\mathbf{H}_2 = \begin{bmatrix} 0.563157226074 & 0 \\ 0 & 0.41405082657 \end{bmatrix},
\]

\[
\mathbf{G}_0 = \begin{bmatrix} -0.042944299775 & -0.042944299775 \\ 0.315743181449 & 0.315743181449 \end{bmatrix},
\]

\[
\mathbf{G}_1 = \begin{bmatrix} -0.25 & -3.13157226074 \\ -1.64008038907 & 2.49974998750 \end{bmatrix},
\]

\[
\mathbf{G}_2 = \begin{bmatrix} 0.141111400451 \\ 0 & 0.563157226074 \end{bmatrix},
\]

and \( \mathbf{H}_j = D_0 \mathbf{H}_{4-j} D_0, G_j = D_0 \mathbf{G}_{4-j} D_0, 3 \leq j \leq 4 \).

For \( N = 5 \):

\[
\mathbf{H}_0 = \begin{bmatrix} -0.15579570720 & 0.06797482939 \\ -0.15580230391 & -0.06795924948 \end{bmatrix},
\]

\[
\mathbf{H}_1 = \begin{bmatrix} 0.0224412948533 & -0.051509845476 \\ -0.024168978389 & 0.05151091732 \end{bmatrix},
\]

\[
\mathbf{H}_2 = \begin{bmatrix} 0.5305441235 & -0.05803727515 \\ 0.249311269502 & -0.058258016680 \end{bmatrix},
\]

and \( \mathbf{H}_j = D_0 \mathbf{H}_{5-j} D_0, 3 \leq j \leq 5, G_k = (-1)^{k+1} \mathbf{H}_j \mathbf{J}, 0 \leq k \leq 5 \).

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for helpful suggestions and comments on this paper.

REFERENCES


Table 1. The areas of the resolution cells of the OPLTF scaling functions and multiwavelets by different objective functions.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Lena</th>
<th>Barbara</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>32:1</td>
<td>64:1</td>
</tr>
<tr>
<td>Or10(1)</td>
<td>34.063</td>
<td>31.018</td>
</tr>
<tr>
<td>Or10(2)</td>
<td>34.077</td>
<td>31.020</td>
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<tr>
<td>Or10(3)</td>
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<td>31.013</td>
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<tr>
<td>Or10(4)</td>
<td>34.084</td>
<td>31.026</td>
</tr>
</tbody>
</table>

Fig. 1. Original Barbara image.

Fig. 2. Reconstructed image using \( L_{\text{asym}} \), compression ratio=32:1, PSNR=26.444dB.

Fig. 3. Reconstructed image using \( S_{\text{biot}_{0.7}} \), compression ratio=32:1, PSNR=26.738dB.  

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Fig. 4. Reconstructed image using optimal multfilter bank $\text{Ort}_0$,
compression ratio=32 : 1, $\text{PSNR}=27.488$ dB.

Fig. 5. A zooming in part of Fig. 1.

Fig. 6. A zooming in part of Fig. 2.

Fig. 7. A zooming in part of Fig. 3.

Fig. 8. A zooming in part of Fig. 4.
Table 3. The parameters for the optimal multiscaler banks and the areas of the resolution cells of the corresponding OPTFR scaling functions and multwavelets.

<table>
<thead>
<tr>
<th>N</th>
<th>(\theta_0/\theta_4)</th>
<th>(\theta_1/\theta_5)</th>
<th>(\theta_2/\theta_6)</th>
<th>(\theta_3/\theta_7)</th>
<th>(N_{\phi^{\pm}})</th>
<th>(N_{\psi^{\pm}})</th>
<th>(N_{\psi^{\pm}})</th>
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<td>.6736</td>
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<td></td>
<td></td>
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<td>.84374</td>
<td>.60237</td>
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</tbody>
</table>

Table 4. Compression results for images 'Lena' and 'Barbara' with filter banks (symmetric extension and periodic extension) and scalar wavelet filters \(D_8\), \(L_{asym_8}\) and \(S_{biorth_{9.7}}\).