On the construction of biorthogonal multiwavelet bases

Qingtang Jiang[†]

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The author is with the Department of Mathematics, National University of Singapore, Singapore 119260 and the Department of Mathematics, Peking University, Beijing 100871 (E-mail:qjiang@haar.math.nus.edu.sg, Fax: 65–7795452).

DRAFT

Current address: Dept. of Math & CS, Univ. of Missouri-St. Louis

Email: jiangq@umsl.edu

Abstract

This paper discusses the construction of compactly supported biorthogonal multiwavelets based on the parametric expressions of symmetric FIR multifilter banks satisfying the perfect reconstruction (PR) conditions. Explicit expressions of M–channel PR multifilter banks for symmetric/antisymmetric scaling functions and biorthogonal multiwavelets are obtained. Expressions of the 2–channel symmetric PR multifilter banks with high sum rule orders and high balancing orders is discussed. In particular, we give the expressions of the 2–channel symmetric PR multifilter banks with the sum rules of order 2 and symmetric PR multifilter banks with balancing order 2. Based on expressions of the symmetric PR multifilter banks, constructions of smooth, high balanced biorthogonal multiwavelets and optimum time–frequency resolution biorthogonal multiwavelets which are more suitable for image applications are discussed.

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Keywords

Approximation order, balanced biorthogonal multiwavelet, biorthogonal multiwavelet, multifilter bank, scaling function, smoothness, symmetry, parametrization, time-frequency resolution.

I. Introduction

Recently, much work has been done on the construction and applications of multiwavelets (see e.g. [1], [3], [4], [6], [11]–[13], [15], [19], [23], [25] and [29]). This paper is about the construction of biorthogonal multiwavelets. Biorthogonal multiwavelets have some advantages over the orthogonal multiwavelets. In particular, biorthogonal multiwavelets will provide flexibility in their construction and applications. So their construction deserves a systematic study.

For integer $M \geq 2$, the construction of biorthogonal multiwavelets of dilation factor M starts with matrix filters $\mathbf{H}_0, \widetilde{\mathbf{H}}_0$ satisfying

$$\sum_{k=0}^{M-1} \mathbf{H}_0(\omega + 2k\pi/M)\widetilde{\mathbf{H}}_0(\omega + 2k\pi/M)^* = \mathbf{I}_r, \quad \omega \in [0, 2\pi). \tag{1}$$

Throughout this paper, $\mathbf{B}^*$ ($\mathbf{B}^T$ respectively) denotes the Hermitian adjoint (transpose respectively) of the matrix $\mathbf{B}$; and $\mathbf{I}_r$ and $\mathbf{0}_r$ denote the $r \times r$ identity matrix and zero matrix respectively. Suppose $\Psi_0 = (\psi_{1,0}, \cdots, \psi_{r,0})^T$ and $\widetilde{\Psi}_0 = (\widetilde{\psi}_{1,0}, \cdots, \widetilde{\psi}_{r,0})^T$ are the $(M, \mathbf{H}_0)$ and $(M, \widetilde{\mathbf{H}}_0)$ refinable vectors respectively, i.e., Ψ_0 and $\widetilde{\Psi}_0$ are vector-valued functions satisfying

$$\Psi_0(x) = M \sum_{k \in Z} \mathbf{h}_0(k)\Psi_0(Mx - k), \quad \widetilde{\Psi}_0(x) = M \sum_{k \in Z} \widetilde{\mathbf{h}}_0(k)\widetilde{\Psi}_0(Mx - k), \tag{2}$$

or equivalently satisfying

$$\widehat{\Psi}_0(\omega) = \mathbf{H}_0(\omega/M)\widehat{\Psi}_0(\omega/M), \quad \widehat{\widetilde{\Psi}}_0(\omega) = \widetilde{\mathbf{H}}_0(\omega/M)\widehat{\widetilde{\Psi}}_0(\omega/M). \tag{3}$$

Suppose $\Psi_0, \widetilde{\Psi}_0 \in L^2(R)^r$. Define the closed subspaces $V_j(\Psi_0)$, $V_j(\widetilde{\Psi}_0)$ of $L^2(R)$ by

$$V_0(\Psi_0) := \overline{\mathrm{span}}\{\psi_{i,0}(x - k), 1 \leq i \leq r, k \in Z\}, \quad V_0(\widetilde{\Psi}_0) := \overline{\mathrm{span}}\{\widetilde{\psi}_{i,0}(x - k), 1 \leq i \leq r, k \in Z\},$$

and $V_j(\Psi_0) := \{f : f(M^{-j}x) \in V_0(\Psi_0)\}$, $V_j(\widetilde{\Psi}_0) := \{f : f(M^{-j}x) \in V_0(\widetilde{\Psi}_0)\}, j \in Z$. Let $\Psi_\ell = (\psi_{1,\ell}, \cdots, \psi_{r,\ell})^T$ and $\widetilde{\Psi}_\ell = (\widetilde{\psi}_{1,\ell}, \cdots, \widetilde{\psi}_{r,\ell})^T$, $1 \leq \ell \leq M - 1$, be the vectors defined by

$$\Psi_\ell(x) = M \sum_{k \in Z} \mathbf{h}_\ell(k)\Psi_0(Mx - k), \quad \widetilde{\Psi}_\ell(x) = M \sum_{k \in Z} \widetilde{\mathbf{h}}_\ell(k)\widetilde{\Psi}_0(Mx - k), \tag{4}$$

or by

$$\widehat{\Psi}_\ell(\omega) = \mathbf{H}_\ell(\omega/M)\widehat{\Psi}_0(\omega/M), \widehat{\widetilde{\Psi}}_\ell(\omega) = \widetilde{\mathbf{H}}_\ell(\omega/M)\widehat{\widetilde{\Psi}}_0(\omega/M) \tag{5}$$

for some matrix filters $\mathbf{H}_\ell$ and $\widetilde{\mathbf{H}}_\ell$. Then the components of Ψ_ℓ and $\widetilde{\Psi}_\ell$ are in $V_1(\Psi_0)$ and $V_1(\widetilde{\Psi}_0)$ respectively. If $(V_j(\Psi_0))$ and $(V_j(\widetilde{\Psi}_0))$ are two multiresolution analyses (MRA) of multiplicity r, then to construct biorthogonal multiwavelets is to find $\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 1 \le \ell \le M-1$ such that $\Psi_\ell, \widetilde{\Psi}_\ell$ defined by (4) satisfy that

$$\{\psi_{j,\ell}(\cdot - k), 1 \le j \le r, 0 \le \ell \le M-1, k \in Z\}$$

and

$$\{\widetilde{\psi}_{j,\ell}(\cdot - k), 1 \le j \le r, 0 \le \ell \le M-1, k \in Z\}$$

form the Riesz bases of $V_1(\Psi_0)$ and $V_1(\widetilde{\Psi}_0)$ respectively with the property that

$$\langle \psi_{j,\ell}, \widetilde{\psi}_{j',\ell'}(\cdot - k)\rangle = \delta(k)\delta(j-j')\delta(\ell-\ell'), \quad 1 \le j, j' \le r, 0 \le \ell, \ell' \le M-1, k \in Z. \tag{6}$$

For such functions $\Psi_\ell, \widetilde{\Psi}_\ell$, $1 \le \ell \le M-1$, $\{\psi_{j,\ell}(M^d x - k), 1 \le j \le r, 1 \le \ell \le M-1, d, k \in Z\}$ and $\{\widetilde{\psi}_{j,\ell}(M^d x - k), 1 \le j \le r, 1 \le \ell \le M-1, d, k \in Z\}$ constitute dual Riesz bases of $L^2(R)$, and we call such functions a set of **biorthogonal multiwavelets** (abbreviated **BIO multiwavelets** in this paper). A necessary condition for $\Psi_\ell, \widetilde{\Psi}_\ell$, $1 \le \ell \le M-1$ to be BIO multiwavelets is that $\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M$ satisfy the **perfect reconstruction (PR) (or biorthogonal) conditions:**

$$\sum_{k=0}^{M-1} \mathbf{H}_\ell(\omega + 2k\pi/M)\widetilde{\mathbf{H}}_{\ell'}(\omega + 2k\pi/M)^* = \delta(\ell-\ell')\mathbf{I}_r, \quad 0 \le \ell, \ell' < M. \tag{7}$$

For matrix filters $\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M$, we say that $\mathbf{H}_0$ ($\widetilde{\mathbf{H}}_0$ respectively) generates a scaling function Ψ_0 ($\widetilde{\Psi}_0$ respectively) if $(V_j(\Psi_0))$ ($(V_j(\widetilde{\Psi}_0))$ respectively) is an MRA, and say $\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M$ generate a set of BIO multiwavelets (or generate biorthogonal multiwavelet bases) if $\mathbf{H}_0, \widetilde{\mathbf{H}}_0$ generate scaling functions $\Psi_0, \widetilde{\Psi}_0$ and $\Psi_\ell, \widetilde{\Psi}_\ell$, $1 \le \ell \le M-1$ defined by (4) are a set of BIO multiwavelets. The set of matrix filters $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ is called an M–channel multiwavelet filter bank (often abbreviated **multifilter bank**), $\{\mathbf{H}_\ell, 0 \le \ell < M\}$ and $\{\widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ are called the analysis bank and synthesis bank respectively.

By (4), Ψ_ℓ ($\widetilde{\Psi}_\ell$ respectively), $1 \le \ell \le M-1$ have the same regularity with Ψ_0 ($\widetilde{\Psi}_0$ respectively). Thus in general, the procedure to construct BIO multiwavelets with good regularity is that, first to find $\mathbf{H}_0$ and $\widetilde{\mathbf{H}}_0$ such that they satisfy (1) and generate scaling functions Ψ_0, $\widetilde{\Psi}_0$ with good regularity, then to find by matrix extension $\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 1 \le \ell < M$ such that $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ is PR. In some cases, e.g., in the construction of multiwavelets with good time-frequency localization, multiwavelets and scaling functions are constructed simultaneously, and hence the analysis/synthesis banks are needed to be provided in terms of some parameters. This paper discusses the parametric expressions of the PR multifilter banks and the construction of BIO multiwavelets based on such expressions.

The rest of this paper is organized as follows. In Section II, we review some results about the biorthogonality, symmetry and regularity of BIO multiwavelets. In the first subsection of Section III, we discuss the parametric expressions of M–channel PR multifilter banks, and present expressions of a group of symmetric PR multifilter banks. Two-channel PR multifilter banks are discussed in more details in the second subsection of Section III. Based on the parametric expressions of the symmetric PR multifilter banks, we discuss in Section IV the construction of the scaling functions and BIO multiwavelets with good smoothness and approximation properties, with high balancing orders and with optimum time-frequency resolution. The proofs of some lemmas and propositions in Section II–IV are presented in Appendix. Matrix filters $\mathbf{H}_\ell$, $\widetilde{\mathbf{H}}_\ell$ discussed in this paper are **finite impulse response (FIR)** filters, and the corresponding filter coefficients are real.

II. Preliminaries

In this section, we review some results about the biorthogonality, symmetry and regularity of the scaling functions and BIO multiwavelets. Most of the results in this section have been known. Propositions 1–3 and Lemma 1 are new, and their proofs are provided in Appendix.

A. Biorthogonality

Recall that an **MRA of multiplicity** r is a nested sequence of closed subspace V_j in $L^2(R)$ satisfying the following conditions (see [5]): (1°) $V_j \subset V_{j+1}, j \in Z$; (2°) $\bigcap_{j\in Z} V_j = \{0\}$, $\bigcup_{j\in Z} V_j$ is dense in $L^2(R)$; (3°)$f \in V_j \Leftrightarrow f(M\cdot) \in V_{j+1}$; (4°) there exist r functions $\varphi_1, \cdots, \varphi_r$ such that the collection of integer translates $\{\varphi_j(\cdot - k) : 1 \le j \le r, k \in Z\}$ is a Riesz basis of V_0. Vector-valued function $\varphi := (\varphi_1, \cdots, \varphi_r)^T$ is called a **scaling function** and said to generate the MRA (V_j).

Let $\mathbf{H}$ be an FIR matrix filter. Assume that $\mathbf{h}(k) = \mathbf{0}$ if $k < 0$ or $k > N$ for some positive integer N. For positive integers N, $M \ge 2$, let $N(M)$ denote the largest integer smaller than $N/(M-1)$. Let $\mathcal{V}_{N(M)}$ denote the space of all $r \times r$ matrices with each entry a trigonometric polynomial whose Fourier coefficients are real and supported in $[-N(M), N(M)]$. The **transition operator** $\mathbf{T_H}$ associated with $\mathbf{H}$ is defined on $\mathcal{V}_{N(M)}$ by

$$\mathbf{T_H}\mathbf{V}(\omega) := \sum_{k=0}^{M-1} \mathbf{H}(\frac{\omega + 2k\pi}{M})\mathbf{V}(\frac{\omega + 2k\pi}{M})\mathbf{H}(\frac{\omega + 2k\pi}{M})^*, \quad \mathbf{V}(\omega) \in \mathcal{V}_{N(M)}. \tag{8}$$

$\mathbf{T_H}$ leaves $\mathcal{V}_{N(M)}$ invariant. It was shown in [21] (see [10] for $M > 2$) that: if φ is L^2-stable, then $\mathbf{T_H}$ satisfies Conditions E and the 1-eigenvector of $\mathbf{T_H}$ is positive (or negative) definite everywhere. For a matrix $\mathbf{B}$ (or an operator $\mathbf{B}$ defined on a finite dimensional linear space), we say $\mathbf{B}$ satisfies **Condition E** if 1 is a simple eigenvalue of $\mathbf{B}$ and other eigenvalues of $\mathbf{B}$ lie inside the open unit disk.

For two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, let $\mathbf{A} \otimes \mathbf{B} := (a_{ij}\mathbf{B})$ denote the Kronecker product of $\mathbf{A}, \mathbf{B}$. Then the representation matrix of the operator $\mathbf{T_H}$ is (see [9], [10] and [18])

$$\mathcal{T}_\mathbf{H} := (M\mathcal{A}_{Mi-j})_{-N(M)\le i,j\le N(M)}, \tag{9}$$

where $\mathcal{A}_j$ is the $r^2 \times r^2$ matrix defined by $\mathcal{A}_j := \sum_{\kappa=0}^{N} \mathbf{h}(\kappa - j) \otimes \mathbf{h}(\kappa)$.

Assume that $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ is a PR FIR multifilter bank. Then the condition that both $\mathcal{T}_\mathbf{H}$ and $\mathcal{T}_{\widetilde{\mathbf{H}}}$ satisfy Condition E is enough for $\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell$ to generate scaling functions and BIO multiwavelets (see [8] for $M = 2$, and this result can be generalized easily to the general case). Thus for a PR FIR multifilter bank $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$, to check it generates biorthogonal multiwavelet bases, we need only to check both $\mathcal{T}_\mathbf{H}$ and $\mathcal{T}_{\widetilde{\mathbf{H}}}$ satisfy Condition E.

B. Symmetry

In this subsection we consider the symmetry of the scaling functions and BIO multiwavelets. For a vector $\mathbf{c} = (c_1, \cdots, c_r)$ with real numbers c_j, denote

$$\mathbf{D_c}(\omega) := \mathrm{diag}(\pm e^{-ic_1\omega/(M-1)}, \pm e^{-ic_2\omega/(M-1)}, \cdots, \pm e^{-ic_r\omega/(M-1)}). \tag{10}$$

We have the following two propositions about the symmetry of the scaling functions and BIO multiwavelets.

Proposition 1: Assume that $\mathbf{H}$ is an FIR matrix filter and $\varphi = (\varphi_1, \cdots, \varphi_r)^T$ is a compactly supported $(M, \mathbf{H})$ refinable vector with $\widehat{\varphi}(0) \neq 0$. If $\mathbf{H}$ satisfies

$$\mathbf{D_c}(M\omega)\mathbf{H}(-\omega)\mathbf{D_c}(-\omega) = \mathbf{H}(\omega), \tag{11}$$

for some $\mathbf{c} = (c_1, \cdots, c_r) \in R^r$, then φ_j is symmetric/antisymmetric about $\frac{c_j}{2(M-1)}$, i.e.

$$\varphi_j(\frac{c_j}{M-1} - x) = \pm\varphi_j(x), \quad 1 \leq j \leq r. \tag{12}$$

Conversely, if φ is L^2–stable, and $Mc_j = c_i \mathrm{mod}(M-1)$, $1 \leq i, j, \leq r$, then (12) implies (11).

Proposition 2: Assume that $\mathbf{H}$ is an FIR matrix filter satisfying (11) for some $\mathbf{c} = (c_1, \cdots, c_r) \in R^r$, and $\varphi = (\varphi_1, \cdots, \varphi_r)^T$ is a compactly supported $(M, \mathbf{H})$ refinable vector with $\widehat{\varphi}(0) \neq 0$. Let $\mathbf{G}$ be an FIR matrix filter, and $\Psi = (\psi_1, \cdots, \psi_r)^T$ be the vector-valued function defined by

$$\Psi(x) = M \sum_k \mathbf{g}(k)\varphi(Mx - k).$$

If $\mathbf{G}$ satisfies

$$\mathbf{D_d}(M\omega)\mathbf{G}(-\omega)\mathbf{D_c}(-\omega) = \mathbf{G}(\omega), \tag{13}$$

for some $\mathbf{d} = (d_1, \cdots, d_r) \in R^r$, then

$$\psi_j(\frac{d_j}{M-1} - x) = \pm\psi_j(x), \quad 1 \leq j \leq r. \tag{14}$$

Conversely, if ϕ is L^2-stable, $Md_j = c_i \mathrm{mod}(M-1)$, $1 \leq i, j, \leq r$, then (14) implies (13).

In Propositions 1 and 2, the sign $+$ (or $-$) in $\mathbf{D_c}$ and $\mathbf{D_d}$ coincide with $+$ (or $-$) in (12) and (14) respectively. For FIR matrix filters $\mathbf{H}, \mathbf{G}$, (11) and (13) also imply that $Mc_j = c_i \mathrm{mod}(M-1)$ and $Md_j = c_i \mathrm{mod}(M-1)$ respectively.

The symmetry of the scaling functions and multiwavelets is also considered in [1] and [24]. Comparing with their results, more conditions such as $Mc_j = c_i \mathrm{mod}(M-1)$ are added here. We find in some cases such conditions cannot be dropped.

C. Approximation and smoothness

The approximation order of a scaling function is related to the sum rules of the corresponding filter. For a given FIR matrix filter $\mathbf{H}$, if there exists a positive integer k and some $1 \times r$ vectors $\mathbf{y}_j, 0 \leq j < k$ with $\mathbf{y}_0 \neq 0$, such that

$$\sum_{0 \leq s \leq j} \binom{j}{s} (iM)^{s-j} \mathbf{y}_s D^{j-s} \mathbf{H}(\frac{2\pi\ell}{M}) = \delta(\ell) M^{-j} \mathbf{y}_j, \quad 0 \leq \ell < M, \tag{15}$$

for all $0 \leq j < k$, we say that the refinement mask $\mathbf{H}$ has the **sum rules** of order k or $\mathbf{H}$ satisfies the **vanishing moment conditions** of order k. Here $D^j\mathbf{H}(\omega)$ denote the matrix formed by the jth derivatives of the entries of $\mathbf{H}(\omega)$. For an FIR matrix filter $\mathbf{H}$, if $\mathbf{H}$ generates a scaling function Φ, then Φ provides approximation of order k if and only if $\mathbf{H}$ has the sum rules of order k (see e.g., [7], [17], and [10]). In this case,

$$\sum_{n \in Z} \sum_{s=0}^{j} \binom{j}{s} n^{j-s} \mathbf{y}_s \Phi(x - n) = x^j, \quad 0 \leq j \leq k. \tag{16}$$

Therefore to construct scaling function with good approximation, we need only to construct $\mathbf{H}$ satisfying addition conditions (15).

It was shown in [2] (see also [14] and [10]) that if an FIR matrix filter $\mathbf{H}$ generates a compactly supported scaling function, then $\mathbf{H}(0)$ satisfies Condition E and $\mathbf{H}$ has the sum rules of order at least 1, i.e. there exists $1\times r$ vector $\mathbf{y}_0$ such that

$$\mathbf{y}_0\mathbf{H}(2\pi k/M)=\delta(k)\mathbf{y}_0, \quad 0\le k\le M-1. \tag{17}$$

Based on this fact we have the following lemma and proposition.

Lemma 1: Suppose $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0\le \ell<M\}$ is a PR FIR multifilter bank generating biorthogonal multiwavelet bases. Then for any right column 1-eigenvectors $\mathbf{v}$ and $\widetilde{\mathbf{v}}$ of $\mathbf{H}_0$ and $\widetilde{\mathbf{H}}_0$ respectively,

$$\widetilde{\mathbf{v}}^T\mathbf{H}_0(2k\pi/M)=\delta(k)\widetilde{\mathbf{v}}^T, \mathbf{v}^T\widetilde{\mathbf{H}}_0(2k\pi/M)=\delta(k)\mathbf{v}^T, \quad 0\le k<M, \tag{18}$$

$$\mathbf{H}_\ell(0)\widetilde{\mathbf{v}}=0, \widetilde{\mathbf{H}}_\ell(0)\mathbf{v}=0, \quad 1\le \ell<M. \tag{19}$$

Proposition 3: Assume that PR FIR multifilter bank $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0\le \ell<M\}$ generates scaling functions $\Psi_0, \widetilde{\Psi}_0$, and BIO multiwavelets $\Psi_\ell, \widetilde{\Psi}_\ell$. Then
i) $\widehat{\Psi}_0(0)^T$, $\widehat{\widetilde{\Psi}}_0(0)^T$ are left 1–eigenvectors of $\widetilde{\mathbf{H}}_0(0)$, $\mathbf{H}_0(0)$ respectively; and $\widehat{\widetilde{\Psi}}_0(0)^T\widehat{\Psi}_0(0)=1$;
ii) each component of BIO multiwavelets Ψ_ℓ, $\widetilde{\Psi}_\ell$, $1\le\ell\le M-1$ is a bandpass functions, i.e., the integral of each component of Ψ_ℓ, $\widetilde{\Psi}_\ell$ is zero.

The smoothness estimates of a refinable vector Φ in terms of the filter $\mathbf{H}$ are discussed in several papers, and the details are not provided here. Here we use the estimate provided in [9].

III. Symmetric PR FIR multifilter banks

In this section, we discuss the symmetric PR FIR multifilter banks. In Section III.A we consider the case for the general M, and in Section III.B we study in more details for $M=2$.

A. M–channel symmetric PR multifilter banks

For $M\ge 2$, denote $W:=e^{-2\pi i/M}$, and $z:=e^{i\omega}$. For an FIR matrix filter $\mathbf{H}$, we use $\mathbf{H}(z)=\sum_{k\in Z}\mathbf{h}(k)z^{-k}$ to denote $\mathbf{H}(\omega)$. Suppose $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0\le\ell<M\}$ is an M–channel FIR multifilter bank. Let $\mathbf{H}_m$ denote the modulation matrix of the analysis bank $\mathbf{H}_\ell, 0\le\ell<M$ defined by (see [22] and [27] for $r=1$)

$$\mathbf{H}_m(\omega):=\left(\mathbf{H}_\ell(zW^k)\right)_{0\le\ell,k\le M-1}, \tag{20}$$

and let $\widetilde{\mathbf{H}}_m$ denote the modulation matrix of the synthesis bank $\widetilde{\mathbf{H}}_\ell, 0\le\ell\le M-1$ defined similarly. Then $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0\le\ell<M\}$ is PR if and only if $\mathbf{H}_m$ and $\widetilde{\mathbf{H}}_m$ satisfy

$$\mathbf{H}_m(\omega)\widetilde{\mathbf{H}}_m(\omega)^*=\mathbf{I}_{rM}. \tag{21}$$

Write

$$\mathbf{H}_\ell(z)=:\sum_{k=0}^{M-1}z^{-k}\mathbf{H}_{(\ell,k)}(z^M), \quad \mathbf{h}_{(\ell,k)}(n)=:\mathbf{h}_\ell(Mn+k), \quad 0\le\ell\le M-1.$$

Define the polyphase matrix $\mathbf{E}_p(\omega)$ of $\mathbf{H}_\ell, 0\le\ell<M$ by

$$\mathbf{E}_p(\omega):=\left(\mathbf{H}_{(\ell,k)}(z)\right)_{0\le\ell,k\le M-1}.$$

The relation of the modulation and polyphase matrices of $\mathbf{H}_\ell, 0 \le \ell < M$ are given by

$$\mathbf{H}_m(\omega) = \sqrt{M}\mathbf{E}_p(M\omega)\mathbf{U}_M(z),$$

where $\mathbf{U}_M(z)$ is the rM by rM unitary matrix defined by

$$\mathbf{U}_M(z) := \frac{\sqrt{M}}{M}\left((zW^k)^{-\ell}\mathbf{I}_r\right)_{0\le \ell,k\le M-1}.$$

Let $\widetilde{\mathbf{E}}_p(\omega)$ be the polyphase matrix of $\widetilde{\mathbf{H}}_\ell, 0 \le \ell < M$, then

$$\widetilde{\mathbf{H}}_m(\omega) = \sqrt{M}\widetilde{\mathbf{E}}_p(M\omega)\mathbf{U}_M(z).$$

Thus $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ is PR if and only if

$$(\sqrt{M}\mathbf{E}_p(\omega))(\sqrt{M}\widetilde{\mathbf{E}}_p(\omega))^* = \mathbf{I}_{rM}. \tag{22}$$

Therefore to construct a PR FIR multifilter bank $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$, we need only to find $\mathbf{E}_p(\omega)$ and $\widetilde{\mathbf{E}}_p(\omega)$ such that they satisfy (22). Assume that $\mathbf{E}_p(\omega)$ and $\widetilde{\mathbf{E}}_p(\omega)$ are defined by

$$\mathbf{E}_p(\omega) = \frac{\sqrt{M}}{M}\mathbf{V}_\gamma(z)\mathbf{V}_{\gamma-1}(z)\cdots\mathbf{V}_1(z)\mathbf{U}_0, \quad \widetilde{\mathbf{E}}_p(\omega) = \frac{\sqrt{M}}{M}\widetilde{\mathbf{V}}_\gamma(z)\widetilde{\mathbf{V}}_{\gamma-1}(z)\cdots\widetilde{\mathbf{V}}_1(z)\widetilde{\mathbf{U}}_0,$$

where $z = e^{i\omega}, \mathbf{U}_0, \widetilde{\mathbf{U}}_0$ are some $rM \times rM$ real matrices; and

$$\mathbf{V}_k(z) := \mathbf{I}_{rM} + (z^{-1}-1)\mathbf{B}_k, \quad \widetilde{\mathbf{V}}_k(z) := \mathbf{I}_{rM} + (z^{-1}-1)\widetilde{\mathbf{B}}_k, \tag{23}$$

for some $rM \times rM$ real matrices $\mathbf{B}_k, \widetilde{\mathbf{B}}_k$. If $\mathbf{U}_0\widetilde{\mathbf{U}}_0^T = \mathbf{I}_{rM}, \mathbf{V}_k(z)(\widetilde{\mathbf{V}}_k(z^{-1}))^T = \mathbf{I}_{rM}, 1 \le k \le \gamma$, then $\mathbf{E}_p(\omega)$ and $\widetilde{\mathbf{E}}_p(\omega)$ satisfy (22). One can show $\mathbf{V}_k(z)(\widetilde{\mathbf{V}}_k(z^{-1}))^T = \mathbf{I}_{rM}$ if and only if $\widetilde{\mathbf{B}}_k = \mathbf{B}_k^T, \mathbf{B}_k^2 = \mathbf{B}_k$. Once $\mathbf{U}_0$ and $\mathbf{B}_k$ are determined, then $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ defined by

$$[\mathbf{H}_\ell(\omega)]_{\ell=M-1}^0 = \frac{\sqrt{M}}{M}\mathbf{V}_\gamma(z^M)\mathbf{V}_{\gamma-1}(z^M)\cdots\mathbf{V}_1(z^M)\mathbf{U}_0\left[z^{-\ell}\mathbf{I}_r\right]_{\ell=M-1}^0, \tag{24}$$

$$\left[\widetilde{\mathbf{H}}_\ell(\omega)\right]_{\ell=M-1}^0 = \frac{\sqrt{M}}{M}\mathbf{V}_\gamma(z^M)^T\mathbf{V}_{\gamma-1}(z^M)^T\cdots\mathbf{V}_1(z^M)^T\mathbf{U}_0^{-T}\left[z^{-\ell}\mathbf{I}_r\right]_{\ell=M-1}^0 \tag{25}$$

is PR, where $z = e^{i\omega}$.

The lattice structures such as (24) and (25) give the factorization of the $M \times M$ FIR lossless systems (see [27], [28] and references therein). The parametrization of orthogonal FIR multifilter banks also can be provided in such a lattice structure, see [12], [13] and [19]. In this paper, we will use such a structure to construct PR FIR multifilter banks.

Theorem 1: Assume that $\mathbf{U}_0$ is a non–singular matrix and $\mathbf{B}_k, 1 \le k \le \gamma$ are matrices satisfying $\mathbf{B}_k^2 = \mathbf{B}_k$. Then $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ defined by (24) and (25) with $\mathbf{V}_k$ defined by (23) is PR.

Next, we consider PR multifilter banks for the symmetry BIO multiwavelets. In the following, we assume that Mr is even, i.e. $rM =: 2n$ for some positive integer n. We will discuss parametric expressions for such PR multifilter banks that if they generate the scaling functions $\Psi_0, \widetilde{\Psi}_0$ and BIO multiwavelets $\Psi_\ell, \widetilde{\Psi}_\ell, 1 \le \ell \le M-1$, then half of all components of $\{\Psi_\ell, 0 \le \ell \le M-1\}$ are symmetric and the other half components are antisymmetric, and the dual scaling functions and BIO multiwavelets $\widetilde{\Psi}_\ell$ have the same symmetric property.

Assume that $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ is a PR FIR multifilter back generating scaling functions $\Psi_0, \widetilde{\Psi}_0$ and BIO multiwavelets $\Psi_\ell, \widetilde{\Psi}_\ell, 1 \le \ell \le M-1$. Suppose $\mathbf{h}_\ell(k) = \mathbf{0}, \widetilde{\mathbf{h}}_\ell(k) = \mathbf{0}$ if $k \notin [0, (\gamma+1)M-1]$ for some

$\gamma \in Z_+\backslash\{0\}$, then Ψ_ℓ and $\widetilde{\Psi}_\ell$ are supported in $[0, 1+\gamma+\frac{\gamma}{M-1}]$. By Proposition 1 and Proposition 2, Ψ_ℓ, $\widetilde{\Psi}_\ell$ are symmetric/antisymmetric about $\frac{1}{2}(1+\gamma+\frac{\gamma}{M-1})$ if and only if $\mathbf{H}_\ell$ and $\widetilde{\mathbf{H}}_\ell$ satisfy

$$z^{-((\gamma+1)M-1)}\mathbf{D}_\ell\mathbf{H}_\ell(-\omega)\mathbf{D}_0 = \mathbf{H}_\ell(\omega), z^{-((\gamma+1)M-1)}\mathbf{D}_\ell\widetilde{\mathbf{H}}_\ell(-\omega)\mathbf{D}_0 = \widetilde{\mathbf{H}}_\ell(\omega), z = e^{i\omega}, \tag{26}$$

or equivalently

$$\mathbf{D}_\ell\mathbf{h}_\ell((\gamma+1)M-1-k)\mathbf{D}_0 = \mathbf{h}_\ell(k), \mathbf{D}_\ell\widetilde{\mathbf{h}}_\ell((\gamma+1)M-1-k)\mathbf{D}_0 = \widetilde{\mathbf{h}}_\ell(k), 0 \le k < (\gamma+1)M, \tag{27}$$

where $\mathbf{D}_\ell, 0 \le \ell \le M-1$, are some $r \times r$ diagonal matrices with diagonal elements ± 1. Note that the sign $+$ (or $-$) in the (j,j) entry of $\mathbf{D}_\ell$ means that the j–th components of Ψ_ℓ, $\widetilde{\Psi}_\ell$ are symmetric (or antisymmetric). Denote

$$\mathbf{S}_0 := \mathrm{diag}(\mathbf{D}_0, \cdots, \mathbf{D}_{\ell-1}).$$

Then by our assumption, the trace of $\mathbf{S}_0$ is zero. Thus there exists an $rM \times rM$ exchange matrix such that

$$\mathbf{M}_0\mathbf{S}_0\mathbf{M}_0^T = \mathrm{diag}(\mathbf{I}_n, -\mathbf{I}_n).$$

For $rM \times rM$ real matrices $\mathbf{U}_0$, define

$$\mathbf{A}_1(\omega) := \mathbf{U}_0[\mathbf{I}_r,\, z^{-1}\mathbf{I}_r,\, \cdots,\, z^{1-M}\mathbf{I}_r]^T, \quad \widetilde{\mathbf{A}}_1(\omega) := \mathbf{U}_0^{-T}[\mathbf{I}_r,\, z^{-1}\mathbf{I}_r,\, \cdots,\, z^{1-M}\mathbf{I}_r]^T. \tag{28}$$

Then $z^{-(M-1)}\mathbf{S}_0\mathbf{A}_1(-\omega)\mathbf{D}_0 = \mathbf{A}_1(\omega)$ if and only if $\mathbf{U}_0$ can be written as

$$\mathbf{U}_0 = [\mathbf{U}_1,\, \mathbf{U}_2,\, \cdots,\, \mathbf{S}_0\mathbf{U}_2\mathbf{D}_0,\, \mathbf{S}_0\mathbf{U}_1\mathbf{D}_0], \tag{29}$$

for some $rM \times r$ real matrices $\mathbf{U}_j$. If we choose $\mathbf{V}_k(z)$ defined by (23) satisfying $z^{-1}\mathbf{S}_0\mathbf{V}_k(z^{-1})\mathbf{S}_0 = \mathbf{V}_k(z)$, or equivalently

$$z^{-1}\mathrm{diag}(\mathbf{I}_n, -\mathbf{I}_n)\mathbf{M}_0\mathbf{V}_k(z^{-1})\mathbf{M}_0^T\mathrm{diag}(\mathbf{I}_n, -\mathbf{I}_n) = \mathbf{M}_0\mathbf{V}_k(z)\mathbf{M}_0^T, \tag{30}$$

and choose $\mathbf{U}_0$ having the form of (29), then $\mathbf{H}_\ell, 0 \le \ell < M$ defined by (24) satisfy (26). In this case, $\widetilde{\mathbf{H}}_\ell$ defined by (25) also satisfy (26).

We now decide $\mathbf{B}_k$ such that $\mathbf{B}_k^2 = \mathbf{B}_k$ and $\mathbf{V}_k(z)$ defined by (23) satisfies (30).

Lemma 2: Assume that $\mathbf{V}_k(z)$ is defined by

$$\mathbf{V}_k(z) := \mathbf{M}_0^T\big(\mathbf{I}_{rM} + (z^{-1}-1)\mathbf{B}_k\big)\mathbf{M}_0, \tag{31}$$

for some $\mathbf{B}_k$ with $\mathbf{B}_k^2 = \mathbf{B}_k$. Then $\mathbf{V}_k(z)$ satisfies (30) if and only if

$$\mathbf{B}_k = \frac{1}{2}\begin{bmatrix} \mathbf{I}_n & \mathbf{b}_k \\ \mathbf{b}_k^{-1} & \mathbf{I}_n \end{bmatrix} \tag{32}$$

for some non-singular $n \times n$ real matrices $\mathbf{b}_k$.

Proof: Clearly if $\mathbf{B}_k$ is given by (32), then $\mathbf{V}_k(z)$ satisfies (30). Conversely, by (30), $\mathbf{B}_k$ can be written as

$$\mathbf{B}_k = \frac{1}{2}\begin{bmatrix} \mathbf{I}_n & \mathbf{b} \\ \mathbf{c} & \mathbf{I}_n \end{bmatrix}.$$

for some $n \times n$ real matrices $\mathbf{b}, \mathbf{c}$. By $\mathbf{B}_k^2 = \mathbf{B}_k$, one has $\mathbf{c} = \mathbf{b}^{-1}$. ∎

Note that if $\mathbf{V}_k(z)$ satisfies (30), then so does $\mathbf{V}_k(z)^T$.

Theorem 2: Assume that $\mathbf{U}_0$ is a non-singular matrix with the form of (29), and $\mathbf{B}_k$ are matrices given by (32) for some non-singular $n \times n$ real matrices $\mathbf{b}_k$, $1 \le k \le \gamma$. Then $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ defined by (24) and (25) with $\mathbf{V}_k$ defined by (31) is PR and satisfies (26). Furthermore if $\mathbf{T}_{\mathbf{H}_0}$ and $\mathbf{T}_{\widetilde{\mathbf{H}}_0}$ satisfy Condition E, then $\{\mathbf{H}_\ell, \widetilde{\mathbf{H}}_\ell, 0 \le \ell < M\}$ generates a set of symmetric/antisymmetric BIO multiwavelets.

Next, we discuss the two-channel symmetric PR multifilter banks in more details.

B. Two-channel symmetric PR multifilter banks

In this subsection we consider the two-channel PR multifilter banks with the sum rules of order 1. In this case, we use $\mathbf{H}, \widetilde{\mathbf{H}}$ to denote the matrix filters for the scaling functions Φ, $\widetilde{\Phi}$, and use $\mathbf{G}$, $\widetilde{\mathbf{G}}$ to denote the matrix filters for BIO multiwavelets Ψ, $\widetilde{\Psi}$.

Suppose $\{\mathbf{H}, \widetilde{\mathbf{H}}, \mathbf{G}, \widetilde{\mathbf{G}}\}$ is an FIR multifilter bank. For arbitrary non-singular real matrices $\mathbf{u}_1, \mathbf{u}_2$, define

$$\left\{ \begin{array}{l} \mathbf{H}_1(\omega) = \mathbf{u}_1 \mathbf{H}(\omega) \mathbf{u}_1^{-1}, \widetilde{\mathbf{H}}_1(\omega) = \mathbf{u}_1^{-T} \widetilde{\mathbf{H}}(\omega) \mathbf{u}_1^T, \\ \mathbf{G}_1(\omega) = \mathbf{u}_2 \mathbf{G}(\omega) \mathbf{u}_1^{-1}, \widetilde{\mathbf{G}}_1(\omega) = \mathbf{u}_2^{-T} \widetilde{\mathbf{G}}(\omega) \mathbf{u}_1^T. \end{array} \right. \tag{33}$$

Then $\{\mathbf{H}, \widetilde{\mathbf{H}}, \mathbf{G}, \widetilde{\mathbf{G}}\}$ is PR if and only if $\{\mathbf{H}_1, \widetilde{\mathbf{H}}_1, \mathbf{G}_1, \widetilde{\mathbf{G}}_1\}$ is PR; and $\{\mathbf{H}, \widetilde{\mathbf{H}}, \mathbf{G}, \widetilde{\mathbf{G}}\}$ generates scaling functions $\Phi, \widetilde{\Phi}$ and BIO multiwavelets $\Psi, \widetilde{\Psi}$ if and only if $\{\mathbf{H}_1, \widetilde{\mathbf{H}}_1, \mathbf{G}_1, \widetilde{\mathbf{G}}_1\}$ generates scaling functions $\Phi_1, \widetilde{\Phi}_1$ and biorthogonal multiwavelets $\Psi_1, \widetilde{\Psi}_1$ with

$$\Phi_1 = \mathbf{u}_1 \Phi, \quad \widetilde{\Phi}_1 = \mathbf{u}_1^{-T} \widetilde{\Phi}, \quad \Psi_1 = \mathbf{u}_2 \Psi, \quad \widetilde{\Psi}_1 = \mathbf{u}_2^{-T} \widetilde{\Psi}.$$

Denote $\mathbf{i}_1 := (1, 0, \cdots, 0)^T \in R^r$.

Lemma 3: Assume that $\mathbf{x}, \mathbf{y} \in R^r$ are two vectors with $\mathbf{x}^T \mathbf{y} = 1$. Then there exists a non-singular real matrix $\mathbf{u}_1$ such that

$$\mathbf{u}_1 \mathbf{i}_1 = \mathbf{x}, \quad \mathbf{u}_1^{-T} \mathbf{i}_1 = \mathbf{y}.$$

By Proposition 3 i), for any biorthogonal scaling functions $\Phi_1, \widetilde{\Phi}_1$, $\widehat{\Phi}_1(0)^T \widehat{\widetilde{\Phi}}_1(0) = 1$. Thus to construct scaling functions $\Phi_1, \widetilde{\Phi}_1$ with $\widehat{\Phi}_1(0) = \mathbf{x}, \widehat{\widetilde{\Phi}}_1(0) = \mathbf{y}$ for some $\mathbf{x}$, $\mathbf{y} \in R^r$, $\mathbf{x}^T \mathbf{y} = 1$, by (33) with $\mathbf{u}_1$ given in Lemma 3, we need only to construct scaling functions $\Phi, \widetilde{\Phi}$ with $\widehat{\Phi}(0) = \mathbf{i}_1, \widehat{\widetilde{\Phi}}(0) = \mathbf{i}_1$. In the following, we give expressions for the PR multifilter banks with the sum rules of order 1 and scaling functions $\Phi, \widetilde{\Phi}$ satisfying $\widehat{\Phi}(0) = \widehat{\widetilde{\Phi}}(0) = \mathbf{i}_1$. In this case by Lemma 1,

$$\mathbf{i}_1^T \mathbf{H}(0) = \mathbf{i}_1^T, \quad \mathbf{i}_1^T \mathbf{H}(\pi) = \mathbf{0}, \quad \mathbf{i}_1^T \widetilde{\mathbf{H}}(0) = \mathbf{i}_1^T, \quad \mathbf{i}_1^T \widetilde{\mathbf{H}}(\pi) = \mathbf{0}, \quad \mathbf{G}(0)\mathbf{i}_1 = \mathbf{0}, \quad \widetilde{\mathbf{G}}(0)\mathbf{i}_1 = \mathbf{0}. \tag{34}$$

Let $\mathbf{H}_m(\omega)$ and $\widetilde{\mathbf{H}}_m(\omega)$ be the modulation matrices of $\mathbf{H}, \mathbf{G}$ and $\widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}$ respectively. By (34),

$$\mathbf{H}_m(0) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix}, \quad \widetilde{\mathbf{H}}_m(0) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{U}}_1 \end{bmatrix},$$

for some $2r-1$ by $2r-1$ real matrices $\mathbf{U}_1, \widetilde{\mathbf{U}}_1$. Thus the polyphase matrices $\mathbf{E}_p(\omega)$ for $\mathbf{H}, \mathbf{G}$, and $\widetilde{\mathbf{E}}_p(\omega)$ for $\widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}$ satisfy

$$\mathbf{E}_p(0) = \frac{1}{2} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_r & \mathbf{I}_r \\ \mathbf{I}_r & -\mathbf{I}_r \end{bmatrix}, \quad \widetilde{\mathbf{E}}_p(0) = \frac{1}{2} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{U}}_1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_r & \mathbf{I}_r \\ \mathbf{I}_r & -\mathbf{I}_r \end{bmatrix}.$$

By $\mathbf{H}_m(0)\widetilde{\mathbf{H}}_m^T(0)=\mathbf{I}_{2r}$, $\mathbf{U}_1\widetilde{\mathbf{U}}_1^T=\mathbf{I}_{2r-1}$, i.e. $\widetilde{\mathbf{U}}_1=\mathbf{U}_1^{-T}$. Suppose $\mathbf{V}_k(z)$ is defined by (23) for $M=2$ and some $\mathbf{B}_k$ with $\mathbf{B}_k^2=\mathbf{B}_k$, then $\{{}_{2\gamma+1}\mathbf{H}, {}_{2\gamma+1}\mathbf{G}, {}_{2\gamma+1}\widetilde{\mathbf{H}}, {}_{2\gamma+1}\widetilde{\mathbf{G}}\}$ defined by

$$\begin{bmatrix} {}_{2\gamma+1}\mathbf{H}(\omega) \\ {}_{2\gamma+1}\mathbf{G}(\omega) \end{bmatrix} = \frac{1}{2}\mathbf{V}_\gamma(z^2)\cdots\mathbf{V}_1(z^2)\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix}\Big(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix}+\begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix}z^{-1}\Big), \tag{35}$$

$$\begin{bmatrix} {}_{2\gamma+1}\widetilde{\mathbf{H}}(\omega) \\ {}_{2\gamma+1}\widetilde{\mathbf{G}}(\omega) \end{bmatrix} = \frac{1}{2}\mathbf{V}_\gamma(z^2)^T\cdots\mathbf{V}_1(z^2)^T\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1^{-T} \end{bmatrix}\Big(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix}+\begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix}z^{-1}\Big) \tag{36}$$

where $z=e^{i\omega}$, is PR and ${}_{2\gamma+1}\mathbf{H}, {}_{2\gamma+1}\widetilde{\mathbf{H}}$ satisfy the sum rules of order 1.

Usually, the support lengths of the scaling functions and BIO multiwavelets constructed based on (35) and (36) are $2\gamma+1$. To construct scaling functions and BIO multiwavelets with even integer support lengths we would use another form of parametric expression for multifilter banks. Let $\{{}_2\mathbf{H}, {}_2\mathbf{G}, {}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}\}$ be PR FIR filter bank defined by (35) and (36) with $\gamma=1$ for some $\mathbf{B}_1$. We want to choose $\mathbf{B}_1$ such that the degrees of ${}_2\mathbf{H}, {}_2\mathbf{G}$ and ${}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ as polynomials of z^{-1} are smaller than 3. To this end, choose $\mathbf{B}_1$ such that

$$\mathbf{B}_1\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix}\begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix}=\mathbf{0}_{2r\times r}, \quad \mathbf{B}_1^T\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1^{-T} \end{bmatrix}\begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix}=\mathbf{0}_{2r\times r}. \tag{37}$$

Then one has that $\mathbf{B}_1^2=\mathbf{B}_1$ and $\mathbf{B}_1$ satisfies (37) if and only if

$$\mathbf{B}_1=\frac{1}{2}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix}\begin{bmatrix} \mathbf{b} & \mathbf{b} \\ \mathbf{b} & \mathbf{b} \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1^{-1} \end{bmatrix} \tag{38}$$

for some $r\times r$ real matrices $\mathbf{b}$ with $\mathbf{b}^2=\mathbf{b}$.

If $\mathbf{B}_1$ is given by (38), then

$$\begin{bmatrix} {}_2\mathbf{H}(\omega) \\ {}_2\mathbf{G}(\omega) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix}\Big(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix}(\mathbf{I}_r-\mathbf{b})+\begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix}z^{-1}+\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix}\mathbf{b}z^{-2}\Big), \tag{39}$$

$$\begin{bmatrix} {}_2\widetilde{\mathbf{H}}(\omega) \\ {}_2\widetilde{\mathbf{G}}(\omega) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1^{-T} \end{bmatrix}\Big(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix}(\mathbf{I}_r-\mathbf{b}^T)+\begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix}z^{-1}+\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix}\mathbf{b}^Tz^{-2}\Big). \tag{40}$$

Thus to construct scaling functions and multiwavelets with even integer support lengths, we use

$$\begin{bmatrix} {}_{2\gamma}\mathbf{H}(\omega) \\ {}_{2\gamma}\mathbf{G}(\omega) \end{bmatrix} = \mathbf{V}_\gamma(z^2)\cdots\mathbf{V}_2(z^2)\begin{bmatrix} {}_2\mathbf{H}(\omega) \\ {}_2\mathbf{G}(\omega) \end{bmatrix}, \tag{41}$$

$$\begin{bmatrix} {}_{2\gamma}\widetilde{\mathbf{H}}(\omega) \\ {}_{2\gamma}\widetilde{\mathbf{G}}(\omega) \end{bmatrix} = \mathbf{V}_\gamma(z^2)^T\cdots\mathbf{V}_2(z^2)^T\begin{bmatrix} {}_2\widetilde{\mathbf{H}}(\omega) \\ {}_2\widetilde{\mathbf{G}}(\omega) \end{bmatrix}, \tag{42}$$

where $z=e^{i\omega}$, $\mathbf{V}_k(z)$ are given by (23) with $M=2$ and $\mathbf{B}_k$ satisfying $\mathbf{B}_k^2=\mathbf{B}_k$, ${}_2\mathbf{H}, {}_2\mathbf{G}$ and ${}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ are defined by (39) and (40) with $r\times r$ real matrices $\mathbf{b}$ satisfying $\mathbf{b}^2=\mathbf{b}$.

In the rest of this subsection, we consider the PR FIR multifilter banks for symmetric BIO multiwavelets. Here for simplity, we consider the case that $r=2r_1$ for some positive integer r_1, and the first half components of the scaling functions and BIO multiwavelets are symmetric while the other half components are antisymmetric. In this case $\mathbf{D}_0=\mathbf{D}_1=$daig$(\mathbf{I}_{r_1}, -\mathbf{I}_{r_1})$, and here we choose the exchange matrix $\mathbf{M}_0$ to be

$$\mathbf{M}_0=\mathrm{diag}(\mathbf{I}_{r_1}, \begin{bmatrix} \mathbf{0} & \mathbf{I}_{r_1} \\ \mathbf{I}_{r_1} & \mathbf{0} \end{bmatrix}, \mathbf{I}_{r_1}). \tag{43}$$

In the following we will provide a group of PR FIR multifilter banks $\mathbf{H}, \mathbf{G}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}$ satisfying

$$\begin{cases} z^{-N}\mathbf{D}_0\mathbf{H}(-\omega)\mathbf{D}_0 = \mathbf{H}(\omega), z^{-N}\mathbf{D}_0\mathbf{G}(-\omega)\mathbf{D}_0 = \mathbf{G}(\omega), \\ z^{-N}\mathbf{D}_0\widetilde{\mathbf{H}}(-\omega)\mathbf{D}_0 = \widetilde{\mathbf{H}}(\omega), z^{-N}\mathbf{D}_0\widetilde{\mathbf{G}}(-\omega)\mathbf{D}_0 = \widetilde{\mathbf{G}}(\omega), \end{cases} \tag{44}$$

where $N \in Z_+ \backslash \{0, 1\}$.

Assume that FIR matrix filters $\mathbf{H}, \mathbf{G}$ are defined by (35) for some positive integer γ and satisfy (44) with $N = 2\gamma + 1$. Then by (44), one has

$$\mathbf{H}(0) = \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{b} \end{bmatrix}, \quad \mathbf{H}(\pi) = \begin{bmatrix} \mathbf{0} & \mathbf{c} \\ \mathbf{d} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}(0) = \begin{bmatrix} \mathbf{e} & \mathbf{0} \\ \mathbf{0} & \mathbf{f} \end{bmatrix}, \quad \mathbf{G}(\pi) = \begin{bmatrix} \mathbf{0} & \mathbf{g} \\ \mathbf{p} & \mathbf{0} \end{bmatrix}$$

for some $r_1 \times r_1$ real matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}$ and $\mathbf{p}$. Thus the non-singular real matrix $\mathbf{U}_1$ in (35) has the form

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{0} & \mathbf{0} & \mathbf{c} \\ \mathbf{0} & \mathbf{b} & \mathbf{d} & \mathbf{0} \\ \mathbf{e} & \mathbf{0} & \mathbf{0} & \mathbf{g} \\ \mathbf{0} & \mathbf{f} & \mathbf{p} & \mathbf{0} \end{bmatrix}. \tag{45}$$

Similarly, if $\mathbf{H}, \mathbf{G}$ are defined by (41) for some positive integer γ and satisfy (44) with $N = 2\gamma$, then the non-singular real matrix $\mathbf{U}_1$ in (39) has the form

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{0} & \mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{b} & \mathbf{0} & \mathbf{d} \\ \mathbf{e} & \mathbf{0} & \mathbf{g} & \mathbf{0} \\ \mathbf{0} & \mathbf{f} & \mathbf{0} & \mathbf{p} \end{bmatrix}. \tag{46}$$

Since we hope that $\widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}$ defined by (36) or by (42) also satifying (44), $\mathbf{U}_1^{-T}$ in (36) and (40) shall have the same form to $\mathbf{U}_1$. Really, one can check that if $\mathbf{U}_1$ has the form (45) or (46), then $\mathbf{U}_1^{-T}$ has the same form to $\mathbf{U}_1$.

In the next lemma we give $\mathbf{b}$ in (39) and (40) such that ${}_2\mathbf{H}, {}_2\mathbf{G}$ and ${}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ satisfy (44) with $N = 2$.

Lemma 4: Suppose ${}_2\mathbf{H}, {}_2\mathbf{G}$, ${}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ are defined by (39) and (40) with $\mathbf{U}_1$ having the form of (46). Then ${}_2\mathbf{H}, {}_2\mathbf{G}$, ${}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ satisfy (44) for $N = 2$ if and only if ${}_2\mathbf{H}, {}_2\mathbf{G}$, ${}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ are given by

$$\begin{bmatrix} {}_2\mathbf{H}(\omega) \\ {}_2\mathbf{G}(\omega) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{bmatrix} \Big(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix} \begin{bmatrix} \mathbf{I}_{r_1} & -\mathbf{w} \\ -\mathbf{w}^{-1} & \mathbf{I}_{r_1} \end{bmatrix} + 2 \begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix} z^{-1} + \begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix} \begin{bmatrix} \mathbf{I}_{r_1} & \mathbf{w} \\ \mathbf{w}^{-1} & \mathbf{I}_{r_1} \end{bmatrix} z^{-2} \Big), \tag{47}$$

$$\begin{bmatrix} {}_2\widetilde{\mathbf{H}}(\omega) \\ {}_2\widetilde{\mathbf{G}}(\omega) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1^{-T} \end{bmatrix} \Big(\begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix} \begin{bmatrix} \mathbf{I}_{r_1} & -\mathbf{w} \\ -\mathbf{w}^{-1} & \mathbf{I}_{r_1} \end{bmatrix}^T + 2 \begin{bmatrix} \mathbf{I}_r \\ -\mathbf{I}_r \end{bmatrix} z^{-1} + \begin{bmatrix} \mathbf{I}_r \\ \mathbf{I}_r \end{bmatrix} \begin{bmatrix} \mathbf{I}_{r_1} & \mathbf{w} \\ \mathbf{w}^{-1} & \mathbf{I}_{r_1} \end{bmatrix}^T z^{-2} \Big) \tag{48}$$

for some non-singular real $r_1 \times r_1$ matrix $\mathbf{w}$, where $z = e^{i\omega}$.

Theorem 3: Suppose FIR filters $\mathbf{H}, \mathbf{G}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}$ are defined by (35) and (36) or by (41) and (42), where $\mathbf{V}_k(z)$ are defined by (31) with $M = 2$ and $\mathbf{B}_k$ given by (32) $(n = r)$ for some non-singular $r \times r$ real matrices $\mathbf{b}_k$, the real matrix $\mathbf{U}_1$ in (35) and (36) has the form (45), and ${}_2\mathbf{H}, {}_2\mathbf{G}, {}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ are defined by (47), (48) for some $r_1 \times r_1$ non-singular real matrix $\mathbf{w}$ and the real matrix $\mathbf{U}_1$ with the form (46). Then $\{\mathbf{H}, \mathbf{G}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}\}$ is PR, has the sum rules of order one and satisfies (44) for $N = 2\gamma + 1$ or $N = 2\gamma$. Further, if $\mathbf{T}_\mathbf{H}$, $\mathbf{T}_{\widetilde{\mathbf{H}}}$ satisfy Condition E, then $\{\mathbf{H}, \mathbf{G}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}\}$ generates symmetric/antisymmetric BIO multiwavelets.

Proof: By Lemma 4, what remains to show is that ${}_1\mathbf{H}, {}_1\mathbf{G}, {}_1\widetilde{\mathbf{H}}, {}_1\widetilde{\mathbf{G}}$ defined by (35) and (36) for $\gamma = 0$ satisfy (44) with $N = 1$, which follows immediately from the fact that $\mathbf{U}_1$ and $\mathbf{U}_1^{-T}$ have the form (45). ■

IV. Biorthogonal multiwavelets

In this section, we discuss the construction of BIO multiwavelets with good approximation and smoothness properties, BIO multiwavelets with high balancing orders and BIO multiwavelets with good time-frequency localization. Here we consider the case $M = 2, r = 2$. In this case, symmetric PR multifilter banks $\{{}_N\mathbf{H}, {}_N\mathbf{G}, {}_N\widetilde{\mathbf{H}}, {}_N\widetilde{\mathbf{G}}\}$ are defined by (35) and (36) if N is odd and by (41) and (42) if N is even, where $\mathbf{M}_0$ is given by (43) with $\mathbf{I}_{r_1} = 1$, $\mathbf{V}_k(z)$ are defined by (31) with $\mathbf{B}_k$ given by (32) (with $n = 2$) for some non-singular 2×2 real matrices $\mathbf{b}_k$, $\mathbf{U}_1$ in (35) and (36) is defined by (45) for some real numbers b, d, f, g, p with $g \neq 0, bp \neq fd$, and ${}_2\mathbf{H}, {}_2\mathbf{G}, {}_2\widetilde{\mathbf{H}}, {}_2\widetilde{\mathbf{G}}$ are defined by (47) and (48) with $\mathbf{I}_{r_1} = 1$ and $\mathbf{w} = w \in R\backslash\{0\}$, and $\mathbf{U}_1$ by (46).

A. Biorthogonal multiwavelets with good regularity

Based on the explicit expressions, we can construct the symmetric/antisymmetric BIO multiwavelets with good regularity. First we consider the sum rules of fileters. The next proposition gives the relation of parameters for the fileters with sum rule order 2. Suppose $\mathbf{b}_k$ is given by

$$\mathbf{b}_k = \begin{bmatrix} x_k & y_k \\ z_k & t_k \end{bmatrix}, \qquad |\mathbf{b}_k| := x_k t_k - y_k z_k \neq 0. \tag{49}$$

Proposition 4: Assume that ${}_{2\gamma+1}\mathbf{H}$ and ${}_{2\gamma}\mathbf{H}$ are the filters defined above with $\mathbf{b}_k$ given by (49) for some x_k, y_k, z_k, t_k. Then ${}_{2\gamma+1}\mathbf{H}$ has the sum rules of order 2 if and only if

$$\begin{bmatrix} \sum_{k=1}^{\gamma} x_k \\ \sum_{k=1}^{\gamma} y_k \end{bmatrix} = \begin{bmatrix} b & f \\ d & p \end{bmatrix}^{-1} \begin{bmatrix} (2b-1)\eta \\ \frac{1}{2} + 2d\eta \end{bmatrix} \tag{50}$$

for some $\eta \in R$ with $\mathbf{y}_0 = (1, 0), \mathbf{y}_1 = (\gamma + \frac{1}{2}, \eta)$; ${}_{2\gamma}\mathbf{H}$ has the sum rules of order 2 if and only if

$$\begin{bmatrix} \sum_{k=2}^{\gamma} x_k \\ \sum_{k=2}^{\gamma} y_k \end{bmatrix} = \begin{bmatrix} b & f \\ d & p \end{bmatrix}^{-1} \begin{bmatrix} (2b-1)\eta - \frac{w}{2} \\ 2d\eta - \frac{w}{2} \end{bmatrix}$$

for some $\eta \in R$ with $\mathbf{y}_0 = (1, 0), \mathbf{y}_1 = (\gamma, \eta)$.

By the relationship of ${}_N\mathbf{H}$ and ${}_N\widetilde{\mathbf{H}}$, one has ${}_{2\gamma+1}\widetilde{\mathbf{H}}$ has 2 order sum rules if and only if

$$\begin{bmatrix} \sum_{k=1}^{\gamma} \frac{t_k}{|\mathbf{b}_k|} \\ \sum_{k=1}^{\gamma} \frac{z_k}{|\mathbf{b}_k|} \end{bmatrix} = \begin{bmatrix} b & d \\ f & p \end{bmatrix} \begin{bmatrix} (2\tilde{b}-1)\tilde{\eta} \\ \frac{1}{2} + 2\tilde{d}\tilde{\eta} \end{bmatrix} \tag{51}$$

for some $\tilde{\eta} \in R$ with $\tilde{\mathbf{y}}_0 = (1, 0), \tilde{\mathbf{y}}_1 = (\gamma + \frac{1}{2}, \tilde{\eta})$; ${}_{2\gamma}\widetilde{\mathbf{H}}$ has 2 order sum rules if and only if

$$\begin{bmatrix} \sum_{k=2}^{\gamma} \frac{t_k}{|\mathbf{b}_k|} \\ \sum_{k=2}^{\gamma} \frac{z_k}{|\mathbf{b}_k|} \end{bmatrix} = \begin{bmatrix} b & d \\ f & p \end{bmatrix} \begin{bmatrix} (2\tilde{b}-1)\tilde{\eta} - \frac{1}{2w} \\ 2\tilde{d}\tilde{\eta} - \frac{1}{2w} \end{bmatrix}$$

for some $\tilde{\eta} \in R$ with $\tilde{\mathbf{y}}_0 = (1,0), \tilde{\mathbf{y}}_1 = (\gamma, \tilde{\eta})$, where

$$\tilde{b} := \frac{p}{bp-df}, \tilde{d} := \frac{-f}{bp-df}.$$

We also can obtain relationship of the parameters for the filters with higher orders of the sum rules, and the details are not provided here. From Propositions 4, we know if both ${}_N\mathbf{H}$ and ${}_N\widetilde{\mathbf{H}}$ have the sum rules of 2, then two parameters for the filters can be expressed by other parameters. In general, if ${}_N\mathbf{H}$ and ${}_N\widetilde{\mathbf{H}}$ have the sum rules of m, then $2(m-1)$ parameters for the filters will be reduced.

When we construct BIO multiwavelets with good approximation and smoothness properties, we find the parameters for the expressions of $\{{}_N\mathbf{H}, {}_N\mathbf{G}, {}_N\widetilde{\mathbf{H}}, {}_N\widetilde{\mathbf{G}}\}$ given above are redundant. In the following we reduce the "redundant" parameters.

Note that for a PR FIR filter bank generating scaling functions and BIO multiwavelets, with the change of the PR FIR banks by (33) for some non–singular matrices $\mathbf{u}_1, \mathbf{u}_2$, the corresponding new scaling functions and BIO multiwavelets have the same approximation and smoothness properties to the old ones. In particular by choosing $\mathbf{u}_1 = \text{diag}(1, u_1)$, $\mathbf{u}_2 = \text{diag}(u_2, u_3)$ for some $u_j \neq 0$, we can assume that d, f, g of $\mathbf{U}_1$ in (45) to be ± 1, and f, g, w in (47) and (48) to be ± 1. Such special choices of d, f, g and f, g, w have no influence to the construction of scaling functions and BIO multiwavelets with good approximation and smoothness properties. In the following we choose

$$\mathbf{U}_1 = \begin{bmatrix} b & 1 & 0 \\ 0 & 0 & 1 \\ 1 & p & 0 \end{bmatrix} \text{(for odd } N) \quad \text{and} \quad \mathbf{U}_1 = \begin{bmatrix} b & 0 & d \\ 0 & 1 & 0 \\ 1 & 0 & p \end{bmatrix} \text{(for even } N) \,, \tag{52}$$

with $\det(\mathbf{U}_1) \neq 0$, and choose w in (47) and (48) to be 1. In this case there are $2N$ ($2N-1$ respectively) free parameters for $\{{}_N\mathbf{H}, {}_N\mathbf{G}, {}_N\widetilde{\mathbf{H}}, {}_N\widetilde{\mathbf{G}}\}$ for odd N (for even N respectively), and $2(N-m+1)$ ($2(N-m)+1$ respectively) free parameters for the filter bank with ${}_N\mathbf{H}$ and ${}_N\widetilde{\mathbf{H}}$ having the sum rules of order m for odd N (for even N respectively).

Let $\{{}_3\mathbf{H}, {}_3\mathbf{G}, {}_3\widetilde{\mathbf{H}}, {}_3\widetilde{\mathbf{G}}\}$ be the PR FIR multifilter bank given above for $\gamma = 1$ with $\mathbf{U}_1$ given by (52). By Proposition 4, the condition for ${}_3\mathbf{H}$ having the sum rules of order 2 is

$$x_1 = (2b-1)(y_1 p - 1/2) - 2y_1;$$

and the addition condition for both ${}_3\mathbf{H}$ and ${}_3\widetilde{\mathbf{H}}$ having the sum rules of order 2 is

$$t_1 = \frac{2\tilde{d}\tilde{f} + (\tilde{b}-1/2)(y_1 - 2\tilde{p})}{\tilde{d} + (\tilde{b}-1/2)x_1} z_1, \tag{53}$$

where $\tilde{b} := \frac{p}{bp-1}, \tilde{d} := \frac{1}{1-bp}, \tilde{f} := \frac{1}{1-bp}, \tilde{p} := \frac{b}{bp-1}$. Thus there are 4 free parameters for ${}_3\mathbf{H}, {}_3\widetilde{\mathbf{H}}$ having the sum rules of order 2. One can check the addition condition for ${}_3\mathbf{H}$ and ${}_3\widetilde{\mathbf{H}}$ having the sum rules of order 3 are $4y_1 + |\mathbf{b}_1| = 0$ and $z_1 = \frac{1}{4}$ respectively. We find if both ${}_3\mathbf{H}$ and ${}_3\widetilde{\mathbf{H}}$ have the sum rules of order 3, we cannot construct the scaling functions with good smoothness. We will construct smooth scaling functions with ${}_3\mathbf{H}$ and ${}_3\widetilde{\mathbf{H}}$ having the sum rules of order 3 and order 2 respectively. In this case, there are 3 free parameters y_1, b, p for the filters. For

$$y_1 = -1/8, \quad b = -3/16, \quad p = 1/16,$$

then $x_1 = 971/1024, z_1 = 18502/36033, t_1 = 241727/2306112$, and the corresponding ${}_3\Phi \in W^{2.3386}(R)$, and ${}_3\widetilde{\Phi} \in W^{1.5182}(R)$. For

$$y_1 = -5/32, \quad b = -7/32, \quad p = 1/64,$$

then $x_1 = 33907/32768$, $z_1 = 1507055/2571237$, $t_1 = 311843975/2632946688$, and the corresponding ${}_3\Phi \in W^{2.0181}(R)$, and ${}_3\widetilde{\Phi} \in W^{1.6569}(R)$. ${}_3\Phi, {}_3\Psi$ and ${}_3\widetilde{\Phi}, {}_3\widetilde{\Psi}$ are shown in Fig. 1 and Fig. 2 respectively. In this way we can construct more BIO multiwavelets with good regularity.

For orthogonal scaling function Φ and multiwavelet Ψ with the same symmetry and support to ${}_3\Phi, {}_3\Psi$, we can only construct $\Phi, \Psi \in W^{1.7668}(R)$ with Φ providing approximation order 2 (see [11]). Thus there are more flexibilities for the construction of BIO multiwavelets than the orthogonal ones.

B. High balanced BIO multiwavelets

In image processing applications, the balanced orthogonal multiwavelets are required (see [15]). In [16], [20], high balanced (k–balanced) orthogonal multiwavelets are introduced. Suppose $\{\mathbf{H}, \mathbf{G}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}\}$ is a PR multifilter bank generating scaling functions and BIO multiwavelets $\Phi, \widetilde{\Phi}$ and $\Psi, \widetilde{\Psi}$. When we use $\{\mathbf{H}, \mathbf{G}\}$ as the analysis filter bank, we say Ψ and $\widetilde{\Psi}$ or $\mathbf{H}$ is k–**balanced** (**balanced** for $k = 1$) if $\mathbf{H}$ has the sum rules of order at least k with some vectors $\mathbf{y}_j = (y_{j,1}, y_{j,2})$ and

$$\int \widetilde{\phi}_1(t)t^j dt = \int \widetilde{\phi}_2(t)(t - \frac{1}{2})^j dt, \quad 0 \le j < k. \tag{54}$$

By (16), $\mathbf{y}_j = \int \widetilde{\phi}(t)^T t^j dt$. Thus (54) is equivalent to (see [16] and [20] for other equivalent forms)

$$y_{j,2} - y_{j,1} = \sum_{0 \le s < j} \binom{j}{s} 2^{s-j} y_{s,1}, \quad 0 \le j < k. \tag{55}$$

In this case, we have

i) $\int \widetilde{\psi}(t)t^j dt = 0, \quad 0 \le j < k$;

ii) $\sum_n (2n)^j \phi_1(t-n) + (2n+1)^j \phi_2(t-n) \in P_j(R)$, the space of polynomials of degree $< j$ over R, $\quad 0 \le j < k$;

iii) $\sum_n ((2n)^j, (2n+1)^j)\mathbf{H}_{\ell-2n} = (p(2\ell), p(2\ell+1))$ for some $p \in P_j(R)$, $\quad 0 \le j < k$;

iv) $\sum_n ((2n)^j, (2n+1)^j)\widetilde{\mathbf{G}}^T_{n-2\ell} = (0, 0), \quad 0 \le j < k$.

Thus a k–balanced multifilter bank has the properties: Annihilation of $P_k(R)$ and $P_k(Z)$; preservation of $P_k(R)$ and $P_k(Z)$ (see also [20]).

For the multifilter bank $\{{}_N\mathbf{H}, {}_N\mathbf{G}, {}_N\widetilde{\mathbf{H}}, {}_N\widetilde{\mathbf{G}}\}$ defined above, it would generate scaling functions with ${}_N\widehat{\Phi}(0) = [1, 0]^T$, ${}_N\widehat{\widetilde{\Phi}}(0) = [1, 0]^T$. So that the biorthogonal multiwavelets ${}_N\Psi, {}_N\widetilde{\Psi}$ cannot be balanced. Let ${}_N\mathbf{H}^b, {}_N\mathbf{G}^b, {}_N\widetilde{\mathbf{H}}^b, {}_N\widetilde{\mathbf{G}}^b$ be the PR multifilter bank defined by

$$\begin{cases} {}_N\mathbf{H}^b(\omega) = \mathbf{R}_0\, {}_N\mathbf{H}(\omega)\mathbf{R}_0^T, & {}_N\widetilde{\mathbf{H}}^b(\omega) = \mathbf{R}_0\, {}_N\widetilde{\mathbf{H}}(\omega)\mathbf{R}_0^T, \\ {}_N\mathbf{G}^b(\omega) = \mathbf{u}_2\, {}_N\mathbf{G}(\omega)\mathbf{R}_0^T, & {}_N\widetilde{\mathbf{G}}^b(\omega) = \mathbf{u}_2^{-T}\, {}_N\widetilde{\mathbf{G}}(\omega)\mathbf{R}_0^T, \end{cases} \tag{56}$$

where $\mathbf{R}_0$ is the matrix defined by

$$\mathbf{R}_0 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

and $\mathbf{u}_2$ is a non-singular matrix. Then the corresponding scaling functions and BIO multiwavelets denoted ${}_N\Phi^b, {}_N\Phi^b$ and ${}_N\Psi^b, {}_N\widetilde{\Psi}^b$ are balanced and satisfy

$${}_N\Phi^b = \mathbf{R}_0\, {}_N\Phi, \quad {}_N\widetilde{\Phi}^b = \mathbf{R}_0\, {}_N\widetilde{\Phi}, \quad {}_N\Psi^b = \mathbf{u}_2\, {}_N\Psi, \quad {}_N\widetilde{\Psi}^b = \mathbf{u}_2^{-T}\, {}_N\widetilde{\Psi}. \tag{57}$$

Suppose ${}_N\mathbf{H}$ has the sum rules of order m with $\mathbf{y}_j, 0 \le m-1$, then ${}_N\mathbf{H}^b$ has the sum rules of order m with $\mathbf{v}_j, 0 \le m-1$ given by $\mathbf{v}_j = \mathbf{y}_j\mathbf{R}_0$. Thus ${}_N\mathbf{H}^b$ is k–balanced ($k \le m$) if and only if

$$y_{j,2} - y_{j,1} = \frac{\sqrt{2}}{2}\sum_{0 \le s < j} \binom{j}{s} 2^{s-j}(y_{s,1} - y_{s,2}), \quad 0 \le j < k.$$

In particular, $y_{1,2} - y_{1,1} = \frac{\sqrt{2}}{4}$ is equivalent to ${}_N\mathbf{H}^b$ being 2–balanced. Thus from Prop. 4, by just letting $\eta = \gamma + \frac{1}{2} + \frac{\sqrt{2}}{4}$ or $\eta = \gamma + \frac{\sqrt{2}}{4}$, we have the parametric expression for the 2–balanced ${}_N\mathbf{H}^b$.

If we want to construct BIO multiwavelets with high balancing orders, again the parameters for ${}_N\mathbf{H}^b$, ${}_N\mathbf{G}^b$, ${}_N\widetilde{\mathbf{H}}^b$, ${}_N\widetilde{\mathbf{G}}^b$ given above are redundant. As in the above subsection, we can let f, g in $\mathbf{U}_1$ to be ± 1.

Let $\{{}_3\mathbf{H}, {}_3\mathbf{G}, {}_3\widetilde{\mathbf{H}}, {}_3\widetilde{\mathbf{G}}\}$ be the PR FIR multifilter bank given above for $\gamma = 1$ with f, g in $\mathbf{U}_1$ to be -1, and let ${}_3\mathbf{H}^b, {}_3\mathbf{G}^b, {}_3\widetilde{\mathbf{H}}^b, {}_3\widetilde{\mathbf{G}}^b$ be the filters defined by (56). If x_1, y_1 are given by (50) (for $\gamma = 1$) with $\eta = \frac{3}{2} + \frac{\sqrt{2}}{4}$, then ${}_3\mathbf{H}^b$ is 2–balanced. Let t_1 be given by (53) with $\tilde{b} := \frac{p}{bp+d}, \tilde{d} := \frac{-1}{bp+d}, \tilde{f} := \frac{d}{bp+d}, \tilde{p} := \frac{b}{bp+d}$, then ${}_3\widetilde{\mathbf{H}}^b$ has the sum rules of order 2. For

$$b = -27/128, \quad d = 17/128, \quad p = -5/8192, \quad z_1 = -31/256,$$

the resulting 2–balanced $\Phi \in W^{1.9988}(R)$, and $\widetilde{\Phi} \in W^{1.5783}(R)$. Based on the parametric expressions of ${}_N\mathbf{H}, {}_N\widetilde{\mathbf{H}}$, we can construct more high order balanced BIO multiwavelets.

C. Biorthogonal multiwavelets with optimum time-frequency resolution

In this subsection, we will construct BIO multiwavelets with better time-frequency resolution. The OPTFR orthonormal multiwavelets were studied and constructed in [11] and [12]. Here we will design the OPTFR BIO multiwavelets which will be more suitable for image processing.

Recall for a window function f, the **time-duration** Δ_f of f is defined by

$$\Delta_f^2 := \int_R (t - \overline{t})^2 |f(t)|^2 dt/E, \quad \text{with } \overline{t} := \int_R t|f(t)|^2 dt/E, \quad E := \int_R |f(t)|^2 dt.$$

The **frequency-bandwidth** of f is defined as $\Delta_{\widehat{f}}$. The product of time-duration and frequency-bandwidth $\Box_f := \Delta_f \Delta_{\widehat{f}}$ is called **resolution cell**.

Since Proposition 3 shows that every component denoted by f_j of BIO multiwavelets is a bandpass function, we shall also consider the frequency-bandwidth $\Delta^b_{\widehat{f_j}}$ of f_j defined by

$$(\Delta^b_{\widehat{f_j}})^2 := \int_0^{+\infty} (\omega - \overline{\omega}_{\widehat{f_j}})^2 |\widehat{f_j}(\omega)|^2 d\omega \Big/ \int_0^{+\infty} |\widehat{f_j}(\omega)|^2 d\omega$$

with

$$\overline{\omega}_{\widehat{f_j}} := \int_0^{+\infty} \omega |\widehat{f_j}(\omega)|^2 d\omega \Big/ \int_0^{+\infty} |\widehat{f_j}(\omega)|^2 d\omega.$$

One can check that for real f_j, $(\Delta^b_{\widehat{f_j}})^2 = \Delta^2_{\widehat{f_j}} - (\overline{\omega}_{\widehat{f_j}})^2$. We denote $\Box^b_f := \Delta_f \Delta^b_{\widehat{f}}$.

For $M = 2$, formulas to compute the time-durations and frequency-bandwidths of scaling functions and BIO multiwavelets are provided in [11]. Such results can be generalized to the case $M > 2$ and we would not give the details here.

Assume that $\{\mathbf{H}, \mathbf{G}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}\}$ is a PR FIR multifilter bank. Then the lowpass frequency responses and highpass frequency responses for this system are (see [26])

$$\begin{cases} h_\alpha(\omega) := \sum_{k\in Z} \mathbf{h}(k)_{\alpha,1} e^{-2ik\omega} + \mathbf{h}(k)_{\alpha,2} e^{-i(2k+1)\omega}, \\ \widetilde{h}_\alpha(\omega) := \sum_{k\in Z} \widetilde{\mathbf{h}}(k)_{\alpha,1} e^{-2ik\omega} + \widetilde{\mathbf{h}}(k)_{\alpha,2} e^{-i(2k+1)\omega}, \quad \alpha = 1, 2, \end{cases} \tag{58}$$

and

$$\begin{cases} q_\alpha(\omega) := \sum_{k\in Z} \mathbf{g}(k)_{\alpha,1} e^{-2ik\omega} + \mathbf{g}(k)_{\alpha,2} e^{-i(2k+1)\omega}, \\ \widetilde{q}_\alpha(\omega) := \sum_{k\in Z} \widetilde{\mathbf{g}}(k)_{\alpha,1} e^{-2ik\omega} + \widetilde{\mathbf{g}}(k)_{\alpha,2} e^{-i(2k+1)\omega}, \quad \alpha = 1, 2, \end{cases} \tag{59}$$

where $\mathbf{A}_{\ell,j}$ denotes the (ℓ,j)-entry of the matrix $\mathbf{A}$. $h_\alpha, \widetilde{h}_\alpha$ act as lowpass filters, while $q_\alpha, \widetilde{q}_\alpha$ act as highpass filters. Thus it is required that

$$|h_\alpha(0)| = |\widetilde{h}_\alpha(0)| = |q_\alpha(\pi)| = |\widetilde{q}_\alpha(\pi)| = 1, \quad h_\alpha(\pi) = \widetilde{h}_\alpha(\pi) = q_\alpha(0) = \widetilde{q}_\alpha(0) = 0, \quad \alpha = 1,2.$$

Assume that $\mathbf{H}$ and $\widetilde{\mathbf{H}}$ generate scaling functions Φ and $\widetilde{\Phi}$ respectively. By Lemma 1 and the facts that $[h_1(0), h_2(0)]^T = \mathbf{H}(0)(1,1)^T, [q_1(0), q_2(0)]^T = \mathbf{G}(0)(1,1)^T$, we have that if $\widehat{\Phi}(0) = c_0(1,1)^T, \widehat{\widetilde{\Phi}}(0) = \tilde{c}_0(1,1)^T$, then $h_\alpha(0) = 1, q_\alpha(0) = 0$, $\alpha = 1,2$. Similarly we have $\widetilde{h}_\alpha(0) = 1, \widetilde{q}_\alpha(0) = 0$, $\alpha = 1,2$.

Proposition 5: Let $\{\mathbf{H}, \mathbf{G}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{G}}\}$ be a PR FIR multifilter bank with $\mathbf{H}, \widetilde{\mathbf{H}}$ balanced. Then

$$h_\alpha(0) = \widetilde{h}_\alpha(0) = 1, \quad q_\alpha(0) = \widetilde{q}_\alpha(0) = 0, \quad \alpha = 1,2.$$

From Proposition 5, what we need to design is the requirements:

$$h_\alpha(\pi) = \widetilde{h}_\alpha(\pi) = 0, \quad |q_\alpha(\pi)| = |\widetilde{q}_\alpha(\pi)| = 1, \quad \alpha = 1,2. \tag{60}$$

Proposition 6: Let ${}_N\mathbf{H}^b$, ${}_N\mathbf{G}^b$ be the filters defined by (56) with $\mathbf{u}_2 = \mathbf{R}_0$ and ${}_N\mathbf{H}$, ${}_N\mathbf{G}$ satisfying (44). Let ${}_Nh^b_\alpha, {}_Nq^b_\alpha, \alpha = 1,2$, be the corresponding frequency responses defined by (58) and (59) respectively, then

$${}_Nh^b_2(\omega) = e^{-i(2N+1)\omega}\,{}_Nh^b_1(-\omega), \quad {}_Nq^b_2(\omega) = e^{-i(2N+1)\omega}\,{}_Nq^b_1(-\omega), \tag{61}$$

and ${}_Nh^b_1(\pi) = {}_N\mathbf{H}(0)_{2,2}$, ${}_Nq^b_1(\pi) = {}_N\mathbf{G}(0)_{2,2}$, here ${}_N\mathbf{H}(0)_{2,2}$ and ${}_N\mathbf{G}(0)_{2,2}$ denote the $(2,2)$-entries of ${}_N\mathbf{H}(0)$ and ${}_N\mathbf{G}(0)$ respectively.

On can show Proposition 6 as in [12].

We now use the parametric expressions for PR FIR multifilter banks provided in Section III.B to construct the balanced BIO multiwavelets with good time-frequency localization under the constrained conditions (60). Assume that the PR multifilter bank $\{{}_N\mathbf{H}, {}_N\mathbf{G}, {}_N\widetilde{\mathbf{H}}, {}_N\widetilde{\mathbf{G}}\}$ defined by (35), (36) or by (41), (42) with $r = 2$, generates ${}_N\Phi, {}_N\Phi$ and ${}_N\Psi, {}_N\widetilde{\Psi}$. Let $\{{}_N\mathbf{H}^b, {}_N\mathbf{G}^b, {}_N\widetilde{\mathbf{H}}^b, {}_N\widetilde{\mathbf{G}}^b\}$ be the PR multifilter bank defined by (56) with $\mathbf{u}_2 = \mathbf{R}_0$; and let ${}_N\Phi^b, {}_N\Phi^b$ and ${}_N\Psi^b, {}_N\widetilde{\Psi}^b$ denote the corresponding balanced scaling functions and BIO multiwavelets. For ${}_N\Phi^b = ({}_N\phi^b_1, {}_N\phi^b_2)^T$ and ${}_N\Psi^b = ({}_N\psi^b_1, {}_N\psi^b_2)^T$, they lost symmetry, but they possess another property: the first components are the reflections of the second components about their center point $N/2$, i.e.

$${}_N\phi^b_2(N-x) = {}_N\phi^b_1(x), \quad {}_N\psi^b_2(N-x) = {}_N\psi^b_1(x).$$

${}_N\widetilde{\Phi}^b$ and ${}_N\widetilde{\Psi}^b$ also possess the same property.

Let ${}_N\widetilde{h}^b_\alpha, {}_N\widetilde{q}^b_\alpha, \alpha = 1,2$, be the frequency responses corresponding to ${}_N\widetilde{\mathbf{H}}^b$, ${}_N\widetilde{\mathbf{G}}^b$. Then by Proposition 6, they also satisfy the relationship (61) and ${}_N\widetilde{h}^b_1(\pi) = {}_N\widetilde{\mathbf{H}}(0)_{2,2}$, ${}_N\widetilde{q}^b_1(\pi) = {}_N\widetilde{\mathbf{G}}(0)_{2,2}$. Thus the requirements (60) for ${}_Nh^b_\alpha, {}_N\widetilde{h}^b_\alpha, {}_Nq^b_\alpha, {}_N\widetilde{q}^b_\alpha$ are

$${}_N\mathbf{H}(0)_{2,2} = {}_N\widetilde{\mathbf{H}}(0)_{2,2} = 0, \quad |{}_N\mathbf{G}(0)_{2,2}| = |{}_N\widetilde{\mathbf{G}}(0)_{2,2}| = 1. \tag{62}$$

Constrained conditions (62) are equivalent to that $\mathbf{U}_1$ in (35), (36) and in (47), (48) with $r = 2$ are given respectively by

$$\mathbf{U}_1 = \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & g \\ \pm 1 & 0 & 0 \end{bmatrix} \text{(for odd } N\text{)} \quad \text{and} \quad \mathbf{U}_1 = \begin{bmatrix} 0 & 0 & d \\ 0 & g & 0 \\ \pm 1 & 0 & 0 \end{bmatrix} \text{(for even } N\text{)}\ . \tag{63}$$

In the following, $\mathbf{U}_1$ in (35), (36) and in (47), (48) are given by (63) for $g \in R\backslash\{0\}$. In this case the number of free parameters for ${}_N\mathbf{H}^b, {}_N\widetilde{\mathbf{H}}^b$, ${}_N\mathbf{G}^b, {}_N\widetilde{\mathbf{G}}^b$ is $2N-1$ if N is odd and is $2N-2$ if N is even. We construct the OPTFR BIO multiwavelets by minimizing the sum

$$ {}_N S := \Box^b_{{}_N\phi_1^b} + \Box^b_{{}_N\psi_1^b} + \Box^b_{{}_N\widetilde{\phi}_1^b} + \Box^b_{{}_N\widetilde{\psi}_1^b}. $$

Let ${}_N\Phi^{bo}, {}_N\widetilde{\Phi}^{bo}$ and ${}_N\Psi^{bo}$, ${}_N\widetilde{\Psi}^{bo}$ denote the resulting scaling functions and BIO multiwavelets. Of course, to construct OPTFR BIO multiwavelets, we can use other objective functions, e.g. $\Box_{{}_N\phi_1^b} + \Box_{{}_N\psi_1^b} + \Box_{{}_N\widetilde{\phi}_1^b} + \Box_{{}_N\widetilde{\psi}_1^b}$, $\Box^b_{{}_N\widetilde{\phi}_1^b} + \Box^b_{{}_N\widetilde{\psi}_1^b}$ or $\Box^b_{{}_N\psi_1^b} + \Box^b_{{}_N\widetilde{\psi}_1^b}$. The minimizations are performed as follows: First based on Propositions 4 and on the parametric expressions of ${}_N\mathbf{H}$, ${}_N\mathbf{G}$, ${}_N\widetilde{\mathbf{H}}$, ${}_N\widetilde{\mathbf{G}}$ with $\mathbf{U}_1$ given by (63), to construct a group of smooth scaling functions ${}_N\Phi, {}_N\widetilde{\Phi}$ and BIO multiwavelets ${}_N\Psi, {}_N\widetilde{\Psi}$ with approximation order 2, then to use these ${}_N\Phi, {}_N\widetilde{\Phi}$ and ${}_N\Psi, {}_N\widetilde{\Psi}$ as starting values for the construction of the OPTFR scaling functions and BIO multiwavelets, and finally to choose the best results. In Table 1, the areas of the resolution cells of ${}_N\Phi^{bo}, {}_N\widetilde{\Phi}^{bo}$, ${}_N\Psi^{bo}$, ${}_N\widetilde{\Psi}^{bo}$ are listed for $3 \le N \le 6$. ${}_3\Phi^{bo}$, ${}_3\Psi^{bo}$ and ${}_3\widetilde{\Phi}^{bo}$, ${}_3\widetilde{\Psi}^{bo}$ are shown in Fig. 3 and Fig. 4 respectively. The optimal filters are available from the author.

V. Conclusion

In this paper, the explicit expressions of M–channel PR FIR multifilter banks for symmetric/antisymmetric scaling functions and biorthogonal multiwavelets are discussed. The expressions of the 2–channel symmetric PR FIR multifilter banks with high sum rule orders and high balancing orders are discussed in more details. In particular, we give the expressions of the 2–channel symmetric PR FIR multifilter banks with the sum rules of order 2 and symmetric PR FIR multifilter banks with balancing order 2. The constructions of smooth, high balanced biorthogonal multiwavelets and the optimum time–frequency resolution biorthogonal multiwavelets which are more suitable for image applications are discussed.

Appendix

Proof of Lemma 1. Since both $\mathbf{H}_0$ and $\widetilde{\mathbf{H}}_0$ generate compactly supported scaling functions, they satisfy the sum rules of order 1, and $\mathbf{H}_0(0)$, $\widetilde{\mathbf{H}}_0(0)$ satisfy Condition E. Thus for any left row 1-eigenvectors $\mathbf{y}_0$ and $\widetilde{\mathbf{y}}_0$ of $\mathbf{H}_0(0)$ and $\widetilde{\mathbf{H}}_0(0)$, respectively,

$$ \mathbf{y}_0\mathbf{H}_0(2\pi k/M) = \delta(k)\mathbf{y}_0, \quad \widetilde{\mathbf{y}}_0\widetilde{\mathbf{H}}_0(2\pi k/M) = \delta(k)\widetilde{\mathbf{y}}_0, \quad 0 \le k \le M-1. $$

Applying $\sum_{k=0}^{M-1}\mathbf{H}_0(2k\pi/M)\widetilde{\mathbf{H}}_\ell(2k\pi/M)^* = \mathbf{I}_r\delta(\ell)$, we have

$$ \mathbf{y}_0\widetilde{\mathbf{H}}_\ell(0)^T = \mathbf{y}_0\delta(\ell), \quad \widetilde{\mathbf{y}}_0\mathbf{H}_\ell(0)^T = \widetilde{\mathbf{y}}_0\delta(\ell). $$

Thus $\mathbf{y}_0^T$ and $\widetilde{\mathbf{y}}_0^T$ are right 1-eigenvectors of $\widetilde{\mathbf{H}}_0(0)$ and $\mathbf{H}_0(0)$ respectively. Since 1 is a simple eigenvalue of $\widetilde{\mathbf{H}}_0(0)$ and $\mathbf{H}_0(0)$, up to a constant, $\mathbf{y}_0^T = \widetilde{\mathbf{v}}$, $\widetilde{\mathbf{y}}_0^T = \mathbf{v}$. Thus (18) and (19) hold. ∎

Proof of Proposition 3. Since $\widehat{\Psi}_0(0)$ ($\widehat{\widetilde{\Psi}}_0(0)$ respectively) is a right 1-eigenvector of $\mathbf{H}_0(0)$ ($\widetilde{\mathbf{H}}_0(0)$ respectively), $\widehat{\Psi}_0(0)^T$ $\widehat{\widetilde{\Psi}}_0(0)^T$ satisfy (18). Thus $\widehat{\Psi}_0(0)^T$, $\widehat{\widetilde{\Psi}}_0(0)^T$ are left 1–eigenvectors of $\widetilde{\mathbf{H}}_0(0)$, $\mathbf{H}_0(0)$ respectively.

Since Ψ_0 and $\widetilde{\Psi}_0$ are biorthogonal to each other, i.e. $\int \Psi_0(x)\widetilde{\Psi}_0(x-k)^T dx = \delta(k)\mathbf{I}_r$, we have by the Poisson summation formula,

$$ \sum_k \widehat{\Psi}_0(2\pi k+\omega)\widehat{\widetilde{\Psi}}_0(2\pi k+\omega)^* = \mathbf{I}_r, \quad \omega \in [0, 2\pi). \tag{64} $$

For any left 1-eigenvector $\mathbf{v}$ of $\mathbf{H}_0(0)$, we have (see [14]) $\mathbf{v}\widehat{\Psi}_0(2\pi k) = 0$ if $k \neq 0$. Thus $\widehat{\widetilde{\Psi}}_0(0)^T \widehat{\Psi}_0(2\pi k) = 0$ for any $k \neq 0$. By (64), $\widehat{\widetilde{\Psi}}_0(0)^T \widehat{\Psi}_0(0) \widehat{\widetilde{\Psi}}_0(0)^T = \widehat{\widetilde{\Psi}}_0(0)^T$. Therefore $\widehat{\widetilde{\Psi}}_0(0)^T \widehat{\Psi}_0(0) = 1$.

By Lemma 1, $\widehat{\Psi}_0(0)$ and $\widehat{\widetilde{\Psi}}_0(0)$ satisfy (19). Thus by (5), $\widehat{\widetilde{\Psi}}_\ell(0) = \widehat{\widetilde{\Psi}}_\ell(0) = 0$, $1 \leq \ell < M$. ■

Proof of Lemma 3. For any $\mathbf{x} \neq 0$, there exists an orthogonal matrix $\mathbf{g}^{(r)}$ such that (see [12]) $\mathbf{x} = \|\mathbf{x}\| \mathbf{g}^{(r)} \mathbf{i}_1$, where $\|\mathbf{x}\|^2 := \sum_{k=1}^r x_k^2$. Let $\mathbf{z} = \|\mathbf{x}\| (\mathbf{g}^{(r)})^T \mathbf{y}$, then $\mathbf{z} = [1, \mathbf{z}_1^T]^T$ for some $\mathbf{z}_1 \in R^{r-1}$. Denote $\mathbf{U}_{00} = \|\mathbf{x}\| \begin{bmatrix} 1 & -\mathbf{z}_1^T \\ 0 & \mathbf{I}_{r-1} \end{bmatrix}$, then $\mathbf{U}_{00}^{-1} = \frac{1}{\|\mathbf{x}\|} \begin{bmatrix} 1 & \mathbf{z}_1^T \\ 0 & \mathbf{I}_{r-1} \end{bmatrix}$. Let $\mathbf{u}_1 = \mathbf{g}^{(r)} \mathbf{U}_{00}$. Then $\mathbf{u}_1 \mathbf{i}_1 = \mathbf{x}, \mathbf{u}_1^{-T} \mathbf{i}_1 = \mathbf{y}$. ■

Proof of Proposition 4. Here we give the proof for ${}_{2\gamma+1}\mathbf{H}$. The proof for ${}_{2\gamma}\mathbf{H}$ is similar. By the definition, we have

$$\mathbf{V}_k(z) = \frac{1}{2} \begin{bmatrix} \mathbf{u}_k & \mathbf{v}_k \\ \mathbf{w}_k & \mathbf{T}_k \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{D}_0 \mathbf{u}_k \mathbf{D}_0 & -\mathbf{v}_k \\ -\mathbf{w}_k & \mathbf{D}_0 \mathbf{T}_k \mathbf{D}_0 \end{bmatrix} z^{-1},$$

with

$$\mathbf{u}_k = \begin{bmatrix} 1 & -x_k \\ -\frac{t_k}{|\mathbf{b}_k|} & 1 \end{bmatrix}, \mathbf{v}_k = \begin{bmatrix} 0 & -y_k \\ \frac{y_k}{|\mathbf{b}_k|} & 0 \end{bmatrix}, \mathbf{w}_k = \begin{bmatrix} 0 & -z_k \\ \frac{z_k}{|\mathbf{b}_k|} & 0 \end{bmatrix}, \mathbf{T}_k = \begin{bmatrix} 1 & -t_k \\ -\frac{x_k}{|\mathbf{b}_k|} & 1 \end{bmatrix}.$$

Let ${}_1\mathbf{H}(\omega), {}_1\mathbf{G}(\omega)$ be the filters defined by (35) with $\gamma = 0$. Then

$$\begin{bmatrix} D\, {}_{2\gamma+1}\mathbf{H}(\ell\pi) \\ D\, {}_{2\gamma+1}\mathbf{G}(\ell\pi) \end{bmatrix} = \sum_{k=1}^{\gamma} D\mathbf{V}_k(0) \begin{bmatrix} {}_1\mathbf{H}(\ell\pi) \\ {}_1\mathbf{G}(\ell\pi) \end{bmatrix} + \begin{bmatrix} D\, {}_1\mathbf{H}(\ell\pi) \\ D\, {}_1\mathbf{G}(\ell\pi) \end{bmatrix}, \ell = 0, 1.$$

Thus

$$D\, {}_{2\gamma+1}\mathbf{H}(\ell\pi) = D\, {}_1\mathbf{H}(\ell\pi) - i \sum_{k=1}^{\gamma} (\mathbf{D}_0 \mathbf{u}_k \mathbf{D}_0\, {}_1\mathbf{H}(\ell\pi) - \mathbf{v}_k\, {}_1\mathbf{G}(\ell\pi)), \ell = 0, 1.$$

Suppose $\mathbf{y}_1 = (\xi, \eta)$ for some $\xi, \eta \in R$. Then ${}_{2\gamma+1}\mathbf{H}$ has the sum rules of order 2 is equivalent to

$$-i\mathbf{y}_0\, D\, {}_{2\gamma+1}\mathbf{H}(\ell\pi) + 2\mathbf{y}_1\, {}_{2\gamma+1}\mathbf{H}(\ell\pi) = \delta(\ell)\mathbf{y}_1,$$

where $\mathbf{y}_0 = (1, 0)$. One can check this is equivalent to

$$\begin{cases} (-\frac{1}{2}, 0) - (\gamma, \sum_{k=1}^{\gamma} b x_k - f y_k) + 2(\xi, b\eta) = (\xi, \eta), \\ (\frac{1}{2}, 0) - (\sum_{k=1}^{\gamma} d x_k + p y_k, 0) + 2(d\eta, 0) = 0. \end{cases}$$

Thus $\xi = \gamma + \frac{1}{2}$ and

$$\begin{cases} b \sum_{k=1}^{\gamma} x_k + f \sum_{k=1}^{\gamma} y_k = (2b - 1)\eta, \\ d \sum_{k=1}^{\gamma} x_k + p \sum_{k=1}^{\gamma} y_k = \frac{1}{2} + 2d\eta. \end{cases}$$

That is x_k, y_k satisfy (50). ■

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