CONSTRUCTION OF BIORTHOGONAL MULTIWAVELETS USING THE LIFTING SCHEME

SAY SONG GOH†, QINGTANG JIANG† AND TAO XIA‡

Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent
Singapore 119260
Republic of Singapore

Abstract:
The lifting scheme has been found to be a flexible method for constructing scalar wavelets with desirable properties. Here it is extended to the construction of multiwavelets. It is shown that any set of compactly supported biorthogonal multiwavelets can be obtained from the Lazy matrix filters with a finite number of lifting steps. As an illustration of the general theory, compactly supported biorthogonal multiwavelets with optimum time-frequency resolution are constructed. In addition, experimental results of applying these multiwavelets to image compression are presented.

1. Introduction

Multiwavelets is a recent topic of active research in the field of wavelets. There is much research conducted on the construction of orthonormal and biorthogonal multiwavelets (see [7], [2], [17], [18], [8], [21], [12]–[15]), and the application of these multiwavelets to signal and image processing is gaining interest as well (see [23], [28], [25]). This paper deals with the construction of biorthogonal multiwavelets using an extension of the lifting scheme in [24], a flexible tool for constructing biorthogonal scalar wavelets.

Fixing notations, let $\mathbb{T}$ be the unit circle, and I_r and 0_r be the $r \times r$ identity matrix and zero matrix respectively. For a matrix B, its conjugate transpose is denoted by B^*, and its (m,μ)-entry is written as $B_{m,\mu}$. We say that B is **standard triangular** if it is a triangular matrix whose diagonal entries are entirely 1's, and B is **standard elementary** if its diagonal entries are entirely 1's, and all, except one, of its off-diagonal entries are zero. The matrix product notation of $\prod_{\ell=L}^{1} B_\ell$ refers to the expansion $B_L B_{L-1} \cdots B_1$.

The construction of biorthogonal multiwavelets of dilation factor 2 begins with a pair of $r \times r$ matrix filters $H, \widetilde{H}$ of the form

$$H(z) = \sum_{k\in\mathbb{Z}} h(k) z^{-k}, \quad \widetilde{H}(z) = \sum_{k\in\mathbb{Z}} \widetilde{h}(k) z^{-k},$$

satisfying

$$H(z)\widetilde{H}(z)^* + H(-z)\widetilde{H}(-z)^* = 2I_r, \quad z \in \mathbb{T}.$$

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Suppose that $\Phi = (\phi_1, \ldots, \phi_r)^T$ and $\widetilde{\Phi} = (\widetilde{\phi}_1, \ldots, \widetilde{\phi}_r)^T$ are vector-valued functions in $L^2(\mathbb{R})^r$ satisfying

$$\Phi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h(k)\Phi(2x-k), \quad \widetilde{\Phi}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \widetilde{h}(k)\widetilde{\Phi}(2x-k).$$

Consider the subspaces

$$V_0(\Phi) := \overline{\mathrm{span}}\{\phi_m(\cdot - k) : 1 \le m \le r, k \in \mathbb{Z}\}, \quad V_0(\widetilde{\Phi}) := \overline{\mathrm{span}}\{\widetilde{\phi}_m(\cdot - k) : 1 \le m \le r, k \in \mathbb{Z}\}$$

of $L^2(\mathbb{R})$. For $j \in \mathbb{Z}$, define closed subspaces $V_j(\Phi)$, $V_j(\widetilde{\Phi})$ of $L^2(\mathbb{R})$ by

$$V_j(\Phi) := \{f \in L^2(\mathbb{R}) : f(2^{-j}\cdot) \in V_0(\Phi)\}, \quad V_j(\widetilde{\Phi}) := \{f \in L^2(\mathbb{R}) : f(2^{-j}\cdot) \in V_0(\widetilde{\Phi})\}.$$

Recall (see [9]) that a **multiresolution analysis (MRA)** of multiplicity r is a sequence of closed subspaces (V_j) in $L^2(\mathbb{R})$ satisfying (1°) $V_j \subset V_{j+1}, j \in \mathbb{Z}$; (2°) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$; (3°) $f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$; and (4°) there exist r functions $\phi_1, \ldots, \phi_r$ in $L^2(\mathbb{R})$ such that the collection of integer translates $\{\phi_m(\cdot - k) : 1 \le m \le r, k \in \mathbb{Z}\}$ forms a Riesz basis of V_0. The vector-valued function $\Phi = (\phi_1, \ldots, \phi_r)^T$ is called a **scaling function**. Suppose that $(V_j(\Phi))$ and $(V_j(\widetilde{\Phi}))$ are two MRAs of multiplicity r with scaling functions Φ and $\widetilde{\Phi}$. Let $\Psi = (\psi_1, \ldots, \psi_r)^T$ and $\widetilde{\Psi} = (\widetilde{\psi}_1, \ldots, \widetilde{\psi}_r)^T$ be the vector-valued functions whose components are in $V_1(\Phi)$ and $V_1(\widetilde{\Phi})$ respectively, and defined by

$$\Psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g(k)\Phi(2x-k), \quad \widetilde{\Psi}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \widetilde{g}(k)\widetilde{\Phi}(2x-k) \tag{1.1}$$

for some $r \times r$ matrix filters $G(z) := \sum_{k\in\mathbb{Z}} g(k)z^{-k}$ and $\widetilde{G}(z) := \sum_{k\in\mathbb{Z}} \widetilde{g}(k)z^{-k}$. Then the construction of biorthogonal multiwavelets involves finding $G, \widetilde{G}$ such that $\{\phi_m(\cdot - k), \psi_m(\cdot - k) : 1 \le m \le r,\ k \in \mathbb{Z}\}$ and $\{\widetilde{\phi}_m(\cdot - k), \widetilde{\psi}_m(\cdot - k) : 1 \le m \le r,\ k \in \mathbb{Z}\}$ form Riesz bases of $V_1(\Phi)$ and $V_1(\widetilde{\Phi})$ respectively, with

$$\begin{cases} \langle \phi_m, \widetilde{\phi}_{m'}(\cdot - k)\rangle = \langle \psi_m, \widetilde{\psi}_{m'}(\cdot - k)\rangle = \delta(k)\delta(m-m'), \\ \langle \phi_m, \widetilde{\psi}_{m'}(\cdot - k)\rangle = \langle \widetilde{\phi}_m, \psi_{m'}(\cdot - k)\rangle = 0, \end{cases}$$

for all $m, m' = 1, \ldots, r,\ k \in \mathbb{Z}$. The corresponding collections $\{2^{j/2}\psi_m(2^j \cdot - k) : 1 \le m \le r,\ j, k \in \mathbb{Z}\}$ and $\{2^{j/2}\widetilde{\psi}_m(2^j \cdot - k) : 1 \le m \le r,\ j, k \in \mathbb{Z}\}$ constitute a pair of dual Riesz bases of $L^2(\mathbb{R})$, and $\Psi, \widetilde{\Psi}$ in (1.1) is said to form a set of **biorthogonal multiwavelets**. Consequently, $H, G, \widetilde{H}, \widetilde{G}$ satisfy the **perfect reconstruction (PR) conditions**:

$$\begin{cases} H(z)\widetilde{H}(z)^* + H(-z)\widetilde{H}(-z)^* = 2I_r, \\ H(z)\widetilde{G}(z)^* + H(-z)\widetilde{G}(-z)^* = 0_r, \\ G(z)\widetilde{H}(z)^* + G(-z)\widetilde{H}(-z)^* = 0_r, \\ G(z)\widetilde{G}(z)^* + G(-z)\widetilde{G}(-z)^* = 2I_r, \quad z \in \mathbb{T}. \end{cases} \tag{1.2}$$

In this case, we say that $H, G, \widetilde{H}, \widetilde{G}$ are PR and form a 2-channel multiwavelet filter bank, or in short, a **multifilter bank**.

A sufficient condition for $\Psi, \widetilde{\Psi}$ to be a set of biorthogonal multiwavelets is available in terms of the transition operators associated to H and $\widetilde{H}$. For an FIR matrix filter H, define $N := \max\{|k| : h(k) \ne 0\}$. Let $\mathcal{V}_N$ denote the space of all $r \times r$ matrices with trigonometric polynomial entries whose Fourier coefficients are supported in $[1-N, N-1]$. Then the **transition operator** T_H associated to H is the linear operator on $\mathcal{V}_N$ defined by

$$T_H V(\omega) := \frac{1}{2}H(e^{i\omega/2})V(\frac{\omega}{2})H(e^{i\omega/2})^* + \frac{1}{2}H(-e^{i\omega/2})V(\frac{\omega}{2}+\pi)H(-e^{i\omega/2})^*, \quad V \in \mathcal{V}_N,\ \omega \in [0, 2\pi).$$

We say that an operator on a finite dimensional linear space satisfies **Condition E** if its spectral radius is 1, and 1 is a simple eigenvalue as well as the unique eigenvalue on the unit circle. Now, suppose that the FIR matrix filters $H, G, \widetilde{H}, \widetilde{G}$ are PR, and the transition operators $T_H, T_{\widetilde{H}}$ associated to H and $\widetilde{H}$ satisfy Condition E. Then $\Psi, \widetilde{\Psi}$ defined by (1.1) form a set of biorthogonal multiwavelets (see [11]).

In [24], the lifting scheme was introduced as a flexible method for constructing biorthogonal scalar wavelets with desirable properties. Subsequently, it was shown (see [5]) that any pair of compactly supported biorthogonal scalar wavelets can be obtained from the Lazy filters with a finite number of lifting steps. This leads to the construction of wavelet transforms that map integers to integers (see [1]).

In Section 2, we extend the lifting scheme in [24] to the multiwavelet setting. Using the Smith factorization theorem on polynomial matrices, we also show that any PR FIR multifilter bank can be factorized into finite steps of lifting, starting from the **Lazy matrix filters** $H^{(0)}, G^{(0)}, \widetilde{H}^{(0)}, \widetilde{G}^{(0)}$ given by

$$H^{(0)}(z) = \widetilde{H}^{(0)}(z) = I_r, \quad G^{(0)}(z) = \widetilde{G}^{(0)}(z) = z^{-1} I_r. \tag{1.3}$$

In other words, any set of compactly supported biorthogonal multiwavelets can be obtained from lifting. In Section 3, we apply the lifting scheme to obtain parametric expressions of PR FIR multifilter banks. These parametric expressions are used to construct biorthogonal multiwavelets with optimum time-frequency localization. The paper concludes in Section 4 with experimental results of applying these multiwavelets to image compression.

2. The lifting scheme for multiwavelets

Assume that $H, G, \widetilde{H}, \widetilde{G}$ form a FIR multifilter bank. Write

$$H(z) = H_e(z^2) + z^{-1} H_o(z^2), \quad G(z) = G_e(z^2) + z^{-1} G_o(z^2),$$

where

$$\begin{cases} H_e(z) := \sum_{k\in\mathbb{Z}} h(2k) z^{-k}, & H_o(z) := \sum_{k\in\mathbb{Z}} h(2k+1) z^{-k}, \\ G_e(z) := \sum_{k\in\mathbb{Z}} g(2k) z^{-k}, & G_o(z) := \sum_{k\in\mathbb{Z}} g(2k+1) z^{-k}. \end{cases}$$

Then the **polyphase matrix** $P(z)$ of H, G is defined by

$$P(z) := \begin{bmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{bmatrix}, \quad z \in \mathbb{T}.$$

The polyphase matrix $\widetilde{P}(z)$ of $\widetilde{H}, \widetilde{G}$ is defined similarly. It is easy to verify that the FIR multifilter bank $H, G, \widetilde{H}, \widetilde{G}$ is PR if and only if

$$P(z)\widetilde{P}(z)^* = I_{2r}, \quad z \in \mathbb{T}. \tag{2.1}$$

As indicated in [26], for FIR matrix filters H, G, there exist FIR matrix filters $\widetilde{H}, \widetilde{G}$ such that (2.1) holds if and only if the determinant of the polyphase matrix $P(z)$ is a monomial in z. If the determinant of $P(z)$ is 1 for all $z \in \mathbb{T}$, the pair (H, G) is said to be **complementary**.

Now, suppose that $H^{(0)}, G^{(0)}, \widetilde{H}^{(0)}, \widetilde{G}^{(0)}$ form a PR FIR multifilter bank with $(H^{(0)}, G^{(0)})$ and $(\widetilde{H}^{(0)}, \widetilde{G}^{(0)})$ both complementary. For $1 \le \ell \le L$, let $S^{(\ell)}(z)$, $\widetilde{S}^{(\ell)}(z)$ be $r \times r$ Laurent polynomial matrices, and $T_1^{(\ell)}(z)$, $T_2^{(\ell)}(z)$ be $r \times r$ standard triangular Laurent polynomial matrices. We shall construct PR FIR multifilters $H^{(\ell)}, G^{(\ell)}, \widetilde{H}^{(\ell)}, \widetilde{G}^{(\ell)}$ via $P^{(\ell)}(z)$ and $\widetilde{P}^{(\ell)}(z)$, the polyphase matrices of $H^{(\ell)}, G^{(\ell)}$ and $\widetilde{H}^{(\ell)}, \widetilde{G}^{(\ell)}$ respectively. For $1 \le \ell \le L$, define

$$P^{(\ell)}(z) := \begin{bmatrix} I_r & 0_r \\ -\widetilde{S}^{(\ell)}(z)^* & I_r \end{bmatrix} \begin{bmatrix} T_1^{(\ell)}(z) & 0_r \\ 0_r & T_2^{(\ell)}(z) \end{bmatrix} \begin{bmatrix} I_r & S^{(\ell)}(z) \\ 0_r & I_r \end{bmatrix} P^{(\ell-1)}(z), \tag{2.2}$$

and

$$\widetilde{P}^{(\ell)}(z) := \begin{bmatrix} I_r & \widetilde{S}^{(\ell)}(z) \\ 0_r & I_r \end{bmatrix} \begin{bmatrix} \widetilde{T}_1^{(\ell)}(z) & 0_r \\ 0_r & \widetilde{T}_2^{(\ell)}(z) \end{bmatrix} \begin{bmatrix} I_r & 0_r \\ -S^{(\ell)}(z)^* & I_r \end{bmatrix} \widetilde{P}^{(\ell-1)}(z), \tag{2.3}$$

where $\widetilde{T}_\nu^{(\ell)}(z) := (T_\nu^{(\ell)}(z)^*)^{-1}$ for $\nu = 1, 2$. Since $\det(T_1^{(\ell)}(z)) = \det(T_2^{(\ell)}(z)) = 1$ for all $z \in \mathbb{T}$, the matrices $\widetilde{T}_1^{(\ell)}(z), \widetilde{T}_2^{(\ell)}(z)$ are still standard triangular Laurent polynomial matrices. Thus for $1 \le \ell \le L$, both $(H^{(\ell)}, G^{(\ell)})$ and $(\widetilde{H}^{(\ell)}, \widetilde{G}^{(\ell)})$ are complementary, and $H^{(\ell)}, G^{(\ell)}, \widetilde{H}^{(\ell)}, \widetilde{G}^{(\ell)}$ are PR. In terms of matrix filters, (2.2) and (2.3) are equivalent to

$$\begin{cases} H^{(\ell)}(z) = T_1^{(\ell)}(z^2)\big(H^{(\ell-1)}(z) + S^{(\ell)}(z^2)G^{(\ell-1)}(z)\big), \\ G^{(\ell)}(z) = T_2^{(\ell)}(z^2)G^{(\ell-1)}(z) - \widetilde{S}^{(\ell)}(z^2)^* H^{(\ell)}(z), \end{cases} \tag{2.4}$$

and

$$\begin{cases} \widetilde{G}^{(\ell)}(z) = \widetilde{T}_2^{(\ell)}(z^2)\left(\widetilde{G}^{(\ell-1)}(z) - S^{(\ell)}(z^2)^* \widetilde{H}^{(\ell-1)}(z)\right), \\ \widetilde{H}^{(\ell)}(z) = \widetilde{T}_1^{(\ell)}(z^2)\widetilde{H}^{(\ell-1)}(z) + \widetilde{S}^{(\ell)}(z^2)\widetilde{G}^{(\ell)}(z). \end{cases} \tag{2.5}$$

Analogous to the terminologies in [5], (2.4) is called **lifting**, while (2.5) is known as **dual lifting**.

Let us begin with the Lazy matrix filters $(H^{(0)}, G^{(0)})$ in (1.3). The polyphase matrix of $H^{(0)}, G^{(0)}$ is I_{2r}, and so $(H^{(0)}, G^{(0)})$ is complementary. Consequently, the matrix filter pair (H, G) with polyphase matrix

$$P(z) = \prod_{\ell=L}^{1} \left(\begin{bmatrix} I_r & 0_r \\ -\widetilde{S}^{(\ell)}(z)^* & I_r \end{bmatrix} \begin{bmatrix} T_1^{(\ell)}(z) & 0_r \\ 0_r & T_2^{(\ell)}(z) \end{bmatrix} \begin{bmatrix} I_r & S^{(\ell)}(z) \\ 0_r & I_r \end{bmatrix} \right) \tag{2.6}$$

is also complementary. The following theorem shows that the polyphase matrix of any complementary FIR matrix filter pair (H, G) can always be factorized into the form (2.6) which is equivalent to a finite number of lifting steps starting from the Lazy matrix filters.

Theorem 2.1. *Given any complementary FIR matrix filter pair (H, G), there exist a nonnegative integer L, $r \times r$ Laurent polynomial matrices $S^{(\ell)}(z)$, $\widetilde{S}^{(\ell)}(z)$, and $r \times r$ standard triangular Laurent polynomial matrices $T_1^{(\ell)}(z)$, $T_2^{(\ell)}(z)$ for $1 \le \ell \le L$, such that the polyphase matrix of H, G can be factorized into the form* (2.6).

The proof of Theorem 2.1 is based on the Smith factorization theorem on polynomial matrices (see, for instance, [26]) which says that any $s \times s$ polynomial matrix $M(z)$ can be factorized into a product of simple $s \times s$ matrices such as triangular and diagonal polynomial matrices. Such simple matrices are obtained by performing elementary row and column operations on the given polynomial matrix. The elementary row operations on $M(z)$ are as follows:

Type 1: Interchange two rows.

Type 2: Multiply a row with a nonzero constant c.

Type 3: Add a polynomial multiple of a row to another row.

Similarly, elementary column operations on $M(z)$ are defined.

An elementary row (column respectively) operation of Type 3 on $M(z)$ is performed by left (right respectively) multiplying $M(z)$ with an $s \times s$ standard elementary polynomial matrix. It is easily seen that the row (column respectively) operation of Type 1 on $M(z)$ is performed by left (right respectively) multiplying $M(z)$ with an $s \times s$ exchange matrix which can be written as a product of a diagonal matrix of the form $\mathrm{diag}(\pm 1, \dots, \pm 1)$ and standard elementary constant matrices.

Proof of Theorem 2.1. Let $P(z)$ be the polyphase matrix of H, G, and n be an integer such that $z^n P(z)$ is a polynomial matrix. Then as in the proof of the Smith factorization theorem (Theorem 13.5.1) in [26], by performing elementary row and column operations of Type 1 and Type 3 on $z^n P(z)$, we see that $z^n P(z)$ can be written in the form

$$z^n P(z) = W(z)\mathrm{diag}(\gamma_1(z), \ldots, \gamma_{2r}(z))\mathrm{diag}(c_1, \ldots, c_{2r})U(z),$$

where $W(z), U(z)$ are products of standard elementary polynomial matrices, $\gamma_1(z), \ldots, \gamma_{2r}(z)$ are polynomials whose highest powers have coefficients 1, and $c_1, \ldots, c_{2r}$ are constants. Since $c_1 \cdots c_{2r}\gamma_1(z)\cdots\gamma_{2r}(z) = \det(z^n P(z)) = z^{2rn}$, we have $c_1 \cdots c_r = 1$, $\gamma_i(z) = z^{n_i}$ for some n_i, and $\sum_{i=1}^{2r} n_i = 2rn$. Consequently,

$$P(z) = W(z)\mathrm{diag}(z^{n_1-n}, \ldots, z^{n_{2r}-n})\mathrm{diag}(c_1, \ldots, c_{2r})U(z).$$

Note that the diagonal matrices $\mathrm{diag}(z^{n_1-n}, \ldots, z^{n_{2r}-n})$ and $\mathrm{diag}(c_1, \ldots, c_{2r})$ can be written as a product of standard elementary matrices. Hence, $P(z)$ is of the form (2.6). □

Remark 2.1. (i) For the case $r = 1$, we have $T_\nu^{(\ell)}(z) = 1$ for $1 \le \ell \le L$, $\nu = 1, 2$, and Theorem 2.1 reduces to the factorization theorem (Theorem 7) in [5].
(ii) The factorization in Theorem 2.1 is not unique. Indeed, even for the case $r = 1$, there can be more than one way of factorizing a polyphase matrix. We refer the reader to [5] for a series of examples on this.
(iii) Although we are only dealing with multiwavelets with dilation factor 2 here, similar arguments can be used to extend our results to multiwavelets with dilation factor M, where M is an integer greater than 1.

In practice, for a PR FIR multifilter bank $H, G, \widetilde{H}, \widetilde{G}$, the matrix filters $\widetilde{H}, \widetilde{G}$ are used to decompose a $r \times 1$ vector-valued signal into two $r \times 1$ vector-valued signals, while the matrix filters H, G are applied in the reconstruction process (see, for instance, [27] or [22]). (Note that the roles of H, G and $\widetilde{H}, \widetilde{G}$ can be interchanged in the implementation of the multifilter bank.) For further decomposition, $\widetilde{H}, \widetilde{G}$ act on the first decomposed $r \times 1$ vector-valued signal. This is different from the effect of a $2r$-channel filter bank in the sense that in a $2r$-channel filter bank, a scalar signal is decomposed into $2r$ scalar signals with further decomposition being carried out on only the first channel.

The lifting and dual lifting steps (2.4) and (2.5) give a recursive, and more efficient, procedure to implement multifilter banks. This is similar to the scalar case in [24]. Indeed, for a given vector-valued signal $\{c^1(k)\}$, first set

$$c^{(0)}(k) := \sum_{n\in\mathbb{Z}} \widetilde{h}^{(0)}(n-2k)c^1(n), \quad d^{(0)}(k) := \sum_{n\in\mathbb{Z}} \widetilde{g}^{(0)}(n-2k)c^1(n).$$

Then for $1 \le \ell \le L$, define

$$\begin{cases} d'(k) := d^{(\ell-1)}(k) - \sum\limits_{n\in\mathbb{Z}} s^{(\ell)}(k-n)^T c^{(\ell-1)}(n), \\ d^{(\ell)}(k) := \sum\limits_{n\in\mathbb{Z}} \widetilde{t}_2^{(\ell)}(n-k)d'(n), \end{cases} \tag{2.7}$$

and

$$\begin{cases} c'(k) := \sum\limits_{n\in\mathbb{Z}} \widetilde{t}_1^{(\ell)}(n-k)c^{(\ell-1)}(n), \\ c^{(\ell)}(k) := c'(k) + \sum\limits_{n\in\mathbb{Z}} \widetilde{s}^{(\ell)}(n-k)d^{(\ell)}(n), \end{cases} \tag{2.8}$$

where $S^{(\ell)}(z) = \sum_{k\in\mathbb{Z}} s^{(\ell)}(k)z^{-k}$, $\widetilde{S}^{(\ell)}(z) = \sum_{k\in\mathbb{Z}} \widetilde{s}^{(\ell)}(k)z^{-k}$, and $\widetilde{T}_\nu^{(\ell)}(z) = \sum_{k\in\mathbb{Z}} \widetilde{t}_\nu^{(\ell)}(k)z^{-k}$ for $\nu = 1, 2$. If $\widetilde{H} = \widetilde{H}^{(L)}$, $\widetilde{G} = \widetilde{G}^{(L)}$ are used to decompose the signal $\{c^1(k)\}$ in a multifilter bank, the decomposed signals $\{c(k)\}$ and $\{d(k)\}$ are given by $c(k) = c^{(L)}(k)$ and $d(k) = d^{(L)}(k)$. The processes described by (2.7) and (2.8) are reversible, and they define the **lifted decomposition and reconstruction multiwavelet algorithms**.

Remark 2.2. By proceeding as in [1], the lifted multiwavelet algorithms yield multiwavelet transforms that map integer vectors into integer vectors. Since the generalization from the scalar case is straightforward, we shall omit the details here.

3. Biorthogonal multiwavelets with optimum time-frequency resolution

The design of optimal time-frequency resolution (OPTFR) wavelets and multiwavelets was studied in [6], [29], [19], [12]–[14]. In this section, we shall use the lifting scheme to construct biorthogonal OPTFR multiwavelets for the vector case of $r = 2$. Thus we are concerned with 2×2 FIR matrix filters $H, G, \widetilde{H}, \widetilde{G}$ that generate scaling functions $\Phi = (\phi_1, \phi_2)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2)^T$ and biorthogonal multiwavelets $\Psi = (\psi_1, \psi_2)^T$, $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2)^T$.

First let us recall some terminologies on time-frequency resolutions. Let $f \in L^2(\mathbb{R})$ be a function with the property that both tf and $\omega\widehat{f}$ are in $L^2(\mathbb{R})$, where $\widehat{f}$ denotes the Fourier transform of f. The **time-duration** Δ_f of f is defined by

$$\Delta_f := \left(\int_{-\infty}^{\infty} (t - \overline{t}_f)^2 |f(t)|^2 \, dt\right)^{1/2} \Big/ \left(\int_{-\infty}^{\infty} |f(t)|^2 \, dt\right)^{1/2}, \tag{3.1}$$

where $\overline{t}_f := \int_{-\infty}^{\infty} t|f(t)|^2 \, dt \Big/ \int_{-\infty}^{\infty} |f(t)|^2 \, dt$. Similarly, the **frequency-bandwidth** $\Delta_{\widehat{f}}$ of f is defined by

$$\Delta_{\widehat{f}} := \left(\int_{-\infty}^{\infty} (\omega - \overline{\omega}_{\widehat{f}})^2 |\widehat{f}(\omega)|^2 \, d\omega\right)^{1/2} \Big/ \left(\int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2 \, d\omega\right)^{1/2}, \tag{3.2}$$

where $\overline{\omega}_{\widehat{f}} := \int_{-\infty}^{\infty} \omega|\widehat{f}(\omega)|^2 \, d\omega \Big/ \int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2 \, d\omega$. In practice, if f is a bandpass function (that is, $\widehat{f}(0) = 0$), then the frequency-bandwidth $\Delta^0_{\widehat{f}}$ of f is defined by

$$\Delta^0_{\widehat{f}} := \left(\int_0^{\infty} (\omega - \overline{\omega}^0_{\widehat{f}})^2 |\widehat{f}(\omega)|^2 \, d\omega\right)^{1/2} \Big/ \left(\int_0^{\infty} |\widehat{f}(\omega)|^2 \, d\omega\right)^{1/2}, \tag{3.3}$$

where $\overline{\omega}^0_{\widehat{f}} := \int_0^{\infty} \omega|\widehat{f}(\omega)|^2 \, d\omega \Big/ \int_0^{\infty} |\widehat{f}(\omega)|^2 \, d\omega$ (see [6], [10]). The product of the time-duration and frequency-bandwidth is the area of the resolution cell of f. We shall use (3.2) to define the frequency-bandwidths of $\phi_1, \phi_2, \widetilde{\phi}_1, \widetilde{\phi}_2$. As for the frequency-bandwidths of $\psi_1, \psi_2, \widetilde{\psi}_1, \widetilde{\psi}_2$, we shall use (3.3) instead because these functions are bandpass functions (see [14]).

The respective low-pass frequency responses and the high-pass frequency responses of a PR FIR multifilter bank $H, G, \widetilde{H}, \widetilde{G}$ are

$$\begin{cases} h_m(\omega) := \sum_{k \in \mathbb{Z}} \left(h(k)_{m,1} e^{-i2k\omega} + h(k)_{m,2} e^{-i(2k+1)\omega}\right), \\ \widetilde{h}_m(\omega) := \sum_{k \in \mathbb{Z}} \left(\widetilde{h}(k)_{m,1} e^{-i2k\omega} + \widetilde{h}(k)_{m,2} e^{-i(2k+1)\omega}\right), \quad m = 1, 2, \end{cases} \tag{3.4}$$

and

$$\begin{cases} g_m(\omega) := \sum_{k \in \mathbb{Z}} \left(g(k)_{m,1} e^{-i2k\omega} + g(k)_{m,2} e^{-i(2k+1)\omega}\right), \\ \widetilde{g}_m(\omega) := \sum_{k \in \mathbb{Z}} \left(\widetilde{g}(k)_{m,1} e^{-i2k\omega} + \widetilde{g}(k)_{m,2} e^{-i(2k+1)\omega}\right), \quad m = 1, 2, \end{cases} \tag{3.5}$$

where $\omega \in [0, 2\pi)$ (see [25]). In image processing, balanced multiwavelets are useful (see [18] or [14]). The biorthogonal multiwavelets Ψ, $\widetilde{\Psi}$ are said to be **balanced** if their corresponding scaling functions Φ, $\widetilde{\Phi}$ satisfy

$\widehat{\Phi}(0) = (1,1)^T/\sqrt{2}$, $\widehat{\widetilde{\Phi}}(0) = (1,1)^T/\sqrt{2}$, where $\widehat{\Phi} := (\widehat{\phi}_1, \widehat{\phi}_2)^T$ and $\widehat{\widetilde{\Phi}} := (\widehat{\widetilde{\phi}}_1, \widehat{\widetilde{\phi}}_2)^T$. In this case,

$$h_m(0) = \widetilde{h}_m(0) = \sqrt{2}, \quad g_m(0) = \widetilde{g}_m(0) = 0, \quad m = 1,2.$$

¿From the practical point of view, one needs to further ensure that

$$h_m(\pi) = 0, \quad |g_m(\pi)| = \sqrt{2}, \quad m = 1,2, \tag{3.6}$$

or

$$\widetilde{h}_m(\pi) = 0, \quad |\widetilde{g}_m(\pi)| = \sqrt{2}, \quad m = 1,2, \tag{3.7}$$

depending on whether H, G or $\widetilde{H}, \widetilde{G}$ are used in the decomposition process.

As shown in [14], balanced multiwavelets can be constructed via symmetric/antisymmetric multiwavelets. The procedure is as follows. Begin with H, G that satisfy

$$z^{-c} D_0 H(z^{-1}) D_0 = H(z), \quad z^{-d} D_0 G(z^{-1}) D_0 = G(z), \tag{3.8}$$

for some integers c, d, where $D_0 := \mathrm{diag}(1, -1)$. Together with some additional conditions, this gives ϕ_1 and ϕ_2 which are symmetric and antisymmetric about $c/2$ respectively, and ψ_1 and ψ_2 which are symmetric and antisymmetric about $d/2$ respectively. Let $\widetilde{H}, \widetilde{G}$ also satisfy (3.8) for some integers $\tilde{c}, \tilde{d}$. It is known (see [3], [16]) that if $H, G, \widetilde{H}, \widetilde{G}$ generate biorthogonal multiwavelets, then $H, \widetilde{H}$ must satisfy the **vanishing moment conditions** of order at least one:

$$\begin{cases} (1,0)H(1) = \sqrt{2}(1,0), & (1,0)H(-1) = (0,0), \\ (1,0)\widetilde{H}(1) = \sqrt{2}(1,0), & (1,0)\widetilde{H}(-1) = (0,0). \end{cases} \tag{3.9}$$

Now, define $H^b, G^b, \widetilde{H}^b, \widetilde{G}^b$ by

$$H^b(z) = RH(z)R^T, \quad G^b(z) = RG(z)R^T, \quad \widetilde{H}^b(z) = R\widetilde{H}(z)R^T, \quad \widetilde{G}^b(z) = R\widetilde{G}(z)R^T,$$

where $R := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then the corresponding biorthogonal multiwavelets $\Psi^b, \widetilde{\Psi}^b$ are balanced, and due to the symmetry/antisymmetry of the original multifilter bank $H, G, \widetilde{H}, \widetilde{G}$, we have

$$\phi_2^b(x) = \phi_1^b(c - x), \quad \psi_2^b(x) = \psi_1^b(d - x), \quad \widetilde{\phi}_2^b(x) = \widetilde{\phi}_1^b(\tilde{c} - x), \quad \widetilde{\psi}_2^b(x) = \widetilde{\psi}_1^b(\tilde{d} - x), \tag{3.10}$$

and

$$|h_2^b(\omega)| = |h_1^b(-\omega)|, \quad |g_2^b(\omega)| = |g_1^b(-\omega)|, \quad |\widetilde{h}_2^b(\omega)| = |\widetilde{h}_1^b(-\omega)|, \quad |\widetilde{g}_2^b(\omega)| = |\widetilde{g}_1^b(-\omega)|, \tag{3.11}$$

where $\omega \in [0, 2\pi)$. Furthermore,

$$\begin{cases} |h_m^b(\pi)| = |H(1)_{2,2}|, & |g_m^b(\pi)| = |G(1)_{2,2}|, \\ |\widetilde{h}_m^b(\pi)| = |\widetilde{H}(1)_{2,2}|, & |\widetilde{g}_m^b(\pi)| = |\widetilde{G}(1)_{2,2}|, \quad m = 1,2. \end{cases} \tag{3.12}$$

In the following, we use ${}_{N,\widetilde{N}}H$, ${}_{N,\widetilde{N}}G$, ${}_{N,\widetilde{N}}\widetilde{H}$, ${}_{N,\widetilde{N}}\widetilde{G}$ to denote the multifilter bank if the filter length of H is N while that of $\widetilde{H}$ is $\widetilde{N}$. Let ${}_{N,\widetilde{N}}\Phi^b$, ${}_{N,\widetilde{N}}\widetilde{\Phi}^b$, ${}_{N,\widetilde{N}}\Psi^b$, ${}_{N,\widetilde{N}}\widetilde{\Psi}^b$ be the scaling functions and biorthogonal multiwavelets corresponding to ${}_{N,\widetilde{N}}H^b$, ${}_{N,\widetilde{N}}G^b$, ${}_{N,\widetilde{N}}\widetilde{H}^b$, ${}_{N,\widetilde{N}}\widetilde{G}^b$. We shall construct biorthogonal OPTFR multiwavelets by minimizing the sum

$$ {}_{N,\widetilde{N}}\Delta := \Delta_{{}_{N,\widetilde{N}}\phi_1^b} \Delta_{{}_{N,\widetilde{N}}\widehat{\phi_1^b}} + \Delta_{{}_{N,\widetilde{N}}\psi_1^b} \Delta^0_{{}_{N,\widetilde{N}}\widehat{\psi_1^b}} + \Delta_{{}_{N,\widetilde{N}}\widetilde{\phi}_1^b} \Delta_{{}_{N,\widetilde{N}}\widehat{\widetilde{\phi}_1^b}} + \Delta_{{}_{N,\widetilde{N}}\widetilde{\psi}_1^b} \Delta^0_{{}_{N,\widetilde{N}}\widehat{\widetilde{\psi}_1^b}}, \tag{3.13}$$

where Δ_f, $\Delta_{\widehat{f}}$ and $\Delta^0_{\widehat{f}}$ are as defined in (3.1)–(3.3), and the minimum is taken over all derived parametric expressions of ${}_{N,\widetilde{N}}H^b$, ${}_{N,\widetilde{N}}G^b$, ${}_{N,\widetilde{N}}\widetilde{H}^b$, ${}_{N,\widetilde{N}}\widetilde{G}^b$. We shall let ${}_{N,\widetilde{N}}H^{bo}$, ${}_{N,\widetilde{N}}G^{bo}$, ${}_{N,\widetilde{N}}\widetilde{H}^{bo}$, ${}_{N,\widetilde{N}}\widetilde{G}^{bo}$ be the optimal multifilter bank, and ${}_{N,\tilde N}\Phi^{bo}$, ${}_{N,\tilde N}\widetilde\Phi^{bo}$, ${}_{N,\tilde N}\Psi^{bo}$, ${}_{N,\tilde N}\widetilde\Psi^{bo}$ be the corresponding scaling functions and multiwavelets. Note that the areas of the resolution cells of $\phi_2^b, \psi_2^b, \widetilde\phi_2^b, \widetilde\psi_2^b$ need not be considered in the sum (3.13) because (3.10) shows that they are the same as those of $\phi_1^b, \psi_1^b, \widetilde\phi_1^b, \widetilde\psi_1^b$.

3.1. **Optimal multifilter banks of odd filter length.** Let $H^{(\ell)}, G^{(\ell)}$, $\widetilde H^{(\ell)}, \widetilde G^{(\ell)}$ be defined by (2.4) and (2.5) with $T_\nu^{(\ell)}(z) = I_2$ for $1 \le \ell \le L$, $\nu = 1, 2$. Then

$$\begin{cases} H^{(\ell)}(z) = H^{(\ell-1)}(z) + S^{(\ell)}(z^2)G^{(\ell-1)}(z), \\ G^{(\ell)}(z) = G^{(\ell-1)}(z) - \widetilde S^{(\ell)}(z^2)^* H^{(\ell)}(z), \end{cases} \tag{3.14}$$

and

$$\begin{cases} \widetilde G^{(\ell)}(z) = \widetilde G^{(\ell-1)}(z) - S^{(\ell)}(z^2)^* \widetilde H^{(\ell-1)}(z), \\ \widetilde H^{(\ell)}(z) = \widetilde H^{(\ell-1)}(z) + \widetilde S^{(\ell)}(z^2)\widetilde G^{(\ell)}(z). \end{cases} \tag{3.15}$$

Now, choose $H^{(0)}, G^{(0)}, \widetilde H^{(0)}, \widetilde G^{(0)}$ to be the Lazy matrix filters in (1.3). Then (3.8) is satisfied with $c = 0$ and $d = 2$. For $1 \le \ell \le L$, we shall construct $H^{(\ell)}, G^{(\ell)}$ which satisfy (3.8) with $c = 0$ and $d = 2$.

Proposition 3.1. *Suppose that $H^{(0)}, G^{(0)}, \widetilde H^{(0)}, \widetilde G^{(0)}$ are as in* (1.3)*. For $1 \le \ell \le L$, let $H^{(\ell)}, G^{(\ell)}$ be defined by* (3.14) *for some Laurent polynomial matrices $S^{(\ell)}(z), \widetilde S^{(\ell)}(z)$. Then $H^{(\ell)}, G^{(\ell)}$ satisfy* (3.8) *with $c = 0$ and $d = 2$ for $1 \le \ell \le L$ if and only if $S^{(\ell)}(z), \widetilde S^{(\ell)}(z)$ satisfy*

$$zD_0S^{(\ell)}(z^{-1})D_0 = S^{(\ell)}(z), \quad zD_0\widetilde S^{(\ell)}(z^{-1})D_0 = \widetilde S^{(\ell)}(z) \tag{3.16}$$

for $1 \le \ell \le L$. Furthermore, if $\widetilde H^{(\ell)}, \widetilde G^{(\ell)}$ are defined by (3.15) *with $S^{(\ell)}(z), \widetilde S^{(\ell)}(z)$ satisfying* (3.16) *for $1 \le \ell \le L$, then $\widetilde H^{(\ell)}, \widetilde G^{(\ell)}$ also satisfy* (3.8) *with $c = 0$ and $d = 2$ for $1 \le \ell \le L$.*

Proof. First, assume that $H^{(\ell)}, G^{(\ell)}$ satisfy (3.8) with $c = 0$ and $d = 2$ for $1 \le \ell \le L$. Then for $1 \le \ell \le L$, it follows from (3.14) that

$$\big(z^2D_0S^{(\ell)}(z^{-2})D_0 - S^{(\ell)}(z^2)\big)G^{(\ell-1)}(\pm z) = 0.$$

Multiplying by $\widetilde G^{(\ell-1)}(\pm z)^*$, we have

$$\big(z^2D_0S^{(\ell)}(z^{-2})D_0 - S^{(\ell)}(z^2)\big)G^{(\ell-1)}(\pm z)\widetilde G^{(\ell-1)}(\pm z)^* = 0.$$

By (1.2), this implies that $zD_0S^{(\ell)}(z^{-1})D_0 = S^{(\ell)}(z)$. Similarly, $\widetilde S^{(\ell)}(z)$ satisfies (3.16) for $1 \le \ell \le L$.

For the converse direction, based on (3.14) and (3.16), we use an inductive argument to conclude that $H^{(\ell)}, G^{(\ell)}$ satisfy (3.8) with $c = 0$ and $d = 2$ for $1 \le \ell \le L$. Similarly, we establish the last part of the proposition on $\widetilde H^{(\ell)}, \widetilde G^{(\ell)}$. □

To obtain final matrix filters $H, \widetilde H$ that satisfy the vanishing moment conditions (3.9), we seek a formula to compute $H^{(\ell)}(1), H^{(\ell)}(-1), \widetilde H^{(\ell)}(1), \widetilde H^{(\ell)}(-1)$. In this connection, the following lemma, which is easily established, will be useful.

Lemma 3.1. *For $1 \le \ell \le L$, let $H^{(\ell)}, G^{(\ell)}, \widetilde H^{(\ell)}, \widetilde G^{(\ell)}$ be defined by* (3.14) *and* (3.15) *for some Laurent polynomial matrices $S^{(\ell)}(z), \widetilde S^{(\ell)}(z)$ satisfying* (3.16)*, where $H^{(0)}, G^{(0)}, \widetilde H^{(0)}, \widetilde G^{(0)}$ are as in* (1.3)*. Then for $1 \le \ell \le L$,*

$$G^{(\ell)}(1) = \widetilde H^{(\ell)}(-1), \quad G^{(\ell)}(-1) = -\widetilde H^{(\ell)}(1), \quad \widetilde G^{(\ell)}(1) = H^{(\ell)}(-1), \quad \widetilde G^{(\ell)}(-1) = -H^{(\ell)}(1).$$

By Lemma 3.1, it follows from (3.14) and (3.15) that for $1 \le \ell \le L$,

$$\begin{bmatrix} H^{(\ell)}(1) & H^{(\ell)}(-1) \\ \widetilde H^{(\ell)}(-1) & -\widetilde H^{(\ell)}(1) \end{bmatrix} = \begin{bmatrix} I_2 & S^{(\ell)}(1) \\ -\widetilde S^{(\ell)}(1) & I_2 - \widetilde S^{(\ell)}(1)S^{(\ell)}(1) \end{bmatrix} \begin{bmatrix} H^{(\ell-1)}(1) & H^{(\ell-1)}(-1) \\ \widetilde H^{(\ell-1)}(-1) & -\widetilde H^{(\ell-1)}(1) \end{bmatrix}.$$

This leads to

Proposition 3.2. *Under the hypothesis of Lemma* 3.1, *we have*

$$\begin{bmatrix} H^{(L)}(1) & H^{(L)}(-1) \\ \widetilde{H}^{(L)}(-1) & -\widetilde{H}^{(L)}(1) \end{bmatrix} = \left(\prod_{\ell=L}^{1} \begin{bmatrix} I_2 & S^{(\ell)}(1) \\ -\widetilde{S}^{(\ell)}(1) & I_2 - \widetilde{S}^{(\ell)}(1) S^{(\ell)}(1) \end{bmatrix} \right) \begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix}. \tag{3.17}$$

Now, for $1 \le \ell \le L$, we choose $S^{(\ell)}(z), \widetilde{S}^{(\ell)}(z)$ satisfying (3.16) to be

$$S^{(\ell)}(z) := \frac{1}{2} \begin{bmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a_\ell & -b_\ell \\ -c_\ell & d_\ell \end{bmatrix} z, \quad \widetilde{S}^{(\ell)}(z) := \frac{1}{2} \begin{bmatrix} \widetilde{a}_\ell & \widetilde{b}_\ell \\ \widetilde{c}_\ell & \widetilde{d}_\ell \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \widetilde{a}_\ell & -\widetilde{b}_\ell \\ -\widetilde{c}_\ell & \widetilde{d}_\ell \end{bmatrix} z.$$

For the case $L = 2$, first consider $\widetilde{S}^{(2)}(z) = 0_2$. Let ${}_{7,5}H, {}_{7,5}G, {}_{7,5}\widetilde{H}, {}_{7,5}\widetilde{G}$ be the corresponding matrix filters. Using (3.17), we see that ${}_{7,5}H, {}_{7,5}\widetilde{H}$ satisfy (3.9) if and only if

$$a_1 = \sqrt{2} - 1, \quad \widetilde{a}_1 = \sqrt{2}/2, \quad a_2 = \sqrt{2} - 1.$$

Since the filter length of ${}_{7,5}\widetilde{H}^b$ is shorter than that of ${}_{7,5}H^b$, we shall use ${}_{7,5}\widetilde{H}^b, {}_{7,5}\widetilde{G}^b$ in the decomposition process. Based on the desired frequency responses of ${}_{7,5}\widetilde{H}^b, {}_{7,5}\widetilde{G}^b$ in (3.7) and the relations in (3.12), we set ${}_{7,5}\widetilde{H}(1)_{2,2} = 0$, ${}_{7,5}\widetilde{G}(1)_{2,2} = \sqrt{2}$ to obtain

$$d_1 = 1 - \sqrt{2}, \quad \widetilde{d}_1 = -\sqrt{2}/2.$$

There are 7 remaining free parameters to determine the balanced biorthogonal multiwavelets. By minimizing ${}_{7,5}\Delta$, we obtain

$$\begin{aligned} &b_1 = .15634620515720, && \widetilde{b}_1 = .54323724572972, && b_2 = .32070154678036, \\ &c_1 = -.58272635112124, && \widetilde{c}_1 = -.94053105759286, && c_2 = -.65586372167406, \\ &d_2 = -.42725496310644. \end{aligned}$$

It can be checked that the transition operators $T_{{}_{7,5}H^{bo}}$, $T_{{}_{7,5}\widetilde{H}^{bo}}$ associated to the optimal matrix filters ${}_{7,5}H^{bo}, {}_{7,5}\widetilde{H}^{bo}$ satisfy Condition E, and therefore the resulting ${}_{7,5}\Psi^{bo}, {}_{7,5}\widetilde{\Psi}^{bo}$ defined by (1.1) form a set of biorthogonal multiwavelets. The areas of the resolution cells for ${}_{7,5}\phi_1^{bo}, {}_{7,5}\psi_1^{bo}, {}_{7,5}\widetilde{\phi}_1^{bo}, {}_{7,5}\widetilde{\psi}_1^{bo}$ are respectively

$$.679146, \quad .637900, \quad .634317, \quad .571908.$$

Next, consider the case $L = 2$ with $\widetilde{S}^{(2)}(z) \neq 0_2$. Let ${}_{7,9}H, {}_{7,9}G, {}_{7,9}\widetilde{H}, {}_{7,9}\widetilde{G}$ denote the corresponding matrix filters. In this case, by (3.17), ${}_{7,9}H, {}_{7,9}\widetilde{H}$ satisfy (3.9) if and only if

$$a_2 = (1 - a_1)/\sqrt{2}, \quad \widetilde{a}_2 = 1 - \sqrt{2}\widetilde{a}_1, \quad (a_1 - 1)\widetilde{a}_1 = 1 - \sqrt{2}.$$

Here, we shall use the matrix filters ${}_{7,9}H^b, {}_{7,9}G^b$ in the decomposition process. To have the desired frequency responses of ${}_{7,9}H^b, {}_{7,9}G^b$ in (3.6), according to (3.12), we select ${}_{7,9}H(1)_{2,2} = 0$, ${}_{7,9}G(1)_{2,2} = \sqrt{2}$. This leads to

$$d_2 + \frac{\sqrt{2}}{2} d_1 = -\frac{\sqrt{2}}{2}, \quad d_2 \widetilde{d}_1 = 1 - \frac{\sqrt{2}}{2}.$$

There are 11 free parameters to determine the balanced biorthogonal multiwavelets. By minimizing ${}_{7,9}\Delta$, we obtain

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} .59934321549133 & .41885175827122 \\ -.63687209098656 & -.52853412945938 \end{bmatrix},$$

$$\begin{bmatrix} \widetilde{a}_1 & \widetilde{b}_1 \\ \widetilde{c}_1 & \widetilde{d}_1 \end{bmatrix} = \begin{bmatrix} 1.03383638662464 & 1.23426452221818 \\ -.90678404033140 & -.87856531777820 \end{bmatrix},$$

$$\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} .28330712925448 & .10410822340904 \\ -.66679368845088 & -.33337671415729 \end{bmatrix},$$

$$\begin{bmatrix} \widetilde{a}_2 & \widetilde{b}_2 \\ \widetilde{c}_2 & \widetilde{d}_2 \end{bmatrix} = \begin{bmatrix} -.46206543923936 & -.87412095509012 \\ -.02184709361176 & .23622223713642 \end{bmatrix}.$$

Again, the resulting ${}_{7,9}\Psi^{bo}, {}_{7,9}\widetilde{\Psi}^{bo}$ defined by (1.1) form a set of biorthogonal multiwavelets. The areas of the resolution cells for ${}_{7,9}\phi_1^{bo}, {}_{7,9}\psi_1^{bo}, {}_{7,9}\widetilde{\phi}_1^{bo}, {}_{7,9}\widetilde{\psi}_1^{bo}$ are respectively

$$.664638, \quad .579071, \quad .656291, \quad .588506.$$

Figure 1 contains the graphs of ${}_{7,9}\phi_1^{bo}, {}_{7,9}\psi_1^{bo}, {}_{7,9}\widetilde{\phi}_1^{bo}, {}_{7,9}\widetilde{\psi}_1^{bo}$, and the plots of the magnitudes of the frequency responses ${}_{7,9}h_1^{bo}, {}_{7,9}g_1^{bo}, {}_{7,9}\widetilde{h}_1^{bo}, {}_{7,9}\widetilde{g}_1^{bo}$ as defined in (3.4) and (3.5). The graphs of ${}_{7,9}\phi_2^{bo}, {}_{7,9}\psi_2^{bo}, {}_{7,9}\widetilde{\phi}_2^{bo}, {}_{7,9}\widetilde{\psi}_2^{bo}$, and the plots of the magnitudes of ${}_{7,9}h_2^{bo}, {}_{7,9}g_2^{bo}, {}_{7,9}\widetilde{h}_2^{bo}, {}_{7,9}\widetilde{g}_2^{bo}$ can be obtained from Figure 1 via (3.10) and (3.11).

3.2. **Optimal multifilter banks of even filter length.** As in the previous case of odd filter length, we use (3.14) and (3.15) to define $H^{(\ell)}, G^{(\ell)}, \widetilde{H}^{(\ell)}, \widetilde{G}^{(\ell)}$ for $1 \le \ell \le L$. However, we begin with the matrix filters $H^{(0)}, G^{(0)}, \widetilde{H}^{(0)}, \widetilde{G}^{(0)}$ given by

$$H^{(0)}(z) = \widetilde{H}^{(0)}(z) = h^{(0)}(0) + h^{(0)}(1)z^{-1}, \quad G^{(0)}(z) = \widetilde{G}^{(0)}(z) = g^{(0)}(-1)z + g^{(0)}(0),$$

where

$$h^{(0)}(0) = g^{(0)}(-1) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad h^{(0)}(1) = g^{(0)}(0) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

These initial matrix filters form a PR multifilter bank, and they satisfy (3.8) with $c = 1$ and $d = -1$. For $1 \le \ell \le L$, select Laurent polynomial matrices $S^{(\ell)}(z), \widetilde{S}^{(\ell)}(z)$ that satisfy

$$z^{-1}D_0 S^{(\ell)}(z^{-1})D_0 = S^{(\ell)}(z), \quad z^{-1}D_0 \widetilde{S}^{(\ell)}(z^{-1})D_0 = \widetilde{S}^{(\ell)}(z).$$

More specifically, let

$$S^{(\ell)}(z) := \frac{1}{2}\begin{bmatrix} a_\ell & b_\ell \\ c_\ell & d_\ell \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a_\ell & -b_\ell \\ -c_\ell & d_\ell \end{bmatrix} z^{-1}, \quad \widetilde{S}^{(\ell)}(z) := \frac{1}{2}\begin{bmatrix} \widetilde{a}_\ell & \widetilde{b}_\ell \\ \widetilde{c}_\ell & \widetilde{d}_\ell \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \widetilde{a}_\ell & -\widetilde{b}_\ell \\ -\widetilde{c}_\ell & \widetilde{d}_\ell \end{bmatrix} z^{-1}.$$

Then proceeding as in Subsection 3.1, we see that $H^{(\ell)}, G^{(\ell)}, \widetilde{H}^{(\ell)}, \widetilde{G}^{(\ell)}$ satisfy (3.8) with $c = 1$ and $d = -1$. The case $L = 2$ with $\widetilde{S}^{(2)}(z) = 0_2$ leads to optimal balanced biorthogonal multiwavelets ${}_{8,6}\Psi^{bo}, {}_{8,6}\widetilde{\Psi}^{bo}$, while the case $L = 2$ with $\widetilde{S}^{(2)}(z) \neq 0_2$ gives ${}_{8,10}\Psi^{bo}, {}_{8,10}\widetilde{\Psi}^{bo}$. All the multifilter banks of these multiwavelets are of even length.

4. Application to image compression

Using the multiwavelet decomposition frame for image compression in [25], we develop a zerotree (see [20]) based algorithm for applying biorthogonal multiwavelets to image compression. In our implementation, we use the symmetric extension transform provided in [27] to extend the original image over its boundaries. All the biorthogonal OPTFR multiwavelets constructed in Section 3 are implemented. Let $\mathrm{Biort}_{N,\widetilde{N}}$ denote the optimal balanced multifilter bank ${}_{N,\widetilde{N}}H^{bo}, {}_{N,\widetilde{N}}G^{bo}, {}_{N,\widetilde{N}}\widetilde{H}^{bo}, {}_{N,\widetilde{N}}\widetilde{G}^{bo}$. Thus the multifilter banks considered are $\mathrm{Biort}_{7,5}$, $\mathrm{Biort}_{7,9}$, $\mathrm{Biort}_{8,6}$, $\mathrm{Biort}_{8,10}$. Since all these multfilter banks are obtained from balanced multiwavelets, no prefiltering is needed in their application (see [18]).

To compare and evaluate the performance of these multifilter banks, we use two standard test images of size 512×512, namely "Lena" and "Barbara". The compression ratios (CR) of "Lena" are 32:1, 64:1 and 100:1, while that of "Barbara" are 16:1, 32:1 and 64:1. We use the Peak Signal to Noise Ratio (PSNR) to measure the quality of the reconstructed image. The reconstruction results are given in Tables 1 and 2. As a basis of comparison, the reconstruction results by several popular scalar wavelets are also included. The scalar wavelets considered are the 8-tap orthonormal Daubechies wavelet (D_8, see [4, p. 195]), the 8-tap least asymmetric orthonormal wavelet (L-asym$_8$, see [4, p. 198]), and the $(9,7)$-tap biorthogonal scalar wavelet (S-biort$_{9,7}$, see [4, p. 279]). Each of these wavelets is chosen because the total length of the corresponding filters is approximately equal to that of the multifilter banks constructed.

The biorthogonal multiwavelets designed in this paper generally perform better than the orthonormal scalar wavelets D$_8$ and L-asym$_8$ (see Tables 1 and 2). In some cases, for instance, images with more high frequency components such as "Barbara", our biorthogonal multiwavelets even outperform the biorthogonal scalar wavelet S-biort$_{9,7}$. Figures 2–4 show the original image, and the reconstructed images of "Barbara" with the scalar wavelet S-biort$_{9,7}$ and the multiwavelet Biort$_{7,9}$ at a compression ratio of 32:1. Note that the reconstructed image using Biort$_{7,9}$ preserves details, such as texture, of the picture better.

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CR	$\text{Biort}_{7,5}$	$\text{Biort}_{7,9}$	$\text{Biort}_{8,6}$	$\text{Biort}_{8,10}$	D_8	L-asym_8	$\text{S-biort}_{9,7}$
32 : 1	34.002	34.234	34.107	34.192	33.086	33.409	34.108
64 : 1	31.000	31.149	31.021	31.080	30.051	30.405	31.154
100 : 1	29.254	29.435	29.334	29.337	28.355	28.894	29.789

TABLE 1. PSNRs (in dB) of compressing the image "Lena" with different multiwavelets and wavelets at compression ratios of 32:1, 64:1, 100:1.

CR	$\text{Biort}_{7,5}$	$\text{Biort}_{7,9}$	$\text{Biort}_{8,6}$	$\text{Biort}_{8,10}$	D_8	L-asym_8	$\text{S-biort}_{9,7}$
16 : 1	31.410	31.672	31.022	31.602	30.174	30.295	30.827
32 : 1	27.643	27.817	27.239	27.701	26.277	26.444	26.738
64 : 1	25.682	25.735	25.673	25.766	24.674	24.988	25.206

TABLE 2. PSNRs (in dB) of compressing the image "Barbara" with different multiwavelets and wavelets at compression ratios of 16:1, 32:1, 64:1.

FIGURE 1. (a) Graphs of (i) ${}_{7,9}\phi_1^{bo}$ (solid line) and ${}_{7,9}\psi_1^{bo}$ (dashed line), (ii) ${}_{7,9}\widetilde{\phi}_1^{bo}$ (solid line) and ${}_{7,9}\widetilde{\psi}_1^{bo}$ (dashed line). (b) Plots of the magnitudes of (i) ${}_{7,9}h_1^{bo}$ (solid line) and ${}_{7,9}g_1^{bo}$ (dashed line), (ii) ${}_{7,9}\widetilde{h}_1^{bo}$ (solid line) and ${}_{7,9}\widetilde{g}_1^{bo}$ (dashed line).

FIGURE 2. The original "Barbara" image.

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FIGURE 3. Reconstructed image using the biorthogonal scalar wavelet S-biort$_{9,7}$ at a compression ratio of 32:1 with PSNR=26.738 dB.

FIGURE 4. Reconstructed image using the biorthogonal multiwavelet Biort$_{7,9}$ at a compression ratio of 32:1 with PSNR=27.817 dB.