# Biorthogonal Wavelets with 4-fold Axial Symmetry for Quadrilateral Surface Multiresolution Processing 

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#### Abstract

Surface multiresolution processing is an important subject in CAGD. It also poses many challenging problems including the design of multiresolution algorithms. Unlike images which are in general sampled on a regular square or hexagonal lattice, the meshes in surfaces processing could have an arbitrary topology, namely, they consist of not only regular vertices but also extraordinary vertices, which requires the multiresolution algorithms have high symmetry.

With the idea of lifting scheme, [1] introduces a novel triangle surface multiresolution algorithm which works for both regular and extraordinary vertices. This method is also successfully used to develop multiresolution algorithms for quad surface and $\sqrt{3}$ triangle surface processing in [35] and [36] respectively. When considering the biorthogonality, these papers do not use the conventional $L^{2}\left(\mathbb{R}^{2}\right)$ inner product, and they do not consider the corresponding lowpass filter, highpass filters, scaling function and wavelets. Hence, some basic properties such as smoothness and approximation power of the scaling functions and wavelets for regular vertices are unclear. On the other hand, the symmetry of subdivision masks (namely, the lowpass filters of filter banks) for surface subdivision is well studied, while the symmetry of the highpass filters for surface processing is rarely considered in the literature.

In this paper we introduce the notion of 4 -fold symmetry for biorthogonal filter banks. We demonstrate that 4 -fold symmetric filter banks result in multiresolution algorithms with the required symmetry for quad surface processing. In addition, we provide 4 -fold symmetric biorthogonal FIR filter banks and construct the associated wavelets, with both the dyadic and $\sqrt{2}$ refinements. Furthermore, we show that some filter banks constructed in this paper result in very simple multiresolution decomposition and reconstruction algorithms as those in $[1,35,36]$. Our method can provide the filter banks corresponding to the multiresolution algorithms in [35] for dyadic multiresolution quad surface processing. Therefore, the properties of the scaling functions and wavelets corresponding to those algorithms can be obtained by analyzing the corresponding filter banks.


Key words and phrases: 4 -fold symmetry, biorthogonal filter banks, biorthogonal wavelets, biorthogonal $\sqrt{2}$-refinement wavelets, surface multiresolution processing, surface multiresolution decomposition/reconstruction.

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## 1 Introduction

This paper is about the design of symmetric filter banks and wavelets for surface multiresolution processing. The filter banks and their symmetry considered in this paper are closely related to

[^0]the surface subdivision masks (local averaging rule templates) and their symmetry. To facilitate the explanation of the symmetry of filter banks required for surface multiresolution processing, first we briefly recall subdivision rules and the symmetry of subdivision masks.

Subdivision is an efficient method to generate smooth surfaces with arbitrary topology and it has been successfully used in animation movie production, see e.g. [30, 37]. To construct a smooth surface, the subdivision process is carried out iteratively, starting from an initial triangle or quadrilateral (quad) mesh, called control mesh/net, to generate a sequence of finer and finer meshes that eventually converges to the desirable limiting surface. The rule how to insert new vertices to a coarse mesh and how to connect the new vertices and old vertices to generate a finer mesh is called the subdivision topological rule. The rule to give the exact positions of the new inserted vertices (and probably to update old vertices) in 3-D space is called the subdivision local averaging rule. The dyadic (or 1-to-4 split) rule is the most commonly used topological rule. Other topological rules include $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and $\sqrt{7}$ refinements, see e.g. $[4,5,12,18,19,20,23$, $24,27,28,33,34]$.

After one iteration of a dyadic quad surface subdivision, a quad in the coarse mesh is split into 4 quads by connecting appropriately the inserted vertices and the updated old vertices. For example, the left of Fig. 1 is a coarse quad mesh. New vertices are added on the edges and faces of quads in the coarse mesh (these new vertices are called edge vertices and face vertices respectively), and then, each face vertex is connected to its nearest four edge vertices to form a finer mesh as shown in the middle of Fig. 1. The exact positions of the added vertices and formula to update the old vertices are given by the local averaging rules which are usually given by templates.


Figure 1: Left: 3-D quad mesh; Middle: Finer mesh; Right: 2-D representation
A vertex is said to be regular if its valence (the number of edges connecting the vertex to other vertices) is 6 for triangle mesh and is 4 for quad mesh. Otherwise, it is called an extraordinary vertex. For example, $E$ in the left picture of Fig. 1 is an extraordinary vertex since its valence is 3 . A vertex and its nearby vertices in a $3-\mathrm{D}$ mesh are expressed locally in a 2 -D plane which makes it easier to express the subdivision topological rule and local averaging rule. For example, the neighborhood of the extraordinary vertex $E$ in the middle of Fig. 1 is expressed in the right picture of Fig. 1.

The local averaging rule (for regular vertices) is associated with some refinement equation. More precisely, the refinement equation

$$
\begin{equation*}
\phi(\mathbf{x})=\sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}} \phi(2 \mathbf{x}-\mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^{2} \tag{1}
\end{equation*}
$$

yields the local averaging rule:

$$
\begin{equation*}
v_{\mathbf{k}}^{j+1}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} v_{\mathbf{n}}^{j} p_{\mathbf{k}-2 \mathbf{n}}, \quad \mathbf{k} \in \mathbf{Z}^{2}, j=0,1, \cdots \tag{2}
\end{equation*}
$$

where $v_{\mathbf{k}}^{j+1}$ are the vertices of the finer mesh obtained after $j+1$ steps of subdivision iterations. The (finite) sequence $\left\{p_{\mathbf{k}}\right\}$ is called the refinement mask or subdivision mask, and the compactly supported function $\phi(\mathbf{x})$ of (1) is called the refinable (or scaling) function. The local averaging rule (2) is sometimes described and represented in the plane with a set of subdivision templates (or stencils). The smoothness of the limiting surface near a regular vertex is determined by that of $\phi$, which in turn can be characterized by the associated refinement mask $\left\{p_{\mathbf{k}}\right\}$. It is common that one first designs the local averaging rule for regular vertices, namely, one constructs $\left\{p_{\mathbf{k}}\right\}$ first. After that, one designs the local averaging rule for extraordinary vertices. In order that one can design compatible local averaging rule for extraordinary vertices, the local averaging rule templates for regular vertices must have high symmetry. Roughly speaking, these templates are independent of the orientations and reflections of vertices. For example, the templates for edge vertices $e_{1}, e_{2}, e_{3}$ around an extraordinary vertex $E$ in the right picture of Fig. 1 must be identical, and that for face vertices $f_{1}, f_{2}, f_{3}$ are the same. The required symmetry of the templates are well-known and the commonly used subdivision schemes do have such a symmetry. For example, Fig. 2 shows the Catmull-Clark scheme [2] to update regular vertices (left), to calculate the positions of the face vertex (middle) and the edge vertex (right). The templates are orientation and reflection invariant.


Figure 2: Catmull-Clark scheme
For a pair of (dyadic refinement) filter banks $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$, the (dyadic refinement) multiresolution decomposition algorithm for an input image/data $\mathcal{C}=\left\{c_{\mathbf{k}}^{0}\right\}$ is

$$
\begin{equation*}
c_{\mathbf{n}}^{j+1}=\frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}-2 \mathbf{n}} c_{\mathbf{k}}^{j}, d_{\mathbf{n}}^{(\ell, j+1)}=\frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} q_{\mathbf{k}-2 \mathbf{n}}^{(\ell)} c_{\mathbf{k}}^{j}, \tag{3}
\end{equation*}
$$

with $\ell=1,2,3, \mathbf{n} \in \mathbf{Z}^{2}$ for $j=0,1, \cdots, J-1$, where $J$ is a positive integer. The multiresolution reconstruction algorithm is given by

$$
\begin{equation*}
\widetilde{c}_{\mathbf{k}}^{j}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{\mathbf{k}-2 \mathbf{n}} \widetilde{c}_{\mathbf{n}}^{j+1}+\sum_{1 \leq \ell \leq 3} \sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{q}_{\mathbf{k}-2 \mathbf{n}}^{(\ell)} d_{\mathbf{n}}^{(\ell, j+1)} \tag{4}
\end{equation*}
$$

with $\mathbf{k} \in \mathbf{Z}^{2}$ for $j=J-1, J-2, \cdots, 0$, where $\widetilde{c}_{\mathbf{n}, J}=c_{\mathbf{n}, J}$. Filter banks $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are said to be the perfect reconstruction filter banks if $\widetilde{c}_{\mathbf{k}}^{j}=c_{\mathbf{k}}^{j}, 0 \leq j \leq$ $J-1$ for any input $\mathcal{C}$. $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ is called the analysis filter bank and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$
the synthesis filter bank. $\left\{c_{\mathbf{k}}^{j}\right\},\left\{d_{\mathbf{k}}^{(\ell, j)}\right\}$ are called the "smooth part" (or "approximation") and the "details" of $\mathcal{C}$.

When (3) and (4) are used for surface multiresolution processing, $\mathcal{C}=\left\{c_{\mathbf{k}}^{0}\right\}$ is the input regular mesh with vertices $c_{\mathbf{k}}^{0}$ in $\mathbb{R}^{3}$. Observe that when $d_{\mathbf{k}}^{(\ell, j+1)}=0$, (4) is reduced to $\widetilde{c}_{\mathbf{k}}^{j}=$ $\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{\mathbf{k}-2 \mathbf{n}} \widetilde{c}_{\mathbf{n}}^{j+1}, j=J-1, J-2, \cdots$. This is the subdivision algorithm with subdivision mask $\left\{\widetilde{p}_{\mathbf{k}}\right\}_{\mathbf{k}}$ starting with the initial control net with vertices $\widetilde{c}_{\mathbf{k}}^{J}$. The decomposition and reconstruction algorithms (3) and (4) are for regular vertices only. One needs to design the corresponding multiresolution algorithms for extraordinary vertices. In order that one can design compatible decomposition and reconstruction algorithms for extraordinary vertices, not only $p$ and $\widetilde{p}$ have high symmetry, but also do $q^{(\ell)}$ and $\widetilde{q}^{(\ell)}, 1 \leq \ell \leq 3$. Clearly, $p$ and $\widetilde{p}$ should possess the same symmetry as that when they are considered as subdivision masks. The question needed to be answered is which kind of symmetry the highpass filters $q^{(\ell)}, \widetilde{q}^{(\ell)}, 1 \leq \ell \leq 3$ should have.

In our study of wavelets for surface multiresolution processing, we realize that for triangle surface, the filter bank should have 6 -fold axial symmetry, while for quad surface, the filter bank should have 4 -fold axial symmetry which is introduced below. The construction of 6 -fold symmetric filter banks and the associated wavelets is considered in our very recent paper [17]. In this paper we consider 4 -fold symmetric wavelets for quad surfaces. Next we give the definition of 4 -fold symmetry.


Figure 3: 4 symmetric axes (lines)

Definition 1. Let $T_{k}, 0 \leq k \leq 3$ be the axes in Fig. 3. A (dyadic refinement) filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ is said to have 4-fold axial (line) symmetry if (i) its lowpass filter $p(\boldsymbol{\omega})$ is symmetric around $T_{k}, 0 \leq k \leq 3$, (ii) $e^{-i\left(\omega_{1}+\omega_{2}\right)} q^{(1)}(\boldsymbol{\omega})$ is symmetric around the axes $T_{k}, 0 \leq k \leq$ 3, (iii) $e^{i \omega_{1}} q^{(2)}(\boldsymbol{\omega})$ is symmetric around the axes $T_{1}$ and $T_{3}$, and (iv) $q^{(3)}(\boldsymbol{\omega})$ is the reflection of $q^{(2)}(\boldsymbol{\omega})$ around the line $\omega_{2}=\omega_{1}$.

In Definition 1 and throughout, for a dyadic refinement mask $\left\{p_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ of real numbers with finitely many $p_{\mathbf{k}}$ nonzero, its corresponding finite impulse response (FIR) filter $p(\boldsymbol{\omega})(p(\boldsymbol{\omega})$ is also called the symbol of $\left\{p_{\mathbf{k}}\right\}$ ) is defined by

$$
p(\boldsymbol{\omega})=\frac{1}{4} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}} e^{-i \mathbf{k} \cdot \boldsymbol{\omega}}
$$

One of the main objectives of this paper is to construct 4-fold axial symmetric biorthogonal FIR filter banks and the associated compactly supported biorthogonal wavelets. The construction
of 4-fold axial symmetric subdivision mask $p$ (namely, $p$ is symmetric around lines $T_{k}, 0 \leq k \leq 3$ ) and its dual mask $\widetilde{p}$ which also has 4 -fold axial symmetry is studied in [8]. To the author's best knowledge, the construction of filter banks which have a symmetry in Definition 1 for both lowpass and highpass filters has not been considered in the literature. We will demonstrate that the 4 -fold symmetry of a filter bank is the symmetry required for quad surface multiresolution processing.

Linear spline and butterfly-scheme related semi-orthogonal wavelets for surface multiresolution processing have been studied in [25, 26], and Doo's subdivision scheme based wavelets for quad surfaces are constructed in [29]. With the idea of lifting scheme, [1] introduces a novel triangle surface multiresolution algorithm which works for both regular and extraordinary vertices. This method is also successfully adopted to develop multiresolution algorithms for quad surface and $\sqrt{3}$-refinement triangle surface processing in [35] and [36] respectively. When considering the biorthogonality, these papers do not use the conventional $L^{2}\left(\mathbb{R}^{2}\right)$ inner product, and therefore, they do not consider the corresponding lowpass filter, highpass filters, scaling function and wavelets. Hence, the properties on smoothness and approximation power of the scaling functions and wavelets for regular vertices are unclear. We show that some filter banks constructed in this paper also result in very simple multiresolution decomposition and reconstruction algorithms as those in $[1,35,36]$. Therefore, the properties of the scaling functions and wavelets corresponding to the quad surface algorithms for regular vertices in [35] can be obtained by analyzing the corresponding filter banks which can be obtained by our method introduced in this paper.


Figure 4: Left: Coarse mesh; Right: Finer mesh after $\sqrt{2}$ refinement
The quad surface subdivision allows not only the dyadic refinement but also $\sqrt{2}$ and $\sqrt{5}$ refinements as well, see $[5,12,24,33,34]$. The $\sqrt{2}$ topological rule is (1) first to insert a vertex on each quad (face) of the coarse mesh, (2) then to remove the edges of the quad of the coarse mesh, and (3) finally, to connect each new vertex to its near four old vertices. See Fig. 4, where the left picture is the coarse mesh and the right picture is the finer mesh with black bullets - denoting the inserted new vertices (called face vertices) and big circles $\odot$ denoting the old vertices (called vertex vertices). Compared with the dyadic refinement, the $\sqrt{2}$-refinement generates more resolutions. Another main objective of this paper is to about the design of $\sqrt{2}$ refinement (quincunx) biorthogonal FIR filter banks for quad surface multiresolution processing. Next, we introduce the 4 -fold symmetry of a $\sqrt{2}$-refinement filter bank $\{p, q\}$.

Definition 2. Let $T_{k}, 0 \leq k \leq 3$ be the axes in Fig. 3. A ( $\sqrt{2}$-refinement) filter bank $\{p, q\}$ is said to have 4 -fold axial (line) symmetry if (i) $p(\boldsymbol{\omega})$ is symmetric around $T_{k}, 0 \leq k \leq 3$, (ii) $e^{-i \omega_{1}} q(\boldsymbol{\omega})$ is symmetric around the axes $T_{k}, 0 \leq k \leq 3$.

The construction of $\sqrt{2}$-refinement wavelets (also called quincunx wavelets) are studied in [21], [22] and [11]. In this paper we construct 4-fold symmetric $\sqrt{2}$-refinement biorthogonal FIR filter banks and the associated compactly supported wavelets. The 4 -fold symmetric filter banks lead to the multiresolution algorithms with the symmetry required for quad surface processing. We show that some of the constructed filter banks yield very simple multiresolution algorithms.

The work on 4 -fold symmetric dyadic refinement filter banks and that on 4 -fold symmetric $\sqrt{2}$-refinement filter banks are carried out in $\S 2$ and $\S 3$ respectively.

## 2 Dyadic refinement wavelets with 4-fold axial symmetry

In this section, we study the dyadic refinement biorthogonal wavelets with 4-fold axial symmetry. This section consists of two subsections. In the first subsection, $\S 2.1$, we obtain families of 4 -fold symmetric biorthogonal FIR filter banks and construct the associated wavelets. We also provide results on the 4 -fold symmetry of filter banks and the associated scaling functions and wavelets in $\S 2.1$. In the second subsection, $\S 2.2$, we show that some filter banks presented in $\S 2.1$ result in simple decomposition and reconstruction algorithms for quad surface multiresolution.

### 2.1 Biorthogonal FIR filter banks and wavelets with 4 -fold axial symmetry

For a filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$, denote

$$
q^{(0)}(\boldsymbol{\omega})=p(\boldsymbol{\omega})
$$

It is well-known that $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are PR filter banks if and only if

$$
\begin{equation*}
\sum_{0 \leq j \leq 3} q^{\left(\ell^{\prime}\right)}\left(\boldsymbol{\omega}+\pi \boldsymbol{\eta}_{j}\right) \overline{\tilde{q}^{(\ell)}\left(\boldsymbol{\omega}+\pi \boldsymbol{\eta}_{j}\right)}=\delta_{\ell^{\prime}-\ell} \tag{5}
\end{equation*}
$$

for $0 \leq \ell, \ell^{\prime} \leq 3, \boldsymbol{\omega} \in \mathbb{R}^{2}$, where $\delta_{k}$ is the kronecker-delta sequence: $\delta_{k}=1$ if $k=0$, and $\delta_{k}=0$ if $k \neq 0$, and $\boldsymbol{\eta}_{j}, 0 \leq j \leq 3$ are the representatives of the group $\mathbf{Z}^{2} /\left(2 \mathbf{Z}^{2}\right)$ :

$$
\begin{equation*}
\boldsymbol{\eta}_{0}=(0,0), \boldsymbol{\eta}_{1}=(-1,-1), \boldsymbol{\eta}_{2}=(1,0), \boldsymbol{\eta}_{3}=(0,1) \tag{6}
\end{equation*}
$$

$\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are also said to be biorthogonal if they satisfy (5).
For a pair of biorthogonal FIR filter banks $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$, let $\phi$ and $\widetilde{\phi}$ be the scaling functions associated with $p$ and $\widetilde{p}$ respectively. Then under ceratin mild conditions (see e.g. $[6,14]$ ), $\phi$ and $\widetilde{\phi}$ of $L^{2}\left(\mathbb{R}^{2}\right)$ are biorthogonal duals: $\int_{\mathbb{R}^{2}} \phi(\mathbf{x}) \widetilde{\phi}(\mathbf{x}-\mathbf{k}) d \mathbf{x}=\delta_{k_{1}} \delta_{k_{2}}, \mathbf{k}=$ $\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2}$. In this case, $\psi^{(\ell)}, \widetilde{\psi}^{(\ell)}, \ell=1,2,3$, defined by

$$
\begin{equation*}
\psi^{(\ell)}(\mathbf{x})=\sum_{\mathbf{k} \in \mathbf{Z}^{2}} q_{\mathbf{k}}^{(\ell)} \phi(2 \mathbf{x}-\mathbf{k}), \widetilde{\psi}^{(\ell)}(\mathbf{x})=\sum_{\mathbf{k} \in \mathbf{Z}^{2}} \widetilde{q}_{\mathbf{k}}^{(\ell)} \widetilde{\phi}(2 \mathbf{x}-\mathbf{k}) \tag{7}
\end{equation*}
$$

are biorthogonal wavelets, namely they generate biorthogonal bases for $L^{2}\left(\mathbb{R}^{2}\right)$.
Let

$$
J_{0}=\left[\begin{array}{ll}
0 & 1  \tag{8}\\
1 & 0
\end{array}\right], J_{1}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right], J_{2}=-J_{0}, J_{3}=-J_{1}
$$

and denote

$$
O_{1}=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

Then for each $j, 0 \leq j \leq 3,\left\{p_{\mathbf{k}}\right\}$ is symmetric around the symmetry axis $T_{j}$ in Fig. 3 if and only if $p_{J_{j} \mathbf{k}}=p_{\mathbf{k}}$; and $\left\{p_{O_{1} \mathbf{k}}\right\}$ is the $\frac{\pi}{2}$ (anticlockwise) rotation of $\left\{p_{\mathbf{k}}\right\}$. By Definition 1 , one can obtain that $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ has 4-fold axial symmetry if and only if

$$
\left\{\begin{array}{l}
p\left(J_{k} \boldsymbol{\omega}\right)=p(\boldsymbol{\omega}), 0 \leq k \leq 3,  \tag{9}\\
q^{(1)}\left(J_{0} \boldsymbol{\omega}\right)=e^{2 i \omega_{2}} q^{(1)}\left(J_{1} \boldsymbol{\omega}\right)=e^{2 i\left(\omega_{1}+\omega_{2}\right)} q^{(1)}\left(J_{2} \boldsymbol{\omega}\right)=e^{2 i \omega_{1}} q^{(1)}\left(J_{3} \boldsymbol{\omega}\right)=q^{(1)}(\boldsymbol{\omega}), \\
q^{(2)}\left(J_{1} \boldsymbol{\omega}\right)=e^{-2 i \omega_{1}} q^{(2)}\left(J_{3} \boldsymbol{\omega}\right)=q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})=q^{(2)}\left(J_{0} \boldsymbol{\omega}\right) .
\end{array}\right.
$$

In the above equations we have used the fact $J_{k}^{-T}=J_{k}, 0 \leq k \leq 3$. This fact and that $O_{1}^{-T}=O_{1}$ will be used again in the rest of this paper.

Observe that

$$
\begin{equation*}
O_{1}=J_{0} J_{3}, \quad J_{k}=O_{1}^{k} J_{0}, 0 \leq k \leq 3 \tag{10}
\end{equation*}
$$

Thus, when we discuss the 4 -fold axial symmetry of a filter bank, we need only consider the operations with $J_{0}$ and $O_{1}$. First we have the following proposition to describe the 4 -fold symmetry.
Proposition 1. A filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ has 4 -fold axial symmetry if and only if it satisfies

$$
\begin{align*}
& {\left[p, q^{(1)}, q^{(2)}, q^{(3)}\right]^{T}\left(O_{1} \boldsymbol{\omega}\right)=\mathcal{M}_{1}(2 \boldsymbol{\omega})\left[p, q^{(1)}, q^{(2)}, q^{(3)}\right]^{T}(\boldsymbol{\omega}),}  \tag{11}\\
& {\left[p, q^{(1)}, q^{(2)}, q^{(3)}\right]^{T}\left(J_{0} \boldsymbol{\omega}\right)=\mathcal{M}_{0}\left[p, q^{(1)}, q^{(2)}, q^{(3)}\right]^{T}(\boldsymbol{\omega}),} \tag{12}
\end{align*}
$$

where

$$
\mathcal{M}_{1}(\boldsymbol{\omega})=\left[\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{13}\\
0 & e^{-i \omega_{1}} & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & e^{i \omega_{1}} & 0
\end{array}\right], \mathcal{M}_{0}=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

Proof. By (10) and direct calculations, one can easily obtain (11) and (12) are equivalent to (9). Hence, the 4 -fold axial symmetry of $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ can be characterized by (11) and (12). $\diamond$

For an FIR filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$, with notation $q^{(0)}(\boldsymbol{\omega})=p(\boldsymbol{\omega})$, write $q^{(\ell)}(\boldsymbol{\omega}), 0 \leq \ell \leq 3$ as

$$
q^{(\ell)}(\boldsymbol{\omega})=\frac{1}{2}\left(q_{0}^{(\ell)}(2 \boldsymbol{\omega})+q_{1}^{(\ell)}(2 \boldsymbol{\omega}) e^{i\left(\omega_{1}+\omega_{2}\right)}+q_{2}^{(\ell)}(2 \boldsymbol{\omega}) e^{-i \omega_{1}}+q_{3}^{(\ell)}(2 \boldsymbol{\omega}) e^{-i \omega_{2}}\right),
$$

where $q_{k}^{(\ell)}(\boldsymbol{\omega})$ are trigonometric polynomials. Then the polyphase matrix of $\left\{p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega})\right.$, $\left.q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})\right\}$ is defined as

$$
\begin{equation*}
V(\boldsymbol{\omega})=\left[q_{k}^{(\ell)}(\boldsymbol{\omega})\right]_{0 \leq \ell, k \leq 3} . \tag{14}
\end{equation*}
$$

Clearly,

$$
\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{2} V(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega}),
$$

where $I_{00}(\boldsymbol{\omega})$ is defined by

$$
\begin{equation*}
I_{00}(\boldsymbol{\omega})=\left[1, e^{i\left(\omega_{1}+\omega_{2}\right)}, e^{-i \omega_{1}}, e^{-i \omega_{2}}\right]^{T} . \tag{15}
\end{equation*}
$$

Observe that 1-tap filter bank $\left\{1, e^{i\left(\omega_{1}+\omega_{2}\right)}, e^{-i \omega_{1}}, e^{-i \omega_{2}}\right\}$ has 4 -fold symmetry. Thus, $I_{00}(\boldsymbol{\omega})$ defined above satisfies (11) and (12). The next proposition presents a characterization of the 4 -fold axial symmetry of a filter bank in terms of its polyphase matrix.

Proposition 2. An FIR filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ has 4-fold axial symmetry if and only if its polyphase matrix $V(\boldsymbol{\omega})$ satisfies

$$
\begin{align*}
& V\left(O_{1} \boldsymbol{\omega}\right)=\mathcal{M}_{1}(\boldsymbol{\omega}) V(\boldsymbol{\omega}) \mathcal{M}_{2}(\boldsymbol{\omega}),  \tag{16}\\
& V\left(J_{0} \boldsymbol{\omega}\right)=\mathcal{M}_{0} V(\boldsymbol{\omega}) \mathcal{M}_{0}, \tag{17}
\end{align*}
$$

where $\mathcal{M}_{1}$ and $\mathcal{M}_{0}$ are given by (13) and $\mathcal{M}_{2}(\boldsymbol{\omega})=\mathcal{M}_{1}(\boldsymbol{\omega})^{-1}$ :

$$
\mathcal{M}_{2}(\boldsymbol{\omega})=\left[\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{18}\\
0 & e^{i \omega_{1}} & 0 & 0 \\
0 & 0 & 0 & e^{-i \omega_{1}} \\
0 & 0 & 1 & 0
\end{array}\right]
$$

Proof. From the definition of $V(\boldsymbol{\omega})$,

$$
\left[p, q^{(1)}, q^{(2)}, q^{(3)}\right]\left(O_{1} \boldsymbol{\omega}\right)=\frac{1}{2} V\left(2 O_{1} \boldsymbol{\omega}\right) I_{00}\left(O_{1} \boldsymbol{\omega}\right)=\frac{1}{2} V\left(2 O_{1} \boldsymbol{\omega}\right) \mathcal{M}_{1}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega}) .
$$

Thus (11) is equivalent to

$$
\frac{1}{2} V\left(2 O_{1} \boldsymbol{\omega}\right) \mathcal{M}_{1}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})=\mathcal{M}_{1}(2 \boldsymbol{\omega}) \frac{1}{2} V(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})
$$

or,

$$
V\left(2 O_{1} \boldsymbol{\omega}\right) \mathcal{M}_{1}(2 \boldsymbol{\omega})=\mathcal{M}_{1}(2 \boldsymbol{\omega}) V(2 \boldsymbol{\omega})
$$

that is

$$
V\left(O_{1} \boldsymbol{\omega}\right)=\mathcal{M}_{1}(\boldsymbol{\omega}) V(\boldsymbol{\omega}) \mathcal{M}_{1}(\boldsymbol{\omega})^{-1}
$$

which is (16).
Similarly, we have that (12) is equivalent to

$$
V\left(2 J_{0} \boldsymbol{\omega}\right) \mathcal{M}_{0}=\mathcal{M}_{0} V(2 \boldsymbol{\omega})
$$

which is (17). Therefore, (11) and (12) are equivalent to (16) and (17). Hence, from Proposition 1, we know $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ has 4-fold axial symmetry if and only if its polyphase matrix $V(\boldsymbol{\omega})$ satisfies both (16) and (17). $\diamond$

In the next proposition we provide the symmetry of the scaling function and wavelets associated with a 4 -fold symmetric filter bank.
Proposition 3. Suppose an FIR filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ has 4 -fold axial symmetry. Let $\phi$ be the associated scaling function and $\psi^{(\ell)}, \ell=1,2,3$ be the functions define by $(7)$ with $q^{(\ell)}$. Then

$$
\begin{align*}
& \phi\left(J_{k} \mathbf{x}\right)=\phi(\mathbf{x}), 0 \leq k \leq 3,  \tag{19}\\
& \psi^{(3)}(\mathbf{x})=\psi^{(2)}\left(J_{0} \mathbf{x}\right),  \tag{20}\\
& \psi^{(2)}\left(J_{1} \mathbf{x}\right)=\psi^{(2)}(\mathbf{x}), \psi^{(2)}\left(J_{3} \mathbf{x}\right)=\psi^{(2)}(\mathbf{x}+(1,0)),  \tag{21}\\
& \left\{\begin{array}{l}
\psi^{(1)}\left(J_{0} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}), \psi^{(1)}\left(J_{1} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}-(0,1)), \\
\psi^{(1)}\left(J_{2} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}-(1,1)), \psi^{(1)}\left(J_{3} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}-(1,0)) .
\end{array}\right. \tag{22}
\end{align*}
$$

Proof. From (1), we have $\widehat{\phi}(\boldsymbol{\omega})=p\left(\frac{\boldsymbol{\omega}}{2}\right) \widehat{\phi}\left(\frac{\boldsymbol{\omega}}{2}\right)$. Thus $\widehat{\phi}(\boldsymbol{\omega})=\Pi_{n=1}^{\infty} p\left(2^{-n} \boldsymbol{\omega}\right) \widehat{\phi}(0)$. Therefore, $p\left(J_{k} \boldsymbol{\omega}\right)=p(\boldsymbol{\omega}), 0 \leq k \leq 3$, imply

$$
\widehat{\phi}\left(J_{k} \boldsymbol{\omega}\right)=\Pi_{n=1}^{\infty} p\left(2^{-n} J_{k} \boldsymbol{\omega}\right) \widehat{\phi}(0)=\Pi_{n=1}^{\infty} p\left(2^{-n} \boldsymbol{\omega}\right) \widehat{\phi}(0)=\widehat{\phi}(\boldsymbol{\omega})
$$

which is (19).
From (7), we have $\widehat{\psi}^{(\ell)}(\boldsymbol{\omega})=q^{(\ell)}\left(\frac{\boldsymbol{\omega}}{2}\right) \widehat{\phi}\left(\frac{\boldsymbol{\omega}}{2}\right), \ell=1,2,3$. Thus

$$
\widehat{\psi}^{(2)}\left(J_{0} \boldsymbol{\omega}\right)=q^{(2)}\left(J_{0} \boldsymbol{\omega} / 2\right) \widehat{\phi}\left(J_{0} \boldsymbol{\omega} / 2\right)=q^{(3)}(\boldsymbol{\omega} / 2) \widehat{\phi}(\boldsymbol{\omega} / 2)=\widehat{\psi}^{(3)}(\boldsymbol{\omega})
$$

which implies (20).
Next we prove $(21)$. From $q^{(2)}\left(J_{1} \boldsymbol{\omega}\right)=q^{(2)}(\boldsymbol{\omega})$ and $\widehat{\phi}\left(J_{1} \mathbf{x}\right)=\widehat{\phi}(\mathbf{x})$, we have

$$
\widehat{\psi}^{(2)}\left(J_{1} \boldsymbol{\omega}\right)=q^{(2)}\left(J_{1} \boldsymbol{\omega} / 2\right) \widehat{\phi}\left(J_{1} \boldsymbol{\omega} / 2\right)=q^{(2)}(\boldsymbol{\omega} / 2) \widehat{\phi}(\boldsymbol{\omega} / 2)=\widehat{\psi}^{(2)}(\boldsymbol{\omega})
$$

which is the first equation in (21). The proof of the second equation is similar. Indeed, $q^{(2)}\left(J_{3} \boldsymbol{\omega}\right)=$ $e^{2 i \omega_{1}} q^{(2)}(\boldsymbol{\omega})$ lead to that

$$
\widehat{\psi}^{(2)}\left(J_{3} \boldsymbol{\omega}\right)=q^{(2)}\left(J_{3} \boldsymbol{\omega} / 2\right) \widehat{\phi}\left(J_{3} \boldsymbol{\omega} / 2\right)=e^{i \omega_{1}} q^{(2)}(\boldsymbol{\omega} / 2) \widehat{\phi}(\boldsymbol{\omega} / 2)=e^{i \omega_{1}} \widehat{\psi}^{(2)}(\boldsymbol{\omega})
$$

Therefore, $\psi^{(2)}\left(J_{3} \mathbf{x}\right)=\psi^{(2)}(\mathbf{x}+(1,0))$, as desired.
Using the formulas for $q^{(1)}$ in (9), one can show (22) similarly. The details are omitted here. $\diamond$

In the rest of this subsection we present biorthogonal FIR filter banks with 4 -fold symmetry, and construct a few sets of the associated biorthogonal wavelets. We are interested in the filter banks which have block structures, namely, they are given by simple blocks and a simple initial filter bank. As mentioned above, 1-tap filter bank $\left\{1, e^{i\left(\omega_{1}+\omega_{2}\right)}, e^{-i \omega_{1}}, e^{-i \omega_{2}}\right\}$ has 4 -fold symmetry and hence, it could be used as the initial filter bank. So the key to obtain the block structures is to find suitable blocks which satisfy both (16) and (17). In the following we present two types of such blocks. First observe that two filter banks $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are biorthogonal if and only if

$$
\begin{equation*}
V(\boldsymbol{\omega}) \tilde{V}(\boldsymbol{\omega})^{*}=I_{4}, \quad \boldsymbol{\omega} \in \mathbb{R}^{2} \tag{23}
\end{equation*}
$$

where $V(\boldsymbol{\omega})$ and $\tilde{V}(\boldsymbol{\omega})$ are their polyphase matrices defined by (14).
In the following we use the notations:

$$
\begin{equation*}
x=e^{-i \omega_{1}}, y=e^{-i \omega_{2}} \tag{24}
\end{equation*}
$$

With $x, y$ in (24), an FIR filter $p(\boldsymbol{\omega})$ can be written as a (Laurent) polynomial of $x, y$. We may use the following block to build the symmetric filter banks:

$$
\begin{align*}
& G(\boldsymbol{\omega})=  \tag{25}\\
& {\left[\begin{array}{cccc}
\kappa+\varrho(x+y)\left(1+\frac{1}{x y}\right)+\iota\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right) & \lambda(1+x)(1+y) & \left(1+\frac{1}{x}\right)\left(\mu+\nu y+\frac{\nu}{y}\right) & \left(1+\frac{1}{y}\right)\left(\mu+\nu x+\frac{\nu}{x}\right) \\
n\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & 1 & 0 & 0 \\
m(1+x) & 0 & 1 & 0 \\
m(1+y) & 0 & 0 & 1
\end{array}\right.}
\end{align*}
$$

where

$$
\kappa=j+4 m \mu+4 n \lambda, \rho=m(\mu+2 \nu)+2 n \lambda, \iota=2 m \nu+n \lambda,
$$

$j, m, n, \lambda, \mu, \nu$ are constants with $j \neq 0$. One can verify that $G(\boldsymbol{\omega})$ satisfies (16) and (17). Thus filter banks built by $G(\boldsymbol{\omega})$ have 4 -fold axial symmetry. For example, the filter bank given by $\frac{1}{4} G(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})$ has 4 -fold axial symmetry. Furthermore, the determinant of $G(\boldsymbol{\omega})$ is $j$, a nonzero constant. Thus, the inverse of $G(\boldsymbol{\omega})$ is a matrix whose entries are also (Laurent) polynomials of $x, y$. More precisely, $\widetilde{G}(\boldsymbol{\omega})=\left(G(\boldsymbol{\omega})^{-1}\right)^{*}$ is given by

$$
\begin{aligned}
& \widetilde{G}(\boldsymbol{\omega})=\frac{1}{j} \times \\
& {\left[\begin{array}{cccc}
1 & -n(1+x)(1+y) & -m\left(1+\frac{1}{x}\right) & -m\left(1+\frac{1}{y}\right) \\
-\lambda\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & A(x, y) & m \lambda\left(1+\frac{1}{x}\right)^{2}\left(1+\frac{1}{y}\right) & m \lambda\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)^{2} \\
-(1+x)\left(\mu+\nu y+\frac{\nu}{y}\right) & n(1+x)^{2}\left((\mu+\nu)(1+y)+\nu y^{2}+\frac{\nu}{y}\right) & B(x, y) & m(1+x)\left((\mu+\nu)\left(1+\frac{1}{y}\right)^{2}+\nu y+\frac{\nu}{y^{2}}\right) \\
-(1+y)\left(\mu+\nu x+\frac{\nu}{x}\right) & n(1+y)^{2}\left((\mu+\nu)(1+x)+\nu x^{2}+\frac{\nu}{x}\right) & m(1+y)\left((\mu+\nu)\left(1+\frac{1}{x}\right)+\nu x+\frac{\nu}{x^{2}}\right) & B(y, x)
\end{array}\right.}
\end{aligned}
$$

where

$$
\begin{aligned}
& A(x, y)=j+4 n \lambda+2 n \lambda(x+y)\left(1+\frac{1}{x y}\right)+n \lambda\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right), \\
& B(x, y)=j+2 m \mu+m \mu\left(x+\frac{1}{x}\right)+m \nu\left(2+x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right) .
\end{aligned}
$$

One can verify that $\widetilde{G}(\boldsymbol{\omega})$ also satisfies (16) and (17).
We may also use block

$$
H(\boldsymbol{\omega})=\left[\begin{array}{cccc}
1 & -o(1+x)(1+y) & -\left(1+\frac{1}{x}\right)\left(g+h y+\frac{h}{y}\right) & -\left(1+\frac{1}{y}\right)\left(g+h x+\frac{h}{x}\right)  \tag{27}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

Actually $H(\boldsymbol{\omega})$ is $G(\boldsymbol{\omega})$ with $j=1, m=n=0, \lambda=-o, \mu=-g, \nu=-h$. In this case $\tilde{H}(\boldsymbol{\omega})=$ $\left(H(\boldsymbol{\omega})^{-1}\right)^{*}$ is given by

$$
\widetilde{H}(\boldsymbol{\omega})=\left[\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{28}\\
o\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) & 1 & 0 & 0 \\
\left.(1+x)\left(g+h y+\frac{h}{y}\right)\right) & 0 & 1 & 0 \\
\left.(1+y)\left(g+h x+\frac{h}{x}\right)\right) & 0 & 0 & 1
\end{array}\right]
$$

Based on the above discussion, we have the following block structure of 4 -fold symmetric biorthogonal FIR filter banks.

Theorem 1. Suppose FIR filter banks $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are given by

$$
\begin{align*}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})\right]^{T}=V_{K}(2 \boldsymbol{\omega}) V_{K-1}(2 \boldsymbol{\omega}) \cdots V_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})}  \tag{29}\\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega}), \widetilde{q}^{(3)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{4} \widetilde{V}_{K}(2 \boldsymbol{\omega}) \widetilde{V}_{K-1}(2 \boldsymbol{\omega}) \cdots \widetilde{V}_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})}
\end{align*}
$$

for some $K \in \mathbf{Z}_{+}$, where $I_{00}(\boldsymbol{\omega})$ is defined by (15), each $V_{k}(\boldsymbol{\omega})$ is a $G(\boldsymbol{\omega})$ in (25) or a $\widetilde{G}(\boldsymbol{\omega})$ in (26) for some parameters $\mu_{k}, \nu_{k}, \lambda_{k}, m_{k}, n_{k}$, or an $H(\boldsymbol{\omega})$ in (27) or a $\widetilde{H}(\boldsymbol{\omega})$ in (28) for some parameters $g_{k}, h_{k}, o_{k}$, and $\widetilde{V}_{k}(\boldsymbol{\omega})=\left(V_{k}(\boldsymbol{\omega})^{-1}\right)^{*}$ is the corresponding $\widetilde{G}(\boldsymbol{\omega})$ in (26) $(G(\boldsymbol{\omega})$ in (25), $\widetilde{H}(\boldsymbol{\omega})$ in (28), $H(\boldsymbol{\omega})$ in (27) accordingly). Then $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are biorthogonal FIR filter banks with 4-fold axial symmetry.

Proof. The polyphase matrices $V(\boldsymbol{\omega})$ and $\widetilde{V}(\boldsymbol{\omega})$ of $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are respectively

$$
V(\boldsymbol{\omega})=2 V_{K}(\boldsymbol{\omega}) V_{K-1}(\boldsymbol{\omega}) \cdots V_{0}(\boldsymbol{\omega}), \tilde{V}(\boldsymbol{\omega})=\frac{1}{2} \widetilde{V}_{K}(\boldsymbol{\omega}) \tilde{V}_{K-1}(\boldsymbol{\omega}) \cdots \tilde{V}_{0}(\boldsymbol{\omega})
$$

Clearly, $V(\boldsymbol{\omega})$ and $\widetilde{V}(\boldsymbol{\omega})$ satisfy (23). Furthermore, since $V_{j}(\boldsymbol{\omega}), \widetilde{V}_{j}(\boldsymbol{\omega})$ satisfy both (16) and (17), so do $V(\boldsymbol{\omega})$ and $\widetilde{V}(\boldsymbol{\omega})$. Hence, $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ are biorthogonal to each other, and both of them have 4 -fold axial symmetry. $\diamond$

The method to build a pair of biorthogonal filter banks $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ from another pair of biorthogonal filter banks with $H(\boldsymbol{\omega})$ or $\widetilde{H}(\boldsymbol{\omega})$ is called the lifting scheme method, see $[32,7]$. In the following we construct biorthogonal wavelets associated with the above filter banks. When we construct biorthogonal wavelets, we will construct one scaling function smoother than its dual scaling function. The filter bank with smoother scaling function should be used as the synthesis filter bank. In this paper we consider the Sobolev smoothness of scaling functions and wavelets. We say a function $f$ on $\mathbb{R}^{2}$ to be in the Sobolev space $W^{s}$ for some $s>0$ if its Fourier transform $\hat{f}$ satisfies $\int_{\mathbb{R}^{2}}\left(1+|\boldsymbol{\omega}|^{2}\right)^{s}|\hat{f}(\boldsymbol{\omega})|^{2} d \boldsymbol{\omega}<\infty$. The Sobolev smoothness of a scaling function $\phi$ can be given by the eigenvalues of the transition operator $\operatorname{matrix} T_{p}$ associated with the corresponding lowpass filter $p$, see $[15,16]$.

To construct a smooth wavelet basis, the corresponding scaling function $\phi$ must have certain approximation power (see e.g. [13]). In addition, if $\phi$ has approximation order $K$, then the decomposition algorithm with highpass filters $q^{(1)}, q^{(2)}, q^{(3)}$ annihilates (discrete) polynomials of total degree $<K$, see e.g. [31, 3]. The approximation order of a refinable $\phi$ can be described by the sum rule order of the associated subdivision mask $p(\boldsymbol{\omega})$, see [13]. We say $p(\boldsymbol{\omega})$ to have sum rule order $K$ if it satisfies that $p(0,0)=1$ and

$$
\begin{equation*}
\frac{\partial^{\alpha_{1}+\alpha_{2}}}{\partial \omega_{1}^{\alpha_{1}} \partial \omega_{2}^{\alpha_{2}}} p\left(\pi \boldsymbol{\eta}_{j}\right)=0,1 \leq j \leq 3 \tag{30}
\end{equation*}
$$

for all $\left(\alpha_{1}, \alpha_{2}\right) \in \mathbf{Z}_{+}^{2}$ with $\alpha_{1}+\alpha_{2}<K$, where $\boldsymbol{\eta}_{j}, 1 \leq j \leq 3$ are defined by (6).
Example 1. Let $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ be the biorthogonal filter banks given by (29) for $K=1$ with

$$
\begin{align*}
{\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})\right]^{T} } & =\widetilde{G}_{1}(2 \boldsymbol{\omega}) \widetilde{G}_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega}),  \tag{31}\\
{\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega}), \widetilde{q}^{(3)}(\boldsymbol{\omega})\right]^{T} } & =\frac{1}{4} G_{1}(2 \boldsymbol{\omega}) G_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega}),
\end{align*}
$$

where $G_{0}(\boldsymbol{\omega})$ and $G_{1}(\boldsymbol{\omega})$ are given by (25) for parameters $j, \mu, \nu, \lambda, m, n$ and $\mu_{1}, \nu_{1}, \lambda_{1}, m_{1}, n_{1}$, $j_{1}=1$ respectively, and $\widetilde{G}_{0}(\boldsymbol{\omega})=\left(G_{0}(\boldsymbol{\omega})^{-1}\right)^{*}, \widetilde{G}_{1}(\boldsymbol{\omega})=\left(G_{1}(\boldsymbol{\omega})^{-1}\right)^{*}$. Solving the system of the equations for sum rule order 1 of both $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$, we have

$$
\begin{align*}
& j=4\left(1+4 n_{1}+4 m_{1}\right) /\left(1+8 \mu_{1} m_{1}+16 n_{1} \lambda_{1}+16 m_{1} \nu_{1}\right), m=-\frac{1}{2}-4 m_{1}, n=-\frac{1}{4}-4 n_{1}, \\
& \lambda=\left(\frac{1}{4}-\lambda_{1}\right) /\left(1+8 \mu_{1} m_{1}+16 n_{1} \lambda_{1}+16 m_{1} \nu_{1}\right)  \tag{32}\\
& \mu=-2 \nu+\left(\frac{1}{2}-2 \nu_{1}-\mu_{1}\right) /\left(1+8 \mu_{1} m_{1}+16 n_{1} \lambda_{1}+16 m_{1} \nu_{1}\right) .
\end{align*}
$$

Because of the symmetry of $p(\boldsymbol{\omega}), \widetilde{p}(\boldsymbol{\omega})$, the conditions in (30) for $p(\boldsymbol{\omega}), \widetilde{p}(\boldsymbol{\omega})$ with $\left(\alpha_{1}, \alpha_{2}\right)=$ $(1,0),(0,1)$ are automatically satisfied. Thus the resulting $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$ actually have sum rule order 2. If we choose

$$
\begin{equation*}
\left[\lambda_{1}, m_{1}, n_{1}, \mu_{1}, \nu, \nu_{1}\right]=\left[\frac{33}{64}, \frac{43}{256}, \frac{5}{32}, \frac{83}{64}, \frac{1}{16},-\frac{9}{64}\right] ; \tag{33}
\end{equation*}
$$

then the resulting $\phi$ is in $W^{1.40893}$, and $\widetilde{\phi}$ in $W^{1.61625}$. If we select

$$
\left[\lambda_{1}, m_{1}, n_{1}, \mu_{1}, \nu, \nu_{1}\right]=\left[\frac{33}{64}, \frac{3}{16}, \frac{17}{128}, \frac{2452781}{1850112}, \frac{776117}{13478355},-\frac{544853}{3700224}\right]
$$

then the resulting $\phi$ and $\widetilde{\phi}$ are in $W^{1.40330}$ and $W^{1.48950}$ respectively. In latter case $\widetilde{p}$ has sum rule order 4. For either case, the resulting $p, \widetilde{p}$ are supported on $[-3,3]^{2}$ and $[-4,4]^{2}$ respectively. Here and below, an FIR filter $h$ being supported on $[-N, N]^{2}$ for some positive integer $N>0$ means that its coefficient $h_{\mathbf{k}}$ satisfy that $h_{\mathbf{k}}=0$ for any $\mathbf{k} \notin[-N, N]^{2}$ ( $h_{\mathbf{k}}$ could be zero for some $\mathbf{k}$ in $\left.[-N, N]^{2}\right)$. One may choose other values for the parameters such that the resulting $\widetilde{\sim}$ is smoother. However, we cannot adjust the parameters such that $\widetilde{\phi}$ has much higher smoothness order with its dual $\phi$ in $L^{2}\left(\mathbb{R}^{2}\right)$.
Example 2. Let $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ be the biorthogonal filter banks given by (29) for $K=1$ with

$$
\begin{align*}
{\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})\right]^{T} } & =H_{0}(2 \boldsymbol{\omega}) \widetilde{G}_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})  \tag{34}\\
{\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega}), \widetilde{q}^{(3)}(\boldsymbol{\omega})\right]^{T} } & =\frac{1}{4} \widetilde{H}_{0}(2 \boldsymbol{\omega}) G_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})
\end{align*}
$$

where $G_{0}(\boldsymbol{\omega})$ and ${\underset{\sim}{H}}_{0}(\boldsymbol{\omega})$ are given by (25) and (27) with parameters $j, \mu, \nu, \lambda, m, n$ and $g_{0}, h_{0}, o_{0}$ respectively, and $\widetilde{G}_{0}(\boldsymbol{\omega})=\left(G_{0}(\boldsymbol{\omega})^{-1}\right)^{*}, \widetilde{H}_{0}(\boldsymbol{\omega})=\left(H_{0}(\boldsymbol{\omega})^{-1}\right)^{*}$. In this case, $p$ and $\widetilde{p}$ are supported on $[-5,5]^{2}$ and $[-2,2]^{2}$ respectively with

$$
\begin{aligned}
& p_{4,4}=p_{-4,4}=p_{-4,-4}=p_{4,-4}=p_{5,5}=p_{-5,5}=p_{-5,-5}=p_{5,-5} \\
& \quad=p_{5,4}=p_{-5,4}=p_{-5,-4}=p_{5,-4}=p_{4,5}=p_{-4,5}=p_{-4,-5}=p_{4,-5}=0 .
\end{aligned}
$$

Solving the system of the equations for sum rule order 1 of both $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$, we have

$$
o_{0}=-\frac{1}{16}-\frac{1}{4} n, g_{0}=-\frac{1}{8}-\frac{1}{4} m-2 h_{0}, \lambda=\frac{1}{4}, \mu=\frac{1}{2}-2 \nu, j=1-4 m-4 n .
$$

Again, because of the symmetry, the resulting $p$ and $\widetilde{p}$ actually have sum rule order 2 . In addition, if $\nu=\frac{1}{16}, n=\frac{1}{8}-\frac{3}{4} m$, then $\widetilde{p}$ has sum rule order 4. Furthermore if $m=\frac{1}{4}$, then $\widetilde{p}(\boldsymbol{\omega})=$ $\frac{1}{256} e^{2 i\left(\omega_{1}+\omega_{2}\right)}\left(1+e^{-i \omega_{1}}\right)^{4}\left(1+e^{-i \omega_{2}}\right)^{4}$, the mask for the Catmull-Clark scheme. However, if $\widetilde{p}$ has sum rule order 4, we cannot choose parameters $h_{0}, m$ (the resulting $p$ depends on $h_{0}, m$ ) such that the corresponding $\phi$ is in $L^{2}\left(\mathbb{R}^{2}\right)$. Thus we consider $p$ and $\widetilde{p}$ with both of them having sum rule order 2 only. If we choose

$$
\begin{equation*}
\left[h_{0}, m, \nu, n\right]=\frac{1}{256}[14,20,9,0] \tag{35}
\end{equation*}
$$

then the resulting $\phi$ is in $W^{0.01991}$ and $\widetilde{\phi}$ in $W^{1.84203}$, while if we choose,

$$
\left[h_{0}, m, \nu, n\right]=\frac{1}{256}[14,16,9,4]
$$

then the resulting $\phi$ and $\widetilde{\phi}$ are in $W^{0.00793}$ and $W^{1.85883}$ respectively. $\diamond$
Example 3. Let $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ be the biorthogonal filter banks given by (29) for $K=2$ with

$$
\begin{align*}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})\right]^{T}=\widetilde{H}_{1}(2 \boldsymbol{\omega}) H_{0}(2 \boldsymbol{\omega}) \widetilde{G}_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})}  \tag{36}\\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega}), \widetilde{q}^{(3)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{4} H_{1}(2 \boldsymbol{\omega}) \widetilde{H}_{0}(2 \boldsymbol{\omega}) G_{0}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega})}
\end{align*}
$$

where $G_{0}(\boldsymbol{\omega})$ is given by (25) with parameters $j, \mu, \nu, \lambda, m, n,{\underset{\sim}{0}}_{0}(\boldsymbol{\omega})$ and $H_{1}(\boldsymbol{\omega})$ are given by (27) with parameters $g_{0}, h_{0}, o_{0}$ and $g_{1}, h_{1}, o_{1}$ respectively, and $\widetilde{G}_{0}(\boldsymbol{\omega})=\left(G_{0}(\boldsymbol{\omega})^{-1}\right)^{*}, \widetilde{H}_{j}(\boldsymbol{\omega})=$
$\left(H_{j}(\boldsymbol{\omega})^{-1}\right)^{*}, j=0,1$. Solving the system of the equations for sum rule orders 1 and 3 of $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$ respectively, we have that the resuting $p$ and $\widetilde{p}$ depend on $h_{0}, m, \nu, \mu_{1}, \nu_{1}$ (other parameters are given in terms of these parameters), and that they actually have sum rule orders 2 and 4 respecively because of their symmetry. Then, we can choose parameters $h_{0}, m, \nu, \mu_{1}, \nu_{1}$ such that the corresponidng $\widetilde{\phi}$ is in $W^{2}$ and $\phi$ has certain smoothness. Below we provide one set of the selected parameters (and also other corresponding parameters for sum rules). If

$$
\begin{align*}
& {\left[j, m, n, \mu, \nu, \lambda, g_{0}, o_{0}, h_{0}, g_{1}, h_{1}, o_{1}\right]} \\
& =\left[\frac{527}{195},-\frac{23}{16}, \frac{1045}{5616},-\frac{63}{160}, \frac{9}{256},-\frac{207}{1280}, \frac{5158981}{29787264},-\frac{2449}{22464}, \frac{1822409}{59574528},-\frac{5}{4}, \frac{1}{64},-\frac{39}{64}\right], \tag{37}
\end{align*}
$$

then the resulting $\phi$ is in $W^{0.52189}$ and $\widetilde{\phi}$ in $W^{2.20781}$ with $p$ and $\widetilde{p}$ having sum rule orders 2 and 4 respecively. We can also choose other numbers such that $\phi$ is in $W^{2}$. For example, if

$$
\begin{aligned}
& {\left[j, m, n, \mu, \nu, \lambda, g_{0}, o_{0}, h_{0}, g_{1}, h_{1}, o_{1}\right]} \\
& =\left[\frac{56}{23},-\frac{5}{2},-\frac{5}{4},-\frac{79}{736}, \frac{31}{1472},-\frac{3}{92}, \frac{17}{32}, \frac{1}{4},-\frac{1}{64},-\frac{121}{112}, \frac{17}{224},-\frac{13}{28}\right],
\end{aligned}
$$

then the resulting $\phi$ and $\tilde{\phi}$ are in $W^{2.06421}$ and $W^{1.06992}$ respectively. In the latter case, both $p$ and $\widetilde{p}$ have sum rule order 4. In the above two cases, $p$ and $\widetilde{p}$ are supported on $[-5,5]^{2}$ and $[-6,6]^{2}$ respectively. $\diamond$

Here we remark that the wavelets constructed in the above are not separable. The projections of $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$ to the $\omega_{1}$ coordinate, $p\left(\omega_{1}, 0\right)$ and $\widetilde{p}\left(\omega_{1}, 0\right)$, result in 1-D FIR filters with both symmetric around 0 . The reader refers to [9] for the relationship between the smoothness of a scaling function and that of its projection.

### 2.2 Dyadic multiresolution algorithms for quad surface processing

As observed above, we have the fact $O_{1}=J_{0} J_{3}$. Thus if a filter $p(\boldsymbol{\omega})$ satisfies $p\left(J_{k} \boldsymbol{\omega}\right)=p(\boldsymbol{\omega})$ for all $k, 0 \leq k \leq 3$, then $p\left(O_{1}^{k} \boldsymbol{\omega}\right)=p(\boldsymbol{\omega})$ for any $0 \leq k \leq 3$. Namely, if $p(\boldsymbol{\omega})$ is reflection invariant, then it is rotation invariant. Thus, the lowpass filter $p(\boldsymbol{\omega})$ of a 4 -fold axial symmetric filter bank $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ is reflection and rotation invariant, which implies that the templates resulted from either decomposition algorithm or reconstruction algorithm with $p$ (in latter case $p$ is a subdivision mask) have the desired symmetry. In this subsection, we demonstrate that the highpass filters $q^{(\ell)}, 1 \leq \ell \leq 3$ also have the symmetry required for quad surface multiresolution processing. The key is to associate appropriately $d_{\mathbf{k}}^{(1,1)}, d_{\mathbf{k}}^{(2,1)}$ and $d_{\mathbf{k}}^{(3,1)}$ to the nodes of $\mathbf{Z}^{2}$ with which a regular quad mesh is represented, where $d_{\mathbf{k}}^{(1,1)}, d_{\mathbf{k}}^{(2,1)}$ and $d_{\mathbf{k}}^{(3,1)}$ are the "details" of the initial vertices $c_{\mathbf{k}}^{0}, \mathbf{k} \in \mathbf{Z}^{2}$ after the decomposition algorithm with highpass filters $q^{(1)}, q^{(2)}$ and $q^{(3)}$ respectively. To this regard, we first separate the nodes of $\mathbf{Z}^{2}$ into four groups.

For the square lattice $\mathbf{Z}^{2}, 2 \mathbf{Z}^{2}=\left\{\left(2 k_{1}, 2 k_{2}\right),\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2}\right\}$ is the set of the labels for the vertices of the coarse mesh. The nodes of $\mathbf{Z}^{2}$ with labels $\left(2 k_{1}, 2 k_{2}\right)$ are called vertex nodes. Next, we separate $\mathbf{Z}^{2} \backslash\left(2 \mathbf{Z}^{2}\right)$ into face nodes with indices in $\{2 \mathbf{k}-(1,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and the edge nodes with indices in $\{2 \mathbf{k}+(1,0), 2 \mathbf{k}+(0,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$. The edge nodes are further separated into two groups with indices in $\{2 \mathbf{k}+(1,0)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and $\{2 \mathbf{k}+(0,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ respectively. See the left of Fig. 5, where the big circles, squares, $\triangle$ and $\nabla$ denote vertex nodes, face nodes, and two groups of edge nodes respectively.

Let $\mathcal{C}=\left\{c_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ be the data sampled on $\mathbf{Z}^{2}$ or a regular quad mesh with vertices $c_{\mathbf{k}}$. Then $\left\{c_{2 \mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of data/vertices associated with vertex nodes, $\left\{c_{2 \mathbf{k}-(1,1)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of
data/vertices associated with face nodes, and $\left\{c_{2 \mathbf{k}+(1,0)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and $\left\{c_{2 \mathbf{k}+(0,1)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ are the sets of data/vertices associated with the above two groups of edge nodes. Denote

$$
\begin{equation*}
v_{\mathbf{k}}=c_{2 \mathbf{k}}, f_{\mathbf{k}}=c_{2 \mathbf{k}-(1,1)}, e_{\mathbf{k}}^{(2)}=c_{2 \mathbf{k}+(1,0)}, e_{\mathbf{k}}^{(3)}=c_{2 \mathbf{k}+(0,1)}, \mathbf{k} \in \mathbf{Z}^{2} \tag{38}
\end{equation*}
$$

Refer to the middle picture of Fig. 5 for these four groups of data/vertices. Observe that each set of nodes with indices $\{2 \mathbf{k}\}_{\mathbf{k} \in \mathbf{Z}^{2}},\{2 \mathbf{k}-(1,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}},\{2 \mathbf{k}+(1,0)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ and $\{2 \mathbf{k}+(0,1)\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ respectively forms a coarse square lattice.


Figure 5: Left: Vertex nodes, face nodes, and two types of edges nodes; Middle: Original data/vertices associated with four groups of nodes; Right: "Smooth part" and "details" associated with four groups of nodes

The multiresolution decomposition algorithm is to decompose the original data/mesh $\mathcal{C}=$ $\left\{c_{\mathbf{k}}\right\}_{\mathbf{k}}$ with the analysis filter bank into the "smooth part" $\left\{c_{\mathbf{k}}^{1}\right\}_{\mathbf{k}}$ and "the details" $\left\{d_{\mathbf{k}}^{(1,1)}\right\}_{\mathbf{k}}$ $\left\{d_{\mathbf{k}}^{(2,1)}\right\}_{\mathbf{k}}$ and $\left\{d_{\mathbf{k}}^{(3,1)}\right\}_{\mathbf{k}}$, while the (prefect) multiresolution reconstruction algorithm is to recover exactly $\mathcal{C}$ from $\left\{c_{\mathbf{k}}^{1}\right\}_{\mathbf{k}},\left\{d_{\mathbf{k}}^{(1,1)}\right\}_{\mathbf{k}}\left\{d_{\mathbf{k}}^{(2,1)}\right\}_{\mathbf{k}}$ and $\left\{d_{\mathbf{k}}^{(3,1)}\right\}_{\mathbf{k}}$ with the synthesis filter bank. Denote

$$
\widetilde{v}_{\mathbf{k}}=c_{\mathbf{k}}^{1}, \tilde{f}_{\mathbf{k}}=d_{\mathbf{k}}^{(1,1)}, \widetilde{e}_{\mathbf{k}}^{(2)}=d_{\mathbf{k}}^{(2,1)}, \widetilde{e}_{\mathbf{k}}^{(3)}=d_{\mathbf{k}}^{(3,1)}
$$

Then, the decomposition algorithm can be written as

$$
\begin{equation*}
\widetilde{v}_{\mathbf{k}}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} p_{\mathbf{k}^{\prime}-2 \mathbf{k}} c_{\mathbf{k}^{\prime}}, \widetilde{f}_{\mathbf{k}}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-2 \mathbf{k}}^{(1)} c_{\mathbf{k}^{\prime}}, \widetilde{e}_{\mathbf{k}}^{(2)}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-2 \mathbf{k}}^{(2)} c_{\mathbf{k}^{\prime}}, \widetilde{e}_{\mathbf{k}}^{(3)}=\frac{1}{4} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-2 \mathbf{k}}^{(3)} c_{\mathbf{k}^{\prime}} \tag{39}
\end{equation*}
$$

for $\mathbf{k} \in \mathbf{Z}^{2}$, and the reconstruction algorithm is

$$
\begin{equation*}
c_{\mathbf{k}}=\sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}}\{\widetilde{p}_{\mathbf{k}-2 \mathbf{k}^{\prime}} \widetilde{v}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-2 \mathbf{k}^{\prime}}^{(1)} \tilde{f}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-2 \mathbf{k}^{\prime}}^{(2)} \widetilde{\epsilon}_{\mathbf{k}^{\prime}}^{(2)}+\widetilde{q}_{\mathbf{k}-2 \mathbf{k}^{\prime}}^{(3)} \overbrace{\mathbf{k}^{\prime}}^{(3)}\}, \mathbf{k} \in \mathbf{Z}^{2} \tag{40}
\end{equation*}
$$

where $p_{\mathbf{k}}, q_{\mathbf{k}}^{(1)}, q_{\mathbf{k}}^{(2)}, q_{\mathbf{k}}^{(3)}, \mathbf{k} \in \mathbf{Z}^{2}$ and $\widetilde{p}_{\mathbf{k}}, \widetilde{q}_{\mathbf{k}}^{(1)}, \widetilde{q}_{\mathbf{k}}^{(2)}, \widetilde{q}_{\mathbf{k}}^{(3)}, \mathbf{k} \in \mathbf{Z}^{2}$ are the coefficients of the analysis filter bank and the synthesis filter banks respectively. Considering $\mathbf{k}$ in (40) with cases $\mathbf{k}=2 \mathbf{j}, 2 \mathbf{j}$ $(1,1), 2 \mathbf{j}+(1,0), 2 \mathbf{j}+(0,1)$, one can write (40) as

$$
\begin{align*}
& v_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{2 \mathbf{n}} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}}^{(1)} \tilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}}^{(2)} \widetilde{\mathrm{c}}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}}^{(3)} \widetilde{\mathrm{c}}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\}, \\
& f_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{\mathbf{Z}}}\left\{\widetilde{p}_{2 \mathbf{n}-(1,1)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}-(1,1)}^{(1)} \tilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}-(1,1)}^{(2)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}-(1,1)}^{(3)} \tilde{e}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\}, \\
& e_{\mathbf{k}}^{(2)}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{2 \mathbf{n}+(1,0)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(1,0)}^{(1)} \tilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(1,0)}^{(2)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}+(1,0)}^{(3)} \widetilde{e}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\},  \tag{41}\\
& e_{\mathbf{k}}^{(3)}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{2 \mathbf{n}+(0,1)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(0,1)}^{(1)} \widetilde{f}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{2 \mathbf{n}+(0,1)}^{(2)} \tilde{\mathrm{e}}_{\mathbf{k}-\mathbf{n}}^{(2)}+\widetilde{q}_{2 \mathbf{n}+(0,1)}^{(3)} \tilde{e}_{\mathbf{k}-\mathbf{n}}^{(3)}\right\} .
\end{align*}
$$

Clearly, the "smooth part" $\left\{\widetilde{v}_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ associates with the vertex nodes of $2 \mathbf{Z}^{2}$. We associate the "detail" $\widetilde{f}_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}^{2}$ with face nodes $\left(2 k_{1}-1,2 k_{2}-1\right)$, and associate $\widetilde{e}_{\mathbf{k}}^{(2)}$ and $\widetilde{e}_{\mathbf{k}}^{(3)}$ with the edge nodes with labels $\left(2 k_{1}+1,2 k_{2}\right)$ and $\left(2 k_{1}, 2 k_{2}+1\right)$ respectively, see the right of Fig. 5. Thus, as the original data/mesh $\mathcal{C}=\left\{v_{\mathbf{k}}, f_{\mathbf{k}}, e_{\mathbf{k}}^{(2)}, e_{\mathbf{k}}^{(3)}\right\}_{\mathbf{k}}$ which associates with $\mathbf{Z}^{2}$, the union of the sets of nodes with which the decomposed data $\left\{\widetilde{v}_{\mathbf{k}}\right\}_{\mathbf{k}},\left\{\tilde{f}_{\mathbf{k}}\right\}_{\mathbf{k}},\left\{\widetilde{e}_{\mathbf{k}}^{(2)}\right\}_{\mathbf{k}},\left\{\widetilde{e}_{\mathbf{k}}^{(3)}\right\}_{\mathbf{k}}$ associate is also $\mathbf{Z}^{2}$. With such association, the multiresolution decomposition and reconstruction algorithms can be described as templates. Furthermore, the 4 -fold symmetric filter banks provided in $\S 2.1$ result in templates with desired symmetry for quad surface multiresolution. Next, let us look at a very simple example to illustrate this.

Let $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ be the filter banks defined by

$$
\begin{aligned}
& {\left[p(\boldsymbol{\omega}), q^{(1)}(\boldsymbol{\omega}), q^{(2)}(\boldsymbol{\omega}), q^{(3)}(\boldsymbol{\omega})\right]^{T}=\widetilde{G}(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega}),} \\
& {\left[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}^{(1)}(\boldsymbol{\omega}), \widetilde{q}^{(2)}(\boldsymbol{\omega}), \widetilde{q}^{(3)}(\boldsymbol{\omega})\right]^{T}=\frac{1}{4} G(2 \boldsymbol{\omega}) I_{00}(\boldsymbol{\omega}),}
\end{aligned}
$$

where $G(\boldsymbol{\omega})$ is defined by (25) with parameters $j, \mu, \nu, \lambda, m, n$. Then the nonzero coefficients of these two filter banks are

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
p_{-1,1} & p_{0,1} & p_{1,1} \\
p_{-1,0} & \mathbf{p}_{0,0} & p_{1,1} \\
p_{-1,-1} & p_{0,-1} & p_{1,-1}
\end{array}\right]=\frac{4}{j}\left[\begin{array}{ccc}
-n & -m & -n \\
-m & \mathbf{1} & -m \\
-n & -m & -n
\end{array}\right] ;} \\
& {\left[\begin{array}{cccc}
q_{-3,1}^{(1)} & q_{-2,1}^{(1)} & \cdots & q_{1,1}^{(1)} \\
q_{-3,0}^{(1)} & q_{-2,0}^{(1)} & \cdots & q_{1,0}^{(1)} \\
\cdots & \cdots & \cdots & \cdots \\
q_{-3,-3}^{(1)} & q_{-2,-3}^{(1)} & \cdots & q_{1,-3}^{(1)}
\end{array}\right]=4\left[\begin{array}{cccc}
t_{4} & t_{5} & t_{2} & t_{5} \\
t_{5} & t_{3} & t_{1} & t_{3} \\
t_{5} \\
t_{2} & t_{1} & \mathbf{t}_{0} & t_{1} \\
t_{2} \\
t_{5} & t_{3} & t_{1} & t_{3} \\
t_{5} \\
t_{4} & t_{5} & t_{2} & t_{5} \\
t_{4}
\end{array}\right] ;} \\
& {\left[\begin{array}{cccc}
q_{-1,3}^{(2)} & q_{-2,3}^{(2)} & \cdots & q_{3,3}^{(2)} \\
q_{-1,2}^{(2)} & q_{-2,2}^{(2)} & \cdots & q_{3,2}^{(2)} \\
\cdots & \cdots & \cdots & \cdots \\
q_{-1,-3}^{(2)} & q_{-2,-3}^{(2)} & \cdots & q_{3,-3}^{(2)}
\end{array}\right]=4\left[\begin{array}{ccccc}
s_{7} & s_{4} & 2 s_{7} & s_{4} & s_{7} \\
s_{4} & s_{6} & 2 s_{4} & s_{6} & s_{4} \\
s_{5} & s_{3} & 2 s_{5} & s_{3} & s_{5} \\
s_{2} & s_{1} & \mathbf{s}_{0} & s_{1} & s_{2} \\
s_{5} & s_{3} & 2 s_{5} & s_{3} & s_{5} \\
s_{4} & s_{6} & 2 s_{4} & s_{6} & s_{4} \\
s_{7} & s_{4} & 2 s_{7} & s_{4} & s_{7}
\end{array}\right] ; q_{j, k}^{(3)}=q_{k, j}^{(2)} ;}
\end{aligned}
$$

and

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
\widetilde{p}_{-2,2} & \cdots & \widetilde{p}_{2,2} \\
\cdots & \cdots & \cdots \\
\widetilde{p}_{-2,-2} & \cdots & \widetilde{p}_{2,-2}
\end{array}\right]=\left[\begin{array}{ccccc}
r_{4} & r_{5} & r_{2} & r_{5} & r_{4} \\
r_{5} & r_{3} & r_{1} & r_{3} & r_{5} \\
r_{2} & r_{1} & r_{0} & r_{1} & r_{2} \\
r_{5} & r_{3} & r_{1} & r_{3} & r_{5} \\
r_{4} & r_{5} & r_{2} & r_{5} & r_{4}
\end{array}\right] ;} \\
& \widetilde{q}_{-1,-1}^{(1)}=1, \widetilde{q}_{q, 0}^{(1)}=\widetilde{q}_{-2,0}^{(1)}=\widetilde{q}_{0,-2}^{(1)}=\widetilde{q}_{-2,-2}^{(1)}=n ; \\
& \widetilde{q}_{1,0}^{(2)}=1, \widetilde{q}_{0,0}^{(2)}=\widetilde{q}_{2,0}^{(1)}=m ; \\
& \widetilde{q}_{0,1}^{(3)}=1, \widetilde{q}_{0,0}^{(3)}=\widetilde{q}_{0,2}^{(3)}=m ;
\end{aligned}
$$

where

$$
\begin{equation*}
t_{0}=1+\frac{4 n \lambda}{j}, t_{1}=\frac{2 m \lambda}{j}, t_{2}=\frac{2 n \lambda}{j}, t_{3}=-\frac{\lambda}{j}, t_{4}=\frac{n \lambda}{j}, t_{5}=\frac{m \lambda}{j} ; \tag{42}
\end{equation*}
$$

$$
\begin{align*}
& s_{0}=1+\frac{2 m \mu}{j}, s_{1}=-\frac{\mu}{j}, s_{2}=\frac{m \mu}{j}, s_{3}=\frac{m(\mu+\nu)}{j}, s_{4}=\frac{m \nu}{j}, s_{5}=\frac{n(\mu+\nu)}{j}, s_{6}=-\frac{\nu}{j}, s_{7}=\frac{n \nu}{j}(43) \\
& r_{0}=j+4 m \mu+4 n \lambda, r_{1}=\mu, r_{2}=2 n \lambda+m(\mu+2 \nu), r_{3}=\lambda, r_{4}=2 m \nu+n \lambda, r_{5}=\nu \tag{44}
\end{align*}
$$

In the above matrices, values for $p_{(0,0)}, q_{(-1,-1)}^{(1)}, q_{(1,0)}^{(2)}, \widetilde{p}_{(0,0)}$ are bold-faced.


Figure 6: From left to right: Templates of decomposition algorithm with $p, q^{(1)}$ and $q^{(2)}$, where $t_{j}, s_{k}$ are defined by (42) and (43)

When $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ is used as the analysis filter bank, the templates of the decomposition algorithm with $p, q^{(1)}$ and $q^{(2)}$ are shown in Fig. 6, while the template with $q^{(3)}$ is the reflection (with the line $\omega_{2}=\omega_{1}$ ) of that with $q^{(2)}$. Clearly, these analysis template are orientation and reflection invariant with respect to the coarse mesh. When $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ is used as the synthesis filter bank, then the reconstruction algorithm is

$$
\begin{aligned}
& v_{\mathbf{k}}= r_{0} \widetilde{v}_{\mathbf{k}}+n\left(\widetilde{f}_{\mathbf{k}}+\widetilde{f}_{\mathbf{k}+(1,0)}+\widetilde{f}_{\mathbf{k}+(1,1)}+\widetilde{f}_{\mathbf{k}+(0,1)}\right)+m\left(\widetilde{e}_{\mathbf{k}}^{(2)}+\widetilde{e}_{\mathbf{k}-(1,0)}^{(2)}+\widetilde{e}_{\mathbf{k}}^{(3)}+\widetilde{e}_{\mathbf{k}-(0,1)}^{(3)}\right) \\
&+r_{2}\left(\widetilde{v}_{\mathbf{k}+(1,0)}+\widetilde{v}_{\mathbf{k}+(0,1)}+\widetilde{v}_{\mathbf{k}-(1,0)}+\widetilde{v}_{\mathbf{k}-(0,1)}\right)+r_{4}\left(\widetilde{v}_{\mathbf{k}+(1,1)}+\widetilde{v}_{\mathbf{k}+(-1,1)}+\widetilde{v}_{\mathbf{k}-(1,1)}+\widetilde{v}_{\mathbf{k}+(1,-1)}\right), \\
& f_{\mathbf{k}}= \widetilde{f}_{\mathbf{k}}+r_{3}\left(\widetilde{v}_{\mathbf{k}}+\widetilde{v}_{\mathbf{k}-(1,0)}+\widetilde{v}_{\mathbf{k}-(1,1)}+\widetilde{v}_{\mathbf{k}-(0,1)}\right) \\
& e_{\mathbf{k}}^{(2)}=\widetilde{e}_{\mathbf{k}}^{(2)}+r_{1}\left(\widetilde{v}_{\mathbf{k}}+\widetilde{v}_{\mathbf{k}+(1,0)}\right)+r_{5}\left(\widetilde{v}_{\mathbf{k}-(0,1)}+\widetilde{v}_{\mathbf{k}+(1,1)}+\widetilde{v}_{\mathbf{k}+(0,1)}+\widetilde{v}_{\mathbf{k}-(1,1)}\right), \\
& e_{\mathbf{k}}^{(3)}= \widetilde{e}_{\mathbf{k}}^{(3)}+r_{1}\left(\widetilde{v}_{\mathbf{k}}+\widetilde{v}_{\mathbf{k}+(0,1)}\right)+r_{5}\left(\widetilde{v}_{\mathbf{k}-(1,0)}+\widetilde{v}_{\mathbf{k}+(1,1)}+\widetilde{v}_{\mathbf{k}+(1,0)}+\widetilde{v}_{\mathbf{k}-(1,1)}\right),
\end{aligned}
$$

where $r_{0}, \cdots, r_{5}$ are given by (44). Thus the reconstruction algorithms can be expressed by the templates to calculate $v, f$ and $e$ as shown in Fig. 7. (Observe that the templates to calculate $e_{\mathbf{k}}^{(2)}$ and $e_{\mathbf{k}}^{(3)}$ are identical.) Clearly the templates are also orientation and reflection invariant. Also observe that if we set details $\widetilde{f}_{k}, e_{\mathbf{k}}^{(2)}$ and $e_{\mathbf{k}}^{(3)}$ to be zero, then these templates are reduced to the subdivision templates of the type of Catmull-Clark's scheme in Fig. 2.

As illustrated by the above simple example, the biorthogonal filter banks provided in $\S 2.1$ result in analysis and synthesis templates both with the required symmetry. Thus, based on these templates one can design the algorithms for extraordinary vertices. The design of the corresponding multiresolution algorithms for extraordinary vertices will be considered elsewhere. Next, we show that multiresolution algorithms resulted from some biorthogonal filter banks in $\S 2.1$ can be described in a simpler way, as that in $[1,35,36]$.

For a pair of 4 -fold symmetric filter banks, since the corresponding decomposition templates to calculate $\widetilde{e}_{\mathbf{k}}^{(2)}$ and $\widetilde{e}_{\mathbf{k}}^{(3)}$ are the same, and the reconstruction templates to recover $e_{\mathbf{k}}^{(2)}$ and $e_{\mathbf{k}}^{(3)}$


Figure 7: From left to right: Templates of reconstruction algorithm to recover $v_{\mathbf{k}}, f_{\mathbf{k}}, e_{\mathbf{k}}^{(2)}, e_{\mathbf{k}}^{(3)}$
are identical, in the following we may simply let $e$ denote the original data/vertices associated with the edge nodes, and use $\tilde{e}$ to denote the "details" associated with edge nodes. Therefore, the decomposition algorithm is to decompose the original data $\{v\} \cup\{f\} \cup\{e\}$ into $\{\widetilde{v}\},\{\widetilde{f}\}$ and $\{\widetilde{e}\}$, and the reconstruction algorithm to recover $\{v\} \cup\{f\} \cup\{e\}$ from $\{\widetilde{v}\},\{\widetilde{f}\}$ and $\{\widetilde{e}\}$.

For given data/mesh $\mathcal{C}$ (or equivalently, for given $\{v\},\{f\}$ and $\{e\}$ ), the multiresolution decomposition algorithm is given by (45)-(48) and shown in Fig. 8, where j, m, $n, \mu, \nu, \lambda, g_{0}, o_{0}, h_{0}, \mu_{1}, \nu_{1}, \lambda_{1}$ are constants to be determined. More precisely, first we replace all $v$ associated with vertex nodes in $2 \mathbf{Z}^{2}$ by $v^{\prime \prime}$ given by formula (45). Then, based on $v^{\prime \prime}$ obtained, we replace all $e, f$ associated with edge and face nodes in $\mathbf{Z}^{2} \backslash\left(2 \mathbf{Z}^{2}\right)$ by $e^{\prime \prime}$ and $f^{\prime \prime}$ given in (46). After that, based on $e^{\prime \prime}$, $f^{\prime \prime}$ obtained in Step 2, all $v^{\prime \prime}$ in Step 1 are updated by $\widetilde{v}$ given in formula (47). Finally, based on $\widetilde{v}$ obtained in Step 3, all $e^{\prime \prime}, f^{\prime \prime}$ in Step 2 are updated by $\widetilde{e}, \tilde{f}$ given in formula (48).

## Decomposition Algorithm:

Step 1. $v^{\prime \prime}=\frac{1}{j}\left\{v-m\left(e_{0}+e_{1}+e_{2}+e_{3}\right)-n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}$;
Step 2. $e^{\prime \prime}=e-\mu\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}\right)-\nu\left(v_{2}^{\prime \prime}+v_{3}^{\prime \prime}+v_{4}^{\prime \prime}+v_{5}^{\prime \prime}\right), f^{\prime \prime}=f-\lambda\left(v_{6}^{\prime \prime}+v_{7}^{\prime \prime}+v_{8}^{\prime \prime}+v_{9}^{\prime \prime}\right)$;
Step 3. $\widetilde{v}=v^{\prime \prime}-g_{0}\left(e_{0}^{\prime \prime}+e_{1}^{\prime \prime}+e_{2}^{\prime \prime}+e_{3}^{\prime \prime}\right)-o_{0}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)$

$$
-h_{0}\left(e_{4}^{\prime \prime}+e_{5}^{\prime \prime}+e_{6}^{\prime \prime}+e_{7}^{\prime \prime \prime}+e_{8}^{\prime \prime}+e_{9}^{\prime \prime}+e_{10}^{\prime \prime}+e_{11}^{\prime \prime}\right) ;
$$

The multiresolution reconstruction algorithm is given by (49)-(52) and shown in Fig. 9, where $j, m, n, \mu, \nu, \lambda, g_{0}, o_{0}, h_{0}, \mu_{1}, \nu_{1}, \lambda_{1}$ are the same constants in decomposition algorithm (45)-(48). The reconstruction algorithm is the reverse algorithm of the decomposition algorithm.

## Reconstruction Algorithm:

Step 1. $e^{\prime \prime}=\widetilde{e}+\mu_{1}\left(\widetilde{v}_{0}+\widetilde{v}_{1}\right)+\nu_{1}\left(\widetilde{v}_{2}+\widetilde{v}_{3}+\widetilde{v}_{4}+\widetilde{v}_{5}\right), f^{\prime \prime}=\widetilde{f}+\lambda_{1}\left(\widetilde{v}_{6}+\widetilde{v}_{7}+\widetilde{v}_{8}+\widetilde{v}_{9}\right) ;$

$$
\text { Step 2. } \begin{gather*}
v^{\prime \prime}=\widetilde{v}  \tag{49}\\
+g_{0}\left(e_{0}^{\prime \prime}+e_{1}^{\prime \prime}+e_{2}^{\prime \prime}+e_{3}^{\prime \prime}\right)+o_{0}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)  \tag{50}\\
\\
+h_{0}\left(e_{4}^{\prime \prime}+e_{5}^{\prime \prime}+e_{6}^{\prime \prime}+e_{7}^{\prime \prime}+e_{8}^{\prime \prime}+e_{9}^{\prime \prime}+e_{10}^{\prime \prime}+e_{11}^{\prime \prime}\right) ;
\end{gather*}
$$

Step 3. $e=e^{\prime \prime}+\mu\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}\right)+\nu\left(v_{2}^{\prime \prime}+v_{3}^{\prime \prime}+v_{4}^{\prime \prime}+v_{5}^{\prime \prime}\right), f=f^{\prime \prime}+\lambda\left(v_{6}^{\prime \prime}+v_{7}^{\prime \prime}+v_{8}^{\prime \prime}+v_{9}^{\prime \prime}\right)$;
Step 4. $v=j v^{\prime \prime}+m\left(e_{0}+e_{1}+e_{2}+e_{3}\right)+n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)$.
When the constants $j, m, n, \mu, \nu, \lambda, g_{0}, o_{0}, h_{0}, \mu_{1}, \nu_{1}, \lambda_{1}$ are appropriately chosen, the decomposed $\{\tilde{v}\}$ is the "smooth part" of the initial data/mesh $\mathcal{C},\{\tilde{f}\}$ and $\{\widetilde{e}\}$ are the "details" of $\mathcal{C}$.


Figure 8: Top-left: Decomposition Alg. Step 1 with each $v$ associated with a vertex node replaced by $v^{\prime \prime}$ given in (45); Top-right: Decomposition Alg. Step 2 with each e and $f$ associated with edge and face nodes replaced by $e^{\prime \prime}$ and $f^{\prime \prime}$ resp. given in (46); Bottom-left: Decomposition Alg. Step 3 with each $v^{\prime \prime}$ obtained in Step 1 replaced by $\widetilde{v}$ given in (47); Bottom-right: Decomposition Alg. Step 4 with each $e^{\prime \prime}$ and $f^{\prime \prime}$ obtained in Step 2 replaced by $\widetilde{e}$ and $\widetilde{f}$ given in (48)

The decomposition algorithm can be applied iteratively to the smooth part to get further smooth part and details of the data. The reconstruction algorithm recovers the original data/mesh from the smooth part and details.

When $\widetilde{f}=0, \widetilde{e}=0$, the reconstruction algorithm is the subdivision algorithm to produce finer and finer meshes from the initial mesh with vertices $\widetilde{v}$. Such subdivision schemes are called the composite schemes. The $\sqrt{3}$ and $\sqrt{7}$ composite subdivision schemes are studied in [28] and [27] respectively.

In the following let us give the filter banks corresponding to the above multiresolution algorithms. First, we consider algorithms (45)-(52) with $h_{0}=0$. With the formulas in (39) and (41), and by careful calculations, we obtain the filter banks $\left\{p, q^{(1)}, q^{(2)}, q^{(3)}\right\}$ and $\left\{\widetilde{p}, \widetilde{q}^{(1)}, \widetilde{q}^{(2)}, \widetilde{q}^{(3)}\right\}$ corresponding to the algorithms (45)-(52) with $h_{0}=0$ are these given by (31) in Example 1 with $m_{1}=g_{0}, n_{1}=o_{0}$. From Example 1, with choices of parameters in (33) and the values for other parameters given by (32), the corresponding $\phi$ and $\widetilde{\phi}$ are in $W^{1.40893}$ and $W^{1.61625}$ respectively.

Next, let us consider algorithms (45)-(52) with $\mu_{1}=\nu_{1}=\lambda_{1}=0$. In this case, the algorithms are reduced to 3 step algorithms (45)-(47) and (50)-(52) with $\widetilde{e}=e^{\prime \prime}, \widetilde{f}=f^{\prime \prime}$. One can obtain the


Figure 9: Top-left: Reconstruction Alg. Step 1 with each $\widetilde{e}$ and $\tilde{f}$ associated with edge and face nodes replaced by $e^{\prime \prime}$ and $f^{\prime \prime}$ given in (49); Top-right: Reconstruction Alg. Step 2 with each $\widetilde{v}$ associated with a vertex node replaced by $v^{\prime \prime}$ given in (50); Bottom-left: Reconstruction Alg. Step 3 with each $e^{\prime \prime}$ and $f^{\prime \prime}$ obtained in Step 1 replaced by e and $f$ given in (51); Bottom-right: Reconstruction Alg. Step 4 with each $v^{\prime \prime}$ obtained in Step 2 replaced by $v$ given in (52)
corresponding filter banks are those given by (34) in Example 2. From Example 2, with choices of parameters in (35), the corresponding $\phi$ and $\widetilde{\phi}$ are in $W^{0.01991}$ and $\widetilde{\phi}$ in $W^{1.84203}$ respectively. Here we provide all selected numbers:

$$
\left[j, m, n, \mu, \nu, \lambda, g_{0}, o_{0}, h_{0}\right]=\left[\frac{11}{16}, \frac{5}{64}, 0, \frac{55}{128}, \frac{9}{256}, \frac{1}{4},-\frac{65}{256},-\frac{1}{16}, \frac{7}{128}\right]
$$

Finally, let us consider the filter banks corresponding to algorithms (45)-(52). In this case, the corresponding filter banks are those given by (36) in Example 3 with

$$
g_{1}=-\mu_{1}, h_{1}=-\nu_{1},, o_{1}=-\lambda_{1}
$$

From Example 3, with the choice of parameters given by (37) (and hence $\left[\mu_{1}, \nu_{1}, \lambda_{1}\right]=$ $\left[\frac{5}{4},-\frac{1}{64}, \frac{39}{64}\right]$ ), we have $\phi$ in $W^{0.52189}$ and $\widetilde{\phi}$ in $W^{2.20781}$ with $p$ and $\widetilde{p}$ having sum rule orders 2 and 4 respectively.

To obtain scaling functions and wavelets with a high approximation order or smooth order, we may use more steps in the above algorithms (45)-(52) with more parameters. The corresponding
filter banks are given as those in Examples 2 and 3 but with more blocks $H(2 \boldsymbol{\omega})$ and/or $\widetilde{H}(2 \boldsymbol{\omega})$. With the filter banks available, then one may use the method discussed in Examples 1-3 to choose suitable parameters.

## $3 \sqrt{2}$-refinement wavelets with 4-fold axial symmetry

In this section, we study $\sqrt{2}$-refinement biorthogonal wavelets with 4 -fold axial symmetry. This section consists of two subsections. In the first subsection, $\S 3.1$, we first obtain characterizations of the 4 -fold symmetry of $\sqrt{2}$-refinement filter banks. Then we provide families of 4 -fold symmetric biorthogonal FIR filter banks. Finally, in this subsection, we construct the associated wavelets. In $\S 3.2$, we show that some filter banks presented in $\S 3.1$ yield simple $\sqrt{2}$-refinement decomposition and reconstruction algorithms for quad surface multiresolution.

### 3.1 Biorthogonal $\sqrt{2}$-refinement FIR filter banks and wavelets with 4-fold axial symmetry

As above, let the 2-D square mesh with vertices $\mathbf{Z}^{2}$ represent a regular quad mesh. Let $\mathbf{Z}_{\sqrt{2}}^{2}$ denote the finer mesh with vertices $\left\{\frac{\sqrt{2}}{2}\left(k_{1}+k_{2}, k_{2}-k_{1}\right)\right\}_{\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2}}$, see the right picture of Fig. 4. To provide the $\sqrt{2}$ multiresolution algorithms, first we need to choose a dilation matrix $M$ such that $M^{-1}$ maps $\mathbf{Z}^{2}$ onto its finer mesh $\mathbf{Z}_{\sqrt{2}}^{2}$. We may choose $M$ to be one of the following matrices:

$$
M_{1}=\left[\begin{array}{cc}
1 & -1  \tag{53}\\
1 & 1
\end{array}\right], M_{2}=\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] .
$$

In this paper, we choose $M=M_{1}$. Since the filter banks we consider in this paper have 4-fold symmetry, from Theorem 2.3 in [10], we know the scaling functions with dilation matrices $M_{1}$ and $M_{2}$ are actually the same. The reader refers to [10] for the details.

For a pair of $\sqrt{2}$-refinement filter banks $\{p(\boldsymbol{\omega}), q(\boldsymbol{\omega})\}$ and $\{\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}(\boldsymbol{\omega})\}$, the multiresolution decomposition algorithm with a dilation matrix $M$ for an input data or regular quad mesh $\mathcal{C}=$ $\left\{c_{\mathbf{k}}^{0}\right\}$ is

$$
\begin{equation*}
c_{\mathbf{n}}^{j+1}=\frac{1}{2} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}-M \mathbf{n}} c_{\mathbf{k}}^{j}, d_{\mathbf{n}}^{j+1}=\frac{1}{2} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} q_{\mathbf{k}-M \mathbf{n}} c_{\mathbf{k}}^{j}, \mathbf{n} \in \mathbf{Z}^{2}, \tag{54}
\end{equation*}
$$

for $j=0,1, \cdots, J-1$, and the multiresolution reconstruction algorithm is given by

$$
\begin{equation*}
\widetilde{c}_{\mathbf{k}}^{j}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{\mathbf{k}-M \mathbf{n}} \widetilde{C}_{\mathbf{n}}^{j+1}+\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{q}_{\mathbf{k}-M \mathbf{n}} d_{\mathbf{n}}^{j+1} \tag{55}
\end{equation*}
$$

with $\mathbf{k} \in \mathbf{Z}^{2}$ for $j=J-1, J-2, \cdots, 0$, where $\widetilde{c}_{\mathbf{n}, J}=c_{\mathbf{n}, J}$. Analogously we say a pair of filter banks $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ to be the perfect reconstruction filter banks or to be biorthogonal if $\widetilde{c}_{\mathbf{k}}^{j}=c_{\mathbf{k}}^{j}$, $0 \leq j \leq J-1$ for any input $\mathcal{C}=\left\{c_{\mathbf{k}}^{0}\right\}$. Again, $\{p, q\}$ is called the analysis filter bank, $\{\widetilde{p}, \widetilde{q}\}$ the synthesis filter bank, and $\left\{c_{\mathbf{k}}^{j}\right\},\left\{d_{\mathbf{k}}^{j}\right\}$ are called the "smooth part" (or "approximation") and the "detail" of $\mathcal{C}$. When $d_{\mathbf{k}}^{j}=0$, (55) is reduced to $\widetilde{c}_{\mathbf{k}}^{j}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}} \widetilde{p}_{\mathbf{k}-M_{\mathbf{n}}} \widetilde{c}_{\mathbf{n}}^{j+1}, j=J-1, J-2, \cdots$. This is the $\sqrt{2}$ subdivision algorithm with subdivision mask $\left\{\widetilde{p}_{\mathbf{k}}\right\}_{\mathbf{k}}$.

For $\sqrt{2}$-refinement, the biorthogonal conditions with $M=M_{1}$ or $M=M_{2}$ are

$$
\begin{equation*}
p(\boldsymbol{\omega}) \overline{\tilde{p}(\boldsymbol{\omega})}+p(\boldsymbol{\omega}+(\pi, \pi)) \overline{\tilde{p}(\boldsymbol{\omega}+(\pi, \pi))}=1, \tag{56}
\end{equation*}
$$

$$
\begin{align*}
& p(\boldsymbol{\omega}) \overline{\tilde{q}(\boldsymbol{\omega})}+p(\boldsymbol{\omega}+(\pi, \pi)) \overline{\tilde{q}(\boldsymbol{\omega}+(\pi, \pi))}=0,  \tag{57}\\
& q(\boldsymbol{\omega}) \overline{\tilde{q}(\boldsymbol{\omega})}+q(\boldsymbol{\omega}+(\pi, \pi)) \overline{\tilde{q}(\boldsymbol{\omega}+(\pi, \pi))}=1, \tag{58}
\end{align*}
$$

where for a $\sqrt{2}$-refinement filter (mask) $\left\{p_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$, the corresponding finite impulse response filter $p(\boldsymbol{\omega})$ is defined by

$$
p(\boldsymbol{\omega})=\frac{1}{2} \sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}} e^{-i \mathbf{k} \cdot \boldsymbol{\omega}} .
$$

We say $p(\boldsymbol{\omega})$ has sum rule order $K$ (with $M=M_{1}$ ) if it satisfies that $p(0,0)=1$ and

$$
\begin{equation*}
\frac{\partial^{\alpha_{1}+\alpha_{2}}}{\partial \omega_{1}^{\alpha_{1}} \partial \omega_{2}^{\alpha_{2}}} p(\pi, \pi)=0, \forall\left(\alpha_{1}, \alpha_{2}\right) \in \mathbf{Z}_{+}^{2}, \alpha_{1}+\alpha_{2}<K . \tag{59}
\end{equation*}
$$

The sum rule order $K$ of $p$ implies that $\phi$ has the approximation order $K$, where $\phi$ is the scaling function satisfying the refinement equation

$$
\begin{equation*}
\phi(\mathbf{x})=\sum_{\mathbf{k} \in \mathbf{Z}^{2}} p_{\mathbf{k}} \phi(M \mathbf{x}-\mathbf{k}) . \tag{60}
\end{equation*}
$$

In this case, the corresponding wavelet $\psi$ is defined by

$$
\begin{equation*}
\psi(\mathbf{x})=\sum_{\mathbf{k} \in \mathbf{Z}^{2}} q_{\mathbf{k}} \phi(M \mathbf{x}-\mathbf{k}) . \tag{61}
\end{equation*}
$$

For a pair of biorthogonal FIR filter banks $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$, under mild conditions, $\phi$ and $\widetilde{\phi}$ are biorthogonal duals, and $\psi$ and $\widetilde{\psi}$ defined by (61) with $q(\boldsymbol{\omega})$ and $\widetilde{q}(\boldsymbol{\omega})$ respectively generate biorthogonal bases for $L^{2}\left(\mathbb{R}^{2}\right)$.

Observe that if two $\sqrt{2}$-refinement lowpass FIR filters $p$ and $\widetilde{p}$ are biorthogonal, that is, they satisfy (56), then with $q$ and $\widetilde{q}$ defined by

$$
q(\boldsymbol{\omega})=s e^{-i \omega_{1}} \overline{\widetilde{p}(\boldsymbol{\omega}+(\pi, \pi))}, \widetilde{q}(\boldsymbol{\omega})=\frac{1}{s} e^{-i \omega_{1}} \overline{p(\boldsymbol{\omega}+(\pi, \pi))},
$$

where $s$ is a nonzero constant, $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ form a pair of $\sqrt{2}$-refinement biorthogonal filter banks. Furthermore, if $p$ and $\widetilde{p}$ are 4 -fold symmetric, namely, they are invariant under $T_{k}, 0 \leq$ $k \leq 3$, then both $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ have 4 -fold symmetry. Thus to construct 4 -fold symmetric $\sqrt{2}$ refinement biorthogonal filter banks, one needs to consider the lowpass filters only. However, here we still use the approach in the above section with both lowpass and highpass filters considered together. Next, we give the characterizations of the 4 -fold symmetry of $\sqrt{2}$-refinement filter banks.

Proposition 4. A ( $\sqrt{2}$-refinement) filter bank $\{p, q\}$ has 4 -fold axial symmetry if and only if it satisfies

$$
\begin{align*}
p\left(O_{1} \boldsymbol{\omega}\right) & =p\left(J_{0} \boldsymbol{\omega}\right)=p(\boldsymbol{\omega}),  \tag{62}\\
q\left(O_{1} \boldsymbol{\omega}\right) & =q\left(J_{0} \boldsymbol{\omega}\right)=e^{i\left(\omega_{1}-\omega_{2}\right)} q(\boldsymbol{\omega}) . \tag{63}
\end{align*}
$$

Proof. Let $h(\boldsymbol{\omega})=e^{i \omega_{1}}(\boldsymbol{\omega})$. Then the definition of 4 -fold symmetry and the fact $O_{1}, J_{0}$ generate $J_{k}, 0 \leq k \leq 3$ imply that $\{p, q\}$ has 4 -fold axial symmetry if and only if

$$
p\left(O_{1} \boldsymbol{\omega}\right)=p\left(J_{0} \boldsymbol{\omega}\right)=p(\boldsymbol{\omega}), h\left(O_{1} \boldsymbol{\omega}\right)=h\left(J_{0} \boldsymbol{\omega}\right)=h(\boldsymbol{\omega}) .
$$

With $O_{1} \boldsymbol{\omega}=\left(\omega_{2},-\omega_{1}\right)$ and $J_{0} \boldsymbol{\omega}=\left(\omega_{2}, \omega_{1}\right)$, one easily get $h\left(O_{1} \boldsymbol{\omega}\right)=h\left(J_{0} \boldsymbol{\omega}\right)=h(\boldsymbol{\omega})$ is equivalent to (63).

For a $\sqrt{2}$-refinement FIR filter bank $\{p, q\}$, let $V(\omega)$ be its polyphase matrix, a $2 \times 2$ trigonometric polynomial matrix, defined by

$$
[p(\boldsymbol{\omega}), q(\boldsymbol{\omega})]^{T}=\frac{1}{\sqrt{2}} V\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})
$$

where $I_{0}(\omega)$ defined by

$$
\begin{equation*}
I_{0}(\boldsymbol{\omega})=\left[1, e^{-i \omega_{1}}\right]^{T}, \boldsymbol{\omega}=\left(\omega_{1}, \omega_{2}\right) \in \mathbb{R}^{2} . \tag{64}
\end{equation*}
$$

Proposition 5. An FIR filter bank $\{p, q\}$ has 4 -fold axial symmetry if and only if its polyphase matrix $V(\boldsymbol{\omega})$ (with dilation matrix $M=M_{1}$ ) satisfies

$$
\begin{align*}
V\left(O_{1} \boldsymbol{\omega}\right) & =\operatorname{diag}\left(1, e^{-i \omega_{2}}\right) V(\boldsymbol{\omega}) \operatorname{diag}\left(1, e^{i \omega_{2}}\right)  \tag{65}\\
V\left(L_{0} \boldsymbol{\omega}\right) & =\operatorname{diag}\left(1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right) V(\boldsymbol{\omega}) \operatorname{diag}\left(1, e^{i\left(\omega_{2}-\omega_{1}\right)}\right) \tag{66}
\end{align*}
$$

Proof. Suppose $\{p, q\}$ has 4-fold axial symmetry. From $M^{T} O_{1}=O_{1} M^{T}$, we have

$$
\begin{aligned}
{[p, q]^{T}\left(O_{1} \boldsymbol{\omega}\right) } & =\frac{1}{\sqrt{2}} V\left(M^{T} O_{1} \boldsymbol{\omega}\right) I_{0}\left(O_{1} \boldsymbol{\omega}\right)=\frac{1}{\sqrt{2}} V\left(O_{1} M^{T} \boldsymbol{\omega}\right)\left[1, e^{-i \omega_{2}}\right]^{T} \\
& =\frac{1}{\sqrt{2}} V\left(O_{1} M^{T} \boldsymbol{\omega}\right) \operatorname{diag}\left(1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right) I_{0}(\boldsymbol{\omega}) .
\end{aligned}
$$

From (62) and (63), we have

$$
[p, q]^{T}\left(O_{1} \boldsymbol{\omega}\right)=\operatorname{diag}\left(1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right)[p, q]^{T}(\boldsymbol{\omega})=\frac{1}{\sqrt{2}} \operatorname{diag}\left(1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right) V\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega}) .
$$

Thus, the 4 -fold symmetry of $\{p, q\}$ implies

$$
\frac{1}{\sqrt{2}} V\left(O_{1} M^{T} \boldsymbol{\omega}\right) \operatorname{diag}\left(1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right) I_{0}(\boldsymbol{\omega})=\frac{1}{\sqrt{2}} \operatorname{diag}\left(1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right) V\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})
$$

or equivalently,

$$
V\left(O_{1} M^{T} \boldsymbol{\omega}\right)=\operatorname{diag}\left(1, e^{i\left(\omega_{1}-\omega_{2}\right)}\right) V\left(M^{T} \boldsymbol{\omega}\right) \operatorname{diag}\left(1, e^{-i\left(\omega_{1}-\omega_{2}\right)}\right)
$$

With $M^{T} \boldsymbol{\omega}$ replaced by $\boldsymbol{\omega}$, the above equation is

$$
V\left(O_{1} \boldsymbol{\omega}\right)=\operatorname{diag}\left(1, e^{-i \omega_{2}}\right) V(\boldsymbol{\omega}) \operatorname{diag}\left(1, e^{i \omega_{2}}\right)
$$

One can show (66) similarly. Indeed, from $J_{0} M^{T}=M^{T} J_{3}$, we have

$$
\begin{aligned}
{[p, q]^{T}\left(J_{3} \boldsymbol{\omega}\right) } & =\frac{1}{\sqrt{2}} V\left(M^{T} J_{3} \boldsymbol{\omega}\right) I_{0}\left(O_{1} \boldsymbol{\omega}\right)=\frac{1}{\sqrt{2}} V\left(J_{0} M^{T} \boldsymbol{\omega}\right)\left[1, e^{i \omega_{1}}\right]^{T} \\
& =\frac{1}{\sqrt{2}} V\left(J_{0} M^{T} \boldsymbol{\omega}\right) \operatorname{diag}\left(1, e^{2 i \omega_{1}}\right) I_{0}(\boldsymbol{\omega}) .
\end{aligned}
$$

From the definition of 4-fold symmetry, one can get $p\left(J_{3} \boldsymbol{\omega}\right)=p(\boldsymbol{\omega}), q\left(J_{3} \boldsymbol{\omega}\right)=e^{2 i \omega_{1}} q(\boldsymbol{\omega})$. Thus

$$
[p, q]^{T}\left(J_{3} \boldsymbol{\omega}\right)=\operatorname{diag}\left(1, e^{2 i \omega_{1}}\right)[p, q]^{T}(\boldsymbol{\omega})=\frac{1}{\sqrt{2}} \operatorname{diag}\left(1, e^{2 i \omega_{1}}\right) V\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega}) .
$$

Therefore, the 4 -fold symmetry of $\{p, q\}$ also implies

$$
\frac{1}{\sqrt{2}} \operatorname{diag}\left(1, e^{2 i \omega_{1}}\right) V\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})=\frac{1}{\sqrt{2}} V\left(J_{0} M^{T} \boldsymbol{\omega}\right) \operatorname{diag}\left(1, e^{2 i \omega_{1}}\right) I_{0}(\boldsymbol{\omega})
$$

or

$$
V\left(J_{0} M^{T} \boldsymbol{\omega}\right)=\operatorname{diag}\left(1, e^{2 i \omega_{1}}\right) V\left(J_{0} M^{T} \boldsymbol{\omega}\right) \operatorname{diag}\left(1, e^{-2 i \omega_{1}}\right)
$$

which is (66) when $M^{T} \boldsymbol{\omega}$ is replaced by $\boldsymbol{\omega}$.
Clearly, the above procedures are reversible. Therefore, the 4 -fold symmetry can be characterized by (65) and (66).

The next proposition provides the symmetry of the scaling function and wavelets associated with a symmetric filter bank.

Proposition 6. Suppose an FIR filter bank $\{p, q\}$ has 4-fold axial symmetry. Let $\phi$ be the associated scaling function with dilation matrix $M=M_{1}$ and $\psi$ be the functions define by (61) with $q$. Then

$$
\begin{equation*}
\phi\left(J_{k} \mathbf{x}\right)=\phi(\mathbf{x}), 0 \leq k \leq 3 \tag{67}
\end{equation*}
$$

and

$$
\begin{array}{ll}
\psi^{(1)}\left(J_{0} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}+(1,-1)), & \psi^{(1)}\left(J_{1} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}-(0,1)), \\
\psi^{(1)}\left(J_{2} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}), & \psi^{(1)}\left(J_{3} \mathbf{x}\right)=\psi^{(1)}(\mathbf{x}+(1,0)) . \tag{68}
\end{array}
$$

Let

$$
\begin{equation*}
\psi_{0}(\boldsymbol{\omega})=\psi\left(\mathbf{x}+\left(\frac{1}{2},-\frac{1}{2}\right)\right) \tag{69}
\end{equation*}
$$

Then (68) is equivalent to $\psi_{0}\left(J_{k} \mathbf{x}\right)=\psi_{0}(\mathbf{x}), 0 \leq k \leq 3$.
Proof. For (67), we need only to prove $\phi\left(O_{1} \mathbf{x}\right)=\phi(\mathbf{x})$ and $\phi\left(J_{0} \mathbf{x}\right)=\phi(\mathbf{x})$. From (60), we have $\widehat{\phi}(\boldsymbol{\omega})=p\left(M^{-T} \boldsymbol{\omega}\right) \widehat{\phi}\left(M^{-T} \boldsymbol{\omega}\right)$. Thus $\widehat{\phi}(\boldsymbol{\omega})=\prod_{k=1}^{\infty} p\left(\left(M^{-T}\right)^{k} \boldsymbol{\omega}\right) \widehat{\phi}(0)$. When $M=M_{1}$ given in (53), $M O_{1}=O_{1} M, M J_{0}=J_{0} O_{1} M$. Thus

$$
M^{k} O_{1}=O_{1} M^{k}, M^{k} J_{0}=J_{0} O_{1}^{k} M^{k}
$$

which implies $\left(M^{-T}\right)^{k} O_{1}=O_{1}\left(M^{-T}\right)^{k},\left(M^{-T}\right)^{k} J_{0}=J_{0} O_{1}^{k}\left(M^{-T}\right)^{k}$. Therefore,

$$
\begin{aligned}
& \widehat{\phi}\left(O_{1} \boldsymbol{\omega}\right)=\Pi_{k=1}^{\infty} p\left(\left(M^{-T}\right)^{k} O_{1} \boldsymbol{\omega}\right) \widehat{\phi}(0) \\
& =\Pi_{k=1}^{\infty} p\left(O_{1}\left(M^{-T}\right)^{k} \boldsymbol{\omega}\right) \widehat{\phi}(0) \\
& =\Pi_{k=1}^{\infty} p\left(\left(M^{-T}\right)^{k} \boldsymbol{\omega}\right) \widehat{\phi}(0)=\widehat{\phi}(\boldsymbol{\omega})
\end{aligned}
$$

and

$$
\begin{aligned}
& \widehat{\phi}\left(J_{0} \boldsymbol{\omega}\right)=\Pi_{k=1}^{\infty} p\left(\left(M^{-T}\right)^{k} J_{0} \boldsymbol{\omega}\right) \widehat{\phi}(0) \\
& =\Pi_{k=1}^{\infty} p\left(J_{0} O_{1}^{k}\left(M^{-T}\right)^{k} \boldsymbol{\omega}\right) \widehat{\phi}(0) \\
& =\Pi_{k=1}^{\infty} p\left(\left(M^{-T}\right)^{k} \boldsymbol{\omega}\right) \widehat{\phi}(0)=\widehat{\phi}(\boldsymbol{\omega})
\end{aligned}
$$

Hence, $\phi\left(O_{1} \mathbf{x}\right)=\phi(\mathbf{x})$ and $\phi\left(J_{0} \mathbf{x}\right)=\phi(\mathbf{x})$. Thus, (67) holds.
Next, we prove (68). It is enough to show $\psi_{0}\left(J_{k} \mathbf{x}\right)=\psi_{0}(\mathbf{x})$, where $\psi_{0}$ is the function defined by (69). Let $h(\boldsymbol{\omega})=e^{i \omega_{1}} q(\boldsymbol{\omega})$. Then

$$
\widehat{\psi}_{0}(\boldsymbol{\omega})=e^{\frac{i}{2}\left(\omega_{1}-\omega_{2}\right)} \widehat{\psi}(\boldsymbol{\omega})=e^{\frac{i}{2}\left(\omega_{1}-\omega_{2}\right)} q\left(M^{-T} \boldsymbol{\omega}\right) \widehat{\phi}\left(M^{-T} \boldsymbol{\omega}\right)=h\left(M^{-T} \boldsymbol{\omega}\right) \widehat{\phi}\left(M^{-T} \boldsymbol{\omega}\right) .
$$

Observe that

$$
M^{-T} J_{0}=J_{1} M^{-T}, M^{-T} J_{1}=J_{2} M^{-T}, M^{-T} J_{2}=J_{3} M^{-T}, M^{-T} J_{3}=J_{0} M^{-T}
$$

Thus for each $k, 0 \leq k \leq 3, M^{-T} J_{k}=J_{k^{\prime}} M^{-T}$ for some $k^{\prime}, 0 \leq k^{\prime} \leq 3$. This and $h\left(J_{k^{\prime}} \boldsymbol{\omega}\right)=h(\boldsymbol{\omega})$ (from the definition of 4 -fold symmetry) imply
$\widehat{\psi}_{0}\left(J_{k} \boldsymbol{\omega}\right)=h\left(M^{-T} J_{k} \boldsymbol{\omega}\right) \widehat{\phi}\left(M^{-T} J_{k} \boldsymbol{\omega}\right)=h\left(J_{k^{\prime}} M^{-T} \boldsymbol{\omega}\right) \widehat{\phi}\left(J_{k^{\prime}} M^{-T} \boldsymbol{\omega}\right)=h\left(M^{-T} \boldsymbol{\omega}\right) \widehat{\phi}\left(M^{-T} \boldsymbol{\omega}\right)=\widehat{\psi}_{0}(\boldsymbol{\omega})$.
Hence $\psi_{0}\left(J_{k} \mathbf{x}\right)=\psi_{0}(\mathbf{x})$, as desired.
Based on Proposition 5, one can easily construct blocks to build symmetric filter banks. For example, one may use

$$
X(\boldsymbol{\omega})=\left[\begin{array}{ll}
j+4 n \lambda+2 n \lambda\left(x+y+\frac{1}{x}+\frac{1}{y}\right)+n \lambda\left(x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right) & \lambda\left(1+\frac{1}{x}+y+\frac{y}{x}\right)  \tag{70}\\
n\left(1+x+\frac{1}{y}+\frac{x}{y}\right) & 1
\end{array}\right]
$$

where $j, n, \lambda$ are constants with $j \neq 0 . X(\boldsymbol{\omega})$ satisfies (65) and (66). Furthermore, $\operatorname{det}(X(\boldsymbol{\omega}))=j$, a nonzero constant. Thus, the inverses of $X(\boldsymbol{\omega})$ is a matrix whose entries are also polynomials of $x, y$. One can easily get that $\widetilde{X}(\boldsymbol{\omega})=\left(X(\boldsymbol{\omega})^{-1}\right)^{*}$ is

$$
\tilde{X}(\boldsymbol{\omega})=\frac{1}{j}\left[\begin{array}{ll}
1 & -n\left(1+\frac{1}{x}+y+\frac{y}{x}\right)  \tag{71}\\
-\lambda\left(1+x+\frac{1}{y}+\frac{x}{y}\right) & j+4 n \lambda+2 n \lambda\left(x+y+\frac{1}{x}+\frac{1}{y}\right)+n \lambda\left(x y+\frac{1}{x y}+\frac{x}{y}+\frac{y}{x}\right)
\end{array}\right]
$$

$\widetilde{X}(\boldsymbol{\omega})$ also satisfies $(65)$ and (66). One may use other blocks. For example, $Y(\boldsymbol{\omega})$ defined by
$Y(\boldsymbol{\omega})=$
$\left[\begin{array}{ll}j+4 n \lambda+2 \lambda(n+m)(x+y)\left(1+\frac{1}{x y}\right)+\lambda(n+2 m)\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\lambda m\left(x+2+\frac{1}{x}\right)\left(y^{2}+\frac{1}{y^{2}}\right)+\lambda m\left(y+2+\frac{1}{y}\right)\left(x^{2}+\frac{1}{x^{2}}\right) & \lambda\left(1+\frac{1}{x}+y+\frac{y}{x}\right) \\ n(1+x)\left(1+\frac{1}{y}\right)+m(1+x)\left(y+\frac{y}{y^{2}}\right)+m\left(1+\frac{1}{y}\right)\left(x^{2}+\frac{1}{x}\right) & 1\end{array}\right]$.
where $j, n, \lambda$ are constants with $j \neq 0$, also satisfies $(65)$ and (66), and the inverses of $Y(\boldsymbol{\omega})$ is a matrix whose entries are also polynomials of $x, y$ with $\tilde{Y}(\boldsymbol{\omega})=\left(Y(\boldsymbol{\omega})^{-1}\right)^{*}$ given by

$$
\begin{aligned}
& \widetilde{Y}(\boldsymbol{\omega})= \\
& {\left[\begin{array}{ll}
\frac{1}{j} & \left.\begin{array}{ll}
1 & -n\left(1+\frac{1}{x}\right)(1+y)-m\left(1+\frac{1}{x}\right)\left(y^{2}+\frac{1}{y}\right)-m(1+y)\left(x+\frac{1}{x^{2}}\right) \\
-\lambda(1+x)\left(1+\frac{1}{y}\right) & j+4 n \lambda+2 \lambda(n+m)\left(\frac{1}{x}+\frac{1}{y}\right)(1+x y)+\lambda(n+2 m)\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\lambda m\left(x+2+\frac{1}{x}\right)\left(y^{2}+\frac{1}{y^{2}}\right)+\lambda m\left(y+2+\frac{1}{y}\right)\left(x^{2}+\frac{1}{x^{2}}\right)
\end{array}\right] .
\end{array} . \quad \begin{array}{ll}
(73)
\end{array}\right.}
\end{aligned}
$$

$\tilde{Y}(\boldsymbol{\omega})$ also satisfies (65) and (66). Therefore, we have the following family of 4 -fold symmetric biorthogonal filter banks.

Theorem 2. Suppose FIR filter banks $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ are given by

$$
\begin{align*}
{[p(\boldsymbol{\omega}), q(\boldsymbol{\omega})]^{T} } & =U_{K}\left(M^{T} \boldsymbol{\omega}\right) U_{K-1}\left(M^{T} \boldsymbol{\omega}\right) \cdots U_{0}\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})  \tag{74}\\
{[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}(\boldsymbol{\omega})]^{T} } & =\frac{1}{2} \widetilde{U}_{K}\left(M^{T} \boldsymbol{\omega}\right) \widetilde{U}_{K-1}\left(M^{T} \boldsymbol{\omega}\right) \cdots \widetilde{U}_{0}\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})
\end{align*}
$$

for some $K \in \mathbf{Z}_{+}$, where $I_{0}(\boldsymbol{\omega})$ is defined by (64), each $U_{k}(\boldsymbol{\omega})$ is an $X(\boldsymbol{\omega})$ in (70) or a $\widetilde{X}(\boldsymbol{\omega})$ in (71) for some parameters $j_{k}, n_{k}, \lambda_{k}$, or a $Y(\boldsymbol{\omega})$ in (72) or a $\widetilde{Y}(\boldsymbol{\omega})$ in (73) for some parameters $j_{k}, n_{k}, m_{k}, \lambda_{k}$, and $\widetilde{U}_{k}(\boldsymbol{\omega})=\left(U_{k}(\boldsymbol{\omega})^{-1}\right)^{*}$ is the corresponding $\widetilde{X}(\boldsymbol{\omega})$ in (71) (or $X(\boldsymbol{\omega})$ in (70), $\widetilde{Y}(\boldsymbol{\omega})$ in (73), $Y(\boldsymbol{\omega})$ in (72) accordingly). Then $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ are biorthogonal FIR filter banks and both have 4-fold axial symmetry.

Example 4. Let $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ be the biorthogonal filter banks given by (74) for $K=1$ with

$$
\begin{align*}
{[p(\boldsymbol{\omega}), q(\boldsymbol{\omega})]^{T} } & =\widetilde{X}_{1}\left(M^{T} \boldsymbol{\omega}\right) \widetilde{X}_{0}\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})  \tag{75}\\
{[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}(\boldsymbol{\omega})]^{T} } & =\frac{1}{2} X_{1}\left(M^{T} \boldsymbol{\omega}\right) X_{0}\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega})
\end{align*}
$$

where $X_{0}(\boldsymbol{\omega})$ and $X_{1}(\boldsymbol{\omega})$ are given by (70) with parameters $j, n, \lambda$ and $n_{1}, \lambda_{1}, j_{1}=1$ respectively, and $\widetilde{X}_{0}(\boldsymbol{\omega}), \widetilde{X}_{1}(\boldsymbol{\omega})$ are given by (71). Solving the system of the equations for sum rule order 1 of both $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$, we have

$$
n=-\frac{1}{4}-2 n_{1}, j=\frac{2\left(1+4 n_{1}\right)}{1+16 n_{1} \lambda_{1}}, \lambda=\frac{1-4 \lambda_{1}}{4\left(1+16 n_{1} \lambda_{1}\right)} .
$$

The resulting $p$ and $\widetilde{p}$ actually have sum rule order 2 because of their symmetry. For this pair of biorthogonal filter banks, $p$ yields smoother scaling functions than $\widetilde{p}$. It is possible to construct $\phi$ associated with $p$ in $W^{2}$ with $\widetilde{\phi}$ associated with $\widetilde{p}$ in $L^{2}\left(\mathbb{R}^{2}\right)$, while it is impossible to construct $\tilde{\phi}$ with such a smoothness order. When $n_{1}=0.14122198675290, \lambda_{1}=0.49080484105305$, the resulting $\phi$ and $\tilde{\phi}$ are in $W^{1.33388}$ and $W^{1.21860}$ respectively, and if $n_{1}=-0.13593823458298, \lambda_{1}=$ 0.09071891834880 , then $\phi \in W^{0.01449}$ and $\widetilde{\phi} \in W^{1.34094}$. It seems that in the latter case $\widetilde{\phi}$ gains the best smoothness with $\phi \in L^{2}\left(\mathbb{R}^{2}\right)$.

In order that $\phi \in W^{2}, p$ must has sum rule order of at least 3. If

$$
\lambda_{1}=\frac{15}{64}+\left(\frac{1}{8}+\frac{1}{16 n_{1}}\right)^{2},
$$

then $p$ has sum rule order 4. If we choose $n_{1}=1$, then the corresponding $\phi$ is in $W^{2.18087}$ and $\widetilde{\phi}$ in $W^{0.03026}$; and if we select $n_{1}=\frac{29}{32}$, then the resulting $\phi$ and $\widetilde{\phi}$ are in $W^{2.16374}$ and $W^{0.05496}$ respectively. For either choice of these $n_{1}$, we should use $\{p, q\}$ as the synthesis filter bank and $\{\widetilde{p}, \widetilde{q}\}$ the analysis filter bank. The resulting $p$ and $\widetilde{p}$ are supported on $[-3,3]^{2}$ and $[-4,4]^{2}$. $\diamond$
Example 5. Let $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ be the biorthogonal filter banks given by (74) for $K=1$ with

$$
\begin{equation*}
[p(\boldsymbol{\omega}), q(\boldsymbol{\omega})]^{T}=\widetilde{Y}_{1}\left(M^{T} \boldsymbol{\omega}\right) \widetilde{X}_{0}\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega}),[\widetilde{p}(\boldsymbol{\omega}), \widetilde{q}(\boldsymbol{\omega})]^{T}=\frac{1}{2} Y_{1}\left(M^{T} \boldsymbol{\omega}\right) X_{0}\left(M^{T} \boldsymbol{\omega}\right) I_{0}(\boldsymbol{\omega}) . \tag{76}
\end{equation*}
$$

where $X_{0}(\boldsymbol{\omega})$ and $Y_{1}(\boldsymbol{\omega})$ are given by (70) and (72) for some parameters $j, n, \lambda$ and $n_{1}, \lambda_{1}, m_{1}$, $j_{1}=1$ respectively, and $\widetilde{X}_{0}(\boldsymbol{\omega}), \widetilde{Y}_{1}(\boldsymbol{\omega})$ are given by (71) and (73).

First we consider the case $\lambda_{1}=0$. Solving the system of the equations for sum rule order 1 for both $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$, we have that if

$$
\begin{equation*}
\lambda=\frac{1}{4}, j=1-4 n, n_{1}=-\frac{1}{8}-\frac{1}{2} n-2 m_{1}, \tag{77}
\end{equation*}
$$

then the resulting $p, \widetilde{p}$ have sum rule order 2. Furthermore, if $n=\frac{1}{8}$, then $\widetilde{p}$ has sum rule order 4 . However, in this case, we cannot choose the remaining parameter $m_{1}$ such that $\phi$ is in $L^{2}\left(\mathbb{R}^{2}\right)$. Thus we consider the $p$ and $\widetilde{p}$ with sum rule order 2 . If $n=\frac{3}{64}, m_{1}=\frac{11}{256}$, then the resulting $\phi$ and $\widetilde{\phi}$ are in $W^{0.00589}$ and $W^{1.91822}$ respectively, with the resulting $p$ and $\widetilde{p}$ supported on $[-4,4]^{2}$ and $[-2,2]^{2}$.

Next let us consider the case $\lambda_{1} \neq 0$. Solving the system of the equations for sum rule orders 1 and 3 for both $p(\boldsymbol{\omega})$ and $\widetilde{p}(\boldsymbol{\omega})$ respectively, we have that if

$$
\begin{align*}
& j=\frac{1-4 n}{1-2 \lambda_{1}-8 n \lambda_{1}}, \lambda=\frac{1-4 \lambda_{1}}{4\left(1-2 \lambda_{1}-8 n \lambda_{1}\right)}, \\
& m_{1}=\frac{1}{32}+\frac{n}{8}+\frac{(8 n-1)\left(2 \lambda 1+8 n \lambda_{1}-1\right)}{512 \lambda_{1}^{2}(1-4 n)}, n_{1}=-\frac{1}{8}-2 m_{1}-\frac{n}{2}, \tag{78}
\end{align*}
$$

then resulting $p$ and $\widetilde{p}$ have sum rule orders 2 and 4 respectively. There are two parameters $n, \lambda_{1}$ left. We can choose different values for $n, \lambda_{1}$ such that $\widetilde{\phi}$ associated with the resulting $\widetilde{p}$ is in $W^{2}$. For example, if $n=0.12577172209778, \lambda_{1}=0.13650403059655$, then the corresponding $\phi \in W^{0.01449}, \widetilde{\phi} \in W^{2.91445}$; and if $n=\frac{1}{8}, \lambda_{1}=\frac{35}{256}$, then the resulting $\phi$ and $\widetilde{\phi}$ are in $W^{0.04400}$ and $W^{2.89734}$ respectively. In either case, the resulting $p$ and $\widetilde{p}$ are supported on $[-4,4]^{2}$ and $[-5,5]^{2}$. $\diamond$

## $3.2 \sqrt{2}$-refinement multiresolution algorithms for quad surface processing



Figure 10: From let to right: Square mesh with nodes $\mathbf{Z}^{2}$ (1st picture), vertex nodes and face nodes (2nd picture), original data/vertices associated with vertex nodes and face nodes (3rd picture), "smooth part" and "detail" associated with vertex nodes and face nodes (4th picture)

As dyadic filter banks, $\sqrt{2}$ multiresolution algorithms (for regular vertices) with the 4 -fold symmetric biorthogonal filter banks given in (74) can be expressed by symmetric templates which are orientation and reflection invariant. Again, the key is to associate appropriately the detail $d_{\mathbf{k}}^{1}$ after decomposition algorithm with the nodes of $\mathbf{Z}^{2}$. To this regard, we separate nodes of $\mathbf{Z}^{2}$ into two groups.

For the square lattice $\mathbf{Z}^{2}$ with which a regular quad mesh is represented, $M \mathbf{Z}^{2}=\{M \mathbf{k}=$ $\left.\left(k_{1}-k_{2}, k_{1}+k_{2}\right),\left(k_{1}, k_{2}\right) \in \mathbf{Z}^{2}\right\}$ is the set of the labels for the vertices of the coarse mesh. For $\sqrt{2}$-refinement, the nodes with labels $M \mathrm{k}$ are called vertex nodes, and the other nodes face nodes. Observe that the set of the labels for face nodes are $M \mathbf{Z}^{2}+(1,0)$. See the second picture of Fig. 10 for these two groups of nodes. Thus, for an initial data/mesh $\mathcal{C}=\left\{c_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}},\left\{c_{M \mathbf{k}}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of data/vertices associated with vertex nodes, $\left\{c_{M \mathbf{k}+(1,0)}\right\}_{\mathbf{k} \in \mathbf{Z}^{2}}$ is the set of data/vertices associated with face nodes. Denote

$$
\begin{equation*}
v_{\mathbf{k}}=c_{M \mathbf{k}}, f_{\mathbf{k}}=c_{M \mathbf{k}+(1,0)}, \mathbf{k} \in \mathbf{Z}^{2} . \tag{79}
\end{equation*}
$$

See the third picture of Fig. 10 for these two groups of data/vertices and the labels for them.
The $\sqrt{2}$ multiresolution decomposition algorithm is to decompose the original data/mesh $\mathcal{C}=\left\{c_{\mathbf{k}}\right\}_{\mathbf{k}}$ with the analysis filter bank into the "smooth part" $\left\{c_{\mathbf{k}}^{1}\right\}_{\mathbf{k}}$ and "the detail" $\left\{d_{\mathbf{k}}^{1}\right\}_{\mathbf{k}}$, while the reconstruction algorithm is to recover $\mathcal{C}$ from $\left\{c_{\mathbf{k}}^{1}\right\}_{\mathbf{k}},\left\{d_{\mathbf{k}}^{1}\right\}_{\mathbf{k}}$ with the synthesis filter bank. Denote

$$
\widetilde{v}_{\mathbf{k}}=c_{\mathbf{k}}^{1}, \widetilde{f}_{\mathbf{k}}=d_{\mathbf{k}}^{1} .
$$

Then, the decomposition and reconstruction algorithms are respectively

$$
\begin{equation*}
\widetilde{v}_{\mathbf{k}}=\frac{1}{2} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} p_{\mathbf{k}^{\prime}-M \mathbf{k}} c_{\mathbf{k}^{\prime}}, \tilde{f}_{\mathbf{k}}=\frac{1}{2} \sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}} q_{\mathbf{k}^{\prime}-M \mathbf{k}^{\prime}} c_{\mathbf{k}^{\prime}}, \mathbf{k} \in \mathbf{Z}^{2}, \tag{80}
\end{equation*}
$$

and

$$
\begin{equation*}
c_{\mathbf{k}}=\sum_{\mathbf{k}^{\prime} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{\mathbf{k}-M \mathbf{k}^{\prime}} \widetilde{v}_{\mathbf{k}^{\prime}}+\widetilde{q}_{\mathbf{k}-M \mathbf{k}^{\prime}} \widetilde{f}_{\mathbf{k}^{\prime}}\right\}, \mathbf{k} \in \mathbf{Z}^{2} \tag{81}
\end{equation*}
$$

Considering $\mathbf{k}$ in (81) with $\mathbf{k}=M \mathbf{j}$ and $\mathbf{k}=M \mathbf{j}+(1,0)$, we can write the reconstruction algorithm (81) further to be

$$
\begin{align*}
& v_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{M \mathbf{n}} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{M \mathbf{n}} \widetilde{f}_{\mathbf{k}-\mathbf{n}}\right\}, \\
& f_{\mathbf{k}}=\sum_{\mathbf{n} \in \mathbf{Z}^{2}}\left\{\widetilde{p}_{M \mathbf{n}+(1,0)} \widetilde{v}_{\mathbf{k}-\mathbf{n}}+\widetilde{q}_{M \mathbf{n}+(1,0)} \tilde{f}_{\mathbf{k}-\mathbf{n}}\right\} . \tag{82}
\end{align*}
$$

Associate the "smooth part" $\widetilde{v}_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}^{2}$ with the vertex nodes $M \mathbf{k}$, and the "detail" $\widetilde{f}_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}^{2}$ with face nodes $M \mathbf{k}+(1,0)$, see the fourth picture of Fig. 10. Thus, the decomposed data $\left\{\widetilde{v}_{\mathbf{k}}\right\}_{\mathbf{k}}$ and $\left\{\widetilde{f}_{\mathbf{k}}\right\}_{\mathbf{k}}$ associate with the whole $\mathbf{Z}^{2}$ as the original data/mesh $\mathcal{C}=\left\{v_{\mathbf{k}}, f_{\mathbf{k}}\right\}_{\mathbf{k}}$. With such association, the $\sqrt{2}$ multiresolution decomposition and reconstruction algorithms can be described as templates. In addition, the 4 -fold symmetric $\sqrt{2}$ filter banks provided in $\S 3.1$ result in templates with desired symmetry for surface multiresolution. This makes it possible to design the corresponding multiresolution algorithms for extraordinary vertices. In the rest of this subsection, we show that some biorthogonal filter banks given in (74) lead to very simple multiresolution algorithms.


Figure 11: Top-left: Decomposition Alg. Step 1 with each $v$ associated with a vertex node replaced by $v^{\prime \prime}$ given in (83); Top-right: Decomposition Alg. Step 2 with each $f$ associated with a face node replaced by $f^{\prime \prime}$ given in (84); Bottom-left: Decomposition Alg. Step 3 with each $v^{\prime \prime}$ obtained in Step 1 replaced by $\widetilde{v}$ given in (85); Bottom-right: Decomposition Alg. Step 4 with each $f^{\prime \prime}$ obtained in Step 2 replaced by $\tilde{f}$ given in (86)

For given $\mathcal{C}$ (or equivalently, for given $\{v\}$ and $\{f\}$ ), the multiresolution decomposition algorithm is given by (83)-(86) and shown in Fig. 11, where $j, n, \lambda, n_{1}, \lambda_{1}$ are constants to be determined. More precisely, first we replace all $v$ associated with vertex nodes of $M \mathbf{Z}^{2}$ by $v^{\prime \prime}$ given by formula (83). Then, based on $v^{\prime \prime}$ obtained, we replace all $f$ associated with face nodes in $M \mathbf{Z}^{2}+(1,0)$ by $f^{\prime \prime}$ given in formula (84). After that, based on $f^{\prime \prime}$ obtained in Step 2, all $v^{\prime \prime}$ in Step 1 are updated by $\widetilde{v}$ given in formula (85). Finally, based on $\widetilde{v}$ obtained in Step 3, all $f^{\prime \prime}$ in Step 2 are updated by $\tilde{f}$ given in formula (86).

## $\sqrt{2}$-refinement Decomposition Algorithm:

Step 1. $v^{\prime \prime}=\frac{1}{j}\left\{v-n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)\right\}$;
Step 2. $f^{\prime \prime}=f-\lambda\left(v_{0}^{\prime \prime}+v_{1}^{\prime \prime}+v_{2}^{\prime \prime}+v_{3}^{\prime \prime}\right)$;

$$
\begin{align*}
& \text { Step 3. } \widetilde{v}=v^{\prime \prime}-n_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)  \tag{85}\\
& \text { Step 4. } \tilde{f}=f^{\prime \prime}-\lambda_{1}\left(\widetilde{v}_{6}+\widetilde{v}_{7}+\widetilde{v}_{8}+\widetilde{v}_{9}\right) \tag{86}
\end{align*}
$$



Figure 12: Top-left: Reconstruction Alg. Step 1 with each $\widetilde{e}$ associated with face node replaced by $f^{\prime \prime}$ given in (87); Top-right: Reconstruction Alg. Step 2 with each $\widetilde{v}$ associated with vertex node replaced by $v^{\prime \prime}$ given in (88); Bottom-left: Reconstruction Alg. Step 3 with each $f^{\prime \prime}$ obtained in Step 1 replaced by $f$ given in (89); Bottom-right: Reconstruction Alg. Step 4 with each $v^{\prime \prime}$ obtained in Step 2 replaced by v given in (90)

The multiresolution reconstruction algorithm to recover $\mathcal{C}$ associated with $\mathbf{Z}^{2}$ (or equivalently, $v$ and $f$ associated with $M \mathbf{Z}^{2}$ and $M \mathbf{Z}^{2}+(1,0)$ respectively) from given $\widetilde{v}$ associated with $M \mathbf{Z}^{2}$ and given $\widetilde{f}$ associated with $M \mathbf{Z}^{2}+(1,0)$. The algorithm is given by (87)-(90) and shown in Fig. 12 , where $j, n, \lambda, n_{1}, \lambda_{1}$ are the same constants in the decomposition algorithm. More precisely, first we update all $\widetilde{f}$ associated with face nodes of $M \mathbf{Z}^{2}+(1,0)$ with the resulting $f^{\prime \prime}$ given by formula (87). Then, we update all $\widetilde{v}$ associated with vertex nodes of $M \mathbf{Z}^{2}$ with the resulting $v^{\prime \prime}$ given by formula (88). After that, based on $v^{\prime \prime}$ obtained, we replace all $f^{\prime \prime}$ obtained in Step 1 by $f$ with formula in (89). Finally, based on $f$ obtained in Step 3, all $v^{\prime \prime}$ in Step 2 are updated with the resulting $v$ given by formula (90).

## $\sqrt{2}$-refinement Reconstruction Algorithm:

Step 1. $f^{\prime \prime}=\widetilde{f}+\lambda_{1}\left(\widetilde{v}_{0}+\widetilde{v}_{1}+\widetilde{v}_{2}+\widetilde{v}_{3}\right)$;
Step 2. $v^{\prime \prime}=\widetilde{v}+n_{1}\left(f_{0}^{\prime \prime}+f_{1}^{\prime \prime}+f_{2}^{\prime \prime}+f_{3}^{\prime \prime}\right)$;
Step 3. $f=f^{\prime \prime}+\lambda\left(v_{6}^{\prime \prime}+v_{7}^{\prime \prime}+v_{8}^{\prime \prime}+v_{9}^{\prime \prime}\right)$;
Step 4. $v=j v^{\prime \prime}+n\left(f_{0}+f_{1}+f_{2}+f_{3}\right)$.
Again, when the constants $j, n, \lambda, n_{1}, \lambda_{1}$ are appropriately chosen, the decomposed $\widetilde{v}$ is the "smooth part" of the initial data/mesh $\mathcal{C}$, and $\tilde{f}$ is the "detail" of $\mathcal{C}$. The decomposition algorithm
can be applied repeatedly to the smooth part to get further smooth part and details of the data, and reconstruction algorithm recovers the original data/mesh from the smooth part and details.

With the formulas in (80) and (82) and careful calculations, we find the filter banks $\{p, q\}$ and $\{\widetilde{p}, \widetilde{q}\}$ corresponding to the algorithms (83)-(90) to be those given by (75) in Example 4. By the results in Example 4, we know we cannot choose parameters such that $\bar{\phi}$ has a desirable smooth order. To obtain smoother $\widetilde{\phi}$, we may use algorithms with more iterative steps. The corresponding filter banks are given similarly to those in Examples 4 but with more blocks $X\left(M^{T} \boldsymbol{\omega}\right)$ and/or $\widetilde{X}\left(M^{T} \boldsymbol{\omega}\right)$. Then, we use the above method to choose the parameters.

$$
\begin{array}{ll}
\text { Decomposition Algorithm Step } 3^{\prime}: & \widetilde{v}=v^{\prime \prime}-n_{1} \sum_{k=0}^{3} f_{k}^{\prime \prime}-m_{1} \sum_{k=4}^{11} f_{k}^{\prime \prime} \\
\text { Reconstruction Algorithm Step } 2^{\prime}: & v^{\prime \prime}=\widetilde{v}+n_{1} \sum_{k=0}^{3} f_{k}^{\prime \prime}+m_{1} \sum_{k=4}^{11} f_{k}^{\prime \prime} \tag{92}
\end{array}
$$



Figure 13: Left: New decomposition Alg. Step $3^{\prime}$ with each $v^{\prime \prime}$ obtained in Step 1 replaced by $\tilde{v}$ given in (91); Right: New reconstruction Alg. Step $2^{\prime}$ with each $\widetilde{v}$ associated with vertex node replaced by $v^{\prime \prime}$ given in (92)

An alterative way to obtain smoother $\tilde{\phi}$ is to modify some algorithms in (83)-(90). For example, the decomposition algorithm Step 3 may be replaced by Step $3^{\prime}$ given in (91) and shown on the left of Fig. 13. In this case, the corresponding reconstruction algorithm Step 2 is replaced by reconstruction Step $2^{\prime}$ given in (92) and shown on the right of Fig. 13. For algorithms $(83)(84)(91)(86)$ and $(87)(92)(89)(90)$, one can obtain the corresponding filter banks to be those given by (76) in Example 5.

First we consider the case $\lambda_{1}=0$. When $\lambda_{1}=0$, the above algorithms are reduced to 3 -step algorithms $(83)(84)(91)$ and $(92)(89)(90)$ with $\tilde{f}=f^{\prime \prime}$. From Example 5, we may use $n=\frac{3}{64}, m_{1}=\frac{11}{256}$ and other values for $j, \lambda, n_{1}$ given by (77). With such choices of parameters, the corresponding $\phi$ and $\widetilde{\phi}$ are in $W^{0.00589}$ and $W^{1.91822}$ respectively with both $p$ and $\widetilde{p}$ having sum rule order 2. Here we provided all selected parameters:

$$
\left[j, n, \lambda, n_{1}, m_{1}\right]=\left[\frac{13}{16}, \frac{3}{64}, \frac{1}{4},-\frac{15}{64}, \frac{11}{256}\right]
$$

When $\lambda_{1} \neq 0$, from Example 5, we may use $n=\frac{1}{8}, \lambda_{1}=\frac{35}{256}$ and other values for $j, \lambda, n_{1}, m_{1}$ given by (78). In this case, as discussed in Example 5, the corresponding $\phi$ and $\widetilde{\phi}$ are in $W^{0.04400}$
and $W^{2.89734}$ with $p$ and $\widetilde{p}$ have sum rule order 2 and 4 respectively. Here we provided all selected parameters:

$$
\left[j, n, \lambda, n_{1}, m_{1}, \lambda_{1}\right]=\left[\frac{128}{151}, \frac{1}{8}, \frac{29}{151},-\frac{9}{32}, \frac{3}{64}, \frac{35}{256}\right] .
$$

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