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Admissible Wavelets on the Siegel Domain of Type One

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Qingtang Jiang and Lizhong Peng

Abstract. Let $Sp(n, R)$ be the symplectic group, and K_n^* its maximal compact subgroup. Then $G = Sp(n, R)/K_n^*$ can be realized as the Siegel domain of type one. The square-integrable representation of G gives the admissible wavelets AW and wavelet transform. The characterization of admissibility condition in terms of the Fourier transform is given. The Bergman kernel follows from the viewpoint of coherent state. With the Laguerre polynomials, Hermite polynomials and Jacobi polynomials, two kinds of orthogonal bases for AW are given, and they then give orthogonal decompositions of L^2-space on the Siegel domain of type one $L^2(\mathcal{H}_n, |y|^\alpha dxdy)$.

1 introduction

Let G be a locally compact group with left Haar measure dx. Let $x \to U(x)(x \in G)$ be an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$. If there is a vector ψ satisfying the following "admissibility condition":

$$0 < c_\psi := \int_G |(\psi, U(x)\psi)|^2 dx/(\psi, \psi) < \infty, \tag{1.1}$$

where $(\cdot, \cdot)$ is the inner product of $\mathcal{H}$, then the representation U is called square-integrable and $f \to (f, U(x)\psi)$ is called "continuous wavelet transform" and ψ is called an admissible wavelet (cf [3]).

A. Grossman and J. Morlet [4] introduced the wavelet transform in the one dimension case, where the group G is the affine group "$ax + b$". It consists of all couples $\{(x, y) : y > 0, x \in R\}$ with the law $(x_1, y_1)(x, y) = (y_1 x + x_1, y_1 y)$. It is a locally compact nonunimodular group with left Haar measure $d\mu_L(x, y) = dxdy/y^2$. In fact it is the quotient group of $SL(2, R)$ by $K := SO(2, R)$, here $SL(2, R)$ is the special linear group.

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The identification is made by (see [13], [16])

$$g = (x, y) \leftrightarrow \begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{bmatrix}, \tag{1.2}$$

and

$$\begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{bmatrix}^{-1} = \begin{bmatrix} 1/\sqrt{y} & -x/\sqrt{y} \\ 0 & \sqrt{y} \end{bmatrix}.$$

Let

$$f_{(x,y)}(x') := U(x, y)f(x') = \frac{1}{\sqrt{y}} f(\frac{x' - x}{y}) \tag{1.3}$$

be the representation of G on the Hardy space $H^2(R)$. Then the (affine) wavelet transform W_ψ for f in $H^2(R)$ associated with an admissible wavelet ψ is given by

$$W_\psi f(b, a) := (f, \psi_{(b.a)}) = \frac{1}{\sqrt{a}} \int_R \overline{\psi}(\frac{x - b}{a}) f(x) dx. \tag{1.4}$$

The map $f \to W_\psi f(b, a)$ gives an isometry (up to a constant) from $H^2(R)$ into $L^2(U, \frac{dbda}{a^2})$, here U is the upper half plane, and the following reconstruction formula holds:

$$f(x) = \frac{1}{c_\psi} \int_U W_\psi f(b, a) \psi_{(b,a)}(x) \frac{dbda}{a^2}. \tag{1.5}$$

R. Murenzi in [12] considered wavelet transform associated to $IG(n)$, the "n-dimensional Euclidean group with dilations". In this paper insteal of $SL(2, R)$ or $IG(n)$, we will consider the continuous wavelet associated the symplectic group $Sp(n, R)$ modulo its maximal subgroup.

Let $\mathcal{P}_n$ denote the space of positive definite $n \times n$ matrices and let

$$\mathcal{H}_n := \{z | z = x + iy, y \in \mathcal{P}_n, x \in GL(n, R) \quad \text{and are symmetric}\}.$$

$\mathcal{H}_n$ is an (unbounded) upper half-plane realization of the classical domain of type three, see [6], [18]. The homomorphism of $\mathcal{H}_n$ is symplectic group $Sp(n, R)$. Let $K_n^* := SO(2n) \cap Sp(n, R)$, then the quotient group $G = Sp(n, R)/K_n^*$ has one to one correspondence to $\mathcal{H}_n$. We will consider the square-integrable representation of G in next section and wavelet transform associated to G in §3. By this wavelet transform, the Bergman kernel of $\mathcal{H}_n$ is given in §4 and the orthogonal decompositions of the function space $L^2(\mathcal{H}_n, |y|^\alpha dxdy)$ are derived in §5.

2 Square-integrable group representation

In order to give the correspondence from the group $G = Sp(n, R)/K_n^*$ onto $\mathcal{H}_n$, let us first give a new look at the arguments from (1.2) to (1.3).

Let $\mathcal{J}P_2 := \{P \in R^{2\times 2} | P \quad \text{positive definite symmetric matrix of deterimant } 1\}$. Then the following maps are identifications and preserve the group action of $SL(2,R)$ on the three homogeneous spaces (see [16, p.125]):

$$\begin{aligned} &SL(2,R)/K \to \mathcal{J}P_2 \to U \\ &Kg \to g^t g = P \to z \in U \quad \text{with} \quad P\begin{bmatrix} 1 \\ -\bar{z} \end{bmatrix} = 0, \end{aligned}$$

where ${}^t g$ is the transpose of g, $SL(2,R)/K$ denotes the homogeneous space of cosets gK, and it is just the affine group G. We can write any element P in the $\mathcal{J}P_2$ as follows:

$$\begin{aligned} P &= \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} [\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}] := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ x/\sqrt{y} & 1/\sqrt{y} \end{bmatrix}. \end{aligned}$$

Let

$$g = \begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{bmatrix},$$

then g is regarded as an element in "$ax+b$", and write $g = \{x, y\}$. In this way we give the identification (1.2). Such correspondence also can be gotten from the Iwasawa decomposition of $SL(2,R)$.

Let $U = \{z = x + iy | y > 0\}$ denote the upper half-plane, its homomorphlism is $SL(2,R)$. For $g \in SL(2,R)$, $g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $SL(2,R)$ gives the action on U by $z \to g(z) = \frac{az+b}{cz+d}$. Especially for $g \in$ "$ax + b''$ given as above it gives an action on R by

$$x' \in R \to g(x') = \frac{\frac{1}{\sqrt{y}}x' - \frac{1}{\sqrt{y}}x}{\sqrt{y}} = \frac{x' - x}{y}.$$

This action induces another action of G on $L^2(R)$ by

$$f(x') \to \{g'(x')\}^{\frac{1}{2}} f(g(x')) = \frac{1}{\sqrt{y}} f(\frac{x'-x}{y}), \tag{2.1}$$

which is (1.3). In fact, (2.1) can be gotten from the (projection) representation of $SL(2,R)$ on $L^2(R)$.

Now let us consider in the similar way about the correspondence from G to $\mathcal{H}_n$. As in §1 $\mathcal{P}_n$ denotes the space of positive definite $n \times n$ real matrices and let S denote the symmetric space of the general linear group $GL(n,R)$ of non-singular $n \times n$ real matrices. S can be considered as the Shilov boundary of the Siegel domain $\mathcal{H}_n$ via $S = \overline{\mathcal{H}}_n \cap (\mathrm{Im} z = 0)$. Denote

$$\begin{aligned} A &= \{a \in GL(n,R) | a \quad \text{is positive and diagonal}\}, \\ N &= \{n \in GL(n,R) | n \quad \text{is upper triangular with ones on the diagonal}\}. \end{aligned}$$

For $y \in \mathcal{P}_n$, y can be written uniquely as

$$y = {}^t(an)(an) = {}^t n a^2 n, \quad \text{with } a \in A, n \in N, \tag{2.2}$$

and a can be given by $a_i^2 = |y_i|/|y_{i-1}|$, here y_k is the upper left hand $k \times k$ corner of the matrix y(cf [17, p.14]).

In the rest part of this paper, still let ${}^t g$ denote the transpose of g and use the notation

$$|y| := \text{determinant}(y).$$

Denote $\mathcal{P}_n^* := \mathcal{P}_{2n} \cap Sp(n,R)$, then the following map gives the identifications of $Sp(n,R)/K_n^*$, $\mathcal{P}_n^*$ and $\mathcal{H}_n$ (see [17, p.286]),

$$\begin{aligned} & Sp(n,R)/K_n^* \to \mathcal{P}_n^* \to \mathcal{H}_n \\ & gK_n^* \to {}^t gg = p \to z \in \mathcal{H}_n \quad \text{with} \quad p \begin{bmatrix} I \\ -\overline{z} \end{bmatrix} = 0. \end{aligned} \tag{2.3}$$

For $p \in \mathcal{P}_n^*$, p can be written

$$p = \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} [\begin{matrix} I & 0 \\ x & I \end{matrix}] = \begin{bmatrix} I & x \\ 0 & I \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ x & I \end{bmatrix},$$

where $y \in \mathcal{P}_n$, and $x \in S$. Then $(I, -\overline{z}) p \begin{pmatrix} I \\ -\overline{z} \end{pmatrix} = 0$ implies $z = x + iy$.

For $y \in \mathcal{P}_n$, it can be written uniquely in the form (2.2). Thus p is expressed as

$$p = \begin{bmatrix} I & x \\ 0 & I \end{bmatrix} \begin{bmatrix} {}^t(an) & 0 \\ 0 & (an)^{-1} \end{bmatrix} \begin{bmatrix} an & 0 \\ 0 & {}^t(an)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ x & I \end{bmatrix},$$

Let

$$g = \begin{bmatrix} I & x \\ 0 & I \end{bmatrix} \begin{bmatrix} {}^t(an) & 0 \\ 0 & (an)^{-1} \end{bmatrix} = \begin{bmatrix} {}^t(an) & x(an)^{-1} \\ 0 & (an)^{-1} \end{bmatrix}, \tag{2.4}$$

then every element in $Sp(n,R)/K_n^*$ can be written uniquely in the form of (2.4) and all such g form a group. Define

$$G := \{ \begin{bmatrix} {}^t(an) & x(an)^{-1} \\ 0 & (an)^{-1} \end{bmatrix} | a \in A, n \in N, x \in S \}.$$

Elements g in G also can be written as

$$g = (a, n, x), \qquad a \in A, n \in N, x \in S, \tag{2.5}$$

with the group operation:

$$g \cdot g_1 = (a, n, x) \cdot (a_1, n_1, x_1) = (aa_1, a^{-1}n_1 an, {}^t(an)x_1 an + x).$$

We can get the left invariant measure $d\mu(g)$ of group G is

$$d\mu(g) = \frac{\alpha(a)}{|a|^{n+1}} dadndx, \tag{2.6}$$

where $\alpha(a) = \Pi_{i=1}^n a_i^{n-2i+1}$, $da = \frac{da_1}{a_1} \cdots \frac{da_n}{a_n}$ and dn, dx are the Lebesque measures (up to a constant) on N, S respectively:

$$dx := c_n \Pi_{1 \le i \le j \le n} dx_{ij}, \quad dn := c_n' \Pi_{1 \le i < j \le n} dx_{ij},$$

where $c_n = 2^n (2\pi)^{\frac{n(n+1)}{4}}$, $c_n' = 2^{n-1}(2\pi)^{\frac{(n-1)n}{4}}$. In the case $n = 1$, G is just the affine group "$ax + b''$. In this way, we can give an identification of G with $\mathcal{H}_n$ by

$$(a, n, x) \leftrightarrow x + {}^t(an)(an)i \in \mathcal{H}_n.$$

For $g \in Sp(n, R)$, with $g^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then g acts on $\mathcal{H}_n$ by

$$z \in \mathcal{H}_n \to g(z) = (Az + B)(Cz + D)^{-1}.$$

For $g \in G$, $g = \begin{bmatrix} {}^t(an) & x(an)^{-1} \\ 0 & (an)^{-1} \end{bmatrix}$, then $g^{-1} = \begin{bmatrix} {}^t(an)^{-1} & -{}^t(an)^{-1}x \\ 0 & an \end{bmatrix}$, it acts on the Shilov boundary S of $\mathcal{H}_n$ by

$$g(x') = {}^t(an)^{-1}(x' - x)(an)^{-1}.$$

Let $L^2(dx) = L^2(S, dx)$ denote the function space square integrable on S with respect to dx. We will consider the wavelet transform associated to G from the square-integrable representation of G on $L^2(dx)$.

Define a unitary representation of G on $L^2(dx)$ by

$$U_g f(x') := |a|^{-\frac{n+1}{2}} f(g(x')) = |a|^{-\frac{n+1}{2}} f({}^t(an)^{-1}(x' - x)(an)^{-1}). \tag{2.7}$$

In fact this unitary representation U_g can be gotten from the (projection) representation of $Sp(n, R)$ (cf [9]). In [9], Kashiwara and Verane considered the irreducible representations of a group similar to G. Now let us give the irreducible representations of G.

For function f on S, let $\hat{f}(\xi)$ denote the Fourier transform of f defined by

$$\hat{f}(\xi) := \int_S e^{-i\mathrm{tr}x\xi} f(x)dx,$$

here $\mathrm{tr}x$ denote the trace of matrix x. Then the representation (2.7) of G is given by

$$(U_g f)^\wedge(\xi) = |a|^{\frac{n+1}{2}} e^{-i\mathrm{tr}x\xi} \hat{f}(an\xi^t(an)). \tag{2.8}$$

Let M denote the subgroup of $\mathcal{P}_n$ consisting of diagonal matrices with entries 1 or -1. For any $m \in M$, with similar denotations in [9], let $\mathcal{O}_m$ denote the open subset of S which consisting of elements being similar to m and let L^2_m denote the subspace of $L^2(dx)$ consisting of functions whose Fourier transforms are supported in $\overline{\mathcal{O}_m}$. We can get L^2_m is an irreducible invariant subspace of the representation of U and let U^m denote the restriction of U to the subspace L^2_m. Then we have

$$U = \sum_{m \in M} \bigoplus U^m,$$

where each representation U^m is an irreducible representation of G and they are different.

For any $m \in M$, if there exists a vector $\psi \in L^2_m$ such that

$$0 < \int_G |(U^m_g \psi, \psi)|^2 d\mu(g) < \infty, \tag{2.9}$$

where $d\mu(g)$ is given in (2.6), then U^m is called square-integrable and the map $f \to (U^m_g \psi, f)$ is called the wavelet transform for $f \in L^2_m$. In fact we can get in next section that all U^m are square-integrable and we would pay more attention to the case $m = I = diag(1, 1, \cdots, 1)$. The admissibility condition (2.9) will be computed and the related wavelet transform will be considered.

3 Wavelet transform

Let $\mathcal{H}_n$ be the Siegel domain given by (1.1), it is of type one and in fact it is the tube of real $n \times n$ symmetric matrices over the cone $\mathcal{P}_n$. The dual cone $\mathcal{P}^*_n$ of $\mathcal{P}_n$ is itself. Thus the Hardy space $H^2(\mathcal{H}_n)$ has the following correspondence to $L^2(\mathcal{P}^*_n)$=$L^2(\mathcal{P}_n)$(see [1], [18]):

$$\begin{aligned} &L^2(\mathcal{P}_n) \to H^2(\mathcal{H}_n) \\ &\phi \to F(z) = \int_{\mathcal{P}_n} e^{i \mathrm{tr} z\lambda} \phi(\lambda) d\lambda. \end{aligned}$$

Thus the function space L^2_I given in §2 is just the boundary values of the Hardy space $H^2(\mathcal{H}_n)$ and L^2_{-I} is the boundary values of the conjugate Hardy space $\overline{H}^2(\mathcal{H}_n)$. Denote

$$H^2 = L^2_I, \quad \overline{H}^2 = L^2_{-I},$$

then $H^2, \overline{H}^2$ are the usual real Hardy space and conjugate Hardy space respectively for the case $n = 1$. In this section the continuous wavelet transform on H^2 will be considered in detail since it connect tight to the Bergman space on $\mathcal{H}_n$.

Let $T = U^I$ be the representation of G on H^2 given in §2. One can get T is square-integrable by calculating the "admissibility condition" (2.9). In fact, for $\psi \in H^2$,

$$\int_G |(\psi, T_g \psi)|^2 d\mu(g) = \int_G | \int \hat{\psi}(\xi) e^{i \mathrm{tr} \xi x} |a|^{\frac{n+1}{2}} \overline{\hat{\psi}(an\xi^t(an))} d\xi|^2 \frac{\alpha(a)}{|a|^{n+1}} dadndx$$

$$
\begin{aligned}
&= \int_A \int_N \int_S |\int \hat{\psi}(\xi)\overline{\hat{\psi}(an\xi^t(an))}e^{i\mathrm{tr}\xi x}d\xi|^2 dx\alpha(a)dxdnda \\
&= \int_A \int_N \int_{\mathcal{P}_n} |\hat{\psi}(\xi)\hat{\psi}(an\xi^t(an))|^2 d\xi\alpha(a)dnda \\
&= \int_{\mathcal{P}_n} \int_A \int_N |\hat{\psi}(an\xi^t(an))|^2\alpha(a)dnda|\hat{\psi}(\xi)|^2 d\xi.
\end{aligned}
$$

The third equation is gotten by the Plancherel formula. For any $\xi \in \mathcal{P}_n$, it can be written uniquely as (cf §2)

$$
\xi = {}^t(a_0n_0)a_0n_0, \qquad \text{with } a_0 \in A, n_0 \in N,
$$

thus

$$
\begin{aligned}
&\int_A \int_N |\hat{\psi}(an\xi^t(an))|^2\alpha(a)dnda = \int_A \int_N |\hat{\psi}(an^tn_0a_0^2n_0{}^tna)|^2\alpha(a)dnda \\
&= \int_A \int_N |\hat{\psi}(ana_0^{2t}na)|^2\alpha(a)dnda = \int_A \int_N |\hat{\psi}(aa_0a_0^{-1}na_0{}^t(a_0^{-1}na)a_0a)|^2\alpha(a)dnda \\
&= \int_A \int_N |\hat{\psi}(aa_0n^tna_0a)|^2\alpha(a)\alpha(a_0)dnda = \int_A \int_N |\hat{\psi}(an^tna)|^2\alpha(a)dnda.
\end{aligned}
$$

In above calculations, the facts $d(n^tn_0) = dn, d(a_0^{-1}na_0) = \alpha(a_0)^{-1}dn, d(a_0a) = da$ are used. Thus the admissibility condition becomes

$$
0 < c_\psi := \int_A \int_N |\hat{\psi}(an^tna)|^2\alpha(a)dnda < \infty. \tag{3.1}
$$

Let $\psi \in H^2$ defined by

$$
\hat{\psi}(\xi) := |\xi|^{\frac{n+1}{2}}e^{-c_0\mathrm{tr}\xi^2},
$$

where c_0 is a positive constant. It is easy to get that ψ satisfies the condition (3.1) and it is called the Mexican hat as in the case $n = 1$. Thus the representation T of G is square-integrable. Let AAW denote the set of all such " analyzing admissible wavelet" , i.e.

$$
AAW := \{\psi : \hat{\psi} \quad \text{is real,} \quad \psi \in H^2 \quad \text{and } \psi \text{ satisfies (3.1)}\}.
$$

For $\psi \in AAW$, the continuous wavelet transform W_ψ for $f \in H^2$ is defined by

$$
f \to W_\psi f(g) := (f, T_g\psi). \tag{3.2}
$$

Theorem 1 For $\psi \in AAW$, then for any $f, h \in H^2$,

$$
< W_\psi f, W_\psi h >= c_\psi(f, h),
$$

where $<,>$ is the inner product of $L^2(G, d\mu)$.

The proof of Theorem 1 is given by using the Plancherel formula as in calculating the admissible condition. In fact,

$$
\begin{aligned}
&< W_\psi f, W_\psi h >= \int_G W_\psi f(x,y)\overline{W_\psi h(x,y)}d\mu(g) \\
&= \int_N\int_A\int_{\mathcal{P}_n} (W_\psi f)^\wedge(a,n,\xi)\overline{(W_\psi h)^\wedge(a,n,\xi)}d\xi\frac{\alpha(a)}{|a|^{n+1}}dadn \\
&= \int_N\int_A\int_{\mathcal{P}_n} \hat{f}(\xi)\hat{\psi}(an\xi^t na)\overline{\hat{h}(\xi)}\hat{\psi}(an\xi^t na)d\xi\alpha(a)dadn \\
&= \int_{\mathcal{P}_n}\int_N\int_A |\hat{\psi}(an\xi^t na)|^2\alpha(a)dadn\hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi \\
&= c_\psi\int_{\mathcal{P}_n}\hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi = c_\psi(f,h).
\end{aligned}
$$

it completes the proof of Theorem 1.

Theorem 2. For $\psi \in AAW$, then there exists the reconstruction formula for any $f \in H^2$

$$f(x) = \frac{1}{c_\psi}\int_G (f, T_g\psi)T_g\psi(x)d\mu(g). \tag{3.3}$$

For $\psi \in AAW$, it is a state in $L^2(dx)$ which can be written as $|\psi >$. Then $\{T_g\psi\}_{g\in G} = \{T_g|\psi >\}_{g\in G}$ is a coherent state system [14]. Denote

$$A_\psi := \{W_\psi f(g) : f \in H^2\} := \{< f|T_g\psi >: f \in H^2\},$$

then A_ψ is a Hilbert space with reproducing kernel.

Theorem 3. Let $K_\psi(g,g')$ be the reproducing kernel of A_ψ, then

$$K_\psi(g,g') = \frac{1}{c_\psi}(T_{g'}\psi, T_g\psi).$$

The reconstruction formula (3.3) in Theorem 2 holds with convergence of the integral at least "in the weak sense", i.e. taking the inner product of both side of (3.3) with any $g \in H^2$, and commuting the inner product with integral over G, leading to a true formula. This is just Theorem 1. In fact, the convergence also holds "in stronger sense" as in [2], we omit details here. Theorem 3 can be gotten by Theorem 2. In fact, from Theorem 2,

$$
\begin{aligned}
&W_\psi f(g) = (f, T_g\psi) = \frac{1}{c_\psi}\int_G (f, T_g'\psi)(T_{g'}\psi, T_g\psi)d\mu(g') \\
&= \frac{1}{c_\psi}\int_G (T_{g'}\psi, T_g\psi)W_\psi f(g')d\mu(g'),
\end{aligned}
$$

thus $K_\psi(g,g') = \frac{1}{c_\psi}(T_{g'}\psi, T_g\psi)$.

For other irreducible representations U^m of G, the admissibility condition (2.9) is similar to (3.1):

$$0 < c_\psi^m := \int_A\int_N |\hat{\psi}(anm^t na)|^2\alpha(a)dnda < \infty,$$

and one can get each U^m is square-integrable. Let AW^m denote the set of the related admissible wavelets. For $\psi \in AW^m$, we can define similarly the related wavelet transform $W_\psi^m f$ for $f \in L_m^2$ and can establish theorems similar to Theorem 1, 2, 3, details omitted here.

4 Coherent state and Bergman kernel

Recalling for $z = x + iy \in \mathcal{H}_n$, z corresponds to an element $g \in G$ given in §2, i.e. it can be written as

$$z = (a, n, x) \in G \quad \text{with } y = {}^t(an)an \tag{4.1}$$

Let $U_z = T_g = U_g^I$ given in §3. For ψ or state $|\psi >\in AAW$, appling U_z to it, we get a coherent state system. In this section the Bergman kernel will be gotten from this viewpoint [14].

For $f \in L^1(\mathcal{P}_n)$, we have (see [17, p.35])

$$\int_{\mathcal{P}_n} f(y)\frac{dy}{|y|^{\frac{n+1}{2}}} = 2^n \int_A \int_N f({}^t n a^2 n)\alpha(a) dn da,$$

here we still denote $\alpha(a) = \Pi_{i=1}^n a_i^{n-2i+1}, da = \frac{da_1}{a_1} \cdots \frac{da_n}{a_n}$ and dn be the Lebesque measures on N as above. Thus for appropriate functions $F(z)$ on $\mathcal{H}_n$, we have

$$2^n \int_{\mathcal{H}_n} F(z)\frac{dz}{|y|^{n+1}} = \int_S \int_A \int_N F(x+i{}^t n a^2 n)\alpha(a)\frac{dndadx}{|a|^{n+1}} = \int_G F(x+i{}^t n a^2 n) d\mu(g). \tag{4.2}$$

Recall for $\alpha > -1$, $L^{\alpha 2}(\mathcal{H}_n)$ denotes function space on $\mathcal{H}_n$ square integrable with respect to the measure $|y|^\alpha dxdy$ and $A^{\alpha 2}$ denotes the homomorphic part of $L^{\alpha 2}(\mathcal{H}_n)$, i.e. the (weighted) Bergman space. For $\psi \in AAW$, let W_ψ be the wavelet transform given by (3.2), then W_ψ is an isometry (up to a constant) from H^2 into $L^2(G, d\mu(g))$.

If we define functions $W_\psi^\alpha f(z)$ on $\mathcal{H}_n$ via this wavelet transform for functions $f \in H^2$:

$$W_\psi^\alpha f(z) := |y|^{-\frac{\alpha+n+1}{2}}(f, U_z\psi), \tag{4.3}$$

then from (4.2), for $\psi \in AAW$, such map is an isometry (up to a constant) from H^2 into $L^{\alpha 2}(\mathcal{H}_n)$. Denote A_ψ^α to be the set

$$A_\psi^\alpha := \{W_\psi^\alpha f(z) = |y|^{-\frac{\alpha+n+1}{2}}(f, U_z\psi) : f \in H^2\}. \tag{4.4}$$

We will choose a special ψ such that $W_\psi^\alpha f(z)$ are homomorpholic functions on $\mathcal{H}_n$ and A_ψ^α is just the Bergman space.

Let $\psi_0(x) \in H^2$ defined by

$$\hat{\psi}_0(\xi) := \begin{cases} |\xi|^s e^{-\mathrm{tr}\xi}, & \text{for} \quad \xi \in \mathcal{P}_n, \\ 0, & \text{elsewhere}, \end{cases} \tag{4.5}$$

with $s = \frac{2\alpha+n+1}{4}$, then for $f \in H^2$,

$$W^\alpha_{\psi_0} f(z) = |y|^{-\frac{\alpha+n+1}{2}}(f, U_z\psi_0) = |y|^{-\frac{\alpha+n+1}{2}} \int \hat{f}(\xi)|a|^{\frac{n+1}{2}} e^{i\mathrm{tr}x\xi}\hat{\psi}_0(an\xi^t na)d\xi$$
$$= |y|^{-\frac{\alpha+n+1}{2}} \int_{\mathcal{P}_n} \hat{f}(\xi)|a|^{\frac{n+1}{2}} e^{i\mathrm{tr}x\xi}|y\xi|^s e^{-\mathrm{tr}y\xi}d\xi = \int \hat{f}(\xi)|\xi|^{\frac{2\alpha+n+1}{4}} e^{-i\mathrm{tr}z\xi}d\xi.$$

Thus $W^\alpha_{\psi_0} f(z)$ is homomorphlic on $\mathcal{H}_n$. In fact $W^\alpha_{\psi_0} f \in A^{\alpha 2}$ and $W^\alpha_{\psi_0}$ is an isometry (up a constant) from H^2 onto $A^{\alpha 2}$, i.e. A_{ψ_0} is just the Bergman space. Thus the Bergman kernel is gotten by calculating the reproducing kernel of A_{ψ_0}.

For $z, z' \in \mathcal{H}_n$, they can be written as in (4.1): $z = (a, n, x), z' = (a', n', x') \in G$. By Theorem 3, the reproducing kernel $K(z, z')$ for $A^\alpha_{\psi_0}$ is given as follows:

$$K(z, z') = c^{-1}_{\psi_0}|y'y|^{-\frac{n+\alpha+1}{2}}(U_{z'}\psi_0, U_z\psi_0).$$

And

$$(U_{z'}\psi_0, U_z\psi_0) = |y|^{\frac{n+1}{4}+s} \int_{\mathcal{P}_n} |\xi|^s (U_{z'}\psi_0)^\wedge(\xi)e^{i\mathrm{tr}(x+iy)\xi}d\xi$$
$$= |y|^{\frac{n+1}{4}+s} \int_{\mathcal{P}_n} |\xi|^s e^{-i\mathrm{tr}x'\xi}|y'|^{\frac{n+1}{4}}\hat{\psi}_0(a'n'\xi^t n'a')e^{i\mathrm{tr}(x+iy)\xi}d\xi$$
$$= |yy'|^{\frac{n+1}{4}+s} \int_{\mathcal{P}_n} |\xi|^{2s} e^{-i\mathrm{tr}x'\xi}e^{-\mathrm{tr}y'\xi}e^{i\mathrm{tr}(x+iy)\xi}d\xi$$
$$= |yy'|^{\frac{n+1}{4}+s} \int_{\mathcal{P}_n} |\xi|^{2s} e^{i\mathrm{tr}(z-\overline{z'})\xi}d\xi.$$

Since $s = \frac{2\alpha+n+1}{4}$, then

$$K(z, z') = \frac{1}{c_{\psi_0}} \int_{\mathcal{P}_n} |\xi|^{\alpha+n+1} e^{i\mathrm{tr}(z-\overline{z'})\xi} \frac{d\xi}{|\xi|^{\frac{n+1}{2}}}.$$

The last integral was given in [1], and finally

$$K(z, z') = C_n|z - \overline{z'}|^{-(\alpha+n+1)},$$

where C_n is a constant. By Cayley transform, $K(z, z')$ is just the Bergman kernel of classical domain of class three gotten by Hua for the case $\alpha = 0$, see [6, p.84].

5 Orthogonal admissible wavelets

Let $H^{\alpha 2} := \{f : f \in L^{\alpha 2}(\mathcal{H}_n), \hat{f}(\xi, y) = 0, \quad \text{for} \quad \xi \notin \mathcal{P}_n\}$. It is obvious that $A^{\alpha 2} \subset H^{\alpha 2}$. If we choose an appropriate orthogonal basis $\{\psi_k\}_k$ of AAW, then each state ψ_k gives a coherent state system $\{U_z|\psi_k >\}_{z\in\mathcal{H}_n}$ and spaces $A^\alpha_{\psi_k} = \{(f, U_z\psi_k) : f \in H^2\}$. We can get $A^\alpha_{\psi_k}$ are orthogonal to each other and $\sum_k \bigoplus A^\alpha_{\psi_k} = H^{\alpha 2}$. Choosing appropriate orthogonal basis of AW^m for each $m \in M$, then can gotten an orthogonal decomposition of $L^{\alpha 2}(\mathcal{H}_n)$.

From (3.1), for two admissible wavelets $\phi(y), \tilde{\phi}(y) \in AAW$, we say that they are orthogonal to each other if they satisfy

$$\int_A \int_N \hat{\tilde{\phi}}(an^tna)\overline{\hat{\tilde{\phi}}}(an^tna)\alpha(a)dnda = 0. \tag{5.1}$$

We now want to give an orthogonal basis of AAW with the first function just being ψ_0 given by (4.5). Let $\phi(y) \in AAW$ defined by

$$\hat{\phi}_{l,k}(an^tna) := g_l(a)h_k(an)|a|^{2s}e^{-tr(^tna^2n)}, \tag{5.2}$$

where $s = \frac{2\alpha+n+1}{4}$, $g_l(a), h_k(n)$ are some real polynomials of n variables $(a_1, \cdots, a_n)$ and $\frac{(n-1)n}{2}$ variables $(n_{ij})_{1\le i<j\le n}$ respectively and here an denotes (without rising confusions)

$$(a_1n_{12}, \cdots, a_1n_{1n}, a_2n_{23}, \cdots, a_2n_{2n}, \cdots, a_{n-1}n_{n-1n}).$$

Then for such wavelets, (5.1) becomes

$$\int_A \int_N g_l(a)h_k(an)g_{l'}(a)h_{k'}(an)|a|^{4s}e^{-tr(^tna^2n)}\alpha(a)dadn = c\delta_{ll'}\delta_{kk'}. \tag{5.3}$$

Use $d(an) = \Pi_{1\le i<j\le n}d(a_in_{ij}) = \Pi_{i=1}^{n-1}a_i^{n-i} \cdot \Pi_{1\le i<j\le n}dn_{ij}$, we know the left hand side of (5.3) is

$$\begin{aligned}
&\int_A \int_N g_l(a)g_{l'}(a)h_k(n)h_{k'}(n)|a|^{4s}\Pi_{i=1}^{n-1}a_i^{1-i}\mathrm{Exp}(-2\sum_{i=1}^n a_i^2 - 2\sum_{1\le i<j\le n} n_{ij}^2)dadn \\
&= \int_{a_i>0} g_l(a)g_{l'}(a)\Pi_{i=1}^{n-1}a_i^{2\alpha+n+1-i}\mathrm{Exp}(-2\sum_{i=1}^n a_i^2)da_1\cdots da_n \cdot \\
&\quad \int_{n_{ij}\in R} h_k(n)h_{k'}(n)\mathrm{Exp}(-2\sum_{1\le i<j\le n} n_{ij}^2)\Pi_{1\le i<j\le n}dn_{ij}.
\end{aligned}$$

Let $L_l^{(\alpha)}(x)$ and $H_k(x)$ be the Laguerre polynomial of degree l and the Hermite polynomial of degree k respectively, i.e.

$$L_l^{(\alpha)}(x) = \sum_{\nu=0}^{l} \binom{l+\alpha}{l-\nu}\frac{(-x)^\nu}{\nu!},$$
$$H_k(x) = k!\sum_{m=0}^{[\frac{k}{2}]}\frac{(-1)^k(2x)^{k-2m}}{m!(k-2m)!}.$$

For $l = (l_1, \cdots, l_n) \in Z_+^n, k = (k_{12}, \cdots, k_{1n}, k_{23}, \cdots, k_{2n}, \cdots, k_{n-1n}) \in Z_+^{\frac{(n-1)n}{2}}$, we define

$$g_l(x) := \Pi_{1\le i\le n}L_{l_i}^{(\alpha+\frac{n-i}{2})}(2a_i^2) \quad h_k(n) := \Pi_{1\le i<j\le n}H_{k_{ij}}(\sqrt{2}n_{ij}). \tag{5.4}$$

Then from (5.4) and the orthogonal properties of Laguerre polynomials and Hermite polynomials, we know $\phi_{l,k}$ defined by (5.2) with $g_l(a), h_k(n)$ given by (5.5) are orthogonal to each other. And by the completeness of Laguerre polynomials and Hermite polynomials in $L^2(R_+, x^\alpha e^{-x}dx)$ and $L^2(R, e^{-x^2}dx)$, we know $\phi_{l,k}, l \in Z_+^n, l \in Z_+^{\frac{(n-1)n}{2}}$, form an orthogonal basis of AAW.

Let

$$A_{l,k} := \{W^\alpha_{\phi_{l,k}} = |y|^{-\frac{\alpha+n+1}{2}}(f, U_z\phi_{l,k}) : f \in H^2\},$$

then $A_{l,k}$ are orthogonal subspaces of the Hilbert space $H^{\alpha 2}$ and in fact they form an orthogonal decomposition of $H^{\alpha 2}$.

For $m \in M$, let $\phi^m_{l,k} \in AW^m$ (with $\phi^I_{l,k} = \phi_{l,k}$) defined by

$$\hat{\phi}^m_{l,k}(anm^tna) := g_l(a)h_k(an)|a|^{\alpha+\frac{n+1}{2}}e^{-tr({}^tna^2n)}, \tag{5.5}$$

where $g_l(a), h_k(n)$ given by (5.5), then $\phi^m_{l,k}, l \in Z_+^n, k \in Z_+^{\frac{(n-1)n}{2}}$, form an orthogonal basis of AW^m. Define $A^m_{l,k}$ similarly

$$A^m_{l,k} := \{W^\alpha_{\phi^m_{l,k}} = |y|^{-\frac{\alpha+n+1}{2}}(f, U_z\phi^m_{l,k}) : f \in L^2_m\}, \tag{5.6}$$

we have

Theorem 4. Let $A^m_{l,k}$ defined by (5.7), then

$$L^{\alpha 2}(\mathcal{H}_n) = \sum_{m\in M}\sum_{l\in Z_+^n}\sum_{k\in Z_+^{\frac{(n-1)n}{2}}}\bigoplus A^m_{k,l}.$$

In [5], similarly decomposition was given, but it is not clear if the first decomposition component is the Bergman space. Here, for $l = \vec{0}$, $k = \vec{0}$, we know from the construction of $\phi^m_{l,k}$ that $\phi_{\vec{0},\vec{0}} = \psi_0$ given in §4. Thus from §4 we know the first component $A_{\vec{0},\vec{0}} = A^I_{\vec{0},\vec{0}}$ is just the Bergman space.

One of the main motivations for the decomposition of L^2 function space on $\mathcal{H}_n$ is for the study of Hankel and Toeplitz type operators, as we did in [7], [8]. We would consider Toeplitz type operators with symbol being rotation invariant or "radial" as in [8]. For this purpose, we need to give another decomposition of L^2 function space via the "polar coordinates" of $y \in \mathcal{P}_n$:

$$y = {}^tk\lambda k, \quad k \in O(n), \quad \lambda \in A. \tag{5.7}$$

In this case, $\lambda_i, 1 \leq i \leq n$, are the eigenvalues of y. In fact, such decomposition of y has been very useful in numerical analysis, multivariate statistics (see [15], [11]). In the rest of this section, we will give an orthogonal decomposition of $L^{\alpha 2}(\mathcal{H}_n)$ by using coordinates (5.8) and will consider the case $n = 2$ for simplicity.

For appropriate function f on $\mathcal{P}_n$, we have (cf [17, p.35])

$$\int_{\mathcal{P}_n} f(y)\frac{dy}{|y|^{\frac{n+1}{2}}} = 2^n\int_A\int_N \hat{f}({}^t(an)^{-1}(an)^{-1})dnda, \tag{5.8}$$

$$\int_{\mathcal{P}_n} f(y)\frac{dy}{|y|^{\frac{n+1}{2}}} = c_n\int_K\int_A \hat{f}({}^tk\lambda k)\gamma(\lambda)d\lambda dk, \tag{5.9}$$

where in (5.10), c_n is a constant, $K = O(n)$, dk is the invariant measure of K and

$$\gamma(\lambda) = \Pi_{i=1}^n \lambda_i^{-\frac{n-1}{2}} \Pi_{1 \le i < j \le n} |\lambda_i - \lambda_j|, \quad \text{for} \quad \lambda = diag(\lambda_1, \cdots, \lambda) \in A.$$

For $\phi(y), \tilde{\phi}(y) \in AAW$, let $\varphi, \tilde{\varphi}$ defined by

$$\hat{\phi}(y) = \alpha^{-\frac{1}{2}}(a)\hat{\varphi}(y^{-1}), \quad \hat{\tilde{\phi}}(y) = \alpha^{-\frac{1}{2}}(a)\hat{\tilde{\varphi}}(y^{-1}), \tag{5.10}$$

with $y = an^t(an), \alpha(a) = \Pi a_i^{n-2i+1}$. If $\phi, \tilde{\phi}$ are orthogonal to each other, then $\varphi, \tilde{\varphi}$ shall satisfy

$$\int_A \int_N \hat{\tilde{\varphi}}({}^t(an)^{-1}(an)^{-1})\overline{\hat{\varphi}({}^t(an)^{-1}(an)^{-1})} dnda = 0. \tag{5.11}$$

From (5.9), (5.10), the left hand side of (5.12) is

$$\int_{\mathcal{P}_n} \hat{\tilde{\varphi}}(y)\overline{\hat{\varphi}(y)} \frac{dy}{|y|^{\frac{n+1}{2}}} = 2^{-n} c_n \int_K \int_A \hat{\tilde{\varphi}}({}^t k\lambda k)\overline{\hat{\varphi}({}^t k\lambda k)}\gamma(\lambda) d\lambda dk. \tag{5.12}$$

For $n = 2$, $y = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} \in \mathcal{P}_2$, we write $y = an^t(an)$ with

$$a_1 = |y|^{\frac{1}{2}}/\sqrt{y_3}, \quad a_2 = \sqrt{y_3}, \quad n_{12} = y_2/\sqrt{|y|},$$

and

$$\alpha(a) = a_1/a_2 = \sqrt{|y|}/y_3. \tag{5.13}$$

We also can give polar coordinates of y by

$$y = {}^t k_\theta \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} k_\theta, \quad \text{with } k_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{5.14}$$

In fact, (5.15) give a one to one correspondence from $\mathcal{P}_2$ to $(\theta, \lambda_1, \lambda_2)$ with $0 \le \theta < \pi, 0 < \lambda_1 < \lambda_2$, and the integral formula of (5.10) will be

$$\int_{\mathcal{P}_2} f(y) \frac{dy}{|y|^{\frac{3}{2}}} = c_n' \int_0^\pi \int_{0<\lambda_1<\lambda_2} \hat{f}({}^t k_\theta \lambda k_\theta)\gamma(\lambda) \frac{d\lambda_1 d\lambda_2 d\theta}{\lambda_1 \lambda_2},$$

with $\gamma(\lambda) = (\lambda_1\lambda_2)^{-\frac{1}{2}}(\lambda_2 - \lambda_1)$. Thus from (5.13), we shall construct $\tilde{\varphi}, \varphi$ satisfying

$$\int_0^\pi \int_{0<\lambda_1<\lambda_2} \hat{\tilde{\varphi}}({}^t k\lambda k)\overline{\hat{\varphi}({}^t k\lambda k)}(\lambda_1\lambda_2)^{-\frac{3}{2}}(\lambda_2 - \lambda_1) d\lambda_1 d\lambda_2 d\theta = 0. \tag{5.15}$$

Let $\hat{\tilde{\varphi}}({}^t k\lambda k), \hat{\varphi}({}^t k\lambda k)$ be the functions of the following forms:

$$h(\theta)q(\lambda_1^{-1}, \lambda_2^{-1})|\lambda_1\lambda_2|^{-\frac{2\alpha+3}{4}} \mathrm{Exp}(-\lambda_1^{-1} - \lambda_2^{-1}),$$

where $h(\theta)$ are real functions of θ and $q(t_1, t_2)$ are real polynomials of t_1, t_2. Then from (5.16), we have

$$\int_0^\pi h(\theta)\tilde{h}(\theta)d\theta \cdot \int_{0<t_2<t_1} q(t_1, t_2)\tilde{q}(t_1, t_2)(t_1t_2)^\alpha e^{-2t_1-2t_2}(t_1 - t_2) dt_1 dt_2 = 0. \tag{5.16}$$

We can choose $h_j^1(\theta) = \cos(2j\theta), h_j^2(\theta) = \sin(2j\theta)$. Then $h_j^i(\theta), i = 1,2, j \in Z$ form an orthogonal basis of $L^2([0,\pi), d\theta)$.

For the radial part, let $q_{k,l}(t_1,t_2)$ defined by

$$q_{k,l}(t_1,t_2) = t_1^k p_k(\frac{t_2}{t_1}) q_l(t_1+t_2),$$

where p_k, q_l are some polynomials of degree k, l respectively. Then p_k, q_l shall satisfy

$$\int_{0<t_2<t_1} p_k(\frac{t_2}{t_1}) q_l(t_1+t_2) p_{k'}(\frac{t_2}{t_1}) q_{l'}(t_1+t_2) t_1^{k+k'+\alpha} t_2^\alpha e^{-2t_1-2t_2}(t_1-t_2) dt_1 dt_2 = C_{kl}\delta_{kk'}\delta_{ll'}.$$

By changes of variables, $t = t_2/t_1, s = t_1+t_2$, we have

$$\int_0^1 p_k(t)p_{k'}(t)\frac{t^\alpha(1-t)}{(1+t)^{2\alpha+k+k'+3}}dt \cdot \int_0^\infty q_l(s)q_{l'}(s)s^{2\alpha+k+k'+2}e^{-2s}ds = C_{kl}\delta_{kk'}\delta_{ll'}.$$

We choose $q_l(t) = L_l^{(2\alpha+2k+2)}(2t)$–the Laguerre polynomial of degree l. And assume that

$$p_k(t) = (1+t)^{2k} g_k(\Big(\frac{1-t}{1+t}\Big)^2),$$

where $g_k(t)$ is a polynomial of degree k, then $g_k(t)$ shall satisfy

$$\int_0^1 g_k(t)g_{k'}(t)(1-t)^\alpha dt = C_k\delta_{kk'}. \tag{5.17}$$

Let $P_k^{(\alpha,\beta)}(t)$ denote the Jacobi polynomial of degree k, they form an orthogonal basis of function space $L^2([-1,1],(1-t)^\alpha(1+t)^\beta dt)$. So that from (5.18), let

$$g_k(t) = P_k^{(\alpha,0)}(2t-1).$$

Finally for $k \in Z_+, l \in Z_+$, let $\psi_{j,k,l}^1, \psi_{j,k,l}^2$ be functions on $\mathcal{P}_2$ defined by

$$\hat{\psi}_{j,k,l}^1(y) := y_3^{\frac{1}{2}}|y|^{\frac{\alpha+1}{2}}\cos(2j\theta)(\mathrm{tr}y)^{2k}e^{-\mathrm{tr}y}L_l^{(2\alpha+2k+2)}(2\mathrm{tr}y)P_k^{(\alpha,0)}(\frac{(\mathrm{tr}y)^2-8|y|}{(\mathrm{tr}y)^2}), \tag{5.18}$$

$$\hat{\psi}_{j,k,l}^2(y) := y_3^{\frac{1}{2}}|y|^{\frac{\alpha+1}{2}}\sin(2j\theta)(\mathrm{tr}y)^{2k}e^{-\mathrm{tr}y}L_l^{(2\alpha+2k+2)}(2\mathrm{tr}y)P_k^{(\alpha,0)}(\frac{(\mathrm{tr}y)^2-8|y|}{(\mathrm{tr}y)^2}),$$

where θ is given by ${}^tk_\theta\lambda k_\theta$ as in (5.15). Then from above construction, we know $\psi_{j,k,l}^i \in AAW$ and all they form an orthogonal basis of AAW. Similarly as above, we define

$$B_{j,k,l}^i := \{|y|^{-\frac{\alpha+3}{2}}W_{\psi_{j,k,l}^i}f(z) : f \in H^2\}, \tag{5.19}$$

then $B_{j,k,l}^i$ are orthogonal to each other with respect to the measure $|y|^\alpha dxdy$ and form an orthogonal decomposition of $H^{\alpha 2}$.

For other $m = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, we can define $\psi^{i,m}_{j,k,l} \in AW^m$ by

$$\hat{\psi}^{i,m}_{j,k,l}({}^t k_\theta m \lambda k_\theta) := \hat{\psi}^{i}_{j,k,l}({}^t k_\theta \lambda k_\theta),$$

and define $B^{i,m}_{j,k,l}$ similarly as (5.20). Then we have

Theorem 5. Let $B^{i,m}_{j,l,k}$ be subspaces of $L^{\alpha 2}(\mathcal{H}_n)$ defined as above with $B^{i,I}_{j,l,k} = B^{i}_{j,l,k}$, then

$$L^{\alpha 2}(\mathcal{H}_n) = \sum_{m\in M} \sum_{i=1,2} \sum_{j\in Z} \sum_{l,k\in Z_+} \bigoplus B^{i,m}_{j,k,l}.$$

For $n > 2$, we also can give an orthogonal decomposition of $L^{\alpha 2}(\mathcal{H}_n)$ as above by using the polar coordinates of $y \in \mathcal{P}_n$. We still assume $\psi, \tilde{\psi}$ given by (5.11), then $\varphi, \tilde{\varphi}$ shall satisfy (having a small change compared to (5.10) or (5.16))

$$\int_{SO(n)} \int_{\lambda_i>0} \hat{\tilde{\varphi}}({}^t k \lambda k) \overline{\hat{\varphi}({}^t k \lambda k)} \gamma(\lambda) \lambda dk = 0.$$

We then assume $\varphi, \tilde{\varphi}$ have the following forms

$$h(k) q(\lambda_1^{-1}, \cdots, \lambda_n^{-1}) (\lambda_1 \cdots \lambda_n)^{-\frac{2\alpha+n+1}{4}} \mathrm{Exp}(-\lambda_1^{-1} - \cdots - \lambda_n^{-1}),$$

then $h, q, \tilde{h}, \tilde{q}$ satisfy

$$\int_{SO(n)} \tilde{h}(k) h(k) dk \cdot \int_{a_i>0} q(a_1, \cdots, a_n) \tilde{q}(a_1, \cdots, a_n) \cdot$$
$$(a_1 \cdots a_n)^\alpha \mathrm{Exp}(-\sum_{i=1}^n a_i) \Pi_{1\le i<j\le n} |a_i - a_j| da_i \cdots da_n = 0.$$

We can choose $h(k) = t^{n\sigma}_{KM}(k)$, an orthogonal basis of $L^2(SO(n), dk)$ as in [19, Ch.10]. For the radial part, let

$$q_{\vec{k}}(a_1, \cdots, a_n) := L^\alpha_{\vec{k}}(a_1, \cdots, a_n),$$

where $L^\alpha_{\vec{k}}(a_1, \cdots, a_n)$ are generalized Laguerre polynomials in [10]. In such way, we can give an orthogonal basis for AAW, even for AW^m, $m \in M$. By theses wavelets and their corresponding wavelet transforms, we define the orthogonal subspaces of $L^{\alpha 2}(\mathcal{H}_n)$ and they all form an orthogonal decomposition of $L^{\alpha 2}(\mathcal{H}_n)$ as above (details omitted).

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Department of Mathematics, National Univ. of Singapore, 10 Kent Ridge Crescent, Singapore 119260
e-mail address: jiang@math.nus.edu.sg

Department of Mathematics, Peking University, Beijing 100871, P. R. China
e-mail address: mathpu@bepc2.ihep.ac.cn