BALANCED MULTI-WAVELETS IN $\mathbb{R}^s$

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Abstract. The notion of K-balancing was introduced a few years ago as a condition for the construction of orthonormal scaling function vectors and multi-wavelets to ensure the property of preservation/annihilation of scalar-valued discrete polynomial data of order K (or degree $K-1$), when decomposed by the corresponding matrix-valued lowpass/highpass filters. While this condition is indeed precise, to the best of our knowledge only the proof for $K=1$ is known. In addition, the formulation of the K-balancing condition for $K \geq 2$ is so prohibitively difficult to satisfy that only a very few examples for $K=2$ and vector dimension $=2$ have been constructed in the open literature. The objective of this paper is to derive various characterizations of the K-balancing condition that include the polynomial preservation property as well as equivalent formulations that facilitate the development of methods for the construction purpose. These results are established in the general multivariate and bi-orthogonal settings for any $K \geq 1$.

1. Introduction

Among the key ingredients to the great success of the wavelet mathematical tools, particularly in applications to signal processing, are the properties of polynomial preservation of the scaling functions, vanishing moments of the wavelets, and their small (compact) support, which contributes to short filters. Other desirable properties include symmetry/anti-symmetry of the scaling functions and wavelets that facilitates linear-phase filtering and taking care of boundary data. While it has been commonly believed that the introduction of the notion of multi-wavelets would add significant values to the wavelet mathematical tool-box for achieving the properties stated above, it is unfortunate that the complications encountered are very difficult to overcome. The main objective of this paper is to establish a mathematical theory that contributes to the understanding of such difficulties, facilitates the development of solutions by providing various useful equivalent formulations, and extends to the higher dimensional setting for the processing of multivariate data, such as images.

To apply vector-valued scaling functions (to be called scaling function vectors later) to model or represent scalar-valued data without manufacturing vector components (which would increase the data file size), the most reasonable approach is to group the data into blocks. For example, for a data sequence $\{x_k\}$, where

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$x_k \in \mathbb{R}$, one may consider the vector-valued data

$$\mathbf{x}_k := [x_{rk}, \cdots, x_{rk+r-1}]^T, \quad k \in \mathbb{Z},$$

to be decomposed by using the scaling function vector/multi-wavelet pair

$$\Phi = [\phi_1, \cdots, \phi_r]^T, \quad \Psi = [\psi_1, \cdots, \psi_r]^T$$

of dimension $r \geq 2$, respectively. For the time being, let us consider orthonormal Φ and Ψ and let the (finite) sequences $\{P_k\}$ and $\{Q_k\}$ of $r \times r$ matrices define the refinement or two-scale relationship, namely:

$$\Phi = \sum_k P_k \Phi(2 \cdot -k), \quad \Psi = \sum_k Q_k \Phi(2 \cdot -k).$$

Then, the lowpass and highpass wavelet decomposition of $\{\mathbf{x}_k\}$ is given by

$$\left\{ \begin{array}{l} \mathbf{y}_n^L = \frac{1}{\sqrt{2}} \sum_k P_{k-2n} \mathbf{x}_k, \\ \mathbf{y}_n^H = \frac{1}{\sqrt{2}} \sum_k Q_{k-2n} \mathbf{x}_k. \end{array} \right. \tag{1.1}$$

Typical examples of such scaling function vectors and multi-wavelets and their corresponding filters $\{P_k\}$ and $\{Q_k\}$ are the GHM multi-wavelets [6, 5] and CL multi-wavelets [4], of dimension $r = 2$. We remark that the GHM and (one of the) CL scaling function vectors have polynomial reproduction order $= 2$, and hence, their corresponding multi-wavelets have vanishing moments of order $= 2$. However, when the data sequence $\{x_k\}$ is perturbed by a polynomial sequence such as

$$\widetilde{\mathbf{x}}_k = \mathbf{x}_k + \mathbf{v}_{s,k,m}$$

where

$$\mathbf{v}_{s,k,m} := [(rk+s)^m, \cdots, (rk+s+r-1)^m]^T, \quad 0 \leq s \leq r-1,$$

then the highpass output data sequence in (1.1) is different from $\{\mathbf{y}_n^H\}$, for $m = 0, 1$ and some s, for both the GHM and CL multi-wavelets (of order 2) mentioned above. More precisely, for these two examples, when $\{x_k\}$ is the zero sequence, we do not have polynomial (of degree m or order $m+1$) output $\{\mathbf{y}_n^L\}$ and zero output $\{\mathbf{y}_n^H\}$, even for $m = 0$ for the input data $\mathbf{v}_{s,k,0} = [1, \cdots, 1]^T$. This is, in general, the main issue of complications in the application of multi-wavelets to process scalar-valued data.

Two approaches have been introduced in the wavelet literature to address this important issue, one indirectly and the other directly. The indirect approach is to apply certain appropriate prefiltering to the input data sequence $\{\mathbf{x}_k\}$ as well as to the lowpass output of each wavelet decomposition level to be used as input for the next level of wavelet decomposition (see [1, 7, 19, 20]). On the other hand, the direct approach is to design Φ and Ψ, so that the decomposition algorithm (1.1) ensures polynomial output $\{\mathbf{y}_k^L\}$ of degree $K-1$ (or order K) and zero output $\{\mathbf{y}_k^H\}$, when the polynomial data sequences $\{\mathbf{x}_k\} = \{\mathbf{v}_{s,k,m}\}$, $k \in \mathbb{Z}$, for $0 \leq s \leq r-1$ and $0 \leq m \leq K-1$, are used as input sequences in (1.1), where $\{P_k\}/\{Q_k\}$ are the refinement (or two-scale) sequences corresponding to the orthonormal Φ and Ψ. More precisely, the notion of **balancing**, meaning that

$$\int_{\mathbb{R}} \phi_1(x)dx = \cdots = \int_{\mathbb{R}} \phi_r(x)dx, \tag{1.2}$$

is imposed on Φ to ensure that the constant data sequence input $\mathbf{x}_k = [1, \cdots, 1]^T$, $k \in \mathbb{Z}$, results in constant sequence output $\{\mathbf{y}_k^L\}$ and zero output $\{\mathbf{y}_k^H\}$ in (1.1). This notion was introduced in [13], and indeed (1.2) is shown to be the precise condition for this purpose. Later, in [16], the balancing condition in (1.2) is extended to the K-balancing condition

$$(1.3)\quad \int_{\mathbb{R}} \phi_1(x) x^m dx = \int_{\mathbb{R}} \phi_2(x) \left(x - \frac{1}{r}\right)^m dx = \cdots = \int_{\mathbb{R}} \phi_r(x) \left(x - \frac{r-1}{r}\right)^m dx,$$

for $m = 0, \cdots, K-1$ (where $[0, \cdots, \frac{r-1}{r}]$ will be called the "center"), to address the issue discussed above for polynomials of degree $K-1$ (or order K). In addition, a stronger condition than (1.3) was introduced in [14]. In [16], and more recently in [12], it was stated that the K-balancing condition (1.3) is the correct condition for addressing this issue in general, but to the best of our knowledge, no proof for $K \geq 2$ exists in the literature (including [16, 17, 14, 12]). In addition, it is important to point out that it has been extremely difficult to construct orthonormal Φ and Ψ to meet the K-balancing requirement (1.3) even for $K = 2$.

The main objective of this paper is to derive various characterizations of the K-balancing condition for any $K \geq 1$, that are valid for the general multivariate and bi-orthogonal settings with arbitrary centers. In addition to addressing the polynomial preservation/annihilation properites by the lowpass/highpass matrix-valued filters, the characterization result also facilitates the development of methods for construction. This paper is organized as follows. Notations and preliminary results are introduced and discussed in Section 2. The main results are stated and studied in Section 3, and the proofs of these results are given in Section 4.

2. Preliminary Results

Let $\mathbb{R}^s$, $s \geq 1$, denote the s-dimensional Euclidean space, $\mathbb{N}$ the set of positive integers, and $\mathbb{Z}_+$ the set of nonnegative integers. An s-tuple $\alpha = (\alpha_1, \cdots, \alpha_s) \in \mathbb{Z}_+^s$ is called a multi-index. The length and factorial of a multi-index α are defined, as usual, by $|\alpha| := \alpha_1 + \cdots + \alpha_s$ and $\alpha! := \alpha_1! \cdots \alpha_s!$, respectively. In comparing multi-indices, $\beta \leq \alpha$ means that $\beta_i \leq \alpha_i$, $i = 1, \cdots, s$, and $\beta < \alpha$ means $\beta \leq \alpha$ but $\beta \neq \alpha$. Also, the binomial coefficients of multi-indices $\beta \leq \alpha$ are $\binom{\alpha}{\beta} := \alpha!/(\beta!(\alpha-\beta)!)$.

Throughout this paper, we will use Greek letters $\alpha, \beta, \gamma, \cdots$ to denote multi-indices. For $\alpha = (\alpha_1, \cdots, \alpha_s) \in \mathbb{Z}_+^s$ and $\mathbf{x} = (x_1, \cdots, x_s) \in \mathbb{R}^s$, we define $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}$. The function $\mathbf{x} \mapsto \mathbf{x}^\alpha$ ($\mathbf{x} \in \mathbb{R}^s$) is called a monomial and its (total) degree is $|\alpha|$. A polynomial of degree m is a linear combination of monomials of degree $\leq m$, namely $q = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ is a polynomial of degree $\deg q := \max\{|\alpha| : c_\alpha \neq 0\}$. For $m \in \mathbb{Z}_+$, we use π_m^s to denote the linear space of all polynomials of (total) degree $\leq m$ in $\mathbb{R}^s$. Denote

$$\mathbf{1} := (1, \cdots, 1), \quad \mathbf{z} := e^{i\omega} = (e^{i\omega_1}, \cdots, e^{i\omega_s}), \quad \omega = (\omega_1, \cdots, \omega_s) \in \mathbb{R}^s.$$

The notation D_i will be used for the partial derivative with respect to the i-th coordinate. Hence, for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_s)$, D^α stands for the differential operator $D_1^{\alpha_1} \cdots D_s^{\alpha_s}$. Let A be an expansive matrix (i.e. all eigenvalues λ satisfying $|\lambda| > 1$) with integer entries. Set $d_{0,0} := 1$ and for $n \in \mathbb{N}$, let $d_{\alpha\beta} \in \mathbb{R}$, $|\alpha| = |\beta| = n$, denote the coefficients of the polynomial

$$(A\mathbf{x})^\alpha = \sum_{|\beta|=n} d_{\alpha\beta} \mathbf{x}^\beta.$$

Then for any sufficiently smooth function f in $\mathbb{R}^s$, we have (see e.g. [11]), for any $|\alpha| = n$,

$$D^\alpha\big(f(A^T\mathbf{x})\big) = \sum_{|\beta|=n} d_{\alpha\beta} D^\beta f(A^T\mathbf{x}). \tag{2.1}$$

In this paper, we also need the notation for a family of sequences of r-dimensional row-vectors $\{\mathbf{v}_\alpha(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^s}$, $r \ge 1$, generated by some constant row-vectors $\mathbf{v}_\alpha$, defined by

$$\mathbf{v}_\alpha(0) := \mathbf{v}_\alpha \quad \text{and} \quad \mathbf{v}_\alpha(\mathbf{k}) := \textstyle\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \mathbf{k}^{\alpha-\beta}\mathbf{v}_\beta, \quad \mathbf{k}\in\mathbb{Z}^s\backslash\{0\}. \tag{2.2}$$

Observe that $\mathbf{v}_0(\mathbf{k}) = \mathbf{v}_0$, $\mathbf{k}\in\mathbb{Z}^s$.

2.1. **Polynomial preservation.** Let ϕ_ℓ, $\ell = 1,\cdots,r$, be compactly supported distributions in $\mathbb{R}^s$, and $\Phi := [\phi_1,\cdots,\phi_r]^T$. We say that Φ has the property of **polynomial preservation** of order m (or $\Phi \in \mathrm{PP}_m$ for short), if there exists a (finite) linear combination φ of integer shifts of $\phi_1,\cdots,\phi_r$, such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^s} q(\mathbf{k})\varphi(\cdot-\mathbf{k}) = q, \quad q\in\pi^s_{m-1},$$

holds in the distribution sense (i.e. equality holds upon taking inner product with any test function). It follows from the Poisson summation formula that the above formulation is equivalent to the (modified) Strang-Fix conditions:

$$D^\alpha\hat{\varphi}(2\pi\mathbf{k}) = \delta_{0,\alpha}\delta_{0,\mathbf{k}} \qquad |\alpha|<m,\ \mathbf{k}\in\mathbb{Z}^s. \tag{2.3}$$

It is also easily seen that Φ satisfies (2.3), if and only if there exists an r-dimensional row-vector $\mathbf{t}(\omega) = \sum_{|\alpha|<m}\mathbf{t}_\alpha e^{-i\alpha\omega}$ of trigonometric polynomials such that

$$D^\alpha(\mathbf{t}\hat{\Phi})(2\pi\mathbf{k}) = \delta_{0,\alpha}\delta_{0,\mathbf{k}}, \qquad |\alpha|<m,\ \mathbf{k}\in\mathbb{Z}^s, \tag{2.4}$$

which, in turn, is equivalent to

$$\mathbf{x}^\alpha = \sum_{\mathbf{k}\in\mathbb{Z}^s}\Big\{\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\mathbf{k}^{\alpha-\beta}\mathbf{y}_\beta\Big\}\Phi(\mathbf{x}-\mathbf{k}),\ |\alpha|<m, \tag{2.5}$$

in the distribution sense, with

$$\mathbf{y}_\alpha := (-iD)^\alpha\mathbf{t}(0), \quad |\alpha|<m. \tag{2.6}$$

For $r=1$, this formula was discussed in [2; Chap. 8], and for the extension to arbitrary $r\ge 1$, the reader is referred to [8].

2.2. **Refinable function vectors.** Let $\{P_\mathbf{k}\}$ be a finite sequence of square matrices of dimension r, and consider its corresponding matrix-valued Laurent polynomial symbol

$$P(\mathbf{z}) := \frac{1}{a}\sum_{\mathbf{k}\in\mathbb{Z}^s} P_\mathbf{k}\mathbf{z}^\mathbf{k}, \quad a := |\det(A)|, \tag{2.7}$$

where A is an expansive matrix with integer entries. Then the matrix $P(\mathbf{1})$ is said to satisfy **Condition E** (or $P(\mathbf{1})\in E$ for short, see [18]), if the value 1 is a simple eigenvalue of the matrix $P(\mathbf{1})$ and all other eigenvalues of $P(\mathbf{1})$ lie in the unit disk $|z|<1$ of the complex plane. Under the assumption that $P(\mathbf{1})\in E$, the infinite product

$$\prod_{j=1}^{\infty} P(e^{-i(A^T)^{-j}\omega})$$

converges uniformly on every compact subset of $\mathbb{R}^s$. Now, let $\mathbf{v}_0$ be a right eigenvector of $P(\mathbf{1})$ corresponding to the eigenvalue 1, i.e.

$$P(\mathbf{1})\mathbf{v}_0 = \mathbf{v}_0,$$

such that $|\mathbf{v}_0| = 1$, and define $\Phi = [\phi_1, \cdots, \phi_r]$ by the inverse Fourier transform of

$$\widehat{\Phi}(\omega) = \prod_{j=1}^{\infty} P(e^{-i(A^T)^{-j}\omega})\mathbf{v}_0. \tag{2.8}$$

Then $\phi_1, \cdots, \phi_r$ are compactly supported distributions, and Φ satisfies the refinement equation

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^s} P_{\mathbf{k}}\Phi(A\mathbf{x} - \mathbf{k}). \tag{2.9}$$

We call the sequence $\{P_{\mathbf{k}}\}$ the mask, and its corresponding symbol $P(\mathbf{z})$, the two-scale symbol of Φ. We also call Φ a normalized solution of the refinement equation (2.9).

On the other hand, for a compactly supported function vector Φ satisfying (2.9) for some finite sequence $\{P_{\mathbf{k}}\}$, if $\Phi \in (L^2)^r := \left(L^2(\mathbb{R}^s)\right)^r$ is L^2-stable, then the symbol $P(\mathbf{z})$ defined by (2.7) satisfies the condition $P(\mathbf{1}) \in E$ (see e.g. [11]). Therefore, in the following, we assume that $P(\mathbf{1}) \in E$ and that Φ is the normalized solution defined by (2.8). (See the survey paper [3] on the conditions under which the compactly supported distribution vector Φ defined by (2.8) is a function vector in $(L^2)^r$.)

2.3. **Sum rules.** For the same expansive matrix A with $|\det A| =: a$, let ω_h, with $\omega_0 = 0$ and $0 \le h < a$, be the representors of $\mathbb{Z}^s/A^T\mathbb{Z}^s$. We say that P satisfies the **sum rule** of order m (or $P \in \mathrm{SR}_m$ for short), if there exist constant vectors $\widetilde{\mathbf{y}}^\alpha$, with $\widetilde{\mathbf{y}}_0 \ne \mathbf{0}$, such that

$$\sum_{\beta\le\alpha}(-1)^{|\beta|}\binom{\alpha}{\beta}\widetilde{\mathbf{y}}_{\alpha-\beta}J_{\beta,\gamma_h} = \sum_{|\tau|=|\alpha|}\left[\sum_{|\beta|=|\alpha|}(A^{-1}\beta)^\beta t^\alpha_{\beta\tau}\right]\frac{\widetilde{\mathbf{y}}_\tau}{\tau!}, \tag{2.10}$$

for all $|\alpha| < m, \gamma_h \in \mathbb{Z}^s/A^T\mathbb{Z}^s$, where

$$J_{\beta,\gamma_h} := \sum_{\mathbf{k}}(\mathbf{k} + A^{-1}\gamma_h)^\beta P_{A\mathbf{k}+\gamma_h},$$

and $\left[t^\alpha_{\beta\tau}\right]$ is the inverse of the matrix $\left[\frac{\beta^\tau}{\tau!}\right]_{|\tau|,|\beta|=|\alpha|}$.

Now, consider an r-vector $\widetilde{\mathbf{t}}(\omega) = \sum_{\mathbf{k}\in\mathbb{Z}^s}\widetilde{\mathbf{t}}_{\mathbf{k}}e^{-i\mathbf{k}\omega}$ of trigonometric polynomials that gives $\widetilde{\mathbf{y}}_\alpha = (-iD)^\alpha\widetilde{\mathbf{t}}(0)$ for all $|\alpha| < m$. It was shown in [8] that (2.10) is equivalent to

$$D^\alpha\left(\widetilde{\mathbf{t}}(A^T\omega)P(e^{-i\omega})\right)|_{\omega=2\pi A^{-T}\omega_h} = \delta_{h,0}D^\alpha\widetilde{\mathbf{t}}(0), \quad |\alpha| < m, \tag{2.11}$$

for all $0 \le h < a$.

The following result shows that if $P \in \mathrm{SR}_m$, then the normalized solution Φ of (2.9) has the property of polynomial preservation of order m.

Theorem 2.1. *Let $P \in SR_m$ for some integer $m \ge 1$ and Φ be a compactly supported normalized solution of (2.9). Then Φ satisfies (2.5) with $\mathbf{y}_\alpha = \widetilde{\mathbf{y}}_\alpha$; or equivalently, $\Phi \in PP_m$. Conversely, if Φ is some compactly supported refinable distribution vector with finite mask $\{P_{\mathbf{k}}\}$ that satisfies PP_m such that the matrices*

$\sum_{\mathbf{k}}(\widehat{\Phi}\widehat{\Phi}^*)(2\mathbf{k}\pi+2\pi A^{-T}\omega_h)$ *are non-singular for all* $0\le h<a$*, then* $P\in SR_m$ *with* $\widetilde{\mathbf{y}}_\alpha=\mathbf{y}_\alpha$.

See the survey paper [8] and references therein for more details.

2.4. **Bi-orthogonal duals.** Let $\Phi=[\phi_1,\cdots,\phi_r]^T\in(L^2)^r$ and $\widetilde{\Phi}=[\widetilde{\phi}_1,\cdots,\widetilde{\phi}_r]^T\in(L^2)^r$ be real-valued compactly supported refinable function vectors with dilation matrix A, and finite masks $\{P_{\mathbf{k}}\}$ and $\{G_{\mathbf{k}}\}$, respectively. We say that Φ and $\widetilde{\Phi}$ are bi-orthogonal duals of each other, if $\{\Phi(\mathbf{x}-\mathbf{k})\}$ and $\{\widetilde{\Phi}(\mathbf{x}-\mathbf{k})\}$ are bi-orthogonal, meaning that

$$\int_{\mathbb{R}^s}\Phi(\mathbf{x})\widetilde{\Phi}(\mathbf{x}-\mathbf{k})^T\,d\mathbf{x}=\delta_{0,\mathbf{k}}I_r,\quad \mathbf{k}\in\mathbb{Z}^s. \tag{2.12}$$

Hence, if $\widetilde{\Phi}=\Phi$, Φ is considered to be orthonormal.

A necessary condition for Φ and $\widetilde{\Phi}$ to be bi-orthogonal duals is that their two-scale symbols satisfy

$$\sum_{n=0}^{a-1}P(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})G(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^*=I_r,\quad \mathbf{z}=e^{-i\omega},\ \omega\in\mathbb{R}^s. \tag{2.13}$$

Conversely, under certain mild conditions, this condition is also sufficient, see e.g. [3].

Suppose $Q^h(\mathbf{z})$ and $H^h(\mathbf{z}), 1\le h<a$, are matrix-valued Laurent polynomials that satisfy

$$\begin{aligned}
&\sum_{0\le n<a}P(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})H^h(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^*=0,\quad 1\le h<a,\\
&\sum_{0\le n<a}Q^h(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})G(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^*=0,\quad 1\le h<a,\\
&\sum_{0\le n<a}Q^h(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})H^\ell(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^*=\delta_{h,\ell}I_r,\quad 1\le h,\ \ell<a,
\end{aligned} \tag{2.14}$$

for $\omega\in\mathbb{R}^s$, where $\mathbf{z}=e^{-i\omega}$. Let $\Psi^h,\widetilde{\Psi}^h$ be the vector-valued functions vectors defined by

$$\widehat{\Psi}^h(A^T\omega)=Q^h(\mathbf{z})\hat{\Phi}(\omega),\quad \hat{\widetilde{\Psi}}^h(A^T\omega)=H^h(\mathbf{z})\hat{\widetilde{\Phi}}(\omega). \tag{2.15}$$

Then if Φ, $\widetilde{\Phi}$ are bi-orthogonal duals, we have

$$\begin{aligned}
&\int_{\mathbb{R}^s}\Phi(\mathbf{x})\widetilde{\Psi}^h(\mathbf{x}-\mathbf{k})^T\,d\mathbf{x}=\mathbf{0},\quad \int_{\mathbb{R}^s}\Psi^h(\mathbf{x})\widetilde{\Phi}(\mathbf{x}-\mathbf{k})^T\,d\mathbf{x}=\mathbf{0},\\
&\int_{\mathbb{R}^s}\Psi^h(\mathbf{x})\widetilde{\Psi}^\ell(\mathbf{x}-\mathbf{k})^T\,d\mathbf{x}=\delta_{0,\mathbf{k}}\delta_{h,\ell}I_r,\quad \mathbf{k}\in\mathbb{Z}^s,1\le h,\ell<a,
\end{aligned} \tag{2.16}$$

and $\{a^{-\frac{n}{2}}\Psi^h(A^{-n}\mathbf{x}-\mathbf{k}),a^{-\frac{n}{2}}\widetilde{\Psi}^h(A^{-n}\mathbf{x}-\mathbf{k}),1\le h<a,n\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^s\}$ is a bi-orthogonal system.

In the following, let Φ and $\widetilde{\Phi}$ be real-valued compactly supported bi-orthogonal dual refinable function vectors in $(L^2)^r$ with dilation matrix A and finite masks $\{P_{\mathbf{k}}\}$ and $\{G_{\mathbf{k}}\}$, respectively, and let $Q^h(\mathbf{z})$ and $H^h(\mathbf{z}), 1\le h<a$, be matrix-valued Laurent polynomials satisfying (2.14). Also, let $\Psi^h,\widetilde{\Psi}^h$ be the function vectors defined by (2.15). Since the bi-orthogonality of Φ and $\widetilde{\Phi}$ assures their stability as well, it follows that both $P(\mathbf{1})$ and $G(\mathbf{1})$ satisfy Condition E.

As usual, we use $\{G, H^h, 1 \le h < a\}$ and $\{P, Q^h, 1 \le h < a\}$ as filters for the purpose of wavelet decomposition and reconstruction respectively. More precisely, the decomposition algorithm is given by

$$(2.17)\qquad \mathbf{c}_{\mathbf{n}}^{(j-1)} = \frac{1}{\sqrt{a}} \sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{n}} \mathbf{c}_{\mathbf{k}}^{(j)}, \quad \mathbf{d}_{\mathbf{n},h}^{(j-1)} = \frac{1}{\sqrt{a}} \sum_{\mathbf{k}} H^h_{\mathbf{k}-A\mathbf{n}} \mathbf{c}_{\mathbf{k}}^{(j)}, \quad \mathbf{n} \in \mathbb{Z}^s,$$

and the reconstruction algorithm is given by

$$(2.18)\qquad \mathbf{c}_{\mathbf{n}}^{(j)} = \frac{1}{\sqrt{a}} \sum_{\mathbf{k}} P^T_{\mathbf{n}-A\mathbf{k}} \mathbf{c}_{\mathbf{k}}^{(j-1)} + \frac{1}{\sqrt{a}} \sum_{1 \le h < a} \sum_{\mathbf{k}} (Q^h_{\mathbf{n}-A\mathbf{k}})^T \mathbf{d}_{\mathbf{k},h}^{(j-1)}, \quad \mathbf{n} \in \mathbb{Z}^s.$$

3. Balanced Refinable Function Vectors

The notion of balancing is generalized to the multivariate bi-orthogonal setting in this section. With balanced bi-orthogonal duals, discrete polynomials are preserved by the low-pass filters and annihilated by the high-pass filters of the corresponding multi-wavelets. This last statement on polynomial annihilation was stated in both [16] and [12] in the one-variable setting but no proof exists in the published literature. In this paper, we give a proof of the general multivariate formulation, which includes the univariate setting as claimed in [16] and [12]. Several important characterization results for these two properties are also discussed in this section.

Let $\mathbf{a}_1, \cdots, \mathbf{a}_r \in \mathbb{R}^s$. A compactly supported vector-valued function $F = [f_1, \cdots, f_r]^T \in (L^2)^r$ is said to be K-**balanced**, $K \ge 1$, relative to the r-tuple $[\mathbf{a}_1, \cdots, \mathbf{a}_r]$, if

$$(3.1)\qquad \int_{\mathbb{R}^s} f_l(\mathbf{x})(\mathbf{x} - \mathbf{a}_l)^\alpha d\mathbf{x} = \int_{\mathbb{R}^s} f_i(\mathbf{x})(\mathbf{x} - \mathbf{a}_i)^\alpha d\mathbf{x}, \quad 1 \le l,\ i \le r,\ |\alpha| < K.$$

We will call $[\mathbf{a}_1, \cdots, \mathbf{a}_r]$ the center of the K-balanced function vector F. This is an extension of the notion of K-balancing introduced for the orthonormal univariate setting by Lebrun and Vetterli [13] for $K = 1$ and Selesnick [16] for $K \ge 1$ with r-tuple center $a_1 = 0,\ a_2 = \frac{1}{r},\ \cdots,\ a_r = (r-1)/r$. For the univariate and orthonormal setting, Selesnick [16] gave a complete characterization of compactly supported K-balanced orthonormal refinable function vector Φ in terms of the roots of some polynomials associated with the two-scale symbol $P(z)$ of Φ.

In this section we will give a necessary and sufficient condition for the validity of (3.1) in terms of an intimate relation between these centers and the vectors $\widetilde{\mathbf{y}}_\alpha$ in (2.10), or equivalently $\mathbf{y}_\alpha$ in (2.5), where $K \le m$. We will also show that K-balancing of $\widetilde{\Phi}$ is equivalent to discrete polynomial preservation of total degree $K-1$ by its mask $\{G_{\mathbf{k}}\}$, and that K-balancing of $\widetilde{\Phi}$ implies discrete polynomial annihilation of total degree $K-1$ by the high-pass filters $\{H^h_{\mathbf{k}}\}$ corresponding to the low-pass filter $\{G_{\mathbf{k}}\}$.

Assume that Φ and $\widetilde{\Phi}$ are compactly supported bi-orthogonal dual refinable function vectors in $(L^2)^r$ with dilation matrix A and finite masks $\{P_{\mathbf{k}}\}$ and $\{G_{\mathbf{k}}\}$, respectively. Suppose Φ satisfies the sum rule of order m, and that $\mathbf{y}_\alpha, |\alpha| < m$, with $\mathbf{y}_0 \ne 0$, are the vectors that satisfy (2.10). In the following, we assume, without loss of generality, that $y_{0,1} \ne 0$. For $\alpha, \beta \in \mathbb{Z}^s_+, |\alpha| < m, |\beta| < m$, we define the coefficients c^α_β as follows. For $\beta \not\le \alpha$, let $c^\alpha_\beta = 0$, and for $\beta \le \alpha$, let c^α_β be the coefficients in the expansion

$$\sum_{\gamma\le\beta\le\alpha} c^\alpha_\beta \binom{\beta}{\gamma} y_{\beta-\gamma,1} = \binom{\alpha}{\gamma}\mathbf{a}_1^{\alpha-\gamma}, \quad \gamma\in\mathbb{Z}^s_+,\ \gamma\le\alpha. \tag{3.2}$$

Under the condition $y_{0,1}\neq 0$, we see that c^α_β are uniquely determined by (3.2). In particular, we have

$$c^\alpha_\alpha = \frac{1}{y_{0,1}}, \quad |\alpha|<m.$$

For $\alpha,\beta\in\mathbb{Z}^s_+, |\alpha|<m, |\beta|<m$, we define the coefficients L^α_β as follows. For $|\beta|>|\alpha|$, let $L^\alpha_\beta=0$, and for $|\beta|\le|\alpha|$, let L^α_β be the coefficients defined by

$$\sum_{\gamma\le\beta, |\beta|\le|\alpha|} L^\alpha_\beta c^\beta_\gamma = \sum_{|\beta|=|\gamma|,\beta\le\alpha} c^\alpha_\beta d_{\beta\gamma}, \quad \gamma\in\mathbb{Z}^s_+,\ |\gamma|\le|\alpha|<m. \tag{3.3}$$

One can easily verify that L^α_β are uniquely determined by (3.3). In particular, $L^0_0=1$, and

$$L^\alpha_\gamma c^\gamma_\gamma = c^\alpha_\alpha d_{\alpha\gamma}, \quad |\gamma|=|\alpha|<m.$$

Since $c^\gamma_\gamma = c^\alpha_\alpha = 1/y_{0,1}$, we have

$$L^\alpha_\gamma = d_{\alpha\gamma}, \quad |\gamma|=|\alpha|<m.$$

Therefore, since the matrix $[d_{\alpha\gamma}]_{|\alpha|=n,|\gamma|=n}$ is non-singular, $[L^\alpha_\gamma]_{|\alpha|=n,|\gamma|=n}$ is also a non-singular matrix. Furthermore, it can be easily shown that (3.3) is equivalent to

$$\sum_{\beta\le\alpha} c^\alpha_\beta \sum_{|\gamma|=|\beta|} d_{\beta\gamma}\mathbf{x}^\gamma = \sum_{|\beta|\le|\alpha|} L^\alpha_\beta \sum_{\gamma\le\beta} c^\beta_\gamma \mathbf{x}^\gamma, \quad \alpha\in\mathbb{Z}^s_+,\ |\alpha|<m,\ \mathbf{x}\in\mathbb{R}^s. \tag{3.4}$$

Observe that $\Big[L^\alpha_\beta\Big]_{|\alpha|<m,|\beta|<m} = \Big[[L^\alpha_\beta]_{|\alpha|=i,|\beta|=n}\Big]_{0\le i<m, 0\le n<m}$ is a lower-triangular block matrix with diagonal blocks $[L^\alpha_\beta]_{|\alpha|=n,|\beta|=n}$. Therefore, $\Big[L^\alpha_\beta\Big]_{|\alpha|<m,|\beta|<m}$ is non-singular as well. Let l^α_β be the entries of the inverse matrix of $\Big[L^\alpha_\beta\Big]_{|\alpha|<m,|\beta|<m}$, namely

$$[l^\alpha_\beta]_{|\alpha|<m,|\beta|<m} := \Big[[L^\alpha_\beta]_{|\alpha|<m,|\beta|<m}\Big]^{-1}. \tag{3.5}$$

Then we have, for $|\beta|>|\alpha|$, $l^\alpha_\beta=0$ and that

$$\sum_{|\gamma|\le|\beta|\le|\alpha|} l^\alpha_\beta L^\alpha_\gamma = \delta_{\alpha\gamma}, \quad |\gamma|\le|\alpha|<m. \tag{3.6}$$

The main result of this paper is the following characterization of K-balancing for the dual $\widetilde{\Phi}$, where $1\le K\le m$.

Theorem 3.1. *Suppose Φ and $\widetilde{\Phi}$ are compactly supported bi-orthogonal dual refinable vector-valued functions in $(L^2)^r$ with dilation matrix A and finite masks $\{P_\mathbf{k}\}$ and $\{G_\mathbf{k}\}$, respectively, and that $Q^h(\mathbf{z})$ and $H^h(\mathbf{z}), 1\le h<a$, are the matrix-valued Laurent polynomials satisfying (2.14). Let $G(\mathbf{z})$ satisfy the sum rule of order m with vectors $\mathbf{y}_\alpha, |\alpha|<m$, in place of $\widetilde{\mathbf{y}}_\alpha$ in the definition (2.10). Then for $K\le m$, the following statements are equivalent with $y_{\alpha,1}$, c^α_β, L^α_β and l^α_β given by (3.2), (3.3) and (3.5).*

(i) *$\widetilde{\Phi}$ is K-balanced relative to $[\mathbf{a}_1,\cdots,\mathbf{a}_r]\subset\mathbb{R}^s$.*

(ii) *The vectors* $\mathbf{y}_\alpha = [y_{\alpha,1}, \cdots, y_{\alpha,r}]$ *satisfy*

$$y_{\alpha,l} = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\mathbf{a}_l - \mathbf{a}_1)^\beta y_{\alpha-\beta,1}, \quad 2 \le l \le r, \quad |\alpha| < K. \tag{3.7}$$

(iii) *For* $|\alpha| < K$,

$$\sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] \Phi(\mathbf{x}-\mathbf{k}) = \sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{x}^\beta, \quad \mathbf{x} \in \mathbb{R}^s. \tag{3.8}$$

(iv) *For* $|\alpha| < K$, $\mathbf{j} \in \mathbb{Z}^s$,

$$\sum_{\mathbf{k}} P_{\mathbf{j}-A\mathbf{k}}^T [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha]^T = \sum_{|\beta|\le|\alpha|} l_\beta^\alpha [(\mathbf{j}+\mathbf{a}_1)^\beta, \cdots, (\mathbf{j}+\mathbf{a}_r)^\beta]^T. \tag{3.9}$$

(v) *For* $|\alpha| < K$, $\mathbf{j} \in \mathbb{Z}^s$,

$$\sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha]^T = a \sum_{|\beta|\le|\alpha|} L_\beta^\alpha [(\mathbf{j}+\mathbf{a}_1)^\beta, \cdots, (\mathbf{j}+\mathbf{a}_r)^\beta]^T. \tag{3.10}$$

Furthermore, if $\widetilde{\Phi}$ *is* K*-balanced, then, for* $1 \le h < a, |\alpha| < K$, $\mathbf{j} \in \mathbb{Z}^s$,

$$\sum_{\mathbf{k}} H_{\mathbf{k}-A\mathbf{j}}^h [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha]^T = [0, \cdots, 0]^T. \tag{3.11}$$

Theorem 3.1 tells us that $\widetilde{\Phi}$ is K-balanced if and only if $\mathbf{y}_\alpha$ can be chosen so that both (2.10) and (3.1) are satisfied for all $|\alpha| < K$. Theorem 3.1 also enables us to decide on the possible balanced order of the bi-orthogonal dual scaling function vectors for a given refinable function vector, and facilities the development of methods for constructing K-balanced bi-orthogonal multi-wavelets as well as the masks of the primal refinable function vectors with K-balanced bi-orthogonal duals for $K \ge 2$. Furthermore, Theorem 3.1 tells us that K-balancing of $\widetilde{\Phi}$ is equivalent to the preservation of π_{K-1}^s by G, and that K-balancing of $\widetilde{\Phi}$ implies the annihilation of π_{K-1}^s by H^h.

Remark 3.2. *Notice that if* $\widetilde{\Phi}$ *satisfies (3.1), then for any* $\mathbf{c}_0 \in \mathbb{R}^s$, *we have*

$$\int_{\mathbb{R}^s} \widetilde{\phi}_l(\mathbf{x})(\mathbf{x}-\mathbf{a}_l-\mathbf{c}_0)^\alpha d\mathbf{x} = \int_{\mathbb{R}^s} \widetilde{\phi}_i(\mathbf{x})(\mathbf{x}-\mathbf{a}_i-\mathbf{c}_0)^\alpha d\mathbf{x}, \quad 1 \le l, i \le r, \ |\alpha| < K.$$

Hence, if $\widetilde{\Phi}$ *is* K*-balanced relative to* $[\mathbf{a}_1, \cdots, \mathbf{a}_r]$, *then for any arbitrary* $\mathbf{c}_0 \in \mathbb{R}^s$, $\widetilde{\Phi}$ *is* K*-balanced relative to* $[\mathbf{a}_1+\mathbf{c}_0, \cdots, \mathbf{a}_r+\mathbf{c}_0]$. *Let* $c_{\beta,\mathbf{c}_0}^\alpha$ *be defined by (3.2) with* $\mathbf{a}_1$ *replaced by* $\mathbf{a}_1+\mathbf{c}_0$. *Define* $L_{\beta,\mathbf{c}_0}^\alpha, l_{\beta,\mathbf{c}_0}^\alpha$ *by (3.3) and (3.5) respectively, with* c_β^α *replaced by* $c_{\beta,\mathbf{c}_0}^\alpha$. *Then we have a result similar to Theorem 3.1 with* $\mathbf{a}_\ell, 1 \le \ell \le r$, $c_\beta^\alpha, L_\beta^\alpha$ *and* l_β^α *in (iii)-(v) and (3.11) replaced by* $\mathbf{a}_\ell + c_0$, $c_{\beta,\mathbf{c}_0}^\alpha, L_{\beta,\mathbf{c}_0}^\alpha$ *and* $l_{\beta,\mathbf{c}_0}^\alpha$, *respectively. In particular,* $\widetilde{\Phi}$ *is* K*-balanced relative to* $[\mathbf{a}_1, \cdots, \mathbf{a}_r]$ *if and only if, for* $|\alpha| < K$, $\mathbf{j} \in \mathbb{Z}^s$, *and* $\mathbf{c}_0 \in \mathbb{R}^s$,

$$\begin{aligned} &\sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} [(\mathbf{k}+\mathbf{a}_1+\mathbf{c}_0)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r+\mathbf{c}_0)^\alpha]^T \\ &\qquad = a \sum_{|\beta|\le|\alpha|} L_{\beta,\mathbf{c}_0}^\alpha [(\mathbf{j}+\mathbf{a}_1+\mathbf{c}_0)^\beta, \cdots, (\mathbf{j}+\mathbf{a}_r+\mathbf{c}_0)^\beta]^T; \end{aligned}$$

and if $\widetilde{\Phi}$ is K-balanced, then

$$\sum_{\mathbf{k}} H_{\mathbf{k}-A\mathbf{j}}[(\mathbf{k}+\mathbf{a}_1+\mathbf{c}_0)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r+\mathbf{c}_0)^\alpha]^T = [0, \cdots, 0]^T, \ \mathbf{c}_0 \in \mathbb{R}^s,$$

for all $|\alpha| < K$, $\mathbf{j} \in \mathbb{Z}^s$.

For the univariate setting, with $s = 1$ and $A = [a]$, the coefficients c^α_β and L^α_β, $\beta \le \alpha$, are computed by simply inverting triangular matrices, as follows:

$$\begin{bmatrix} \binom{0}{0}y_{0,1} & \binom{1}{0}y_{1,1} & \binom{2}{0}y_{2,1} & \cdots & \binom{\alpha}{0}y_{\alpha,1} \\ 0 & \binom{1}{1}y_{0,1} & \binom{2}{1}y_{1,1} & \cdots & \binom{\alpha}{1}y_{\alpha-1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \binom{\alpha}{\alpha}y_{0,1} \end{bmatrix} \begin{bmatrix} c^\alpha_0 \\ c^\alpha_1 \\ c^\alpha_2 \\ \vdots \\ c^\alpha_\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad 0 \le \alpha < m;$$

and

$$\begin{bmatrix} c^0_0 & c^1_0 & c^2_0 & \cdots & c^\alpha_0 \\ 0 & c^1_1 & c^2_1 & \cdots & c^\alpha_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c^\alpha_\alpha \end{bmatrix} \begin{bmatrix} L^\alpha_0 \\ L^\alpha_1 \\ L^\alpha_2 \\ \vdots \\ L^\alpha_\alpha \end{bmatrix} = a \begin{bmatrix} c^\alpha_0 \\ c^\alpha_1 \\ c^\alpha_2 \\ \vdots \\ c^\alpha_\alpha \end{bmatrix}, \quad 0 \le \alpha < m.$$

As a consequence of the above theorem, we have the following univariate version with r-tuple center $[0, \frac{1}{r}, \cdots, \frac{r-1}{r}]$.

Corollary 3.3. *Suppose Φ and $\widetilde{\Phi}$ are compactly supported bi-orthogonal dual refinable vector-valued functions in $(L^2(\mathbb{R}))^r$ with integer dilation $a > 1$, and finite masks $\{P_\mathbf{k}\}$ and $\{G_\mathbf{k}\}$, respectively, and that $Q(\mathbf{z})$ and $H(\mathbf{z})$ are the matrix-valued Laurent polynomials satisfying (2.14). Let $G(\mathbf{z})$ satisfy the sum rule of order m with vectors $\mathbf{y}_\alpha, 0 \le \alpha < m$. Then for $1 \le K \le m$, the following statements are equivalent.*

(i) *$\widetilde{\Phi}$ is K-balanced relative to $[0, \frac{1}{r} \cdots, \frac{r-1}{r}]$.*

(ii) $y_{\alpha,l} = \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta}(\frac{l-1}{r})^\beta y_{\alpha-\beta,1}, \quad 1 \le l \le r,\ 0 \le \alpha < K.$

(iii) *For all $0 \le \alpha < K, x \in \mathbb{R}$,*

$$\sum_k [(rk)^\alpha, (rk+1)^\alpha, \cdots, (rk+r-1)^\alpha]\Phi(x-k) = r^\alpha \sum_{0\le\beta\le\alpha} c^\alpha_\beta x^\beta.$$

(iv) *For all $0 \le \alpha < K, j \in \mathbb{Z}$,*

$$\sum_k P^T_{j-ak}[(rk)^\alpha, (rk+1)^\alpha, \cdots, (rk+r-1)^\alpha]^T$$
$$= r^\alpha \sum_{0\le\beta\le\alpha} l^\alpha_\beta r^{-\beta}[(rj)^\beta, (rj+1)^\beta, \cdots, (rj+r-1)^\beta]^T.$$

(v) *For all $0 \le \alpha < K, j \in \mathbb{Z}^s$,*

$$\sum_k G_{k-aj}[(rk)^\alpha, (rk+1)^\alpha, \cdots, (rk+r-1)^\alpha]^T$$
$$= ar^\alpha \sum_{0\le\beta\le\alpha} L^\alpha_\beta r^{-\beta}[(rj)^\beta, (rj+1)^\beta, \cdots, (rj+r-1)^\beta]^T.$$

Furthermore if $\widetilde{\Phi}$ is K-balanced, then

$$\sum_k H_{k-aj}[(rk)^\alpha, (rk+1)^\alpha, \cdots, (rk+r-1)^\alpha]^T = [0, 0, \cdots, 0]^T, \quad 0 \le \alpha < K, \; j \in \mathbb{Z}.$$

The proof of Theorem 3.1 will be given in the next section. We remark that the proof of (ii) $\Leftrightarrow$ (iii) for this univariate setting in Corollary 3.3 was given in [15] for orthonormal multi-wavelets.

Before going on to prove our results in this paper, let us first give a relatively simple example.

Example 3.4. Let $\triangle^2$ denote the four-directional mesh with grid lines given by $x = j, y = k, x + y = \ell$, and $x - y = m$, where $j, k, \ell, m \in \mathbb{Z}$. Let ϕ_1 denote the bivariate piecewise linear hat function with $\phi_1(0,0) = 1$ and support given by the square with vertices $(1,0), (0,1), (-1,0)$ and $(0,-1)$, and ϕ_2 be the bivariate piecewise linear hat function with $\phi_2(\frac{1}{2}, \frac{1}{2}) = 1$ and support given by the square $[0,1] \times [0,1]$. Then $\{\phi_\ell(\cdot - \mathbf{k}) : \ell = 1, 2, \mathbf{k} \in \mathbb{Z}^2\}$ is a basis of the space $S_1^0(\triangle^2)$ of continuous bivariate piecewise linear splines on $\triangle^2$, and the 2-vector $\Phi = [\phi_1, \phi_2]^T$ is a refinable function vector with respect to the quincunx dilation matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Furthermore, the two-scale symbol of Φ is given by

$$P(\mathbf{z}) = \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{2}(1 + z_1^{-1})(1 + z_2^{-1}) \\ z_1 & 0 \end{bmatrix},$$

(see [3] for details.) One can also easily verify that P satisfies the sum rule of order 2 with

$$\mathbf{y}_{0,0} = [1, 1], \quad \mathbf{y}_{0,1} = [0, \frac{1}{2}], \quad \mathbf{y}_{1,0} = [0, \frac{1}{2}].$$

By Theorem 3.1, it is possible to construct a 2-balanced bi-orthogonal dual $\widetilde{\Phi}$ with balanced centers $\mathbf{a}_1 = \mathbf{0}, \mathbf{a}_2 = (\frac{1}{2}, \frac{1}{2})$. Indeed we can construct such $\widetilde{\Phi}$ in the Sobolev space $W^{1.5419}(\mathbb{R}^2)$ with mask $\{G_\mathbf{k}\}$ given by

$$G_{-2,-2} = \begin{bmatrix} 0.01575013951780 & -0.03150027903560 \\ -0.01892905803343 & 0.03785811606686 \end{bmatrix},$$

$$G_{-2,-1} = \begin{bmatrix} 0 & -0.15843358760708 \\ 0 & 0.02993505709487 \end{bmatrix},$$

$$G_{-2,0} = \begin{bmatrix} 0.06460052596114 & 0.02923253568480 \\ 0.00014059892203 & -0.03021625493894 \end{bmatrix},$$

$$G_{-2,1} = \begin{bmatrix} 0 & -0.05028090150119 \\ 0 & 0.02405271011087 \end{bmatrix},$$

$$G_{-2,2} = \begin{bmatrix} 0.04922428668434 & -0.04816767186750 \\ -0.02414572239367 & 0.02423873467647 \end{bmatrix},$$

$$G_{-1,-2} = \begin{bmatrix} 0 & 0.01190628331457 \\ 0 & 0.05913115448101 \end{bmatrix},$$

$$G_{-1,-1} = \begin{bmatrix} -0.02575399262375 & 0.22953556857562 \\ 0.02293468166412 & -0.17279369097098 \end{bmatrix},$$

$$G_{-1,0} = \begin{bmatrix} 0 & 0.28620535439550 \\ 0 & 0.09444462118874 \end{bmatrix},$$

$$G_{-1,1} = \begin{bmatrix} -0.16550273213986 & 0.06584847570061 \\ -0.06926045101368 & 0.05023982566669 \end{bmatrix},$$
$$G_{-1,2} = \begin{bmatrix} 0 & 0.04816767186750 \\ 0 & -0.02423873467647 \end{bmatrix},$$
$$G_{0,-2} = \begin{bmatrix} 0.10248315697941 & -0.21687259727340 \\ -0.10818354117373 & 0.15723592786645 \end{bmatrix},$$
$$G_{0,-1} = \begin{bmatrix} 0 & 0.35917518862102 \\ 0 & 0.26814929951511 \end{bmatrix},$$
$$G_{0,0} = \begin{bmatrix} 1.43960560960969 & 0.24587266918848 \\ -0.09653889199693 & 0.00327755426099 \end{bmatrix},$$
$$G_{0,1} = \begin{bmatrix} 0 & -0.02543054977000 \\ 0 & -0.11383454731172 \end{bmatrix},$$
$$G_{0,2} = \begin{bmatrix} -0.04043821079348 & -0.00770917621114 \\ 0.01734837476648 & 0.05313670678855 \end{bmatrix},$$
$$G_{1,-2} = \begin{bmatrix} 0 & 0.03703506051931 \\ 0 & -0.22509408720712 \end{bmatrix},$$
$$G_{1,-1} = \begin{bmatrix} -0.00252158534948 & -0.17429448116797 \\ -0.41689600004767 & 0.63350085992090 \end{bmatrix},$$
$$G_{1,0} = \begin{bmatrix} 0 & 0.04485722059429 \\ 2 & 0.05330498824457 \end{bmatrix},$$
$$G_{1,1} = \begin{bmatrix} -0.14646204991027 & 0.02762475980777 \\ 0.14370248944396 & -0.23015297408176 \end{bmatrix},$$
$$G_{1,2} = \begin{bmatrix} 0 & 0.00770917621114 \\ 0 & -0.05313670678855 \end{bmatrix},$$
$$G_{2,-2} = \begin{bmatrix} -0.01701860785442 & -0.00299784481048 \\ -0.00804852291226 & 0.24119113303164 \end{bmatrix},$$
$$G_{2,-1} = \begin{bmatrix} 0 & 0.00299784481048 \\ 0 & -0.24119113303164 \end{bmatrix},$$
$$G_{2,0} = \begin{bmatrix} 0.05871715343166 & 0.00900510889987 \\ -0.21347232093243 & -0.01867007326897 \end{bmatrix},$$
$$G_{2,1} = \begin{bmatrix} 0 & -0.00900510889987 \\ 0 & 0.01867007326897 \end{bmatrix},$$
$$G_{2,2} = \begin{bmatrix} -0.01316441355952 & 0 \\ 0.13230980380067 & 0 \end{bmatrix}, \quad G_{\mathbf{k}} = \mathbf{0}, \quad \text{for } |k_1| > 2 \text{ or } |k_2| > 2.$$

Here we use the Sobolev smoothness formula provided in [9] for $\widetilde{\Phi}$.

4. Proofs

We will prove Theorem 3.1 by showing that (i) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii), and (iii)$\Rightarrow$ (3.11).

Recall that $\mathbf{y}_\alpha, |\alpha| < m$, are the real-valued vectors for the sum rules of order m of $G(\mathbf{z})$, namely $\mathbf{y}_\alpha, |\alpha| < m$, satisfy (2.10) (or equivalently (2.5)). Let $\mathbf{y}_\alpha(\mathbf{k}), |\alpha| < m, \mathbf{k} \in \mathbb{Z}^s$, be the vectors defined by (2.2) in terms of $\mathbf{y}_\alpha$. Then (2.5) and (2.12) imply that

$$\mathbf{y}_\alpha(\mathbf{k}) = \int_{\mathbb{R}^s} (\mathbf{x}+\mathbf{k})^\alpha \widetilde{\Phi}(\mathbf{x})^T d\mathbf{x}, \quad \mathbf{k}\in\mathbb{Z}^s,\ |\alpha|<m.$$

In particular, we have

$$\mathbf{y}_\alpha = \int_{\mathbb{R}^s} \mathbf{x}^\alpha \widetilde{\Phi}(\mathbf{x})^T d\mathbf{x}, \quad |\alpha|<m. \tag{4.1}$$

From (2.5) and the bi-orthogonality condition (2.16), we know that $\widetilde{\Psi}^h = [\widetilde{\psi}_1^h, \cdots, \widetilde{\psi}_r^h]^T$ has vanishing moments of order m, i.e.,

$$\int_{\mathbb{R}^s} \mathbf{x}^\alpha \widetilde{\psi}_\ell^h(\mathbf{x}) d\mathbf{x} = 0, \quad |\alpha|<m,\ \ 1\le \ell\le r. \tag{4.2}$$

4.1. **Proof of (i) $\Leftrightarrow$ (ii).** Let us first establish (i) $\Rightarrow$ (ii). By (4.1), we have

$$\begin{aligned}
y_{\alpha,l} &= \int_{\mathbb{R}^s} \widetilde{\phi}_l(\mathbf{x})\mathbf{x}^\alpha d\mathbf{x} = \int_{\mathbb{R}^s} \widetilde{\phi}_l(\mathbf{x})(\mathbf{x}-\mathbf{a}_l+\mathbf{a}_l)^\alpha d\mathbf{x}\\
&= \int_{\mathbb{R}^s} \widetilde{\phi}_l(\mathbf{x}) \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \mathbf{a}_l^\beta (\mathbf{x}-\mathbf{a}_l)^{\alpha-\beta} d\mathbf{x}\\
&= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \mathbf{a}_l^\beta \int_{\mathbb{R}^s} \widetilde{\phi}_1(\mathbf{x})(\mathbf{x}-\mathbf{a}_1)^{\alpha-\beta} d\mathbf{x}\\
&= \int_{\mathbb{R}^s} \widetilde{\phi}_1(\mathbf{x})(\mathbf{x}+\mathbf{a}_l-\mathbf{a}_1)^\alpha d\mathbf{x}\\
&= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\mathbf{a}_l-\mathbf{a}_1)^\beta \int_{\mathbb{R}^s} \widetilde{\phi}_1(\mathbf{x})\mathbf{x}^{\alpha-\beta} d\mathbf{x}\\
&= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\mathbf{a}_l-\mathbf{a}_1)^\beta y_{\alpha-\beta,1},
\end{aligned}$$

and hence, (ii) holds.

Next, we will prove (ii) $\Rightarrow$ (i). By (4.1), it follows, for any $1\le l\le r$, that

$$\begin{aligned}
\int_{\mathbb{R}^s} \widetilde{\phi}_l(\mathbf{x})\mathbf{x}^\alpha d\mathbf{x} &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\mathbf{a}_l-\mathbf{a}_1)^\beta \int_{\mathbb{R}^s} \widetilde{\phi}_1(\mathbf{x})\mathbf{x}^{\alpha-\beta} d\mathbf{x}\\
&= \int_{\mathbb{R}^s} \widetilde{\phi}_1(\mathbf{x})(\mathbf{x}+\mathbf{a}_l-\mathbf{a}_1)^\alpha d\mathbf{x},
\end{aligned}$$

so that

$$\begin{aligned}
\int_{\mathbb{R}^s} \widetilde{\phi}_1(\mathbf{x})(\mathbf{x}-\mathbf{a}_1)^\alpha d\mathbf{x} &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (-\mathbf{a}_l)^{\alpha-\beta} \int_{\mathbb{R}^s} \widetilde{\phi}_1(\mathbf{x})(\mathbf{x}+\mathbf{a}_l-\mathbf{a}_1)^\beta d\mathbf{x}\\
&= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (-\mathbf{a}_l)^{\alpha-\beta} \int_{\mathbb{R}^s} \widetilde{\phi}_l(\mathbf{x})\mathbf{x}^\beta d\mathbf{x} = \int_{\mathbb{R}^s} \widetilde{\phi}_l(\mathbf{x})(\mathbf{x}-\mathbf{a}_l)^\alpha d\mathbf{x}.
\end{aligned}$$

Therefore (3.1) holds.

4.2. **Proof of (ii) $\Rightarrow$ (iii).** We need the following two lemmas. Let $\mathbf{v}_\alpha = [v_{\alpha,1}, \cdots, v_{\alpha,r}], |\alpha| < K$, be the vectors defined by

$$v_{\alpha,l} := \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\mathbf{a}_l - \mathbf{a}_1)^\beta y_{\alpha-\beta,1}. \tag{4.3}$$

Notice that $v_{\alpha,1} = y_{\alpha,1}, |\alpha| < K$.

Lemma 4.1. *Suppose that c^α_β and $\mathbf{v}_\alpha$ are the scalars and vectors defined by (3.2) and (4.3), respectively. Then*

$$\sum_{\gamma\le\beta\le\alpha} c^\alpha_\beta \binom{\beta}{\gamma} v_{\beta-\gamma,l} = \binom{\alpha}{\gamma} \mathbf{a}_l^{\alpha-\gamma}, \quad 1 \le l \le r,\ \gamma \le \alpha,\ |\alpha| < K; \tag{4.4}$$

and

$$\sum_{\beta\le\alpha} c^\alpha_\beta \mathbf{v}_\beta(\mathbf{k}) = [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha], \quad |\alpha| < K,\ \mathbf{k} \in \mathbb{Z}^s. \tag{4.5}$$

Proof. First we show that (4.4) holds. By the definition of $v_{\alpha,l}$, we have

$$\begin{aligned}
&\sum_{\gamma\le\beta\le\alpha} c^\alpha_\beta \binom{\beta}{\gamma} v_{\beta-\gamma,l} \\
&= \sum_{\gamma\le\beta\le\alpha} c^\alpha_\beta \binom{\beta}{\gamma} \sum_{\lambda\le\beta-\gamma} \binom{\beta-\gamma}{\lambda} (\mathbf{a}_l - \mathbf{a}_1)^\lambda y_{\beta-\gamma-\lambda,1} \\
&= \sum_{\lambda\le\beta-\gamma} \sum_{\gamma+\lambda\le\beta\le\alpha} c^\alpha_\beta \binom{\beta}{\gamma+\lambda} \binom{\gamma+\lambda}{\lambda} (\mathbf{a}_l - \mathbf{a}_1)^\lambda y_{\beta-\gamma-\lambda,1} \\
&= \sum_{\lambda\le\beta-\gamma} \binom{\gamma+\lambda}{\lambda} (\mathbf{a}_l - \mathbf{a}_1)^\lambda \sum_{\gamma+\lambda\le\beta\le\alpha} c^\alpha_\beta \binom{\beta}{\gamma+\lambda} y_{\beta-\gamma-\lambda,1} \\
&= \sum_{\lambda\le\beta-\gamma} \binom{\gamma+\lambda}{\lambda} \binom{\alpha}{\gamma+\lambda} (\mathbf{a}_l - \mathbf{a}_1)^\lambda \mathbf{a}_1^{\alpha-\gamma-\lambda} \\
&= \sum_{\lambda\le\beta-\gamma} \binom{\alpha}{\gamma} \binom{\alpha-\gamma}{\lambda} (\mathbf{a}_l - \mathbf{a}_1)^\lambda \mathbf{a}_1^{\alpha-\gamma-\lambda} \\
&= \binom{\alpha}{\gamma} \sum_{\lambda\le\beta-\gamma} \binom{\alpha-\gamma}{\lambda} (\mathbf{a}_l - \mathbf{a}_1)^\lambda \mathbf{a}_1^{\alpha-\gamma-\lambda} \\
&= \binom{\alpha}{\gamma} (\mathbf{a}_l - \mathbf{a}_1 + \mathbf{a}_1)^{\alpha-\gamma} = \binom{\alpha}{\gamma} \mathbf{a}_l^{\alpha-\gamma},
\end{aligned}$$

where the fourth equality follows from (3.2). Thus, (4.4) is valid.

To prove (4.5), we have, by (4.4),

$$\begin{aligned}
(\mathbf{k}+\mathbf{a}_l)^\alpha &= \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \mathbf{a}_l^{\alpha-\gamma} \mathbf{k}^\gamma = \sum_{\gamma\le\alpha} \sum_{\gamma\le\beta\le\alpha} c^\alpha_\beta \binom{\beta}{\gamma} v_{\beta-\gamma,l} \mathbf{k}^\gamma \\
&= \sum_{\beta\le\alpha} c^\alpha_\beta \sum_{\gamma\le\beta} \binom{\beta}{\gamma} v_{\beta-\gamma,l} \mathbf{k}^\gamma,
\end{aligned}$$

for $1 \le l \le r$, so that

$$[(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] = \sum_{\beta\le\alpha} c_\beta^\alpha \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \mathbf{v}_{\beta-\gamma}\mathbf{k}^\gamma = \sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{v}_\beta(\mathbf{k}).$$

That is, (4.5) holds. □

Lemma 4.2. *Suppose that c_β^α and $\mathbf{v}_\alpha$ are the scalars and vectors defined by (3.2) and (4.3), respectively. Then for any $n \le K$, the following statements are equivalent.*

(a) $\mathbf{y}_\alpha, |\alpha| < n$, *satisfy (3.7) with $K = n$.*
(b) $\mathbf{y}_\alpha = \mathbf{v}_\alpha, \quad |\alpha| < n$.
(c) $\mathbf{y}_\alpha, |\alpha| < n$, *satisfy*

$$[(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] = \sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{y}_\beta(\mathbf{k}), \quad |\alpha| < n, \ \mathbf{k} \in \mathbb{Z}^s. \tag{4.6}$$

Proof. The equivalence (a) ⇔ (b) follows from the definition of $\mathbf{v}_\alpha, \ |\alpha| < K$, and the implication (b) ⇒ (c) follows from (4.5). We next give the proof of (c) ⇒ (b). By (4.6), we have

$$[(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] = \sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{y}_\beta(\mathbf{k}) = \sum_{\beta\le\alpha} c_\beta^\alpha \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \mathbf{y}_{\beta-\gamma}\mathbf{k}^\gamma.$$

Thus, for any $1 \le l \le r$, we have

$$\begin{aligned}\sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \mathbf{a}_l^{\alpha-\gamma}\mathbf{k}^\gamma &= \sum_{\gamma\le\alpha}\sum_{\gamma\le\beta\le\alpha} c_\beta^\alpha \binom{\beta}{\gamma} y_{\beta-\gamma,l}\mathbf{k}^\gamma \\ &= \sum_{\gamma\le\alpha}\Big(\sum_{\gamma\le\beta\le\alpha} c_\beta^\alpha \binom{\beta}{\gamma} y_{\beta-\gamma,l}\Big)\mathbf{k}^\gamma, \quad \mathbf{k}\in\mathbb{Z}^s,\end{aligned}$$

so that

$$\sum_{\gamma\le\beta\le\alpha} c_\beta^\alpha \binom{\beta}{\gamma} y_{\beta-\gamma,l} = \binom{\alpha}{\gamma} \mathbf{a}_l^{\alpha-\gamma}, \quad \gamma, \ \alpha\in\mathbb{Z}_+^s, \ \gamma\le\alpha, \ |\alpha|<n. \tag{4.7}$$

Hence, $\mathbf{y}_\alpha$ are solutions of the equation (4.4) for $\mathbf{v}_\alpha$ with $K = n$. The fact that the solution of (4.4) for $\mathbf{v}_\alpha$ is unique yields $\mathbf{y}_\alpha = \mathbf{v}_\alpha$. □

We now return to the proof of (ii) ⇒ (iii). By Lemma 4.2, we see that $\mathbf{y}_\alpha, |\alpha| < K$, satisfy (4.6). Thus, by (2.5), we have, for any $|\alpha| < K$,

$$\begin{aligned}\sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{x}^\beta &= \sum_{\beta\le\alpha} c_\beta^\alpha \sum_{\mathbf{k}} \mathbf{y}_\beta(\mathbf{k})\Phi(\mathbf{x}-\mathbf{k}) \\ &= \sum_{\mathbf{k}}\sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{y}_\beta(\mathbf{k})\Phi(\mathbf{x}-\mathbf{k}) \\ &= \sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha]\, \Phi(\mathbf{x}-\mathbf{k}).\end{aligned}$$

Therefore (3.8) is valid.

4.3. **Proof of (iii) $\Rightarrow$ (iv).** We need the following two lemmas. Denote $e_{0,0} := 1$. For $n \in \mathbb{N}$, let $e_{\alpha\beta}$, $|\alpha| = |\beta| = n$, be determined by

$$(A^{-1}\mathbf{x})^\alpha =: \sum_{|\beta|=n} e_{\alpha\beta}\mathbf{x}^\beta.$$

Lemma 4.3. *$[e_{\alpha\beta}]_{|\alpha|=n,|\beta|=n}$ is the inverse of the matrix $[d_{\alpha\beta}]_{|\alpha|=n,|\beta|=n}$.*

Proof. For $|\alpha| = n$, we have,

$$\begin{aligned}\mathbf{x}^\alpha &= \sum_{|\beta|=n} e_{\alpha\beta}(A\mathbf{x})^\beta = \sum_{|\beta|=n} e_{\alpha\beta} \sum_{|\gamma|=n} d_{\beta\gamma}\mathbf{x}^\gamma \\ &= \sum_{|\gamma|=n}\sum_{|\beta|=n} e_{\alpha\beta}d_{\beta\gamma}\mathbf{x}^\gamma, \quad \mathbf{x} \in \mathbb{R}^s.\end{aligned}$$

Thus, $\sum_{|\beta|=n} e_{\alpha\beta}d_{\beta\gamma} = \delta_{\alpha\gamma}$ for all α, γ with $|\alpha| = |\gamma| = n$. □

Lemma 4.4. *Let l^α_β be defined by (3.6). Then*

$$\sum_{\gamma\le\beta,|\beta|\le|\alpha|} l^\alpha_\beta c^\beta_\gamma = \sum_{|\beta|=|\gamma|,\beta\le\alpha} c^\alpha_\beta e_{\beta\gamma}, \quad |\gamma| \le |\alpha| < K. \tag{4.8}$$

Proof. For any $\lambda \in \mathbb{Z}^s_+$, with $|\lambda| \le |\alpha|$, we have

$$\begin{aligned}\sum_{|\gamma|=|\lambda|}\sum_{|\beta|\le|\alpha|} l^\alpha_\beta c^\beta_\gamma d_{\gamma\lambda} &= \sum_{|\beta|\le|\alpha|} l^\alpha_\beta \sum_{|\gamma|=|\lambda|} c^\beta_\gamma d_{\gamma\lambda} \\ &= \sum_{|\beta|\le|\alpha|} l^\alpha_\beta \sum_{|\gamma|\le|\beta|} L^\beta_\gamma c^\gamma_\lambda \quad \text{(by (3.3))} \\ &= \sum_{|\gamma|\le|\alpha|}\sum_{|\gamma|\le|\beta|\le|\alpha|} l^\alpha_\beta L^\beta_\gamma c^\gamma_\lambda = \sum_{|\gamma|\le|\alpha|} \delta_{\alpha\gamma}c^\gamma_\lambda = c^\alpha_\lambda.\end{aligned}$$

Thus, for any $\alpha \in \mathbb{Z}^s_+, |\alpha| = |\gamma|$, it follows that

$$\begin{aligned}\sum_{|\lambda|=|\alpha|} c^\alpha_\lambda e_{\lambda\alpha} &= \sum_{|\lambda|=|\alpha|} \Big(\sum_{|\gamma|=|\lambda|}\sum_{|\beta|\le|\alpha|} l^\alpha_\beta c^\beta_\gamma d_{\gamma\lambda}\Big) e_{\lambda\alpha} \\ &= \sum_{|\gamma|=|\lambda|} \Big(\sum_{|\beta|\le|\alpha|} l^\alpha_\beta c^\beta_\gamma\Big) \sum_{|\lambda|=|\alpha|} d_{\gamma\lambda}e_{\lambda\alpha} \\ &= \sum_{|\gamma|=|\lambda|} \Big(\sum_{|\beta|\le|\alpha|} l^\alpha_\beta c^\beta_\gamma\Big)\delta_{\gamma\alpha} = \sum_{|\beta|\le|\alpha|} l^\alpha_\beta c^\beta_\alpha.\end{aligned}$$

Therefore (4.8) holds. □

We are now ready to prove (iii) $\Rightarrow$ (iv). By the refinement property of Φ, we have

$$\Phi(\mathbf{x} - \mathbf{k}) = \sum_{\mathbf{j}} P_{\mathbf{j}-A\mathbf{k}}\Phi(A\mathbf{x} - \mathbf{j}),$$

which, together with (3.8), implies that

$$\sum_{\mathbf{k}} [(\mathbf{k} + \mathbf{a}_1)^\alpha, \cdots, (\mathbf{k} + \mathbf{a}_r)^\alpha] \sum_{\mathbf{j}} P_{\mathbf{j}-A\mathbf{k}}\Phi(A\mathbf{x} - \mathbf{j}) = \sum_{\beta\le\alpha} c^\alpha_\beta \mathbf{x}^\beta.$$

Multiplying both sides of the above equation to the right by $a\widetilde{\Phi}(A\mathbf{x}-\mathbf{j})$ and then integrating over $\mathbb{R}^s$, we have

$$
\begin{aligned}
&\sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha]\, P_{\mathbf{j}-A\mathbf{k}} \\
&\quad = a \sum_{\beta\le\alpha} c_\beta^\alpha \int_{\mathbb{R}^s} \mathbf{x}^\beta \widetilde{\Phi}(A\mathbf{x}-\mathbf{j})^T d\mathbf{x} \\
&\quad = \sum_{\beta\le\alpha} c_\beta^\alpha \int_{\mathbb{R}^s} \left(A^{-1}(\mathbf{x}+\mathbf{j})\right)^\beta \widetilde{\Phi}(\mathbf{x})^T d\mathbf{x} \\
&\quad = \sum_{\beta\le\alpha} c_\beta^\alpha \sum_{|\gamma|=|\beta|} e_{\beta\gamma} \int_{\mathbb{R}^s} (\mathbf{x}+\mathbf{j})^\gamma \widetilde{\Phi}(\mathbf{x})^T d\mathbf{x} \\
&\quad = \sum_{\beta\le\alpha} \sum_{|\gamma|=|\beta|} c_\beta^\alpha e_{\beta\gamma} \mathbf{y}_\gamma(\mathbf{j}) = \sum_{|\gamma|\le|\alpha|} \Big(\sum_{|\beta|=|\gamma|} c_\beta^\alpha e_{\beta\gamma} \Big) \mathbf{y}_\gamma(\mathbf{j}) \\
&\quad = \sum_{|\gamma|\le|\alpha|} \Big(\sum_{|\gamma|\le|\beta|\le|\alpha|} l_\beta^\alpha c_\gamma^\beta \Big) \mathbf{y}_\gamma(\mathbf{j}) = \sum_{|\beta|\le|\alpha|} l_\beta^\alpha \sum_{|\gamma|\le|\beta|} c_\gamma^\beta \mathbf{y}_\gamma(\mathbf{j}) \\
&\quad = \sum_{|\beta|\le|\alpha|} l_\beta^\alpha \left[(\mathbf{j}+\mathbf{a}_1)^\beta, \cdots, (\mathbf{j}+\mathbf{a}_r)^\beta\right],
\end{aligned}
$$

where the sixth equality follows from (4.8), and the last equality follows from (2.5) and (3.8). Indeed, by (2.5) and (3.8), we have

$$
\sum_{\mathbf{k}} \{[(\mathbf{k}+\mathbf{a}_1)^\alpha, \dots, (\mathbf{k}+\mathbf{a}_r)^\alpha] - \sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{y}_\beta(\mathbf{k})\} \Phi(\mathbf{x}-\mathbf{k}) = 0, \quad |\alpha| < K.
$$

Since Φ is stable and compactly supported, we have, for any $\mathbf{k}\in\mathbb{Z}^s$,

$$
\sum_{\beta\le\alpha} c_\beta^\alpha \mathbf{y}_\beta(\mathbf{k}) = [(\mathbf{k}+\mathbf{a}_1)^\alpha, \dots, (\mathbf{k}+\mathbf{a}_r)^\alpha], \quad |\alpha| < K.
$$

Therefore (3.9) holds.

4.4. **Proof of (iv) $\Rightarrow$ (v).** By the bi-orthogonality condition (2.13), we have

$$
\sum_{\mathbf{k}} G_{\mathbf{k}} P_{\mathbf{k}-A\mathbf{j}}^T = a\delta_{0\mathbf{j}} I_r, \quad \mathbf{j}\in\mathbb{Z}^s,
$$

so that

$$
\begin{aligned}
&\sum_{\mathbf{j}'} \sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} P_{\mathbf{k}-A\mathbf{j}'}^T \left[(\mathbf{j}'+\mathbf{a}_1)^\beta, \cdots, (\mathbf{j}'+\mathbf{a}_r)^\beta\right]^T \\
&\qquad = a\left[(\mathbf{j}+\mathbf{a}_1)^\beta, \cdots, (\mathbf{j}+\mathbf{a}_r)^\beta\right]^T, \quad |\beta| < K.
\end{aligned}
$$

This, together with (3.9), implies that

$$
\begin{aligned}
&\sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} \sum_{|\gamma|\le|\beta|} l_\gamma^\beta [(\mathbf{k}+\mathbf{a}_1)^\gamma, \cdots, (\mathbf{k}+\mathbf{a}_r)^\gamma]^T \\
&\qquad = a\left[(\mathbf{j}+\mathbf{a}_1)^\beta, \cdots, (\mathbf{j}+\mathbf{a}_r)^\beta\right]^T, \quad |\beta| < K.
\end{aligned}
$$

Multiplying both sides of the above equation by L^α_β and taking the summation over β, we have

$$\begin{aligned} a &\sum_{|\beta|\le|\alpha|} L^\alpha_\beta \left[(\mathbf{j}+\mathbf{a}_1)^\beta, \cdots, (\mathbf{j}+\mathbf{a}_r)^\beta\right]^T \\ &= \sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} \sum_{|\beta|\le|\alpha|} L^\alpha_\beta \sum_{|\gamma|\le|\beta|} l^\beta_\gamma \left[(\mathbf{k}+\mathbf{a}_1)^\gamma, \cdots, (\mathbf{k}+\mathbf{a}_r)^\gamma\right]^T \\ &= \sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} \sum_{|\gamma|\le|\alpha|} \delta_{\alpha\gamma} \left[(\mathbf{k}+\mathbf{a}_1)^\gamma, \cdots, (\mathbf{k}+\mathbf{a}_r)^\gamma\right]^T \\ &= \sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} \left[(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha\right]^T. \end{aligned}$$

Thus (3.10) holds.

4.5. **Proof of (v) $\Rightarrow$ (ii).** First we need the following result.

Proposition 4.5. *Suppose P, G are dual masks as defined by (2.13). Assume that P satisfies the sum rules of order m with $\mathbf{y}_\alpha, |\alpha| < m$. Then*

$$\sum_{\mathbf{k}} G_{\mathbf{k}} \mathbf{y}_\alpha(\mathbf{k})^T = a \sum_{|\beta|=|\alpha|} d_{\alpha\beta} \mathbf{y}^T_\beta, \quad \alpha \in \mathbb{Z}^s_+,\ |\alpha| < m. \tag{4.9}$$

Proof. Let $\mathbf{t}(\omega)$ be a vector-valued trigonometric polynomial that satisfies

$$(-iD)^\alpha \mathbf{t}(0) = \mathbf{y}_\alpha. \tag{4.10}$$

Then $\mathbf{t}(\omega)$ also satisfies (2.11). By (2.13), we see that

$$\sum_{n=0}^{a-1} \mathbf{t}(A^T\omega) P(\mathbf{z}e^{-i2\pi A^{-T}\omega_n}) G(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^* = \mathbf{t}(A^T\omega).$$

This, together with (2.11) for $\mathbf{t}(\omega)$, leads to

$$\begin{aligned} D^\alpha\big(\mathbf{t}(A^T\omega)\big)|_{\omega=0} &= \sum_{n=0}^{a-1} D^\alpha\big(\mathbf{t}(A^T\omega)P(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})G(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^*\big)|_{\omega=0} \\ &= \sum_{n=0}^{a-1}\sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta\big(\mathbf{t}(A^T\omega)P(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})\big)|_{\omega=0} D^{\alpha-\beta}\big(G(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^*\big)\big)|_{\omega=0} \\ &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \mathbf{t}(0) D^{\alpha-\beta}\big(G(\mathbf{z})^*\big)|_{\omega=0} \\ &= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \mathbf{t}(0) i^{|\alpha-\beta|} D^{\alpha-\beta} G^T(\mathbf{1}), \end{aligned}$$

for any α with $|\alpha| < K$, where (2.11) has been used. Thus, it follows by applying (4.10) that

$$\begin{aligned} \sum_{\beta\le\alpha} \binom{\alpha}{\beta}\, i^{|\beta|}\mathbf{y}_\beta\, i^{|\alpha-\beta|}\ D^{\alpha-\beta}G^T(\mathbf{1}) &= D^\alpha\big(\mathbf{t}(A^T\omega)\big)|_{\omega=0} \\ &= \sum_{|\beta|=|\alpha|} d_{\alpha\beta} D^\beta \mathbf{t}(0) = \sum_{|\beta|=|\alpha|} i^{|\alpha|}\ d_{\alpha\beta}\mathbf{y}_\beta. \end{aligned}$$

Therefore, we have

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\mathbf{y}_\beta\ D^{\alpha-\beta}G^T(\mathbf{1})=\sum_{|\beta|=|\alpha|}d_{\alpha\beta}\mathbf{y}_\beta,$$

or equivalently,

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\ \mathbf{y}_\beta\ \frac{1}{a}\sum_{\mathbf{k}}\mathbf{k}^{\alpha-\beta}G_{\mathbf{k}}^T=\sum_{|\beta|=|\alpha|}d_{\alpha\beta}\mathbf{y}_\beta.$$

Hence, we may conclude that

$$a\sum_{|\beta|=|\alpha|}d_{\alpha\beta}\mathbf{y}_\beta=\sum_{\mathbf{k}}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\mathbf{y}_\beta\mathbf{k}^{\alpha-\beta}\ G_{\mathbf{k}}^T=\sum_{\mathbf{k}}\mathbf{y}_\beta(\mathbf{k})G_{\mathbf{k}}^T.$$

That is, (4.9) holds. $\square$

We are now ready to establish (v) $\Rightarrow$ (ii). The proof of (3.7) is by induction on $|\alpha|$. When $\alpha=0$, we have, by (3.10) with $\mathbf{j}=0$,

$$\sum_{\mathbf{k}}G_{\mathbf{k}}\mathbf{1}^T=aL_0^0\mathbf{1}^T.$$

Since $L_0^0=1$, we have $\sum_{\mathbf{k}}G_{\mathbf{k}}\mathbf{1}^T=a\mathbf{1}^T$, or equivalently,

$$G(\mathbf{1})\mathbf{1}^T=\mathbf{1}^T.$$

On the other hand, by (4.9) with $\alpha=0$, we have

$$G(\mathbf{1})\mathbf{y}_0^T=\mathbf{y}_0^T.$$

Since 1 is a simple eigenvalue of $G(\mathbf{1})$, we see that $\mathbf{y}_0=y_{0,1}\mathbf{1}$. Therefore $\mathbf{y}_0$ satisfies (3.7).

Let $n<K$ be a positive integer. Suppose for any $\alpha\in\mathbb{Z}_+^s$, $|\alpha|<n$, $\mathbf{y}_\alpha$ satisfies (3.7). We will show that for all multi-indices α with $|\alpha|=n$, $\mathbf{y}_\alpha$ also satisfy (3.7). This will complete the proof of the induction step.

Let $\mathbf{v}_\alpha,|\alpha|<m$, be the vectors defined by (4.3). By Lemma 4.2, to show that $\mathbf{y}_\alpha,|\alpha|=n$, satisfy (3.7), it is sufficient to show that $\mathbf{y}_\alpha=\mathbf{v}_\alpha$ for $|\alpha|=n$. By the induction hypothesis, it follows from Lemma 4.2 again that $\mathbf{y}_\beta=\mathbf{v}_\beta,|\beta|<n$. Therefore for $|\beta|<n$, we have $\mathbf{v}_\beta(\mathbf{k})=\mathbf{y}_\beta(\mathbf{k}),\mathbf{k}\in\mathbb{Z}^s$. For $|\alpha|=n$,

$$\begin{aligned}\mathbf{v}_\alpha(\mathbf{k})&=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\mathbf{k}^{\alpha-\beta}\mathbf{v}_\beta=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\mathbf{k}^{\alpha-\beta}\mathbf{y}_\beta+\mathbf{v}_\alpha-\mathbf{y}_\alpha\\&=\mathbf{y}_\alpha(\mathbf{k})+\mathbf{v}_\alpha-\mathbf{y}_\alpha.\end{aligned}\tag{4.11}$$

By (3.10) with $\mathbf{j}=0$, we have

$$\begin{aligned}
&\sum_{\mathbf{k}} G_{\mathbf{k}}\left[(\mathbf{k}+\mathbf{a}_1)^{\alpha},\cdots,(\mathbf{k}+\mathbf{a}_r)^{\alpha}\right]^T\\
&\qquad = a\sum_{|\beta|\le|\alpha|} L^{\alpha}_{\beta}\left[\mathbf{a}_1^{\beta},\cdots,\mathbf{a}_r^{\beta}\right]^T\\
&\qquad = a\sum_{|\beta|\le|\alpha|} L^{\alpha}_{\beta}\sum_{\gamma\le\beta} c^{\beta}_{\gamma}\mathbf{v}_{\gamma}^T\\
&\qquad = a\sum_{\beta\le\alpha} c^{\alpha}_{\beta}\sum_{|\gamma|=|\beta|} d_{\beta\gamma}\mathbf{v}_{\gamma}^T\\
&\qquad = a\sum_{\beta\le\alpha} c^{\alpha}_{\beta}\sum_{|\gamma|=|\beta|} d_{\beta\gamma}\mathbf{y}_{\gamma}^T + ac^{\alpha}_{\alpha}\sum_{|\gamma|=|\alpha|} d_{\alpha\gamma}(\mathbf{v}_{\gamma}-\mathbf{y}_{\gamma})^T.
\end{aligned}$$

Here, the second equality is achieved by applying (4.5) with $\mathbf{k}=0$, and the third equality follows from (3.4).

On the other hand, since

$$\begin{aligned}
&\sum_{\mathbf{k}} G_{\mathbf{k}}\left[(\mathbf{k}+\mathbf{a}_1)^{\alpha},\cdots,(\mathbf{k}+\mathbf{a}_r)^{\alpha}\right]^T\\
&\quad = \sum_{\mathbf{k}} G_{\mathbf{k}}\sum_{\beta\le\alpha} c^{\alpha}_{\beta}\mathbf{v}_{\beta}(\mathbf{k})^T = \sum_{\beta\le\alpha} c^{\alpha}_{\beta}\sum_{\mathbf{k}} G_{\mathbf{k}}\mathbf{v}_{\beta}(\mathbf{k})^T\\
&\quad = \sum_{\beta<\alpha} c^{\alpha}_{\beta}\sum_{\mathbf{k}} G_{\mathbf{k}}\mathbf{v}_{\beta}(\mathbf{k})^T + c^{\alpha}_{\alpha}\sum_{\mathbf{k}} G_{\mathbf{k}}\mathbf{v}_{\alpha}(\mathbf{k})^T\\
&\quad = \sum_{\beta<\alpha} c^{\alpha}_{\beta}\sum_{\mathbf{k}} G_{\mathbf{k}}\mathbf{y}_{\beta}(\mathbf{k})^T + c^{\alpha}_{\alpha}\sum_{\mathbf{k}} G_{\mathbf{k}}\mathbf{y}_{\alpha}(\mathbf{k})^T + c^{\alpha}_{\alpha}\sum_{\mathbf{k}} G_{\mathbf{k}}(\mathbf{v}_{\alpha}-\mathbf{y}_{\alpha})^T\\
&\quad = \sum_{\beta\le\alpha} c^{\alpha}_{\beta}\sum_{\mathbf{k}} G_{\mathbf{k}}\mathbf{y}_{\beta}(\mathbf{k})^T + c^{\alpha}_{\alpha}a\, G(\mathbf{1})(\mathbf{v}_{\alpha}-\mathbf{y}_{\alpha})^T\\
&\quad = \sum_{\beta\le\alpha} c^{\alpha}_{\beta}a\sum_{|\gamma|=|\beta|} d_{\beta\gamma}\mathbf{y}_{\gamma}^T + c^{\alpha}_{\alpha}a\, G(\mathbf{1})(\mathbf{v}_{\alpha}-\mathbf{y}_{\alpha})^T,
\end{aligned}$$

where (4.11) and (4.9) have been used, we have

$$ac^{\alpha}_{\alpha}\sum_{|\gamma|=|\alpha|} d_{\alpha\gamma}(\mathbf{v}_{\gamma}-\mathbf{y}_{\gamma})^T = c^{\alpha}_{\alpha}a\, G(\mathbf{1})(\mathbf{v}_{\alpha}-\mathbf{y}_{\alpha})^T,\quad |\alpha|=n.$$

Also, since $c^{\alpha}_{\alpha}\ne 0$, we have

$$\left[I_{d_n}\otimes G(\mathbf{1}) - [d_{\alpha\gamma}]_{|\alpha|=n,|\gamma|=n}\otimes I_r\right]\left[\mathbf{v}_{\gamma}^T-\mathbf{y}_{\gamma}^T\right]_{|\gamma|=n} = 0, \tag{4.12}$$

where $d_n := \binom{n+s-1}{s-1}$ and $A\otimes B=[a_{ij}B]$ denotes the Kronecker product of two matrices $A=[a_{ij}]$ and B. It is known (see, for example, Lemma 2.3 in [10]) that if $G(\mathbf{1})$ satisfies Condition E, then for $n>0$, the matrix

$$I_{d_n}\otimes G(\mathbf{1}) - [d_{\alpha\beta}]_{|\alpha|=n,|\beta|=n}\otimes I_r$$

is non-singular. Thus, it follows from (4.12), that $\mathbf{y}_{\gamma}=\mathbf{v}_{\gamma}, |\gamma|=n$. Therefore $\mathbf{y}_{\gamma}$ satisfy (3.7) for $|\gamma|=n$. By the induction on $|\alpha|$, we conclude that $\mathbf{y}_{\alpha}$ satisfy (3.7) for all $|\alpha|<K$.

4.6. **Proof of (iii) $\Rightarrow$ (3.11).** Suppose $\alpha \in \mathbb{Z}_+^s$ is a multi-index with $|\alpha| < K$. Let $\mathbf{b_j}, \mathbf{c_j}$ be the $1 \times r$ vectors, depending on α, that satisfy

$$\begin{aligned} &\sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha]\, \Phi(\mathbf{x}-\mathbf{k}) \\ &= \sum_{\mathbf{j}} \mathbf{b_j} \Phi(A^{-1}\mathbf{x}-\mathbf{j}) + \mathbf{c_j} \Psi(A^{-1}\mathbf{x}-\mathbf{j}). \end{aligned} \tag{4.13}$$

By the bi-orthogonality condition (2.16) and the definition for $\widetilde{\Psi}^h$ (2.15), we see that

$$\begin{aligned} a\mathbf{c_j} &= \sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] \int_{\mathbb{R}^s} \Phi(\mathbf{x}-\mathbf{k}) \widetilde{\Psi}^h (A^{-1}\mathbf{x}-\mathbf{j})^T d\mathbf{x} \\ &= \sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] \int_{\mathbb{R}^s} \Phi(\mathbf{x}-\mathbf{k}) \widetilde{\Phi}(\mathbf{x}-\mathbf{k})^T d\mathbf{x} \big(H^h_{\mathbf{k}-A\mathbf{j}}\big)^T \\ &= \sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] \left(H^h_{\mathbf{k}-A\mathbf{j}}\right)^T. \end{aligned}$$

On the other hand, by (3.8), the left-hand side of (4.13) is simply $\sum_{\beta \le \alpha} c^\alpha_\beta \mathbf{x}^\beta$. Thus, we have, by (4.2),

$$\begin{aligned} a\mathbf{c_j} &= \sum_{\beta \le \alpha} \mathbf{c}^\alpha_\beta \int_{\mathbb{R}^s} \mathbf{x}^\beta \widetilde{\Psi}^h (A^{-1}\mathbf{x}-\mathbf{j})^T d\mathbf{x} \\ &= a \sum_{\beta \le \alpha} \mathbf{c}^\alpha_\beta \int_{\mathbb{R}^s} \big(A(\mathbf{x}+\mathbf{j})\big)^\beta \widetilde{\Psi}^h(\mathbf{x})^T d\mathbf{x} \\ &= a \sum_{\beta \le \alpha} \mathbf{c}^\alpha_\beta \sum_{|\gamma|=|\beta|} d_{\beta\gamma} \int_{\mathbb{R}^s} (\mathbf{x}+\mathbf{j})^\gamma \widetilde{\Psi}^h(\mathbf{x})^T d\mathbf{x} \\ &= \mathbf{0}. \end{aligned}$$

Therefore (3.11) holds.

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