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Multivariate Balanced Vector-Valued Refinable Functions

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Abstract

Vanishing moments of sufficiently high order and compact supports of reasonable size contribute to the great success of wavelets in various areas of applications, particularly in signal and image processing. However, for multi-wavelets, polynomial preservation of the refinable function vectors does not necessarily imply annihilation of discrete polynomials by the high-pass filters of the corresponding orthogonal or bi-orthogonal multi-wavelets. This led to the introduction of the notion of "balanced" multi-wavelets by Lebrun and Vertterli, and later, generalization to higher-order balancing by Selesnick. Selesnick's work is concerned only with orthonormal refinable function vectors and orthonormal multi-wavelets. In this paper after giving a brief overview of the state-of-the-art of vector-valued refinable functions in the preliminary section, we will discuss our most recent contribution to this research area. Our goal is to derive a set of necessary and sufficient conditions that characterize the balancing property of any order for the general multivariate matrix-dilation setting. We will end the second section by demonstrating our theory with examples of univariate splines and bivariate splines on the four-directional mesh.

Introduction

Orthonormal multi-wavelets associated with certain refinable function vectors (also called scaling function vectors) have several advantages in comparison with scalar wavelets. For example, an orthonormal multi-wavelet can possess all the desirable properties of orthogonality, short support, high order of smoothness and vanishing moments, and symmetry/anti-symmetry [8, 10, 11, 12, 33]. Thus, orthonormal multi-wavelets offer the possibility of superior performance for signal/image processing applications, particularly for feature extraction, pattern recognition, and noise reduction (removal) [52, 53].

However, various mathematical difficulties arise in applying multi-wavelets to process scalar-valued data. Although pre-filtering methods have been suggested and work well for some applications [1, 16, 37, 52, 53, 57, 58], the extra effort requires additional computational cost and often leads to other complications. More recently, the notion of "balanced" orthonormal multi-wavelets was introduced in [40] in an attempt to eliminate the need of pre-filtering. After a follow-up work [48] was completed, the terminology "balancing" in [40] is now called "1-balancing," since this notion is generalized to "K-balancing" for any $K \geq 1$ for better performance. However, with the exception of a few examples constructed in [48, 49], all the balanced orthonormal multi-wavelets in the open literature are only 1-balanced [19, 43, 54, 55, 56].

Nonetheless, as compared with scalar-valued wavelets, even 1-balanced multi-wavelets already achieve at least comparable results when applied to image compression. For example, it was demonstrated in [19, 43, 56], that 1-balanced orthonormal multi-wavelets already perform better than scalar-valued orthogonal wavelets in several comparison tests among a wide variety of natural images, and even outperform the bi-orthogonal 9/7 wavelet for certain images such as "Barbara". It is therefore convincing that K-balanced wavelets, for $K \geq 2$, should provide a much more powerful tool. In this regard, it should be mentioned that the bi-orthogonal 9/7 wavelet was adopted by the JPEG-2000 standard for image compression and by the MPEG-4 standard as an option for compression of I-frames for videos.

In this paper we generalize the notion of K-balanced multi-wavelets from the orthonormal setting in [48] to the bi-orthogonal and multivariate settings in order to give more flexibility for construction (particularly for the univariate setting) and to allow for a broader range of applications (when multivariate multi-wavelets are preferred). More precisely, suppose that $\mathbf{a}_1, \cdots, \mathbf{a}_r \in \mathbb{R}^s$. A compactly supported vector-valued function $F = [f_1, \cdots, f_r]^T \in (L^2)^r := (L^2(\mathbb{R}^s))^r$ is said to be K**-balanced** relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r)$, if

$$\int_{\mathbb{R}^s} f_l(\mathbf{x})(\mathbf{x} - \mathbf{a}_l)^\alpha d\mathbf{x} = \int_{\mathbb{R}^s} f_i(\mathbf{x})(\mathbf{x} - \mathbf{a}_i)^\alpha d\mathbf{x},$$

for $1 \leq l, i \leq r, |\alpha| < K$. Let $\Phi, \widetilde{\Phi}$ be refinable function vectors that are bi-orthogonal duals to each other. We will give several characterizations for K-balancing of $\widetilde{\Phi}$. In particular, a characterization formulation in terms of the vectors $\widetilde{\mathbf{y}}_\alpha$ in the definition of the sum rule order of the mask of Φ (see §1.3 about sum rule order) will be given. More precisely, we will see that $\widetilde{\Phi}$ is K-balanced relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r) \subset \mathbb{R}^s$ if and only if $\widetilde{\mathbf{y}}_\alpha = (y_{\alpha,1}, \cdots, y_{\alpha,r})$ satisfy

$$y_{\alpha,l} = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\mathbf{a}_l - \mathbf{a}_1)^\beta y_{\alpha-\beta,1}, \quad 2 \leq l \leq r, |\alpha| < K.$$

This condition enables us to decide easily the balancing order K of the bi-orthogonal dual of a refinable function vector, and helps us to construct orthonormal and biothogonal refinable function vectors.

This paper is organized as follows. We will first discuss preliminary results on refinable function vectors in Section 1. K-balancing of the refinable function vectors and its characterizations are discussed in Section 2, where some examples of balanced refinable function vectors are also given.

1 Preliminary Results

The study of polynomial preservation (or polynomial reproduction) in the linear algebraic span of integer-shifts of a finite number of compactly supported functions has been a popular area of investigation in the Approximation Theory community. We first give a very brief summary in §1.1 of several equivalent statements of this problem in the distribution setting in $\mathbb{R}^s$, $s \geq 1$. A general approach for constructing such compactly supported distributions that are refinable relative to a given expansive dilation matrix and some matrix-valued Laurent polynomial symbol $P(\mathbf{z})$ is discussed in §1.2. The notion of sum rules to be satisfied by $P(\mathbf{z})$ is introduced in §1.3, where sum rules and polynomial preservation are shown to be intimately related for compactly supported refinable distributions. In §1.4, we show that these distributions are indeed functions in L^2, provided that the transition operator associated with $P(\mathbf{z})$ satisfies Condition E and that $P(\mathbf{z})$ itself satisfies the sum rule of at least the first order. Smoothness of these L^2 functions and methods, along with available software, to determine the order of Sobolev and Hölder smoothness are discussed in §1.5. Finally, the characterization of bi-orthogonal duals by the bi-orthogonal condition of their corresponding two-scale Laurent polynomial symbols is formulated in terms of the sum rule and Condition E in §1.6.

1.1 Polynomial preservation

Let $\phi_\ell, \ell = 1, \cdots, r$, be compactly supported distributions in $\mathbb{R}^s$, $s \geq 1$, and $\Phi := [\phi_1, \cdots, \phi_r]^T$. We say that Φ has the property of **polynomial preservation** of order m (or $\Phi \in \mathrm{PP}_m$ for short) if there exists a (finite) linear combination φ of integer shifts of $\phi_1, \cdots, \phi_r$, such that

$$\sum_{\mathbf{k} \in \mathbf{Z}^s} q(\mathbf{k}) \varphi(\cdot - \mathbf{k}) = q, \quad q \in \pi_{m-1}^s,$$

holds in the distribution sense (i.e. equality holds upon taking inner product with any test function), where π_{m-1}^s denotes the space of all polynomials of total degree $< m$ in $\mathbb{R}^s$. It follows from the Poisson summation formula that the above formulation is equivalent to the (modified) Strang-Fix conditions:

$$D^\alpha \hat{\varphi}(2\pi\mathbf{k}) = \delta_{0,\alpha}\delta_{0,\mathbf{k}} \qquad |\alpha| < m, \quad \mathbf{k} \in \mathbb{Z}^s. \tag{1.1}$$

It is also easily seen that Φ satisfies (1.1) if and only if there exists a $1 \times r$ vector $\mathbf{t}(\omega) = \sum_{|\alpha|<m} \mathbf{t}_\alpha e^{-i\alpha\omega}$ of trigonometric polynomials such that

$$D^\alpha(\mathbf{t}\hat{\Phi})(2\pi\mathbf{k}) = \delta_{0,\alpha}\delta_{0,\mathbf{k}}, \qquad |\alpha| < m, \mathbf{k} \in \mathbb{Z}^s, \tag{1.2}$$

which, in turn, is equivalent to

$$\mathbf{x}^\alpha = \sum_{\mathbf{k}\in\mathbf{Z}^s} \{\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \mathbf{k}^{\alpha-\beta}\mathbf{y}_\beta\}\Phi(\mathbf{x}-\mathbf{k}), \ |\alpha| < m, \tag{1.3}$$

in the distribution sense, with

$$\mathbf{y}_\alpha := (-iD)^\alpha \mathbf{t}(0), \quad |\alpha| < m. \tag{1.4}$$

For $r = 1$, this formula was discussed in [6], and for arbitrary $r \geq 1$, the reader is referred to [25].

1.2 Construction of compactly supported Φ

Let A be an expansive matrix (i.e. all eigenvalues λ of A satisfy $|\lambda| > 1$) with integer entries. For any finite sequence $\{P_\mathbf{k}\}$ of square matrices of dimension r, we consider its corresponding matrix-valued Laurent polynomial symbol

$$P(\mathbf{z}) := \frac{1}{a}\sum_{\mathbf{k}\in\mathbf{Z}^s} P_\mathbf{k}\mathbf{z}^\mathbf{k}, \quad a := |\det(A)|, \tag{1.5}$$

and say that $P(\mathbf{1})$, $\mathbf{1} := (1, \cdots, 1)$, satisfies **Condition E** (or $P(\mathbf{1}) \in E$ for short [51]), if the value 1 is a simple eigenvalue of the matrix $P(\mathbf{1})$ and all other eigenvalues of $P(\mathbf{1})$ lie in $|z| < 1$. Under the assumption that $P(\mathbf{1}) \in E$, it follows that the infinite product

$$\prod_{j=1}^{\infty} P(e^{-i(A^T)^{-j}\omega})$$

converges uniformly on every compact subset of $\mathbb{R}^s$. This was first proved in [17] for $s = 1$ and $A = [2]$, and extended to $A = 2I_s$ in [42]. The proof for our more general matrix A follows the same argument given in [42]. Let $\mathbf{v}_0$ be the right eigen-vector of $P(\mathbf{1})$ corresponding to the eigenvalue 1, i.e.

$$P(\mathbf{1})\mathbf{v}_0 = \mathbf{v}_0,$$

such that $|\mathbf{v}_0| = 1$, and define

$$\widehat{\Phi}(\omega) = \prod_{j=1}^{\infty} P(e^{-i(A^T)^{-j}\omega})\mathbf{v}_0. \tag{1.6}$$

Then its inverse Fourier transform Φ is a compactly supported distribution that satisfies the so-called refinement equation

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Z}^s} P_{\mathbf{k}}\Phi(A\mathbf{x} - \mathbf{k}). \tag{1.7}$$

We call the sequence $\{P_{\mathbf{k}}\}$ and its corresponding symbol $P(\mathbf{z})$ the mask and two-scale symbol of Φ, respectively. We also call Φ a normalized solution of the refinement equation (1.7).

On the other hand, if Φ is a compactly supported distribution vector that satisfies (1.7) for some finite sequence $\{P_{\mathbf{k}}\}$, we say that Φ is a refinable distribution vector. For such Φ, if each entry of the matrix $\sum_{\mathbf{k}\in\mathbf{Z}^s}(\widehat{\Phi}\widehat{\Phi}^*)(\omega)$ is a bounded function and that the matrix

$$\sum_{\mathbf{k}\in\mathbf{Z}^s}(\widehat{\Phi}\widehat{\Phi}^*)(2\mathbf{k}\pi)$$

is non-singular, then the symbol $P(\mathbf{z})$ defined by (1.5) satisfies $P(\mathbf{1}) \in E$ (see e.g. [34]).

1.3 Sum rules

In this section, we will always assume that $P(\mathbf{1}) \in E$. We will study conditions on the finite sequence $\{P_{\mathbf{k}}\}$ under which the compactly supported distribution Φ introduced in (1.6) satisfies $\Phi \in \mathrm{PP}_m$ as discussed in §1.1.

Let A be defined as in §1.2, and let ω_h, with $\omega_0 = 0$ and $0 \le h < a$, be the representors of $\mathbb{Z}^s/A^T\mathbb{Z}^s$. We say that P satisfies the **sum rule** of order m (or $P \in \mathrm{SR}_m$ for short) if there exist constant vectors $\tilde{\mathbf{y}}^\alpha$, with $\tilde{\mathbf{y}}_0 \neq \mathbf{0}$, such that

$$\sum_{\beta\le\alpha}(-1)^{|\beta|}\binom{\alpha}{\beta}\tilde{\mathbf{y}}_{\alpha-\beta}J_{\beta,\gamma_h} = \sum_{|\tau|=|\alpha|}\left[\sum_{|\beta|=|\alpha|}(A^{-1}\beta)^\beta t^\alpha_{\beta\tau}\right]\frac{\tilde{\mathbf{y}}_\tau}{\tau!}, \tag{1.8}$$

for all $|\alpha| < m, \gamma_h \in \mathbb{Z}^s / A^T \mathbb{Z}^s$, where

$$J_{\beta,\gamma_h} := \sum_{\mathbf{k}} (\mathbf{k} + A^{-1}\gamma_h)^\beta P_{A\mathbf{k}+\gamma_h},$$

and $\left[t^{\alpha}_{\beta\tau}\right]$ is the inverse of the matrix $\left[\frac{\beta^\tau}{\tau!}\right]_{|\tau|,|\beta|=|\alpha|}$. For the case $A = 2I_s$, (1.8) can be simplified to be

$$\sum_{\beta \le \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} \widetilde{\mathbf{y}}_{\alpha-\beta} J_{\beta,\gamma} = 2^{-\alpha} \widetilde{\mathbf{y}}_\alpha, \tag{1.9}$$

for all $\gamma \in \{0,1\}^s, |\alpha| < m$, with

$$J_{\beta,\gamma} := \sum_{\mathbf{k}} (\mathbf{k} + 2^{-1}\gamma)^\beta P_{2\mathbf{k}+\gamma}.$$

Of course when $r = 1$, we can further simplify (1.9) to

$$\begin{cases} \sum_{\mathbf{k}\in\mathbb{Z}^s} P_{\mathbf{k}} = 2^s, \\ \sum_{\mathbf{k}\in\mathbb{Z}^s} (2\mathbf{k}+\gamma)^\alpha P_{2\mathbf{k}+\gamma} = \sum_{\mathbf{k}\in\mathbb{Z}^s} (2\mathbf{k})^\alpha P_{2\mathbf{k}}, \quad \gamma \in \{0,1\}^s, |\alpha| < m, \end{cases}$$

for some suitable choice of $\{\widetilde{y}_\alpha\}$.

Now, consider an r-vector $\widetilde{\mathbf{t}}(\omega) = \sum_{\mathbf{k}\in\mathbb{Z}^s} \widetilde{\mathbf{t}}_{\mathbf{k}} e^{-i\mathbf{k}\omega}$ of trigonometric polynomials that gives $\widetilde{\mathbf{y}}_\alpha = (-iD)^\alpha \widetilde{\mathbf{t}}(0)$ for all $|\alpha| < m$. It was shown in [25] that (1.8) is equivalent to

$$D^\alpha(\widetilde{\mathbf{t}}(A^T\omega)P(e^{-i\omega}))|_{\omega=2\pi A^{-T}\omega_h} = \delta_{h,0} D^\alpha \widetilde{\mathbf{t}}(0), \quad |\alpha| < m, \tag{1.10}$$

for all $0 \le h < a$.

The following result shows that if $P \in \mathrm{SR}_m$, then the normalized solution Φ of (1.7) has the property of polynomial preservation of order m ($\Phi \in \mathrm{PP}_m$).

Theorem 1.1. *Let $P \in SR_m$ for some integer $m \ge 1$ and Φ be a compactly supported normalized solution of (1.7). Then Φ satisfies (1.3) with $\mathbf{y}_\alpha = \widetilde{\mathbf{y}}_\alpha$; or equivalently, $\Phi \in PP_m$.*

Conversely, if Φ is some compactly supported refinable distribution vector with finite mask $\{P_{\mathbf{k}}\}$ that satisfies PP_m such that the matrices $\sum_{\mathbf{k}}(\widehat{\Phi}\widehat{\Phi}^*)(2\mathbf{k}\pi + 2\pi A^{-T}\omega_h)$ are non-singular for all $0 \le h < a$, then $P \in SR_m$ with $\widetilde{\mathbf{y}}_\alpha = \mathbf{y}_\alpha$.

Theorem 1.1 was obtained in [21] for $r = 1$, and independently in [2] and [34] for any $r \ge 1$. The equivalence of PP_m and SR_m in the vector setting were derived earlier under certain stronger assumptions on Φ, such as linear independence of $\{\phi_\ell(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^s, \ell = 1, \cdots, r\}$ in [18] for $s = 1$ and [3, 4] for $s \geq 1$, and stability of Φ in [46] for $s = 1$.

For a given finite mask $\{P_\mathbf{k}\}$ and dilation matrix A, (1.8) can be used to determine the maximum sum rule order m of P and to find the vectors $\widetilde{\mathbf{y}}_\alpha$, and hence, the maximum polynomial preservation order and the vectors $\mathbf{y}_\alpha = \widetilde{\mathbf{y}}_\alpha$ in (1.3) for generating π^s_{m-1}. Matlab routines for calculating sum rule orders can be downloaded from the website of the second author at `www.math.umsl.edu/~jiang`.

1.4 From compactly supported distributions to function vectors

To assure that the compactly supported distribution vector Φ constructed in §1.2 is a function vector in $(L^2)^r$, we consider the transition operator T_P associated with the symbol $P(\mathbf{z})$ in (1.5). Let $C_0(\mathbb{T}^s)^{r\times r}$ denote the space of all $r \times r$ matrices with trigonometric polynomial entries. The transition operator T_P associated with P is defined on $C_0(\mathbb{T}^s)^{r\times r}$ as

$$T_P F(\omega) = \sum_{0\leq h<a} P(e^{-iA^{-T}(\omega+2\pi\omega_h)}) F(A^{-T}(\omega + 2\pi\omega_h)) P(e^{-iA^{-T}(\omega+2\pi\omega_h)})^*,$$

The representation matrix of T_P, also denoted by T_P for convenience, is

$$T_P = [B_{A\mathbf{k}-\mathbf{j}}]_{\mathbf{k},\mathbf{j}}, \tag{1.11}$$

where

$$B_\mathbf{j} = \frac{1}{a} \sum_\mathbf{k} P_{\mathbf{k}-\mathbf{j}} \otimes \overline{P}_\mathbf{k},$$

and $\otimes$ denotes the Kronecker product of A and B, namely $A \otimes B = [a_{ij}B]$ (see e.g. [34]).

Next, consider the refinement operator

$$U_P\mathbf{f} := \sum_{\mathbf{k}\in\mathbf{Z}^s} P_\mathbf{k} \mathbf{f}(A\cdot - \mathbf{k}), \quad \mathbf{f} \in (L^2)^r, \tag{1.12}$$

that defines the so-called **cascade algorithm**

$$\begin{cases} \Phi_n := U_P\Phi_{n-1},\ n = 1, 2, \cdots, \\ \Phi_0 \in (L^2)^r. \end{cases} \tag{1.13}$$

It is known (see e.g. [5]) that for the cascade algorithm (1.13) with compactly supported initial function vector $\Phi_0 \in (L^2)^r$ to converge in the $(L^2)^r$-norm, it is necessary that Φ_0 satisfies

$$\sum_{\mathbf{k}\in\mathbb{Z}^s} \mathbf{y}_0 \Phi_0(\cdot - \mathbf{k}) = c_0 \neq 0, \tag{1.14}$$

where c_0 is a nonzero constant and $\mathbf{y}_0$ is the left eigenvector of $P(\mathbf{1})$ corresponding to the eigenvalue 1. The significance of the convergence of the cascade algorithm is that the limit Φ, which is necessary an $(L^2)^r$ function vector, is a refinable function vector that satisfies (1.7). The following result is established in [5] for the more general Sobolev setting.

Theorem 1.2. *A necessary and sufficient condition for the cascade algorithm to converge in the $(L^2)^r$-norm for any initial compactly supported $\Phi_0 \in (L^2)^r$ that satisfies (1.14) is that*

$$P \in SR_1 \text{ and } T_P \in E. \tag{1.15}$$

We remark that special cases of this theorem have been considered in earlier works [14, 28, 39, 50, 51]. In particular, Theorem 1.2 itself was already established in [50] for $A = 2I_s$. The convergence in the L^p-norm has also been studied in [15, 20, 28].

The cascade algorithm (1.13) is closely related to the theory of stationary vector subdivision. The vector subdivision operator S_P

$$(S_P\mathbf{u})_{\mathbf{k}} := \sum_{\mathbf{j}\in\mathbb{Z}^s} \mathbf{u}_{\mathbf{j}} P_{\mathbf{k}-A\mathbf{j}}, \quad \mathbf{k} \in \mathbb{Z}^s, \tag{1.16}$$

defined on $\ell_0(\mathbb{Z}^s)^{1\times r}$, where $\mathbf{u} = (\mathbf{u}_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^s}$, is used to generate $\mathbf{u}^n := S_P\mathbf{u}^{n-1}$, $n = 1, 2, \cdots$, with initial $\mathbf{u}^0 \in \ell_0(\mathbb{Z}^s)^{1\times r}$. The characterization of L^p convergence of $(\mathbf{u}^n)$ (for $s = 1, r \geq 1$) was studied in [45].

To verify (1.15), the difficulty is to study $T_P \in E$. In this regard, it is sufficient to consider certain truncation of T_P. More precisely, consider

$$\Omega := \{\sum_{k=1}^{\infty} A^{-k} x_k : \quad x_k \in [-N, N]^s, k \in \mathbb{N}\},$$

where $N = N_2 - N_1$, with N_1 and N_2 denoting the lower and upper coordinate degrees of P, respectively. Also, consider the invariant subspace

$$H_\Omega := \{F(\omega) \in C_0(\mathbb{T}^s)^{r\times r} : \quad F(\omega) = \sum_{\mathbf{k}\in\Omega\cap\mathbb{Z}^s} F_{\mathbf{k}} e^{-i\mathbf{k}\omega}\},$$

of $C_0(\mathbb{T}^s)$ under T_P. It was shown in [22] and [34] that eigenfunctions of T_P corresponding to the nonzero eigenvalues lie in H_Ω. Thus, to study the eigenvalues and eigenfunctions of T_P, one needs only consider those of the restriction $T_{P,\Omega}$ of T_P to the finite-dimensional space H_Ω, or equivalently those of

$$T_{P,\Omega} = [B_{A\mathbf{k}-\mathbf{j}}]_{\mathbf{k},\mathbf{j}\in(\Omega\cap\mathbf{Z}^s)}.$$

For $\Phi = [\phi_1, \cdots, \phi_r]^T \in (L^2)^r$, we say that Φ is $(L^2)^r$-stable, provided that there exist positive constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1\|\mathbf{c}\|_{\ell^2(\mathbf{Z}^s)^{1\times r}} \le \|\sum_{\mathbf{k}\in\mathbf{Z}^s} \mathbf{c}_\mathbf{k}\Phi(\cdot - \mathbf{k})\|_{L^2(\mathbb{R}^s)} \le C_2\|\mathbf{c}\|_{\ell^2(\mathbf{Z}^s)^{1\times r}}, \quad \forall \mathbf{c} \in \ell^2(\mathbb{Z}^s)^{1\times r}.$$

The following theorem is concerned with the stability of the refinable vector-valued functions (see [50] for $A = 2I_s$ and a straightforward generalization to arbitrary A in [34]).

Theorem 1.3. *The normalized solution Φ of (1.7) is $(L^2)^r$-stable if and only if (1.15) is satisfied and that the matrix-valued eigenfunction of T_P corresponding to the eigenvalue* 1 *is either positive or negative definite.*

From Theorems 1.2 and 1.3, we see that stability implies the convergence of the cascade algorithm.

1.5 Smoothness of refinable function vectors

The Sobolev smoothness of the refinable vector-valued functions was discussed in [9, 27, 29, 34, 44, 47, 50]. The results for $s = 1$ in [9, 44] are based on the factorization of the symbol $P(z)$. In general, it is not possible to generalize these results to the multivariate setting. On the other hand, the smoothness estimates in [34, 50] for $\mathbb{R}^s$ are related to both the eigenvalues and the corresponding eigenvectors of the transition operator T_P, while those in [27, 47] are related to the spectral radius of the operator T_P restricted to an invariant subspace of T_P. Since computation of the eigenvectors of a large matrix is not as stable as that of the eigenvalues, it is difficulty to decide whether an eigenvalue of $T_{P,\Omega}$ should be kept or thrown away when the methods in [34, 50] are used for matrices $T_{P,\Omega}$ of high dimensions. On the other hand, due to roundoff error, it is difficulty to find the invariant subspaces in [27, 29, 47] when the size of matrix $T_{P,\Omega}$ is large. A characterization of the Sobolev smoothness in terms solely of the eigenvalues of the transition operator is desired. For the scalar case, such a characterization is available in [31] when the dilation matrix A is isotropic, meaning that all eigenvalues of A have the same magnitude. Recently a characterization of the smoothness for vector-valued Φ in terms of the eigenvalues of T_P was obtained in [26].

More precisely, suppose that A is istropic with eigenvalues $\sigma_1, \cdots \sigma_s$, and that the mask $\{P_{\mathbf{k}}\}$ satisfies the sum rule of order m. Let $\sigma = (\sigma_1, \cdots, \sigma_s)$, and $\lambda_j, 1 \le j \le r$ with $\lambda_1 = 1$ and $|\lambda_j| < 1, j = 2, \cdots r$, be the eigenvalues of $P(\mathbf{1})$. Set

$$S_m := \mathrm{spec}(T_{P,\Omega}) \backslash \widetilde{S}_m \; , \tag{1.17}$$

with $\widetilde{S}_m$ being the set of all

$$\sigma^{-\alpha}\overline{\lambda_j}, \overline{\sigma^{-\alpha}}\lambda_j, \sigma^{-\beta}, \quad \alpha, \beta \in \mathbb{Z}_+^s, |\alpha| < m, |\beta| < 2m, 2 \le j \le r,$$

where multiplicities of the above $\sigma^{-\alpha}\overline{\lambda_j}, \overline{\sigma^{-\alpha}}\lambda_j, \sigma^{-\beta}$ are taken into account in the definition of $\widetilde{S}_m$. Also let $\nu_2(\Phi)$ denote its critical Sobolev exponent, defined by

$$\nu_2(\Phi) := \sup\{\nu : \ \phi_j \in W^\nu(\mathbb{R}^s), j = 1, \cdots, r\},$$

where $W^\nu(\mathbb{R}^s)$ is the Sobolev space defined by

$$W^\nu(\mathbb{R}^s) := \{f : \quad \int_{\mathbb{R}^s} (1 + |\omega|^2)^\nu |\hat{f}(\omega)|^2 d\omega < \infty\}.$$

Then we have following theorem [26].

Theorem 1.4. *Let $\Phi \in (L^2)^r$ be the normalized solution of (1.7) with mask $\{P_{\mathbf{k}}\}$ and some isotropic dilation matrix A. Also, let m be the largest positive integer for which $P \in SR_m$, and $\rho_0 = \max\{|\lambda| : \lambda \in S_m\}$, where S_m is defined in* (1.17). *Then*

$$\nu_2(\Phi) \ge -\frac{s}{2} \log_a \rho_0 \; . \tag{1.18}$$

If, in addition, Φ is $(L^2)^r$-stable, then

$$\nu_2(\Phi) = -\frac{s}{2} \log_a \rho_0 \; . \tag{1.19}$$

For the scalar case $(r = 1)$, $\widetilde{S}_m$ is given by

$$\widetilde{S}_m = \{\sigma^{-\beta} : |\beta| < 2m\}.$$

Theorem 1.4 for $r = 1$ and $s \ge 1$ is contained in [31]. This theorem leads to some efficient ways for computing highly accurate values for the Sobolev smoothness $\nu_2(\Phi)$ by standard eigenvalue solvers.

The Hölder smoothness of Φ can be characterized in terms of the uniform joint spectral radius of the family $\mathcal{A}_m := \{A_{P,\gamma}|_{V_m}, \gamma \in \Gamma\}$, where Γ is a fixed set of representators for $\mathbb{Z}^s/A\mathbb{Z}^s$, and $A_{P,\gamma}, \gamma \in \Gamma$ are the operators defined by

$$(A_{P,\gamma}\mathbf{v})_{\mathbf{k}} = \sum_{\mathbf{j}\in\mathbf{Z}^s} P_{\gamma+A\mathbf{k}-\mathbf{j}}\mathbf{v}_{\mathbf{j}}\ , \qquad \mathbf{k} \in \mathbb{Z}^s, \quad v \in \ell_0(\mathbb{Z}^s)^r,$$

and V_m is the subspace of $\ell_0(\mathbb{Z}^s)^r$ consisting of $\{\mathbf{v}_{\mathbf{k}}\}$ with support in an appropriate finite set $\Omega_1 \cap \mathbb{Z}^s$, and satisfying

$$D^\alpha(\widetilde{\mathbf{t}}(\omega)\mathbf{v}(\omega))|_{\omega=0} = 0, \quad |\alpha| < m, \tag{1.20}$$

where $\mathbf{v}(\omega) := \sum_{\mathbf{k}\in\Omega_1\cap\mathbf{Z}^s} \mathbf{v}_{\mathbf{k}} e^{-i\mathbf{k}\omega}$ and $\widetilde{\mathbf{t}}(\omega)$ is the vector in (1.10). (See Theorem 5 in [36] for the details.) For $s = 1$, a different characterization of Hölder smoothness of Φ was obtained in [29] using an invariant subspace of $A_{P,\gamma}$ different from V_m. For the subspace V_m defined in (1.20), since we know $\widetilde{\mathbf{t}}(\omega)$, we can easily give its basis. Therefore we can give the representating matrices of the operators $A_{P,\gamma}|_{V_m}$. The reader is referred to [36] for the Matlab routines for calculation of the order of smoothness.

1.6 Bi-orthogonal duals

Let $\Phi = [\phi_1, \cdots, \phi_r]^T \in (L^2)^r$ and $\widetilde{\Phi} = [\widetilde{\phi}_1, \cdots, \widetilde{\phi}_r]^T \in (L^2)^r$ be compactly supported refinable vector-valued functions with dilation matrix A and with finite masks $\{P_{\mathbf{k}}\}$ and $\{G_{\mathbf{k}}\}$, respectively. We say that Φ and $\widetilde{\Phi}$ are bi-orthogonal duals of each other if $\{\Phi(\mathbf{x}-\mathbf{k})\}$ and $\{\widetilde{\Phi}(\mathbf{x}-\mathbf{k})\}$ are bi-orthogonal, meaning that

$$\int_{\mathbb{R}^s} \Phi(\mathbf{x})\widetilde{\Phi}(\mathbf{x}-\mathbf{k})^*\,d\mathbf{x} = \delta_{0,\mathbf{k}} I_r, \quad \mathbf{k} \in \mathbb{Z}^s. \tag{1.21}$$

Hence, if $\widetilde{\Phi} = \Phi$, we say that Φ is orthonormal.

A necessary condition for Φ and $\widetilde{\Phi}$ to be bi-orthogonal duals is that their two-scale symbols satisfy

$$\sum_{n=0}^{m-1} P(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})G(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^* = I_r, \quad \mathbf{z} = e^{-i\omega}, \omega \in \mathbb{R}^s. \tag{1.22}$$

Under certain mild conditions, this condition is also sufficient, as follows.

Theorem 1.5. *A necessary and sufficient condition for Φ and $\widetilde{\Phi}$, with finite maskes $\{P_{\mathbf{k}}\}$ and $\{G_{\mathbf{k}}\}$, respectively, to be bi-orthogonal duals is that* *(i) $P(\mathbf{z})$ and $Q(\mathbf{z})$ are bi-orthogonal matrix-valued Laurent polynomials as defined in (1.22),*

(ii) $P(\mathbf{1}), G(\mathbf{1}) \in E$,

(iii) $P, G \in SR_1$, and

(iv) $T_P, T_G \in E$.

The proof this theorem can be formulated as a generalization of the proof in [23] for $s = 1, A = [2]$, and the proof in [24] for $s \geq 1$. When $\Phi = \widetilde{\Phi}$ and $P = G$, Theorem 1.5 gives a characterization of the orthonormality of Φ in terms of its mask $\{P_{\mathbf{k}}\}$.

Suppose $Q^h(\mathbf{z})$ and $H^h(\mathbf{z}), 1 \leq h < a$, are matrix-valued Laurent polynomials that satisfy

$$\left\{ \begin{array}{l} \sum_{0 \leq n < a} P(\mathbf{z}e^{-i2\pi A^{-T}\omega_n}) H^h(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^* = 0, \quad 1 \leq h < a \\ \sum_{0 \leq n < a} Q^h(\mathbf{z}e^{-i2\pi A^{-T}\omega_n}) G(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^* = 0, \quad 1 \leq h < a \\ \sum_{0 \leq n < a} Q^h(\mathbf{z}e^{-i2\pi A^{-T}\omega_n}) H^\ell(\mathbf{z}e^{-i2\pi A^{-T}\omega_n})^* = \delta_{h,\ell} I_r, \quad 1 \leq h, \ell < a, \end{array} \right. \tag{1.23}$$

for $\omega \in \mathbb{R}^s$, where $\mathbf{z} = e^{-i\omega}$. Let $\Psi^h, \widetilde{\Psi}^h$ be the vector-valued functions defined by

$$\widehat{\Psi}^h(A^T\omega) = Q^h(\mathbf{z})\hat{\Phi}(\omega), \quad \hat{\widetilde{\Psi}}^h(A^T\omega) = H^h(\mathbf{z})\hat{\widetilde{\Phi}}(\omega). \tag{1.24}$$

Then $\{a^{-\frac{n}{2}}\Psi^h(A^{-n}\mathbf{x} - \mathbf{k}), a^{-\frac{n}{2}}\widetilde{\Psi}^h(A^{-n}\mathbf{x} - \mathbf{k}), 1 \leq h < a, n \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^s\}$ forms a bi-orthogonal system (cf. [30]).

We use $\{G, H^h, 1 \leq h < a\}$ and $\{P, Q^h, 1 \leq h < a\}$ for wavelet decomposition and reconstruction respectively. More precisely, the decomposition algorithm is given by

$$\mathbf{c}_{\mathbf{n}}^{(j-1)} = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{n}} \mathbf{c}_{\mathbf{k}}^{(j)}, \quad \mathbf{d}_{\mathbf{n},h}^{(j-1)} = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} H^h_{\mathbf{k}-A\mathbf{n}} \mathbf{c}_{\mathbf{k}}^{(j)}, \quad \mathbf{n} \in \mathbb{Z}^s, \tag{1.25}$$

and the reconstruction algorithm is given by

$$\mathbf{c}_{\mathbf{n}}^{(j)} = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} P^T_{\mathbf{n}-A\mathbf{k}} \mathbf{c}_{\mathbf{k}}^{(j-1)} + \frac{1}{\sqrt{m}} \sum_{1 \leq h < a} \sum_{\mathbf{k}} (Q^h_{\mathbf{n}-A\mathbf{k}})^T \mathbf{d}_{\mathbf{k},h}^{(j-1)}, \quad \mathbf{n} \in \mathbb{Z}^s. \tag{1.26}$$

2 Balanced Refinable Vectors

The notion of balancing is generalized to the multivariate bi-orthogonal setting in this section. With balanced bi-orthogonal duals, discrete polynomials are preserved by the lowpass filters and annihilated by the highpass filters of the corresponding multi-wavelets. Several important characterization results for these two properties are discussed in this section. Examples are given in some details, first for univariate quadratic and cubic spline function vectors, then for continuous piecewise linear polynomials on the four-directional mesh $\triangle^2$ in $\mathbb{R}^2$, and finally for C^1 cubic bivariate splines also on $\triangle^2$. Both the dilation matrix $2I_2$ and the quincunx matrix are considered in the bivariate examples.

2.1 The notion of balancing

Let $\mathbf{a}_1, \cdots, \mathbf{a}_r \in \mathbb{R}^s$. A compactly supported vector-valued function $F = [f_1, \cdots, f_r]^T \in (L^2)^r$ is said to be ***K*-balanced**, $K \geq 1$, relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r)$, if

$$\int_{\mathbb{R}^s} f_l(\mathbf{x})(\mathbf{x} - \mathbf{a}_l)^\alpha d\mathbf{x} = \int_{\mathbb{R}^s} f_i(\mathbf{x})(\mathbf{x} - \mathbf{a}_i)^\alpha d\mathbf{x}, \tag{2.1}$$

for $1 \leq l, i \leq r, |\alpha| < K$. This is an extension of the notion of K-balancing introduced for the orthonormal univariate setting by Lebrun and Vetterli [40] for $K = 1$ and Selesnick [48] for $K \geq 1$. (See their other work [41, 49].) The motivation of their consideration is for processing scalar-valued digital data by $r \times r$ matrix-valued lowpass filters $\{P_k\}$ without the need of pre-filtering as studied in [1, 16, 37, 52, 53, 57, 58], so that independent of shifts in the formulation of the r-vector blocks of the scalar-valued input data, the filtering process yields polynomial output upon lowpass filtering of any polynomial input data in $\pi_{K-1} := \pi^1_{K-1}$. This is significant in wavelet signal processing to allow for the discrete wavelet transform (DWT), with vanishing moments of order K or higher, to extract detailed features of the input signal. Our extension from $\mathbb{R}^1$ to $\mathbb{R}^s$ and from orthogonal Φ to the bi-orthogonal setting facilitates the study of arbitrary dilation matrix A and achieving certain desirable filter and multi-wavelet design criteria. For example, the dilation matrix A with determinant of A equal to ± 2 reduces implementation complexity and contributes to memory and cost reduction.

For the univariate and orthonormal setting, Selesnick [48] gave a complete characterization of compactly supported K-balanced orthonormal refinable function vector Φ with two-scale symbol $P(z) = [p_{j,k}(z)]$ in terms of an associated polynomial

$$\sum_{j=0}^{r-1} j(j+r)\cdots(j+(K-1)r)\Big\{\sum_{\ell}^{K-1} \frac{(-1)^\ell}{(j+\ell r)}\binom{K-1}{\ell} z^{r\ell}\Big\} \sum_{k=1}^{r} p_{(j+1),k}(z^r) z^{k-1} \tag{2.2}$$

in that Φ is K-balanced relative to centers

$$a_1 = 0, \quad a_2 = \frac{1}{r}, \quad \cdots, \quad a_r = \frac{r-1}{r}$$

if and only if the polynomial (2.2) is divisible by $(1 + z + \cdots z^{2r-1})^K$. Hence, to construct $P(z)$ and hence Φ, it is necessary to solve for the coefficients of $p_{j,k}(z)$ so that the QMF condition

$$P(z)P(z)^* + P(-z)P(-z)^* = I_r, \quad |z| = 1,$$

is satisfied.

In the next section we will give a necessary and sufficient condition for (2.1) in terms of an intimate relation between these centers and the vectors $\mathbf{y}_\alpha$ in (1.8), or equivalently $\mathbf{y}_\alpha$ in (1.3), where $K \le m$. We will also show that K-balancing of $\widetilde{\Phi}$ is equivalent to discrete polynomial preservation of total degree $K-1$ by its mask $\{G_{\mathbf{k}}\}$, and that K-balancing of $\widetilde{\Phi}$ implies discrete polynomial annihilation of total degree $K-1$ by the highpass filters $\{H^h_{\mathbf{k}}\}$ corresponding to their lowpass filter $\{G_{\mathbf{k}}\}$.

2.2 Characterization Theorems

Suppose that Φ and $\widetilde{\Phi}$ are compactly supported bi-orthogonal dual refinable vector-valued functions in $(L^2)^r$ with dilation matrix A and finite masks $\{P_{\mathbf{k}}\}$ and $\{G_{\mathbf{k}}\}$, respectively. Let $Q^h(\mathbf{z})$ and $H^h(\mathbf{z}), 1 \le h < a$, be the matrix-valued Laurent polynomials satisfying (1.23) and that $\Psi^h, \widetilde{\Psi}^h$ be the vector-valued functions defined by (1.24).

Suppose Φ satisfies the sum rule with order m and $\mathbf{y}_\alpha, |\alpha| < m$ in (1.8). We have the following characterizations of K-balancing for $\widetilde{\Phi}$, $K \le m$.

Theorem 2.1. *$\widetilde{\Phi}$ is K-balanced relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r) \subset \mathbb{R}^s$ if and only if $\mathbf{y}_\alpha = [y_{\alpha,1}, \cdots, y_{\alpha,r}]$ satisfy*

$$y_{\alpha,l} = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\mathbf{a}_l - \mathbf{a}_1)^\beta y_{\alpha-\beta,1}, \quad 2 \le l \le r, |\alpha| < K. \tag{2.3}$$

For the case $s = 1$ with $a_\ell = \frac{\ell-1}{r}$, (2.3) reduces to

$$y_{\alpha,l} = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\frac{\ell-1}{r})^\beta y_{\alpha-\beta,1}, \quad 2 \le l \le r, 0 \le \alpha < K. \tag{2.4}$$

Theorem 2.1 tells us that $\widetilde{\Phi}$ is K-balanced if and only if $\mathbf{y}_\alpha$ can be so chosen that both (1.8) and (2.3) are satisfied for $|\alpha| < K$. Theorem 2.1 also enables us to decide the possible balanced order of the bi-orthogonal dual scaling vectors for a given refinable function vector, and tells us how to construct high balancing orthogonal multi-wavelets and the masks of the primal refinable function vectors with high balanced bi-orthogonal duals. We also have have the following:

Theorem 2.2. *$\widetilde{\Phi}$ is K-balanced relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r) \subset \mathbb{R}^s$ if and only if*

$$\sum_{\mathbf{k}} [(\mathbf{k}+\mathbf{a}_1)^\alpha, \cdots, (\mathbf{k}+\mathbf{a}_r)^\alpha] \Phi(\mathbf{x}-\mathbf{k}) \in \pi^s_{|\alpha|},$$

for all $|\alpha| < K$.

The following result tells us that K-balancing of $\widetilde{\Phi}$ implies the preservation of π^s_K for G and the annihilation of π^s_K for H^h.

Theorem 2.3. *$\widetilde{\Phi}$ is K-balanced relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r) \subset \mathbb{R}^s$ if and only if*

$$\sum_{\mathbf{k}} G_{\mathbf{k}-A\mathbf{j}} \begin{bmatrix} (\mathbf{k}+\mathbf{a}_1)^\alpha \\ \vdots \\ (\mathbf{k}+\mathbf{a}_r)^\alpha \end{bmatrix} = \begin{bmatrix} p_\alpha(\mathbf{j}+\mathbf{a}_1) \\ \vdots \\ p_\alpha(\mathbf{j}+\mathbf{a}_r) \end{bmatrix}, \quad |\alpha| < K, \mathbf{j} \in \mathbb{Z}^s,$$

where $p_\alpha \in \pi^s_{|\alpha|}$. If $\widetilde{\Phi}$ is K-balanced relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r) \subset \mathbb{R}^s$, then

$$\sum_{\mathbf{k}} H^h_{\mathbf{k}-A\mathbf{j}} \begin{bmatrix} (\mathbf{k}+\mathbf{a}_1)^\alpha \\ \vdots \\ (\mathbf{k}+\mathbf{a}_r)^\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |\alpha| < K, \mathbf{j} \in \mathbb{Z}^s, 1 \le h < a.$$

We also have the following result that gives the intimate relationship of polynomial preservation of total degree $K-1$ by the mask of Φ and K-balancing of its bi-orthogonal dual $\widetilde{\Phi}$.

Theorem 2.4. *$\widetilde{\Phi}$ is K-balanced relative to $(\mathbf{a}_1, \cdots, \mathbf{a}_r) \subset \mathbb{R}^s$ if and only if*

$$\sum_{\mathbf{k}} P^T_{\mathbf{j}-A\mathbf{k}} \begin{bmatrix} (\mathbf{k}+\mathbf{a}_1)^\alpha \\ \vdots \\ (\mathbf{k}+\mathbf{a}_r)^\alpha \end{bmatrix} = \begin{bmatrix} q_\alpha(\mathbf{j}+\mathbf{a}_1) \\ \vdots \\ q_\alpha(\mathbf{j}+\mathbf{a}_r) \end{bmatrix}, \quad \mathbf{j} \in \mathbb{Z}^s,$$

for all $|\alpha| < K$, where $q_\alpha \in \pi^s_{|\alpha|}$.

The proofs of Theorems 2.1–2.4 are given in our paper [7].

2.3 Univariate B-splines

In this section we discuss univariate balanced bi-orthogonal dual $\widetilde{\Phi}$ (relative to $a_1 = 0, a_2 = \frac{1}{2}$) of Φ with spline components. We will use (1.8) to determine the sum rule order of the masks. For bi-orthogonal filters P, G, we will apply Theorem 1.5 to decide whether they generate bi-orthogonal refinabe function vectors, and Theorem 1.4 to calculate the Sobolev smoothness of the refinable function vectors.

Example 1. Let $N_{3,0}, N_{3,1}$ be the normalized quadratic B-splines with knots $0, 0, 1, 1$ and $0, 1, 1, 2$, respectively, i.e.,

$$N_{3,0}(x) = 2x(1-x)\chi_{[0,1]}, \quad N_{3,1}(x) = x^2\chi_{[0,1]} + (2-x)^2\chi_{[1,2]}.$$

Then $\Phi_1 := [N_{3,0}, N_{3,1}]^T$ is refinable with two-scale symbol given by

$$P_1(z) = \frac{1}{8}(\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} z^2),$$

which satisfies the sum rule of order 3 with

$$\mathbf{y}_0 = [1, 1], \quad \mathbf{y}_1 = [\frac{1}{2}, 1], \quad \mathbf{y}_2 = [0, 1],$$

(cf. [46]). It is clear that these vectors don't satisfy (2.4) with $K = 3$, but $\mathbf{y}_0, \mathbf{y}_1$ satisfy (2.4) with $K = 2$. Thus it is possible to construct a 2-balanced (but not 3-balanced) bi-orthogonal dual of Φ_1 by applying Theorem 2.1.

On the other hand, for by

$$\phi_1 = 3N_{3,0} + N_{3,1}, \quad \phi_2 = -N_{3,0} + N_{3,1},$$

$\Phi_2 := [\phi_1, \phi_2]^T$ is refinable with two-scale symbol given by

$$P_2(z) = \frac{1}{32}(\begin{bmatrix} 13 & 15 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} 12 & 4 \\ 4 & 12 \end{bmatrix} z + \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} z^2),$$

which satisfies the sum rule of order 3 with

$$\mathbf{y}_0 = [1, 1], \quad \mathbf{y}_1 = [\frac{3}{4}, \frac{5}{4}], \quad \mathbf{y}_2 = [\frac{1}{2}, \frac{3}{2}].$$

One can easily verify that these vectors satisfy (2.4) with $K = 3$. Therefore it is possible to construct a 3-balanced bi-orthogonal dual $\widetilde{\Phi}$ of Φ. Indeed, we can construct such a $\widetilde{\Phi}_2$ in $W^{1.1442}(\mathbb{R})$ with supp$(\widetilde{\Phi}_2)$=$[-1,3]$. Its mask $\{G_k\}$ is given by

$$G_{-1} = \begin{bmatrix} -0.18960780661003 & -0.18960780661003 \\ 0.24341593393929 & 0.24341593393929 \end{bmatrix},$$
$$G_0 = \begin{bmatrix} 0.81782419542379 & 0.58025231948912 \\ -1.08301023392189 & -0.64562424126293 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} 1.07469506607298 & -0.12915190410120 \\ 1.14314871878723 & 1.68587976841759 \end{bmatrix},$$
$$G_2 = \begin{bmatrix} 0.12120349037547 & -0.07884298044345 \\ 1.38787180492559 & -0.86142245396542 \end{bmatrix},$$
$$G_3 = \begin{bmatrix} -0.04574279673035 & 0.03897822313370 \\ -0.58328696638954 & 0.46961173553079 \end{bmatrix},$$

where $G_k = \mathbf{0}$ for $k < -1$ or $k > 3$. $\Box$

Example 2. Let $N_{4,0}, N_{4,1}$ be the normalized cubic B-splines with knots $0,0,1,1,2$ and $0,1,1,2,2$, respectively, i.e.,

$$N_{4,0}(x) = \frac{1}{2}x^2(3-\frac{5}{2}x)\chi_{[0,1]} + \frac{1}{4}(2-x)^3\chi_{[1,2]},$$
$$N_{4,1}(x) = \frac{1}{4}x^3\chi_{[0,1]} + \frac{1}{2}(2-x)^2(\frac{5}{2}x-2)\chi_{[1,2]}.$$

Then $\Phi_3 := [N_{4,0}, N_{4,1}]^T$ is refinable with two-scale symbol given by

$$P_3(z) = \frac{1}{8}(\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} z^2),$$

which satisfies the sum rule of order 4 with

$$\mathbf{y}_0 = [1,1], \quad \mathbf{y}_1 = [\frac{2}{3}, \frac{4}{3}], \quad \mathbf{y}_2 = [\frac{1}{3}, \frac{5}{3}], \quad \mathbf{y}_3 = [0,2].$$

However, it is clear that $\mathbf{y}_0, \mathbf{y}_1$ do not satisfy (2.4) with $K = 2$, but $\mathbf{y}_0$ does with $K = 1$. Thus we can only construct a 1-balanced (but not 2-balanced) bi-orthogonal dual.

On the other hand, for

$$\varphi_1 = 2N_{4,0} + 2N_{4,1}, \quad \varphi_2 = \frac{2}{3}(N_{4,1} - N_{4,0}),$$

that constitute the canonical Hermite cubic basis, the refinement mask corresponding to $\Phi_4 := [\varphi_1, \varphi_2]^T$ satisfies the sum rule of order 4 with

$$\mathbf{y}_0 = [1,0], \quad \mathbf{y}_1 = [1,1], \quad \mathbf{y}_2 = [1,2], \quad \mathbf{y}_3 = [1,3].$$

Hence, it follows from Theorem 2.1 that Φ_4 does not have a balanced bi-orthogonal dual.

Next, consider the splines

$$\phi_1 = \frac{1}{6}(7N_{4,0} - N_{4,1}), \quad \phi_2 = \frac{1}{6}(-N_{4,0} + 7N_{4,1}),$$

and $\Phi_5 := [\phi_1, \phi_2]^T$ with two-scale symbol given by

$$P_5(z) = \frac{1}{32}(\begin{bmatrix} 11 & 21 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 24 & 8 \\ 8 & 24 \end{bmatrix} z + \begin{bmatrix} 1 & -1 \\ 21 & 11 \end{bmatrix} z^2),$$

which satisfies the sum rule of order 4 with

$$\mathbf{y}_0 = [1,1], \quad \mathbf{y}_1 = [\frac{3}{4}, \frac{5}{4}], \quad \mathbf{y}_2 = [\frac{1}{2}, \frac{3}{2}], \quad \mathbf{y}_3 = [\frac{1}{4}, \frac{7}{4}].$$

One can easily verify that these vectors don't satisfy (2.4) with $K = 4$, but $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2$ satisfy (2.4) with $K = 3$. It is therefore possible to construct a 3-balanced bi-orthogonal dual $\widetilde{\Phi}_5$ of Φ_5. Indeed we can construct such a $\widetilde{\Phi}_5$ in $W^{0.8252}(\mathbb{R})$ with supp$(\widetilde{\Phi}_5)$=$[-1,3]$ and mask given by

$$G_{-1} = \frac{1}{512}\begin{bmatrix} 133 & -277 \\ -29 & 61 \end{bmatrix}, \quad G_0 = \frac{1}{512}\begin{bmatrix} -161 & 815 \\ 33 & -175 \end{bmatrix},$$

$$G_1 = \frac{1}{256}\begin{bmatrix} 395 & -83 \\ -83 & 395 \end{bmatrix}, \quad G_2 = \frac{1}{512}\begin{bmatrix} -175 & 33 \\ 815 & -161 \end{bmatrix},$$

$$G_3 = \frac{1}{512}\begin{bmatrix} 61 & -29 \\ -277 & 133 \end{bmatrix}, \quad G_k = \mathbf{0}, k \in \mathbb{Z}\backslash\{-1,0,1,2,3\}.$$

□

2.4 The parametric approach

The following two examples are based on the parametric expression of the bi-orthogonal two-scale symbols $P(z), G(z)$ in [32]. For the parametric expression of the symmetric filters which could generate high balanced orthonormal Φ, the reader is referred to [35].

Example 3. Consider the bi-orthogonal two-scale symbols

$$P(z)=\frac{1}{2}\begin{bmatrix}\widetilde{a}\\ -\widetilde{b}\end{bmatrix}[d,-c]+\frac{1}{2}\begin{bmatrix}\widetilde{e}&0\\ 0&\widetilde{f}\end{bmatrix}z+\frac{1}{2}\begin{bmatrix}\widetilde{a}\\ \widetilde{b}\end{bmatrix}[d,c]z^2,$$

$$G(z)=\frac{1}{2}\begin{bmatrix}a\\ -b\end{bmatrix}[c,-d]+\frac{1}{2}\begin{bmatrix}e&0\\ 0&f\end{bmatrix}z+\frac{1}{2}\begin{bmatrix}a\\ b\end{bmatrix}[c,d]z^2,$$

with $b=7/(16dc\widetilde{b})$. If we choose

$$e=\widetilde{e}=1, f=\widetilde{f}=\frac{1}{2}, ac=\widetilde{a}d=\frac{1}{2}, c=d/4/(d\widetilde{b}-1),$$

then P,G satisfy the sum rule of order 2 with the vectors $\mathbf{y}_0,\mathbf{y}_1$ for P given by

$$\mathbf{y}_0=[1,0],\quad \mathbf{y}_1=[c_0,\frac{1}{4}],$$

for some $c_0\in\mathbb{R}$. Let

$$R_0:=\frac{\sqrt{2}}{2}\begin{bmatrix}1&-1\\ 1&1\end{bmatrix}.$$

Then for suitable choices of $\widetilde{b},d$, the symbols $R_0G(z)R_0^T$, $R_0P(z)R_0^T$ generate a 2-balanced bi-orthogonal dual $\widetilde{\Phi}$ of a 1-balanced Φ. For $(\widetilde{b},d)=-(6/13,3/2)$, $\widetilde{\Phi}\in W^{0.736924}(\mathbb{R}),\Phi\in W^{1.356559}(\mathbb{R})$, and for $(\widetilde{b},d)=(3/10,13/6)$, $\widetilde{\Phi}\in W^{0.4421}(\mathbb{R}),\Phi\in W^{1.6027}(\mathbb{R})$. $\square$

Example 4. Let $P=\frac{1}{2}\sum_{k=0}^{3}P_kz^k, G=\frac{1}{2}\sum_{k=0}^{3}G_kz^k$ be bi-orthogonal two-scale symbols with

$$P_0=\frac{1}{2}\begin{bmatrix}1-\frac{x_3}{by}&\frac{1}{4y}\\ 1/b-x_1&-\frac{1}{4}\end{bmatrix},\quad P_1=\frac{1}{2}\begin{bmatrix}1+\frac{x_3}{by}&\frac{1}{4y}\\ -1/b-x_1&\frac{1}{4}\end{bmatrix},$$

$$P_2=\frac{1}{2}\begin{bmatrix}1+\frac{x_3}{by}&-\frac{1}{4y}\\ 1/b+x_1&\frac{1}{4}\end{bmatrix},\quad P_3=\frac{1}{2}\begin{bmatrix}1-\frac{x_3}{by}&-\frac{1}{4y}\\ -1/b+x_1&-\frac{1}{4}\end{bmatrix},$$

$$G_0=\frac{1}{2}\begin{bmatrix}1-x_1b&-x_2\\ b-x_3/y&x_2/y\end{bmatrix}, G_1=\frac{1}{2}\begin{bmatrix}1+x_1b&-x_2\\ -b-x_3/y&-x_2/y\end{bmatrix},$$

$$G_2=\frac{1}{2}\begin{bmatrix}1+x_1b&x_2\\ b+x_3/y&-x_2/y\end{bmatrix}, G_3=\frac{1}{2}\begin{bmatrix}1-x_1b&x_2\\ -b+x_3/y&x_2/y\end{bmatrix}.$$

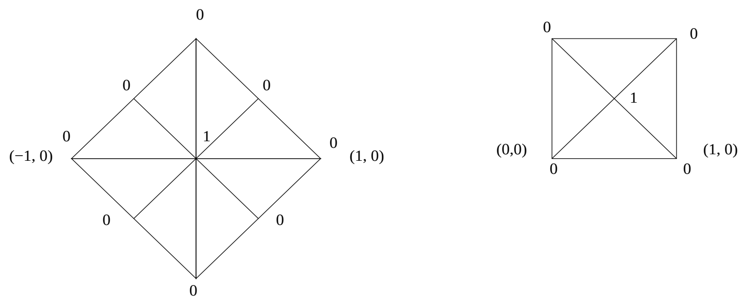

Figure 1: Bézier-nets for ϕ_1 and ϕ_2

By setting

$$x_3 = \frac{b+1}{2}, \quad x_1 = \frac{16b+1}{2b(4b+5)}, \quad x_2 = -\frac{3b-1}{b(4b+5)}, \quad y = x_1x_3 - \frac{x_2}{4},$$

P, G satisfy the sum rule of order 3 and 2 respectively, and the vectors $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2$ for $R_0PR_0^T$ have the structure (2.3) for $K = 3$. With $b = 28/13$, $\widetilde{\Phi} \in W^{0.7222}(\mathbb{R}), \Phi \in W^{2.5524}(\mathbb{R})$, and with $b = 2$, $\widetilde{\Phi} \in W^{0.7857}(\mathbb{R}), \Phi \in W^{2.4559}(\mathbb{R})$. In both cases, $R_0\widetilde{\Phi}$ is 3-balanced. □

2.5 Bivariate linear splines

Example 5. Let $\triangle^2$ denote the four-directional mesh with grid lines given by $x = j, y = k, x + y = \ell$, and $x - y = m$, where $j, k, \ell, m \in \mathbb{Z}$. Let ϕ_1 denote the bivariate piecewise linear hat function with $\phi_1(0,0) = 1$ and support given by the square with vertices $(1,0), (0,1), (-1,0)$ and $(0,-1)$, and ϕ_2 be the bivariate piecewise linear hat function with $\phi_2(\frac{1}{2}, \frac{1}{2}) = 1$ and support given by the square $[0,1] \times [0,1]$. See Fig.1 for the Bézier-nets of ϕ_1, ϕ_2. Then $\{\phi_\ell(\cdot - \mathbf{k}) : \ \ell = 1, 2, \mathbf{k} \in \mathbb{Z}^2\}$ is a basis of the space $S_1^0(\triangle^2)$ of continuous bivariate piecewise linear splines on $\triangle^2$. Then $\Phi = [\phi_1, \phi_2]^T$ is refinable with respect to the quincunx dilation matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.5}$$

See Fig.2 for the Bézier-nets of $\phi_1(A^{-1}\cdot), \phi_2(A^{-1}\cdot)$. The two-scale symbol of Φ is given by

$$P_A(\mathbf{z}) = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{2}(1+z_1^{-1})(1+z_2^{-1}) \\ z_1 & 0 \end{bmatrix}.$$

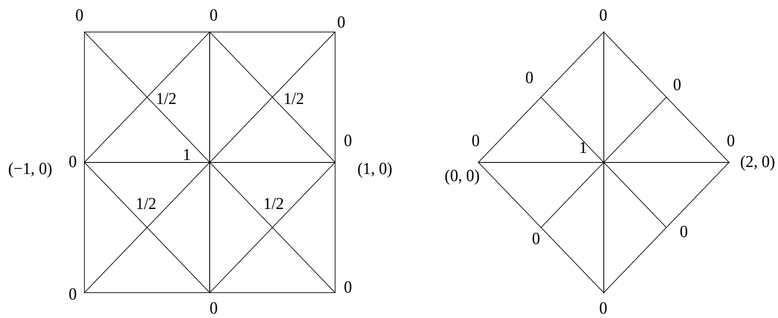

Figure 2: Bézier-nets for $\phi_1(A^{-1}\cdot), \phi_2(A^{-1}\cdot)$

One can easily verify that P satisfies the sum rule of order 2 with

$$\mathbf{y}_{0,0} = [1, 1], \quad \mathbf{y}_{0,1} = [0, \frac{1}{2}], \quad \mathbf{y}_{1,0} = [0, \frac{1}{2}].$$

By Theorem 2.1, it is possible to construct a 2-balanced bi-orthogonal dual $\tilde{\Phi}$ with balanced centers $\mathbf{a}_1 = \mathbf{0}, \mathbf{a}_2 = (\frac{1}{2}, \frac{1}{2})$. Indeed we can construct such $\tilde{\Phi}$ in $W^{1.5420}(\mathbb{R}^2)$ with mask $\{G_{\mathbf{k}}\}$ supported in $[-2, 2]^2 \cap \mathbb{Z}^2$. The mask is not provided here. $\square$

Example 6. Consider the two-scale symbol $P_A(\mathbf{z})$ in Example 5, and observe that $\Phi = [\phi_1, \phi_2]^T$ is also refinable with respect to the dilation matrix $2I_2$ (cf. [13]), with two-scale symbol $P(\mathbf{z})$ given by

$$P(\mathbf{z}) = P_A(e^{-iA\omega})P_A(\mathbf{z}), \quad \mathbf{z} = e^{-i\omega}.$$

Then P satisfies the sum rule of order 2 with the same $\mathbf{y}_\alpha, |\alpha| < 2$. Φ has a 2-balanced bi-orthogonal dual in $W^{0.2783}(\mathbb{R}^2)$ with mask $\{G_{\mathbf{k}}\}$ supported in $[-2, 2]^2 \cap \mathbb{Z}^2$. $\square$

2.6 Bivariate C^1 cubic splines

Example 7. Let $S_3^1(\triangle^2)$ denote the bivariate C^1 cubic spline space on the four-directional mesh $\triangle^2$ and let $\phi_1, \cdots, \phi_5$ be its basis functions, as constructed in [38]. See Fig.3 for the Bézier-nets for ϕ_j, $1 \le j \le 5$. In Fig.3A and Fig.3B, only portions of the Bézier-nets of ϕ_1, ϕ_2 are displayed due to the symmetry

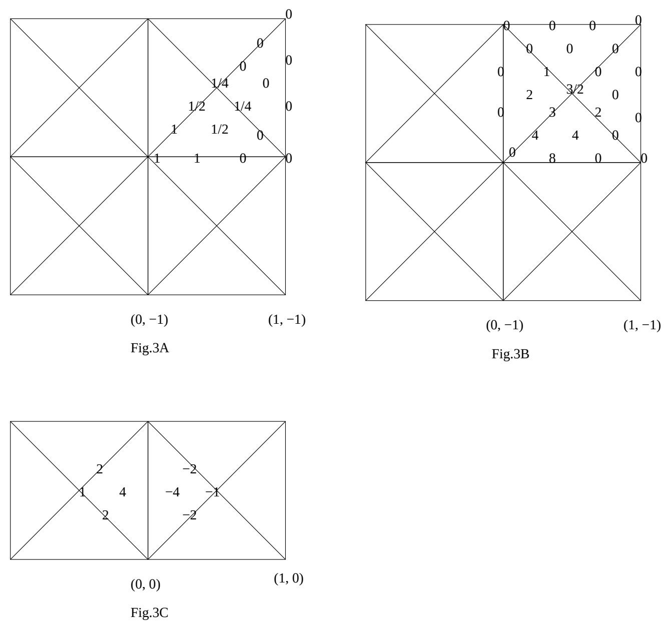

Figure 3: Bézier-nets for ϕ_1, $24\phi_2$ and $96\phi_4$ with $\phi_3(x,y) = \phi_2(y,x)$, $\phi_5(x,y) = -\phi_4(-y,-x)$

of ϕ_1, ϕ_2: Here, ϕ_1 is symmetric about lines $x = y, y = -x$ and $x = 0$, $y = 0$, and ϕ_2 is symmetric about the x-axis and anti-symmetric about the y-axis. In Fig. 3C, the Bézier coefficients that are not displayed are supposed to be zeros. Then $\Phi = [\phi_1, \cdots, \phi_5]^T$ is refinable with respect to the quincunx dilation matrix A given in (2.5). See Fig.4, Fig.5 and Fig.6 for the Bézier-nets for $\phi_1(A^{-1}\cdot)$, $\phi_2(A^{-1}\cdot)$ and $\phi_4(A^{-1}\cdot)$. Again due to the symmetric properties of ϕ_1, ϕ_2, only portions of the Bézier-nets of $\phi_1(A^{-1}\cdot)$ and $\phi_2(A^{-1}\cdot)$ are displayed in Fig.4 and Fig.5, while in Fig.6, the Bézier coefficients that are not displayed are supposed to be zeros. This mask is given by

$$P_{0,-2} = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, P_{-1,0} = \frac{1}{16}\begin{bmatrix} 4 & 12 & 0 & -24 & 0 \\ -1 & -3 & 1 & 2 & 2 \\ -1 & -3 & -1 & 10 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & 1 & -1 \end{bmatrix},$$

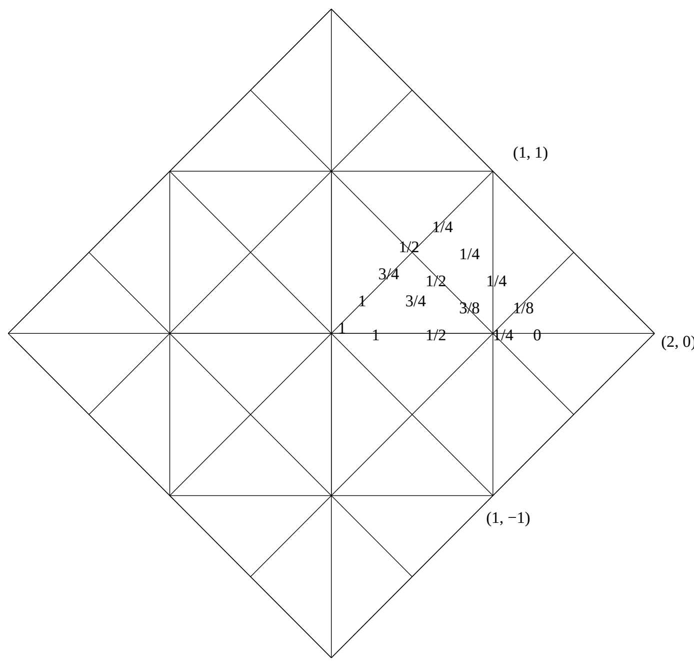

Figure 4: Bézier-nets for $\phi_1(A^{-1}\cdot)$

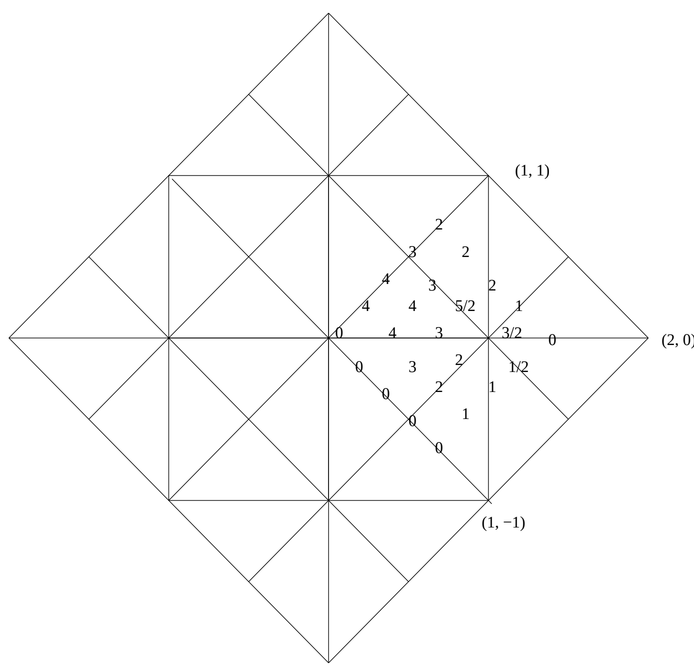

Figure 5: Bézier-nets for $24\phi_2(A^{-1}\cdot)$

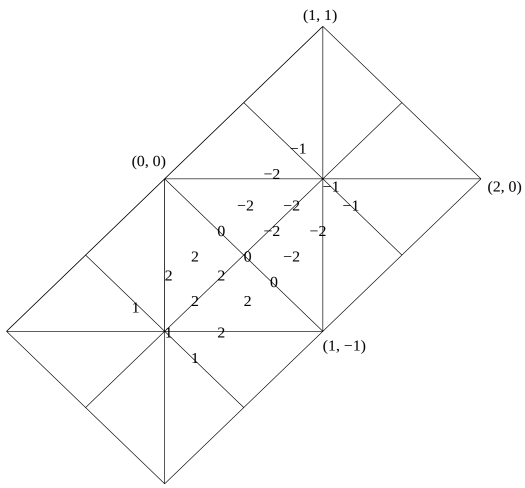

Figure 6: Bézier-nets for $96\phi_4(A^{-1}\cdot)$

$$P_{0,-1}=\frac{1}{16}\begin{bmatrix} 4 & 0 & 12 & 0 & -24\\ -1 & 1 & -3 & 10 & 10\\ 1 & 1 & 3 & 10 & -2\\ \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & -3 & 1\\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & -3 & 3 \end{bmatrix}, P_{-1,-1}=\frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & -24 & 0\\ 0 & 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -3 & 0 \end{bmatrix},$$

$$P_{1,-1}=\frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 24 & -24\\ 0 & 0 & 0 & 2 & 2\\ 0 & 0 & 0 & 10 & -10\\ 0 & 0 & 0 & -3 & -3\\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},\quad P_{0,0}=\frac{1}{16}\begin{bmatrix} 16 & 0 & 0 & 0 & 0\\ 0 & 8 & 8 & 10 & 10\\ 0 & 8 & -8 & 10 & -10\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$P_{0,1}=\frac{1}{16}\begin{bmatrix} 4 & 0 & -12 & 0 & 24\\ 1 & 1 & -3 & 2 & 2\\ -1 & 1 & 3 & 2 & -10\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\quad P_{1,1}=\frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 24\\ 0 & 0 & 0 & 0 & 10\\ 0 & 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_{1,0}=\frac{1}{16}\begin{bmatrix} 4 & -12 & 0 & 24 & 0\\ 1 & -3 & 1 & 10 & 10\\ 1 & -3 & -1 & 2 & -10\\ -\frac{1}{6} & \frac{1}{2} & \frac{1}{2} & 1 & -3\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\quad P_{2,0}=\frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $P_{\mathbf{k}}=\mathbf{0}$ for other indices $\mathbf{k}$.

Remark 2.1. *Observe that for each j, supp($\phi_j(\cdot - \mathbf{k})$ for $\mathbf{k} \in \mathbb{Z}^2$ with $|\mathbf{k}| = 1$ is not a subset of $\cup_{j=1}^{5} supp(\phi_j(A^{-1}\cdot))$. However Φ is still refinable with a finite mask.*

Of course $\Phi = [\phi_1, \cdots, \phi_5]^T$ is also refinable with $2I_2$. In this case the two-scale symbol, denoted by $R(\mathbf{z})$, is given by $R(\mathbf{z}) = P(e^{-iA\omega})P(\mathbf{z})$, where P is the two-scale symbol for the quincunx dilation matrix A given above, and the nonzero matrix coefficients $R_{\mathbf{k}}$ of $R(\mathbf{z})$ are given by

$$R_{-1,-2} = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R_{-1,-1} = \frac{1}{8}\begin{bmatrix} 2 & 3 & 3 & -6 & 6 \\ -\frac{1}{2} & -\frac{1}{2} & -1 & 1 & -2 \\ -\frac{1}{2} & -1 & -\frac{1}{2} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix},$$

$$R_{-1,0} = \frac{1}{8}\begin{bmatrix} 4 & 6 & 0 & -6 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}, R_{0,-1} = \frac{1}{8}\begin{bmatrix} 4 & 0 & 6 & 0 & -6 \\ 0 & 2 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$R_{-1,1} = \frac{1}{8}\begin{bmatrix} 2 & 3 & -3 & 6 & -6 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & -1 & 2 \\ \frac{1}{2} & 1 & -\frac{1}{2} & 2 & -1 \\ \frac{1}{12} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{12} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}, R_{0,1} = \frac{1}{8}\begin{bmatrix} 4 & 0 & -6 & 0 & 6 \\ 0 & 2 & 0 & 0 & -2 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$R_{1,-2} = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R_{1,-1} = \frac{1}{8}\begin{bmatrix} 2 & -3 & 3 & 6 & -6 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 1 & -2 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_{1,0} = \frac{1}{8}\begin{bmatrix} 4 & -6 & 0 & 6 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R_{1,1} = \frac{1}{8}\begin{bmatrix} 2 & -3 & -3 & -6 & 6 \\ \frac{1}{2} & -\frac{1}{2} & -1 & -1 & 2 \\ \frac{1}{2} & -1 & -\frac{1}{2} & -2 & 1 \\ -\frac{1}{12} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_{2,-1} = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R_{2,1} = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_{0,0} = \mathrm{diag}(1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{8}, \tfrac{1}{8}).$$

One can also verify that both P and R satisfy the sum rule of order 4 with the same vectors

$$\begin{aligned}
&\mathbf{y}_{0,0} = [1,0,0,0,0], \quad \mathbf{y}_{1,0} = [0,1,0,0,0], \quad \mathbf{y}_{0,1} = [0,0,1,0,0],\\
&\mathbf{y}_{2,0} = \mathbf{y}_{1,1} = \mathbf{y}_{2,0} = \mathbf{y}_{3,0} = \mathbf{y}_{0,3} = [0,0,0,0,0],\\
&\mathbf{y}_{2,1} = [0,0,0,0,2], \quad \mathbf{y}_{1,2} = [0,0,0,2,0].
\end{aligned}$$

However, from the structure of $\mathbf{y}_\alpha$, we know that Φ does not have a balanced bi-orthogonal dual. $\square$

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