Approximation Power of Refinable Vectors of Functions †

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Abstract

In this paper we survey recent results on approximation power of refinable vectors of functions. Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be an $r \times 1$ vector of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. The first part of this paper is devoted to an investigation of approximation power of $\mathbb{S}(\Phi)$, the shift-invariant space generated from Φ. We review results on characterizations of the approximation order of $\mathbb{S}(\Phi)$ and describe approximation schemes that achieve the optimal approximation order. We also give a self-contained treatment of various equivalent forms of the Strang-Fix conditions.

We say that Φ is refinable if $\Phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\Phi(M \cdot - \alpha)$, where M is an expansive $s \times s$ integer matrix, and the refinement mask a is finitely supported. The second part of this paper is dedicated to a study of accuracy of Φ. We review results on characterizations of the accuracy of Φ in terms of the mask in both time and frequency domains. We also discuss the relationship between the accuracy of Φ and the sum rules associated with the mask. Examples are provided to illustrate the general theory.

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§1. Introduction

In this paper we survey recent results on approximation power of refinable vectors of functions. Since wavelets are generated from refinable functions, this study plays an important role in wavelet analysis. Our goal is to give a characterization of the approximation order of a refinable vector of functions in a compact form, which can be easily applied to concrete problems.

Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}^s$ the s-dimensional Euclidean space. An element of $\mathbb{R}^s$ is also viewed as an $r \times 1$ vector of real numbers. The inner product of two vectors x and y in $\mathbb{R}^s$ is denoted by $x \cdot y$.

Let f be a (Lebesgue) measurable function from $\mathbb{R}^s$ to $\mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers. For $1 \le p < \infty$, let

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p \, dx \right)^{1/p}.$$

For $p = \infty$, let $\|f\|_\infty$ be the essential supremum of f on $\mathbb{R}^s$. By $L_p(\mathbb{R}^s)$ we denote the Banach space of all measurable functions f such that $\|f\|_p < \infty$. A function f is said to be integrable if f lies in $L_1(\mathbb{R}^s)$. If $1 \le p, p' \le \infty$ and $1/p + 1/p' = 1$, then p and p' are called conjugate exponents. Suppose $f \in L_p(\mathbb{R}^s)$ and $g \in L_{p'}(\mathbb{R}^s)$, where p and p' are conjugate exponents. We define

$$\langle f, g \rangle := \int_{\mathbb{R}^s} f(x) g(x) \, dx.$$

Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0$ the set of nonnegative integers. An s-tuple $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is called a **multi-index**. The length of μ is $|\mu| := \mu_1 + \cdots + \mu_s$, and the factorial of μ is $\mu! := \mu_1! \cdots \mu_s!$. For $\mu, \nu \in \mathbb{N}_0^s$, $\nu \le \mu$ means $\nu_j \le \mu_j$, $j = 1, \ldots, s$. If $\nu \le \mu$ and $\nu \ne \mu$, we write $\nu < \mu$. For $\nu \le \mu$, we define

$$\binom{\mu}{\nu} := \frac{\mu!}{\nu!(\mu - \nu)!}.$$

Let $\mathbb{Z}$ denote the set of integers. By $\ell(\mathbb{Z}^s)$ we denote the linear space of all sequences on $\mathbb{Z}^s$. A sequence b on $\mathbb{Z}^s$ is said to be finitely supported if $b(\alpha) \ne 0$ only for finitely many α. Let $\ell_0(\mathbb{Z}^s)$ denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$.

A square integer matrix M is said to be **expansive** if its spectrum lie outside the closed unit disc. The transpose of M is denoted by M^T.

For a vector $y \in \mathbb{R}^s$, we use D_y to denote the differential operator given by

$$D_y f(x) := \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}, \qquad x \in \mathbb{R}^s.$$

Also, we use ∇_y to denote the difference operator given by

$$\nabla_y f = f - f(\cdot - y).$$

Let $e_1, \ldots, e_s$ be the unit coordinate vectors in $\mathbb{R}^s$. For $j = 1, \ldots, s$, we write D_j for D_{e_j}. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, D^μ stands for the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$.

For a positive integer k and $1 \le p \le \infty$, the Sobolev semi-norm $|\cdot|_{k,p}$ is defined by

$$|f|_{k,p} := \sum_{|\mu|=k} \|D^\mu f\|_p.$$

The Sobolev space $W_p^k(\mathbb{R}^s)$ consists of all functions $f \in L_p(\mathbb{R}^s)$ such that $|f|_{k,p} < \infty$. We use $C(\mathbb{R}^s)$ to denote the space of continuous functions on $\mathbb{R}^s$. The space $C^k(\mathbb{R}^s)$ consists of all continuous functions f for which $D^\mu f \in C(\mathbb{R}^s)$ for all $|\mu| \le k$.

The Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^s,$$

where i denotes the imaginary unit. The domain of the Fourier transform can be naturally extended to compactly supported distributions.

For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$, define

$$x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}.$$

The function $x \mapsto x^\mu$ ($x \in \mathbb{R}^s$) is called a monomial and its (total) degree is $|\mu|$. A polynomial is a linear combination of monomials. The degree of a polynomial $q = \sum_\mu c_\mu x^\mu$ is defined to be $\deg q := \max\{|\mu| : c_\mu \ne 0\}$. By $q(D)$ we denote the differential operator $\sum_\mu c_\mu D^\mu$. Let Π denote the linear space of all polynomials, and let Π_k denote the linear space of all polynomials of degree at most k. By convention, $\Pi_{-1} = \{0\}$.

Now let $\Phi = \{\phi_1, \ldots, \phi_r\}$ be a collection of compactly supported functions in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$). The first part of this paper is devoted to an investigation of approximation power of $\mathbb{S}(\Phi)$, the shift-invariant space generated from Φ. Under the mild condition that the sequences $(\hat{\phi}_j(2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $j = 1, \ldots, r$, are linearly independent, it was shown in [15] that $\mathbb{S}(\Phi)$ provides approximation order k if and only if $\mathbb{S}(\Phi) \supset \Pi_{k-1}$. In Section 2 we shall review this result and discuss approximation schemes which achieve the optimal approximation order. With the help of the Poisson summation formula, polynomial reproducibility of $\mathbb{S}(\Phi)$ can be described as conditions on the Fourier transforms of $\phi_1, \ldots, \phi_r$ at $2\pi\beta$, $\beta \in \mathbb{Z}^s$. These conditions are often called the Strang-Fix conditions (see [30]). In Section 3 we will give a self-contained treatment of various equivalent forms of the Strang-Fix conditions. As far as polynomial reproducibility is concerned, $\phi_1, \ldots, \phi_r$ are allowed to be compactly supported distributions. We say that Φ has **accuracy** k if $\mathbb{S}(\Phi) \supset \Pi_{k-1}$.

For compactly supported distributions $\phi_1, \ldots, \phi_r$ on $\mathbb{R}^s$, we use the same letter Φ to denote the $r \times 1$ vector $(\phi_1, \ldots, \phi_r)^T$. We say that Φ is **refinable** if Φ satisfies the following vector refinement equation

$$\Phi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\Phi(M\cdot - \alpha),$$

where M is an expansive $s \times s$ integer matrix, and each $a(\alpha)$ is an $r \times r$ complex matrix. We call a the (refinement) **mask** and assume that a is finitely supported. The second part of this paper is dedicated to a study of accuracy of Φ. In Section 4, by using the Fourier analysis technique, we shall review results on accuracy of Φ in terms of the symbol of the mask. In Section 5 we will give a characterization of the accuracy of Φ in terms of the so-called sum rules associated with the mask. Finally, in Section 6, examples will be provided to show the reader how this characterization is applied to concrete problems.

§2. Approximation Power of Shift-invariant Spaces

A linear space S of functions from $\mathbb{R}^s$ to $\mathbb{C}$ is called **shift-invariant** if it is invariant under multi-integer translates, *i.e.*,

$$f \in S \Longrightarrow f(\cdot - \alpha) \in S \qquad \forall\, \alpha \in \mathbb{Z}^s.$$

Let Φ be a finite set of functions from $\mathbb{R}^s$ to $\mathbb{C}$. We denote by $\mathbb{S}_0(\Phi)$ the linear span of shifts of the functions in Φ. Then $\mathbb{S}_0(\Phi)$ is the smallest shift-invariant space containing Φ.

Given a function $\phi : \mathbb{R}^s \to \mathbb{C}$ and a sequence $b \in \ell(\mathbb{Z}^s)$, the **semi-convolution** $\phi *' b$ is the sum

$$\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) b(\alpha).$$

This sum makes sense if either ϕ is compactly supported or a is finitely supported. Let Φ be a finite collection of compactly supported functions from $\mathbb{R}^s$ to $\mathbb{C}$. We use $\mathbb{S}(\Phi)$ to denote the linear space of functions of the form $\sum_{\phi \in \Phi} \phi *' b_\phi$, where b_ϕ $(\phi \in \Phi)$ are sequences on $\mathbb{Z}^s$.

For a subset E of $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ and $f \in L_p(\mathbb{R}^s)$, define the distance from f to E by

$$\operatorname{dist}(f, E)_p := \inf_{g \in E} \{\|f - g\|_p\}.$$

Let S be a closed shift-invariant subspace of $L_p(\mathbb{R}^s)$. For $h > 0$, let σ_h be the scaling operator given by $\sigma_h f := f(\cdot/h)$ for functions f on $\mathbb{R}^s$. Let $S^h := \sigma_h(S)$. For a positive integer k, we say that S provides L_p**-approximation order** k if, for every sufficiently smooth function f in $L_p(\mathbb{R}^s)$,

$$\operatorname{dist}(f, S^h)_p \le C_f h^k \qquad \forall\, h > 0,$$

where C_f is a constant independent of h.

Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. Then $\mathbb{S}(\Phi) \cap L_p(\mathbb{R}^s)$ is closed in $L_p(\mathbb{R}^s)$ (see [15, Theorem 3.1]). We say that $\mathbb{S}(\Phi)$ provides approximation order k if $\mathbb{S}(\Phi) \cap L_p(\mathbb{R}^s)$ does.

The following result was established in [19, Theorem 4.1].

Theorem 2.1. *Let ϕ be a compactly supported function in $L_p(\mathbb{R}^s)$. If*

$$\sum_{\alpha \in \mathbb{Z}^s} q(\alpha)\phi(\cdot - \alpha) = q \qquad \forall\, q \in \Pi_{k-1}, \tag{2.1}$$

then $\mathbb{S}(\phi)$ provides L_p-approximation order k.

Let us review the approximation scheme given in [19]. First, choose a compactly supported nonnegative function ρ in $C^k(\mathbb{R}^s)$ such that $\int_{\mathbb{R}^s} \rho(x)\,dx = 1$. For $h > 0$, let $\rho_h(x) := \rho(x/h)/h$, $x \in \mathbb{R}^s$. Second, for given $f \in L_p(\mathbb{R}^s)$, set

$$f_h(x) := \int_{\mathbb{R}^s} (f - \nabla_y^k f)(x)\rho_h(y)\,dy, \quad x \in \mathbb{R}^s.$$

Then $f_h \in C^k(\mathbb{R}^s)$. Let

$$s_h(x) := \sum_{\gamma \in \mathbb{R}^s} f_h(h\gamma)\phi(x/h - \gamma), \quad x \in \mathbb{R}^s.$$

Suppose $f \in W_p^k(\mathbb{R}^s)$. If ϕ satisfies (2.1), then it was proved in [19, Theorem 4.1] that

$$\|f - s_h\|_p \le C|f|_{k,p}h^k, \quad h > 0,$$

where C is a constant independent of h, p, and f.

Another approximation scheme was discussed by Lei, Jia, and Cheney in [27]. The scheme is based on quasi-projections (see [26] and [14]). Suppose $\tilde{\phi}$ is a compactly supported function in $L_{p'}(\mathbb{R}^s)$, where p' is the exponent conjugate to p, *i.e.*, $1/p + 1/p' = 1$. Let T be the linear operator given by

$$Tf := \sum_{\gamma \in \mathbb{Z}^s} \langle f, \tilde{\phi}(\cdot - \gamma)\rangle \phi(\cdot - \gamma), \quad f \in L_p(\mathbb{R}^s).$$

T is called a quasi-projection operator. For $h > 0$, we denote by T^h the operator $\sigma_h T \sigma_{1/h}$, where σ_h is the scaling operator given by $\sigma_h f = f(\cdot/h)$. Suppose $f \in W_p^k(\mathbb{R}^s)$. It was proved in [27, Theorem 2.1] that there exists a positive constant C independent of h, p, and f such that

$$\|T^h f - f\|_p \le C|f|_{k,p}h^k, \quad h > 0,$$

provided $Tq = q$ for all $q \in \Pi_{k-1}$.

Now let $\Phi = \{\phi_1, \ldots, \phi_r\}$ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$). If there exists some $\phi \in \mathbb{S}_0(\Phi)$ satisfying (2.1), then Theorem 2.1 tells us that $\mathbb{S}(\Phi)$ provides approximation order k. Is the same conclusion still valid under the weaker condition $\mathbb{S}(\Phi) \supset \Pi_{k-1}$? The answer is a surprising no. The first counterexample was given by de Boor and Höllig in [3] by considering bivariate C^1-cubics. Their results can be described as follows. Denote by h the hat function given by

$$h(x) := \max\{1 - |1 - x|, 0\}, \quad x \in \mathbb{R}.$$

Let ϕ_1 and ϕ_2 be the functions on $\mathbb{R}^2$ given by

$$\phi_1(x_1, x_2) := \int_0^1 h(x_1 - t)h(x_2 - t)\,dt \quad \text{and} \quad \phi_2(x_1, x_2) := \int_0^1 h(x_2 - t)h(x_1 - x_2 + t)\,dt,$$

where $(x_1, x_2) \in \mathbb{R}^2$. In [3], de Boor and Höllig proved that $\mathbb{S}(\phi_1, \phi_2) \supset \Pi_3$ but $\mathbb{S}(\phi_1, \phi_2)$ does not provide L_∞-approximation order 4. In fact, the optimal L_∞-approximation order provided by $\mathbb{S}(\phi_1, \phi_2)$ is 3. This conclusion was also established in [2] by using a different method. Note that in this example

$$\sum_{\alpha\in\mathbb{Z}^2} \phi_1(x-\alpha) = 1 = \sum_{\alpha\in\mathbb{Z}^2} \phi_2(x-\alpha) \quad \forall\, x \in \mathbb{R}^2.$$

Suppose Φ satisfies the following additional condition: For $c_1, \ldots, c_r \in \mathbb{C}$,

$$\sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^s} c_j \phi_j(\cdot - \alpha) = 0 \quad \Longrightarrow \quad c_1 = \cdots = c_r = 0.$$

In terms of the Fourier transform, this condition is equivalent to saying that the sequences $(\hat{\phi}_j(2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $j = 1, \ldots, r$, are linearly independent. Under this condition, $\mathbb{S}(\Phi) \supset \Pi_{k-1}$ implies the existence of a function $\psi \in \mathbb{S}_0(\Phi)$ such that

$$\sum_{\alpha\in\mathbb{Z}^s} q(\alpha)\psi(\cdot - \alpha) = q \qquad \forall\, q \in \Pi_{k-1}. \tag{2.2}$$

Therefore we have the following result (see [15, Theorem 8.2]).

Theorem 2.2. *Let $\Phi = \{\phi_1, \ldots, \phi_r\}$ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. Suppose that the sequences $(\hat{\phi}_j(2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $j = 1, \ldots, r$, are linearly independent. For a positive integer k, the following statements are equivalent:*
(a) *$\mathbb{S}(\Phi)$ provides L_p-approximation order k.*
(b) *$\mathbb{S}(\Phi) \supset \Pi_{k-1}$.*
(c) *There exists an element $\psi \in \mathbb{S}_0(\Phi)$ such that (2.2) is valid.*

In this situation, quasi-projection also can be used to construct an approximation scheme. Suppose $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$ are compactly supported functions in $L_{p'}(\mathbb{R}^s)$, where p' is the exponent conjugate to p. Let T be the linear operator on $L_p(\mathbb{R}^s)$ given by

$$Tf := \sum_{j=1}^{r} \sum_{\gamma\in\mathbb{Z}^s} \langle f, \tilde{\phi}_j(\cdot - \gamma)\rangle \phi_j(\cdot - \gamma), \quad f \in L_p(\mathbb{R}^s).$$

Set $T^h := \sigma_h T \sigma_{1/h}$. Then T^h can be expressed as follows:

$$T^h f := \sum_{j=1}^{r} \sum_{\gamma\in\mathbb{Z}^s} \frac{1}{h} \langle f, \tilde{\phi}_j(\cdot/h - \gamma)\rangle \phi_j(\cdot/h - \gamma), \quad f \in L_p(\mathbb{R}^s).$$

Suppose $f \in W_p^k(\mathbb{R}^s)$. It was proved in [27, Theorem 2.1] that there exists a positive constant C independent of h, p, and f such that

$$\|T^h f - f\|_p \le C|f|_{k,p} h^k, \quad h > 0,$$

provided $Tq = q$ for all $q \in \Pi_{k-1}$.

Suppose Φ is a finite collection of compactly supported distributions on $\mathbb{R}^s$. Then the definitions of $\mathbb{S}_0(\Phi)$ and $\mathbb{S}(\Phi)$ still make sense. Moreover, if the sequences $(\hat{\phi}_j(2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $j = 1, \ldots, r$, are linearly independent, then condition (b) and condition (c) in Theorem 2.2 are equivalent.

§3. The Strang-Fix Conditions

Let ϕ be a compactly supported integrable function on $\mathbb{R}^s$. With the help of the Poisson summation formula it can be shown that the conditions in (2.1) are equivalent to the following conditions:

$$D^\mu \hat{\phi}(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \quad \forall\, |\mu| < k \quad \text{and} \quad \beta \in \mathbb{Z}^s, \tag{3.1}$$

where δ denotes the Kronecker symbol. These conditions were formulated by Strang and Fix [30]. As a matter of fact, in the univariate case ($s = 1$), these conditions were known to Schoenberg [29].

The equivalence between (2.1) and (3.1) is a special case of the more general results in the following lemma. These results were stated in [30]. Also see [4] for more detailed discussions. In what follows, by Δ_k we denote the set $\{\mu \in \mathbb{N}_0^s : |\mu| < k\}$.

Lemma 3.1. *Let $\phi_1, \ldots, \phi_r$ be compactly supported integrable functions on $\mathbb{R}^s$. For each $\mu \in \Delta_k$, suppose ψ_μ is a linear combination of $\phi_1, \ldots, \phi_r$. The following statements are equivalent:*

(a) $$\frac{x^\mu}{\mu!} = \sum_{\nu \le \mu} \sum_{\alpha \in \mathbb{Z}^s} \frac{\alpha^\nu}{\nu!} \psi_{\mu-\nu}(x - \alpha) \quad \forall\, \mu \in \Delta_k.$$

(b) $$\sum_{\nu \le \mu} \sum_{\alpha \in \mathbb{Z}^s} \frac{(\alpha - x)^\nu}{\nu!} \psi_{\mu-\nu}(x - \alpha) = \delta_{0\mu} \quad \forall\, \mu \in \Delta_k.$$

(c) $$\sum_{\nu \le \mu} \frac{(-iD)^\nu}{\nu!} \hat{\psi}_{\mu-\nu}(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \quad \forall\, \mu \in \Delta_k \text{ and } \beta \in \mathbb{Z}^s.$$

Proof. (a) $\Rightarrow$ (b): By the Binomial Theorem we have

$$\frac{(\alpha - x)^\nu}{\nu!} = \sum_{\tau \le \nu} \frac{(-x)^\tau}{\tau!} \frac{\alpha^{\nu-\tau}}{(\nu - \tau)!}.$$

It follows that

$$\sum_{\nu \le \mu} \sum_{\alpha \in \mathbb{Z}^s} \frac{(\alpha - x)^\nu}{\nu!} \psi_{\mu-\nu}(x - \alpha) = \sum_{\tau \le \mu} \frac{(-x)^\tau}{\tau!} \sum_{\tau \le \nu \le \mu} \frac{\alpha^{\nu-\tau}}{(\nu - \tau)!} \psi_{(\mu-\tau)-(\nu-\tau)}(x - \alpha). \tag{3.2}$$

Hence, for $\mu \in \Delta_k$, condition (a) implies

$$\sum_{\nu \le \mu} \sum_{\alpha \in \mathbb{Z}^s} \frac{(\alpha - x)^\nu}{\nu!} \psi_{\mu-\nu}(x - \alpha) = \sum_{\tau \le \mu} \frac{(-1)^\tau x^\tau}{\tau!} \frac{x^{\mu-\tau}}{(\mu - \tau)!} = \frac{x^\mu}{\mu!} \sum_{\tau \le \mu} (-1)^\tau \binom{\mu}{\tau}.$$

Consequently, condition (b) is valid, since

$$\sum_{\tau \le \mu} (-1)^\tau \binom{\mu}{\tau} = \begin{cases} 1 & \text{for } \mu = 0, \\ (1-1)^\mu = 0 & \text{for } |\mu| > 0. \end{cases}$$

(b) $\Rightarrow$ (a): The proof proceeds by induction on $|\mu|$. For $\mu = 0$, (b) implies

$$\sum_{\alpha\in\mathbb{Z}^s} \psi_0(x-\alpha) = 1.$$

Suppose $|\mu| > 0$ and (a) is true for all multi-indices μ' with $|\mu'| < |\mu|$. Then for $0 < \tau \le \mu$ we have

$$\frac{x^{\mu-\tau}}{(\mu-\tau)!} = \sum_{\tau\le\nu\le\mu} \sum_{\alpha\in\mathbb{Z}^s} \frac{\alpha^{\nu-\tau}}{(\nu-\tau)!} \psi_{(\mu-\tau)-(\nu-\tau)}(x-\alpha). \tag{3.3}$$

Moreover, condition (b) implies

$$\sum_{\nu\le\mu} \sum_{\alpha\in\mathbb{Z}^s} \frac{(\alpha-x)^\nu}{\nu!} \psi_{\mu-\nu}(x-\alpha) = 0.$$

This together with (3.2) and (3.3) yields

$$\sum_{0<\tau\le\mu} \frac{(-x)^\tau}{\tau!} \frac{x^{\mu-\tau}}{(\mu-\tau)!} + \sum_{\nu\le\mu} \sum_{\alpha\in\mathbb{Z}^s} \frac{\alpha^\nu}{\nu!} \psi_{\mu-\nu}(x-\alpha) = 0. \tag{3.4}$$

On the other hand,

$$\sum_{0<\tau\le\mu} \frac{(-x)^\tau}{\tau!} \frac{x^{\mu-\tau}}{(\mu-\tau)!} + \frac{x^\mu}{\mu!} = \frac{x^\mu}{\mu!} \sum_{\tau\le\mu} \binom{\mu}{\tau} (-1)^\tau = 0. \tag{3.5}$$

Comparing (3.4) with (3.5), we obtain

$$\sum_{\nu\le\mu} \sum_{\alpha\in\mathbb{Z}^s} \frac{\alpha^\nu}{\nu!} \psi_{\mu-\nu}(x-\alpha) = \frac{x^\mu}{\mu!}, \quad 0 < |\mu| < k.$$

(b) $\Rightarrow$ (c): Let

$$g_\mu(x) := \sum_{\nu\le\mu} \sum_{\alpha\in\mathbb{Z}^s} \frac{(\alpha-x)^\nu}{\nu!} \psi_{\mu-\nu}(x-\alpha), \quad x \in \mathbb{R}^s.$$

Then g_μ is 1-periodic, *i.e.*, $g_\mu(\cdot+\gamma) = g_\mu$ for all $\gamma \in \mathbb{Z}^s$. For $\beta \in \mathbb{Z}^s$ we have

$$\begin{aligned}\int_{[0,1]^s} g_\mu(x) e^{-i2\pi\beta\cdot x} dx &= \sum_{\nu\le\mu} \sum_{\alpha\in\mathbb{Z}^s} \int_{[0,1]^s} \frac{(\alpha-x)^\nu}{\nu!} \psi_{\mu-\nu}(x-\alpha) e^{-i2\pi\beta\cdot x}\, dx \\ &= \sum_{\nu\le\mu} \int_{\mathbb{R}^s} \frac{(-x)^\nu}{\nu!} \psi_{\mu-\nu}(x) e^{-i2\pi\beta\cdot x}\, dx = \sum_{\nu\le\mu} \frac{(-iD)^\nu}{\nu!} \hat{\psi}_{\mu-\nu}(2\beta\pi).\end{aligned}$$

It follows from (b) that $g_\mu = \delta_{0\mu}$ for $|\mu| < k$. Hence,

$$\int_{[0,1]^s} g_\mu(x) e^{-i2\pi\beta\cdot x} dx = \delta_{0\mu}\delta_{0\beta}. \tag{3.6}$$

Consequently,

$$\sum_{\nu\le\mu} \frac{(-iD)^\nu}{\nu!} \hat{\psi}_{\mu-\nu}(2\beta\pi) = \delta_{0\mu}\delta_{0\beta}, \quad |\mu| < k.$$

(c) $\Rightarrow$ (b): (c) implies (3.6). Hence, the Fourier coefficients of the 1-periodic function g_μ are $\delta_{0\mu}\delta_{0\beta}$, $\beta \in \mathbb{Z}^s$. It follows that $g_\mu = \delta_{0\mu}$, $|\mu| < k$. $\square$

A trigonometric polynomial h is a function from $\mathbb{R}^s$ to $\mathbb{C}$ having the form

$$h(\xi) = \sum_{\gamma\in\mathbb{Z}^s} c_\gamma e^{i\gamma\cdot\xi}, \quad \xi \in \mathbb{R}^s,$$

where c_γ are complex numbers and $c_\gamma = 0$ except for finitely many γ.

Recall that $\Delta_k = \{\mu \in \mathbb{N}_0^s : |\mu| < k\}$. It is known that the matrix

$$(\gamma^\mu)_{\gamma,\mu\in\Delta_k}$$

is invertible (see, *e.g.*, [1, §4]).

Lemma 3.2. *For given complex numbers b_μ, $\mu \in \Delta_k$, there exists a unique trigonometric polynomial*

$$h : \xi \mapsto \sum_{\gamma\in\Delta_k} c_\gamma e^{i\gamma\cdot\xi}, \quad \xi \in \mathbb{R}^s,$$

such that

$$(-iD)^\mu h(0) = b_\mu \quad \forall\, \mu \in \Delta_k. \tag{3.7}$$

Proof. We have

$$(-iD)^\mu h(0) = \sum_{\gamma\in\Delta_k} c_\gamma \gamma^\mu.$$

Hence, (3.7) is equivalent to the following system of linear equations:

$$\sum_{\gamma\in\Delta_k} c_\gamma \gamma^\mu = b_\mu \quad \forall\, \mu \in \Delta_k.$$

Since the matrix $(\gamma^\mu)_{\gamma,\mu\in\Delta_k}$ is invertible, this system of linear equations has a unique solution for $(c_\gamma)_{\gamma\in\Delta_k}$. $\square$

Let Φ be an $r \times 1$ vector $(\phi_1, \ldots, \phi_r)^T$, where $\phi_1, \ldots, \phi_r$ are compactly supported integrable functions on $\mathbb{R}^s$. We say that Φ satisfies the Strang-Fix conditions of order k if there exists a (finite) linear combination ψ of shifts of $\phi_1, \ldots, \phi_r$ such that

$$D^\mu \hat{\psi}(2\beta\pi) = \delta_{0\mu}\delta_{0\beta} \qquad \forall\, \mu \in \Delta_k \text{ and } \beta \in \mathbb{Z}^s.$$

Such a function is often called a **superfunction**. Note that $\hat{\psi}(\xi) = B(\xi)\hat{\Phi}(\xi)$, $\xi \in \mathbb{R}^s$, where B is a $1 \times r$ vector of trigonometric polynomials. Thus, Φ satisfies the Strang-Fix conditions of order k if and only if there exists a $1\times r$ vector B of trigonometric polynomials such that $D^\mu(B\hat{\Phi})(2\pi\beta) = \delta_{0\mu}\delta_{0\beta}$ for all $\mu \in \Delta_k$ and $\beta \in \mathbb{Z}^s$.

Theorem 3.3. *Let Φ be an $r \times 1$ vector $(\phi_1, \ldots, \phi_r)^T$, where $\phi_1, \ldots, \phi_r$ are compactly supported integrable functions on $\mathbb{R}^s$. Then the following statements are equivalent.*

(a) *Φ satisfies the Strang-Fix conditions of order k.*

(b) *There exist $\psi_\mu \in \mathrm{span}\{\phi_1, \ldots, \phi_r\}$, $\mu \in \Delta_k$, such that*

$$\sum_{\nu \le \mu} \frac{(-iD)^\nu}{\nu!} \hat{\psi}_{\mu-\nu}(2\beta\pi) = \delta_{0\mu}\delta_{0\beta} \quad \forall\, \mu \in \Delta_k \text{ and } \beta \in \mathbb{Z}^s.$$

(c) *There exist $\psi_\mu \in \mathrm{span}\{\phi_1, \ldots, \phi_r\}$, $\mu \in \Delta_k$, such that*

$$\frac{x^\mu}{\mu!} = \sum_{\nu \le \mu} \sum_{\alpha \in \mathbb{Z}^s} \frac{\alpha^\nu}{\nu!} \psi_{\mu-\nu}(x - \alpha) \quad \forall\, \mu \in \Delta_k.$$

Proof. It was proved in Lemma 2.1 that (b) and (c) are equivalent. Hence, it remains to show that (a) and (b) are equivalent. Suppose Φ satisfies the Strang-Fix conditions of order k. Then there exists a $1 \times r$ vector B of trigonometric polynomials such that $D^\mu(B\hat{\Phi})(2\beta\pi) = \delta_{0\mu}\delta_{0\beta}$ for all $\mu \in \Delta_k$ and $\beta \in \mathbb{Z}^s$. Let

$$\psi_\mu := \frac{(-iD)^\mu B(0)\,\Phi}{\mu!} \qquad \mu \in \Delta_k.$$

By the Leibniz formula for differentiation we obtain

$$\begin{aligned} \frac{(-iD)^\mu (B\hat{\Phi})(2\beta\pi)}{\mu!} &= \sum_{\nu \le \mu} \frac{(-iD)^{\mu-\nu} B(0)}{(\mu-\nu)!} \, \frac{(-iD)^\nu \hat{\Phi}(2\beta\pi)}{\nu!} \\ &= \sum_{\nu \le \mu} \frac{(-iD)^\nu \hat{\psi}_{\mu-\nu}(2\beta\pi)}{\nu!}. \end{aligned} \tag{3.8}$$

But $D^\mu(B\hat{\Phi})(2\beta\pi) = \delta_{0\mu}\delta_{0\beta}$ for all $\mu \in \Delta_k$ and $\beta \in \mathbb{Z}^s$. Hence, (b) is true.

Conversely, suppose (b) is true. Then for each $\mu \in \Delta_k$, $\psi_\mu = B_\mu \Phi$, where B_μ is a $1 \times r$ vector of complex numbers. By Lemma 3.2 there exists a $1 \times r$ vector B of trigonometric polynomials such that

$$\frac{(-iD)^\mu B(0)}{\mu!} = B_\mu \quad \forall\, \mu \in \Delta_k.$$

Hence, condition (b) and (3.8) together yield

$$\frac{(-iD)^\mu}{\mu!} (B\hat{\Phi})(2\beta\pi) = \sum_{\nu \le \mu} \frac{(-iD)^\nu}{\nu!} \hat{\psi}_{\mu-\nu}(2\beta\pi) = \delta_{0\mu}\delta_{0\beta} \quad \forall\, \mu \in \Delta_k \text{ and } \beta \in \mathbb{Z}^s.$$

Therefore, Φ satisfies the Strang-Fix conditions of order k. $\square$

We remark that Theorem 3.3 is still valid if $\phi_1, \ldots, \phi_r$ are compactly supported distributions on $\mathbb{R}^s$. Recall that Φ has accuracy k if $\mathbb{S}(\Phi) \supset \Pi_{k-1}$.

§4. Accuracy of Refinable Vectors of Functions

Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be an $r \times 1$ vector of compactly supported distributions on $\mathbb{R}^s$ such that $\hat{\Phi}(0) \neq 0$. Suppose Φ satisfies the following vector refinement equation

$$\Phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\Phi(M \cdot - \alpha), \tag{4.1}$$

where M is an expansive integer matrix, and the mask a is finitely supported.

In the univariate case ($s = 1$), accuracy of refinable vectors of functions was investigated by Heil, Strang, and Strela [11], and by Plonka [28]. They characterized the accuracy of Φ in terms of the mask a under the assumption of linear independence or stability of Φ.

In the multivariate case, approximation by refinable vectors of square-integrable functions was studied by de Boor, DeVore, and Ron [2], and by Jiang [24]. They characterized the approximation order of Φ in terms of the symbol of a by using the Fourier analysis technique. Without using the Fourier analysis technique, Cabrelli, Heil, and Molter [5, 6] determined the accuracy of Φ in terms of the mask a. Moreover, they determined explicitly the coefficients in the expansion of polynomials into a series of multi-integer translates of Φ. In [8], Chen, Sheng, and Xiao combined techniques in both frequency and time domains and improved some earlier results on accuracy of Φ.

Taking Fourier transform of both sides of (4.1), we obtain

$$\hat{\Phi}(\xi) = A\big((M^T)^{-1}\xi\big)\hat{\Phi}\big((M^T)^{-1}\xi\big), \quad \xi \in \mathbb{R}^s, \tag{4.2}$$

where

$$A(\xi) = \frac{1}{|\det M|} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha\cdot\xi}. \tag{4.3}$$

It follows from (4.2) that

$$\hat{\Phi}(0) = A(0)\hat{\Phi}(0).$$

In other words, $\hat{\Phi}(0)$ is a right eigenvector of $A(0)$ corresponding to eigenvalue 1.

Let Ω be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M^T\mathbb{Z}^s$. Without loss of any generality, we may assume $0 \in \Omega$.

The following two theorems extend the results in [2, Theorem 5.2] and [24, Theorem 3.2] to the case when $\phi_1, \ldots, \phi_r$ are compactly supported distributions on $\mathbb{R}^s$.

Theorem 4.1. *Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be an $r \times 1$ vector of compactly supported distributions on $\mathbb{R}^s$ such that $\hat{\Phi}(0) \neq 0$. Suppose Φ is a solution of the refinement equation (4.1). Then Φ satisfies the Strang-Fix conditions of order k, provided there exists a $1 \times r$ vector B of trigonometric polynomials such that, with G given by $G(\xi) = B(M^T\xi)A(\xi)$, $\xi \in \mathbb{R}^s$, the following three conditions are satisfied:*

(a) $B(0)\hat{\Phi}(0) = 1$;

(b) $D^\mu G(2\pi(M^T)^{-1}\omega) = 0$ *for all* $\mu \in \Delta_k$ *and* $\omega \in \Omega \setminus \{0\}$;

(c) $D^\mu G(0) = D^\mu B(0)$ *for all* $\mu \in \Delta_k$.

Proof. Let $f(\xi) := B(\xi)\hat{\Phi}(\xi)$, $\xi \in \mathbb{R}^s$. Then condition (a) tells us $f(0) = B(0)\hat{\Phi}(0) = 1$. By (4.2) we have

$$f(\xi) = B(\xi)A((M^T)^{-1}\xi)\hat{\Phi}((M^T)^{-1}\xi) = G((M^T)^{-1}\xi)\hat{\Phi}((M^T)^{-1}\xi), \quad \xi \in \mathbb{R}^s. \tag{4.4}$$

Condition (b) implies $D^\mu(G\circ(M^T)^{-1})(2\pi\omega)=0$ for all $\mu\in\Delta_k$ and $\omega\in\Omega\setminus\{0\}$. Suppose $\beta\in\mathbb{Z}^s\setminus M^T\mathbb{Z}^s$. Then $\beta=\omega+M^T\gamma$ for some $\omega\in\Omega\setminus\{0\}$ and $\gamma\in\mathbb{Z}^s$. By the Leibniz formula for differentiation we obtain

$$D^\mu f(2\pi\beta)=\sum_{\nu\le\mu}\binom{\mu}{\nu}D^\nu\big(G\circ(M^T)^{-1}\big)(2\pi\beta)\,D^{\mu-\nu}\big(\hat\Phi\circ(M^T)^{-1}\big)(2\pi\beta).$$

But condition (b) tells us that

$$D^\nu\big(G\circ(M^T)^{-1}\big)(2\pi\beta)=D^\nu\big(G\circ(M^T)^{-1}\big)(2\pi\omega+2\pi M^T\gamma)=D^\nu\big(G\circ(M^T)^{-1}\big)(2\pi\omega)=0$$

for all $\omega\in\Omega\setminus\{0\}$ and $\nu\in\Delta_k$. Hence,

$$D^\mu f(2\pi\beta)=0\qquad\forall\mu\in\Delta_k\ \text{and}\ \beta\in\mathbb{Z}^s\setminus M^T\mathbb{Z}^s.$$

It follows from (4.4) that

$$f(M^T\xi)=G(\xi)\hat\Phi(\xi),\quad \xi\in\mathbb{R}^s.$$

Since both G and B are 2π-periodic, condition (c) gives $D^\mu G(2\pi\beta)=D^\mu B(2\pi\beta)$ for $\mu\in\Delta_k$ and $\beta\in\mathbb{Z}^s$. By the Leibniz formula for differentiation we obtain

$$\begin{aligned}D^\mu(f\circ M^T)(2\pi\beta)&=\sum_{\nu\le\mu}\binom{\mu}{\nu}D^\nu G(2\pi\beta)D^{\mu-\nu}\hat\Phi(2\pi\beta)\\&=\sum_{\nu\le\mu}\binom{\mu}{\nu}D^\nu B(2\pi\beta)D^{\mu-\nu}\hat\Phi(2\pi\beta)=D^\mu f(2\pi\beta).\end{aligned}\tag{4.5}$$

By using this relation repeatedly we conclude that $D^\mu f(2\pi\beta)=0$ for all $\mu\in\Delta_k$ and $\beta\in(M^T)^n\mathbb{Z}^s\setminus(M^T)^{n+1}\mathbb{Z}^s$, $n=0,1,\ldots$. Note that the set $\mathbb{Z}^s\setminus\{0\}$ is the union of the sets $(M^T)^n\mathbb{Z}^s\setminus(M^T)^{n+1}\mathbb{Z}^s$, $n=0,1,\ldots$. Therefore,

$$D^\mu f(2\pi\beta)=0\qquad\forall\,\mu\in\Delta_k\ \text{and}\ \beta\in\mathbb{Z}^s\setminus\{0\}.$$

Let $g:=f\circ M^T$. It follows from (4.5) that

$$D^\mu g(0)=D^\mu(f\circ M^T)(0)=D^\mu f(0)\quad\forall\,\mu\in\Delta_k.$$

Consequently, $D^\mu(g-f)(0)=0$ for all $\mu\in\Delta_k$. By using the Chain Rule for differentiation we deduce that

$$D^\mu\big((g-f)\circ(M^T)^{-n}\big)(0)=0\qquad\forall\,\mu\in\Delta_k\ \text{and}\ n\in\mathbb{N}.$$

But $g\circ(M^T)^{-n}=f\circ(M^T)^{-n+1}$. Hence,

$$D^\mu\big(f\circ(M^T)^{-n+1}\big)(0)=D^\mu\big(f\circ(M^T)^{-n}\big)(0)\qquad\forall\,\mu\in\Delta_k\ \text{and}\ n\in\mathbb{N}.$$

It follows that

$$D^{\mu} f(0) = D^{\mu}\big(f \circ (M^T)^{-n}\big)(0) \qquad \forall\, \mu \in \Delta_k \text{ and } n \in \mathbb{N}.$$

Therefore, for $\mu \in \Delta_k \setminus \{0\}$ we have

$$D^{\mu} f(0) = \lim_{n\to\infty} D^{\mu}\big(f \circ (M^T)^{-n}\big)(0) = 0.$$

We have shown that $D^{\mu}(B\hat{\Phi})(2\pi\beta) = \delta_{0\mu}\delta_{0\beta}$ for all $\mu \in \Delta_k$ and $\beta \in \mathbb{Z}^s$. In other words, Φ satisfies the Strang-Fix conditions of order k. $\square$

For the scalar case ($r = 1$), the above theorem can be simplified as follows: If $D^{\mu} A(2\pi(M^T)^{-1}\omega) = 0$ for all $\mu \in \Delta_k$ and $\omega \in \Omega \setminus \{0\}$, then Φ has accuracy k (see [16, Theorem 3.1]). The converse of this statement is true under the additional condition that, for each $\omega \in \Omega$, there exists some $\beta \in \mathbb{Z}^s$ such that $\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) \neq 0$. Without this additional condition, there is a counterexample (see [16, Example 4.2]).

For the vector case ($r > 1$), the converse of Theorem 4.1 is true under a similar additional condition. In what follows, for a subset E of a vector space, span E denotes the linear span of E.

Theorem 4.2. *Suppose span$\{\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$ for all $\omega \in \Omega$. If Φ has accuracy k, then there exists a $1 \times r$ vector B of trigonometric polynomials such that conditions (a), (b), and (c) in Theorem 4.1 are satisfied.*

Proof. Since Φ has accuracy k and $\operatorname{span}\{\hat{\Phi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, by Theorem 2.2 we see that Φ satisfies the Strang-Fix conditions of order k. Hence, there exists a $1 \times r$ vector B of trigonometric polynomials such that the function f given by $f(\xi) := B(\xi)\hat{\Phi}(\xi)$, $\xi \in \mathbb{R}^s$, satisfies

$$D^{\mu} f(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \quad \forall\, \mu \in \Delta_k \text{ and } \beta \in \mathbb{Z}^s.$$

In particular, $B(0)\hat{\Phi}(0) = f(0) = 1$. This verifies condition (a). We claim that B also satisfies conditions (b) and (c).

Let $G(\xi) := B(M^T\xi)A(\xi)$, $\xi \in \mathbb{R}^s$. Then $f(M^T\xi) = G(\xi)\hat{\Phi}(\xi)$, $\xi \in \mathbb{R}^s$.

We first establish (b) by induction on $|\mu|$. It follows from $f(M^T\xi) = G(\xi)\hat{\Phi}(\xi)$ that for $\omega \in \Omega \setminus \{0\}$,

$$0 = f(2\pi\omega + 2\pi M^T\beta) = G(2\pi(M^T)^{-1}\omega)\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) \quad \forall\, \beta \in \mathbb{Z}^s.$$

Since $\operatorname{span}\{\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, we must have

$$G(2\pi(M^T)^{-1}\omega) = 0 \quad \forall\, \omega \in \Omega \setminus \{0\}.$$

This proves (b) for $\mu = 0$. Suppose $0 < m < k$. Assume that (b) is valid for all $|\nu| < m$. Let $|\mu| = m$. For $\omega \in \Omega \setminus \{0\}$ and $\beta \in \mathbb{Z}^s$, we have

$$\begin{aligned} 0 &= D^{\mu}(f \circ M^T)\big(2\pi(M^T)^{-1}\omega + 2\pi\beta\big) \\ &= \sum_{\nu \le \mu} D^{\nu} G\big(2\pi(M^T)^{-1}\omega\big) D^{\mu-\nu}\hat{\Phi}\big(2\pi(M^T)^{-1}\omega + 2\pi\beta\big) \\ &= D^{\mu} G\big(2\pi(M^T)^{-1}\omega\big)\hat{\Phi}\big(2\pi(M^T)^{-1}\omega + 2\pi\beta\big). \end{aligned}$$

This is true for all $\beta \in \mathbb{Z}^s$. Since $\operatorname{span}\{\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, we obtain $D^\mu G(2\pi(M^T)^{-1}\omega) = 0$ for all $\omega \in \Omega \setminus \{0\}$. This completes the induction procedure.

Next, let us prove (c). This will be done by induction on $|\mu|$. It follows from the equation $f(M^T\xi) = G(\xi)\hat{\Phi}(\xi)$ that

$$1 = G(0)\hat{\Phi}(0) \quad \text{and} \quad 0 = f(2\pi M^T\beta) = G(0)\hat{\Phi}(2\pi\beta), \quad \beta \in \mathbb{Z}^s \setminus \{0\}.$$

On the other hand, the equation $f(\xi) = B(\xi)\hat{\Phi}(\xi)$ yields

$$1 = B(0)\hat{\Phi}(0) \quad \text{and} \quad 0 = f(2\pi\beta) = B(0)\hat{\Phi}(2\pi\beta), \quad \beta \in \mathbb{Z}^s \setminus \{0\}.$$

It follows that $[G(0) - B(0)]\hat{\Phi}(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s$. Since $\{\hat{\Phi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, we must have $G(0) - B(0) = 0$, *i.e.*, $G(0) = B(0)$. This proves (c) for $\mu = 0$.

Suppose $0 < m < k$. Assume that (c) is true for all $|\nu| < m$. Let $|\mu| = m$. For $\beta \in \mathbb{Z}^s$, it follows from the equation $f(M^T\xi) = G(\xi)\hat{\Phi}(\xi)$ that

$$\begin{aligned} 0 = D^\mu(f \circ M^T)(2\pi\beta) &= \sum_{\nu \le \mu} \binom{\mu}{\nu} D^\nu G(0) D^{\mu-\nu}\hat{\Phi}(2\beta\pi) \\ &= D^\mu G(0)\hat{\Phi}(2\pi\beta) + \sum_{\nu < \mu} \binom{\mu}{\nu} D^\nu G(0) D^{\mu-\nu}\hat{\Phi}(2\beta\pi). \end{aligned}$$

On the other hand, the equation $f(\xi) = B(\xi)\hat{\Phi}(\xi)$ yields

$$\begin{aligned} 0 = D^\mu f(2\pi\beta) &= \sum_{\nu \le \mu} \binom{\mu}{\nu} D^\nu B(0) D^{\mu-\nu}\hat{\Phi}(2\beta\pi) \\ &= D^\mu B(0)\hat{\Phi}(2\pi\beta) + \sum_{\nu < \mu} \binom{\mu}{\nu} D^\nu B(0) D^{\mu-\nu}\hat{\Phi}(2\beta\pi). \end{aligned}$$

By the induction hypothesis, $D^\nu G(0) = D^\nu B(0)$ for $|\nu| < m$. Therefore, we have

$$[D^\mu G(0) - D^\mu B(0)]\hat{\Phi}(2\pi\beta) = 0 \quad \forall \beta \in \mathbb{Z}^s.$$

Since $\{\hat{\Phi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, we must have $D^\mu G(0) - D^\mu B(0) = 0$. This completes the induction procedure and thereby the proof of the theorem. □

We note that the condition

$$\operatorname{span}\{\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r \quad \forall \omega \in \Omega$$

is a consequence of linear independence or stability (see [20]). A viable way for checking this condition in terms of the mask was provided in [12]. In the univariate case ($s = 1$), without requiring such a condition, Jia, Riemenschneider, and Zhou [21] gave a characterization of the accuracy of Φ strictly in terms of the refinement mask.

For the scalar case ($r = 1$), Theorem 4.1 is valid under conditions (a), (b), and $G(0) = B(0)$. Is this true for the vector vase ($r > 1$)? The following example gives a negative answer.

Example 4.3. Let $s=1$, $r=2$, and $M=(2)$. Suppose the mask a is supported in $[0,1]$,

$$a(0)=\begin{bmatrix}1 & 0\\ -1/2 & 1/4\end{bmatrix} \quad \text{and} \quad a(1)=\begin{bmatrix}1 & 0\\ 1/2 & 1/4\end{bmatrix}.$$

Then [22, Example 6.1] tells us that there exist functions ϕ_1 and ϕ_2 in $L_2(\mathbb{R})$ supported on $[0,1]$ such that $\Phi=(\phi_1,\phi_2)^T$ satisfies the refinement equation

$$\Phi = a(0)\Phi(2\cdot) + a(1)\Phi(2\cdot -1)$$

and $\hat{\Phi}(0)=(1,0)^T$. We observe that $1/2$ is not an eigenvalue of the block matrix

$$\begin{bmatrix}a(0) & 0\\ 0 & a(1)\end{bmatrix}.$$

Hence, by [21, Theorem 2.1], Φ does not have accuracy 2.

We have

$$A(\xi)=\frac{1}{2}\left[a(0)+a(1)e^{-i\xi}\right]=\begin{bmatrix}(1+e^{-i\xi})/2 & 0\\ (-1+e^{-i\xi})/4 & (1+e^{-i\xi})/8\end{bmatrix}, \quad \xi\in\mathbb{R}.$$

If we choose $B(\xi)=(1,(1-e^{-i\xi})/2)$, $\xi\in\mathbb{R}$, then $B(0)=(1,0)$. Hence, $B(0)\hat{\Phi}(0)=1$ and $B(0)A(0)=B(0)$. Let $G(\xi)=B(2\xi)A(\xi)$, $\xi\in\mathbb{R}$. It can be easily checked that $G(\pi)=0$ and $G'(\pi)=0$. Consequently, B satisfies condition (b) in Theorem 4.1 for $k=2$. $\square$

§5. Accuracy and the Sum Rules

Let Φ be an $r\times 1$ vector of compactly supported distributions satisfying the refinement equation (4.1). In this section our goal is to give a characterization of the accuracy of Φ in terms of the sum rules associated with the refinement mask.

Let Γ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$. Without loss of any generality we may assume $0\in\Gamma$. For $\gamma\in\Gamma$ and $\mu\in\mathbb{N}_0^s$, define

$$K_{\mu,\gamma} := \frac{1}{\mu!}\sum_{\beta\in\mathbb{Z}^s}(M\beta+\gamma)^{\mu}a(M\beta+\gamma). \tag{5.1}$$

Suppose $r=1$. We say that a satisfies the **basic sum rule** if

$$\sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta) \quad \forall\gamma\in\Gamma.$$

The basic sum rule was employed by Cavaretta, Dahmen, and Micchelli [7] in their study of stationary subdivision. More generally, a is said to satisfy the **sum rules** of order k if

$$K_{\mu,\gamma}=K_{\mu,0} \qquad \forall\mu\in\Delta_k \ \text{ and } \ \gamma\in\Gamma.$$

If the mask a satisfies the sum rules of order k, then it was proved in [16] that Φ has accuracy k. This extends an earlier result of Daubechies and Lagarias [9] on univariate refinable functions. Conversely, if Φ has accuracy k, and if the sequence $(\hat{\Phi}(2\pi(M^T)^{-1}\omega+2\pi\beta))_{\beta\in\mathbb{Z}^s}$ is nonzero for each $\omega\in\Omega$, then the mask a satisfies the sum rules of order k.

For the vector case ($r>1$), a certain form of the sum rules was discussed by Jetter and Plonka [13].

Our starting point is the following lemma.

Lemma 5.1. *Let a be a finitely supported sequence of $r \times r$ complex matrices, and let*

$$A(\xi) := \frac{1}{m} \sum\nolimits_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s,$$

where $m := |\det M| = \#\Omega = \#\Gamma$. Then for polynomials $p_1, \ldots, p_r$ of s variables, the following two statements are equivalent:

(i) $[p_1(iD), \ldots, p_r(iD)] A(2\pi (M^T)^{-1}\omega) = 0$ *for all* $\omega \in \Omega \setminus \{0\}$.

(ii) $\sum_{\beta \in \mathbb{Z}^s} [p_1(M\beta), \ldots, p_r(M\beta)] a(M\beta) = \sum_{\beta \in \mathbb{Z}^s} [p_1(M\beta+\gamma), \ldots, p_r(M\beta+\gamma)] a(M\beta+\gamma)$ *for all* $\gamma \in \Gamma$.

Proof. For the scalar case ($r = 1$), this result was established in [16, Lemma 3.3]. For the general case, the proof is almost the same. Let us sketch the proof.

From the expression of $A(\xi)$ we have

$$m[p_1(iD), \ldots, p_r(iD)] A(\xi) = \sum_{\alpha \in \mathbb{Z}^s} [p_1(\alpha), \ldots, p_r(\alpha)] a(\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s.$$

An element $\alpha \in \mathbb{Z}^s$ can be written uniquely as $M\beta + \gamma$ with $\beta \in \mathbb{Z}^s$ and $\gamma \in \Gamma$. Observe that, for $\xi := 2\pi (M^T)^{-1}\omega$, $\omega \in \Omega$,

$$-i\alpha\cdot\xi = -i(M\beta + \gamma)\cdot 2\pi (M^T)^{-1}\omega = -i\,2\pi\beta\cdot\omega - i\,2\pi M^{-1}\gamma\cdot\omega.$$

Consequently,

$$m[p_1(iD), \ldots, p_r(iD)] A(2\pi (M^T)^{-1}\omega) = \sum_{\gamma \in \Gamma} b(\gamma) e^{-i2\pi M^{-1}\gamma\cdot\omega}, \quad \omega \in \Omega,$$

where

$$b(\gamma) := \sum_{\beta \in \mathbb{Z}^s} [p_1(M\beta + \gamma), \ldots, p_r(M\beta + \gamma)] a(M\beta + \gamma), \quad \gamma \in \Gamma.$$

Note that the matrix $(e^{-i2\pi M^{-1}\gamma\cdot\omega}/\sqrt{m})_{\gamma\in\Gamma,\omega\in\Omega}$ is a unitary matrix (see [16, Lemma 3.2]). Therefore, $\sum_{\gamma\in\Gamma} b(\gamma) e^{-i2\pi M^{-1}\gamma\cdot\omega} = 0$ for all $\omega \in \Omega \setminus \{0\}$ if and only if $b(\gamma) = b(0)$ for all $\gamma \in \Gamma$. This establishes the equivalence between (i) and (ii). □

Lemma 5.1 enables us to transfer conditions (a), (b), and (c) in Theorem 4.1 into certain sum rules associated with the mask. Let us first consider the case when M is a diagonal integer matrix: $M = \operatorname{diag}(\sigma_1, \ldots, \sigma_s)$, where $|\sigma_j| > 1$, $j = 1, \ldots, s$. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s$, we write σ^μ for $\sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s}$. Suppose that there exists a $1 \times r$ vector B of trigonometric polynomials such that conditions (b) and (c) in Theorem 4.1 are satisfied. Recall that $G(\xi) = B(M^T\xi)A(\xi)$, $\xi \in \mathbb{R}^s$. Clearly,

$$D_j G(\xi) = \sigma_j D_j B(M^T\xi) A(\xi) + B(M^T\xi) D_j A(\xi), \quad j = 1, \ldots, s.$$

More generally, for $\mu \in \mathbb{N}_0^s$ we have

$$(-iD)^\mu G(\xi) = \sum_{\nu \le \mu} \binom{\mu}{\nu} \sigma^{\mu-\nu} (-iD)^{\mu-\nu} B(M^T\xi)(-iD)^\nu A(\xi), \quad \xi \in \mathbb{R}^s.$$

Hence, condition (b) is equivalent to saying that, for all $\omega \in \Omega \setminus \{0\}$ and $\mu \in \Delta_k$,

$$0 = (-iD)^\mu G(2\pi(M^T)^{-1}\omega) = \sum_{\nu \le \mu} \binom{\mu}{\nu} \sigma^{\mu-\nu} (-iD)^{\mu-\nu} B(0)\,(-iD)^\nu A(2\pi(M^T)^{-1}\omega).$$

With $B_\mu := (-iD)^\mu B(0)/\mu!$, $\mu \in \Delta_k$, we see that condition (b) is equivalent to

$$\sum_{\nu \le \mu} \sigma^{\mu-\nu} B_{\mu-\nu} (-iD)^\nu A(2\pi(M^T)^{-1}\omega)/\nu! = 0 \quad \forall\, \mu \in \Delta_k \text{ and } \omega \in \Omega \setminus \{0\}.$$

By Lemma 5.1 we conclude that the above equation is equivalent to the following:

$$\sum_{\nu \le \mu} (-1)^{|\nu|} \sigma^{\mu-\nu} B_{\mu-\nu} K_{\nu,\gamma} = \sum_{\nu \le \mu} (-1)^{|\nu|} \sigma^{\mu-\nu} B_{\mu-\nu} K_{\nu,0} \quad \forall\, \mu \in \Delta_k \text{ and } \gamma \in \Gamma \setminus \{0\}. \tag{5.2}$$

Furthermore, we observe that condition (c) is equivalent to

$$(-iD)^\mu B(0) = (-iD)^\mu G(0) = \sum_{\nu \le \mu} \binom{\mu}{\nu} \sigma^{\mu-\nu} (-iD)^{\mu-\nu} B(0)\,(-iD)^\nu A(0) \quad \forall\, \mu \in \Delta_k.$$

This in turn is equivalent to

$$B_\mu = \sum_{\nu \le \mu} \sigma^{\mu-\nu} B_{\mu-\nu} (-1)^{|\nu|} \frac{1}{m} \sum_{\gamma \in \Gamma} K_{\nu,\gamma} \quad \forall\, \mu \in \Delta_k. \tag{5.3}$$

Equations (5.2) and (5.3) together are equivalent to

$$\sum_{\nu \le \mu} (-1)^{|\nu|} \sigma^{\mu-\nu} B_{\mu-\nu} K_{\nu,\gamma} = B_\mu \quad \forall\, \mu \in \Delta_k \text{ and } \gamma \in \Gamma. \tag{5.4}$$

We are in a position to establish the following theorem, which generalizes Theorem 4.1 in [17] from the univariate case ($s = 1$) to the multivariate case.

Theorem 5.2. *Let Φ be an $r \times 1$ vector of compactly supported distributions satisfying the refinement equation (4.1) associated with an expansive matrix $M = \mathrm{diag}(\sigma_1, \ldots, \sigma_s)$ and a finitely supported mask a. If there exist $1 \times r$ vectors $B_\mu \in \mathbb{C}^{1\times r}$, $\mu \in \Delta_k$, such that $B_0\hat{\Phi}(0) = 1$ and (5.4) holds true, then Φ has accuracy k. Moreover,*

$$\frac{x^\mu}{\mu!} = \sum_{\nu \le \mu} \sum_{\alpha \in \mathbb{Z}^s} \frac{\alpha^\nu}{\nu!} B_{\mu-\nu} \Phi(x - \alpha) \quad \forall\, \mu \in \Delta_k. \tag{5.5}$$

Conversely, if Φ has accuracy k, and if span$\{\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$ *for all* $\omega \in \Omega$, *then there exist* $1 \times r$ *vectors* $B_\mu \in \mathbb{C}^{1\times r}$, $\mu \in \Delta_k$, *such that* $B_0\hat{\Phi}(0) = 1$ *and (5.4) holds true.*

Proof. Suppose B_μ ($\mu \in \Delta_k$) are $1 \times r$ vectors of complex numbers such that (5.4) holds true and $B_0\hat{\Phi}(0) = 1$. By Lemma 3.2 there exists a $1 \times r$ vector B of trigonometric polynomials such that

$$\frac{(-iD)^\mu B(0)}{\mu!} = B_\mu \quad \forall\, \mu \in \Delta_k. \tag{5.6}$$

It follows that $B(0)\hat{\Phi}(0) = B_0\hat{\Phi}(0) = 1$. Let $G(\xi) = B(M^T\xi)A(\xi)$, $\xi \in \mathbb{R}^s$. Then conditions (b) and (c) in Theorem 4.1 are satisfied. Hence, Φ has accuracy k. Moreover, from the proof of Theorem 4.1 we see that $(-iD)^\mu(B\hat{\Phi})(2\beta\pi) = \delta_{0\mu}\delta_{0\beta}$ for all $\mu \in \Delta_k$ and $\beta \in \mathbb{Z}^s$. Let $\psi_\mu := B_\mu\Phi$, $\mu \in \Delta_k$. Then it follows from (3.8) that

$$\sum_{\nu\le\mu} \frac{(-iD)^\nu}{\nu!}\hat{\psi}_{\mu-\nu}(2\pi\beta) = \delta_{0\mu}\delta_{0\beta} \quad \forall\, \mu \in \Delta_k \text{ and } \beta \in \mathbb{Z}^s.$$

In light of Lemma 3.1, this implies (5.5).

Conversely, if Φ has accuracy k, and if span$\{\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$ for all $\omega \in \Omega$, then there exists a $1 \times r$ vector B of trigonometric polynomials such that conditions (a), (b), and (c) in Theorem 4.1 are satisfied. Let B_μ ($\mu \in \Delta_k$) be given by (5.6). Then $B_0\hat{\Phi}(0) = B(0)\hat{\Phi}(0) = 1$. Since conditions (b) and (c) together are equivalent to (5.4), we conclude that (5.4) holds true. □

Next, we consider the case when M is a general dilation matrix. Recall that $e_1, \ldots, e_s$ are the unit coordinate vectors in $\mathbb{R}^s$. We may view e_j as the jth column of the $s \times s$ identity matrix. Let $v_j := (M^T)^{-1}e_j$, $j = 1, \ldots, s$. We have

$$D_{v_j}(f \circ M^T)(\xi) = \lim_{t\to 0} \frac{f(M^T\xi + te_j) - f(M^T\xi)}{t} = D_j f(M^T\xi), \quad \xi \in \mathbb{R}^s.$$

Suppose that there exists a $1\times r$ vector B of trigonometric polynomials such that conditions (a), (b), and (c) in Theorem 4.1 are satisfied. Recall that $G(\xi) = B(M^T\xi)A(\xi)$, $\xi \in \mathbb{R}^s$. Clearly,

$$D_{v_j}G(\xi) = D_jB(M^T\xi)A(\xi) + B(M^T\xi)D_{v_j}A(\xi).$$

More generally, with $D_V^\mu := D_{v_1}^{\mu_1} \cdots D_{v_s}^{\mu_s}$, we have

$$D_V^\mu G(\xi) = \sum_{\nu\le\mu} \binom{\mu}{\nu} D^{\mu-\nu}B(M^T\xi)D_V^\nu A(\xi), \quad \xi \in \mathbb{R}^s. \tag{5.7}$$

Consequently, condition (b) is equivalent to saying that, for $\omega \in \Omega \setminus \{0\}$ and $\mu \in \Delta_k$,

$$0 = (-iD_V)^\mu G(2\pi(M^T)^{-1}\omega) = \sum_{\nu\le\mu} \binom{\mu}{\nu} (-iD)^{\mu-\nu}B(0)\,(-iD_V)^\nu A(2\pi(M^T)^{-1}\omega).$$

With B_μ ($\mu \in \Delta_k$) given by (5.6) we see that condition (b) is equivalent to

$$\sum_{\nu \le \mu} B_{\mu-\nu} \frac{(-iD_V)^\nu A(2\pi(M^T)^{-1}\omega)}{\nu!} = 0 \quad \forall\, \omega \in \Delta_k \text{ and } \omega \in \Omega \setminus \{0\}. \tag{5.8}$$

For $\nu = (\nu_1, \ldots, \nu_s)$, it follows from (4.3) that

$$(iD_V)^\nu A(\xi) = \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} (v_1^T \alpha)^{\nu_1} \cdots (v_s^T \alpha)^{\nu_s} a(\alpha) e^{-i\alpha\cdot\xi}.$$

An element $\alpha \in \mathbb{Z}^s$ can be written uniquely as $M\beta + \gamma$ with $\beta \in \mathbb{Z}^s$ and $\gamma \in \Gamma$. Hence, we obtain

$$\frac{(iD_V)^\nu A(2\pi(M^T)^{-1}\omega)}{\nu!} = \frac{1}{m} \sum_{\gamma \in \Gamma} J_{\nu,\gamma} e^{-i2\pi\gamma\cdot(M^T)^{-1}\omega},$$

where

$$J_{\nu,\gamma} := \frac{1}{\nu!} \sum_{\beta \in \mathbb{Z}^s} \big(v_1^T(M\beta+\gamma)\big)^{\nu_1} \cdots \big(v_s^T(M\beta+\gamma)\big)^{\nu_s} a(M\beta+\gamma).$$

Note that

$$v_j^T(M\beta+\gamma) = e_j^T M^{-1}(M\beta+\gamma) = e_j^T(\beta + M^{-1}\gamma).$$

Hence,

$$J_{\nu,\gamma} = \frac{1}{\nu!} \sum_{\beta \in \mathbb{Z}^s} (\beta + M^{-1}\gamma)^\nu a(M\beta+\gamma). \tag{5.9}$$

By Lemma 5.1 we conclude that (5.8) is equivalent to the following:

$$\sum_{\nu \le \mu} (-1)^{|\nu|} B_{\mu-\nu} J_{\nu,\gamma} = \sum_{\nu \le \mu} (-1)^{|\nu|} B_{\mu-\nu} J_{\nu,0} \quad \forall\, \mu \in \Delta_k \text{ and } \gamma \in \Gamma. \tag{5.10}$$

Furthermore, we observe that condition (c) is equivalent to $D_V^\mu G(0) = D_V^\mu B(0)$ for all $\mu \in \Delta_k$. By (5.7) we have

$$D_V^\mu G(0) = \sum_{\nu \le \mu} \binom{\mu}{\nu} D^{\mu-\nu} B(0) D_V^\nu A(0).$$

Therefore, condition (c) is equivalent to

$$\frac{(-iD_V)^\mu B(0)}{\mu!} = \sum_{\nu \le \mu} (-1)^{|\nu|} B_{\mu-\nu} \frac{1}{m} \sum_{\gamma \in \Gamma} J_{\nu,\gamma} \quad \forall\, \mu \in \Delta_k. \tag{5.11}$$

Equations (5.10) and (5.11) together are equivalent to

$$\frac{(-iD_V)^\mu B(0)}{\mu!} = \sum_{\nu \le \mu} (-1)^{|\nu|} B_{\mu-\nu} J_{\nu,\gamma} \quad \forall\, \mu \in \Delta_k \text{ and } \gamma \in \Gamma.$$

Suppose $|\mu| = n$ and

$$\frac{D_V^\mu}{\mu!} = \sum_{|\tau|=n} c_{\mu\tau} \frac{D^\tau}{\tau!}.$$

Let us determine the coefficients $c_{\mu\tau}$ ($|\mu| = n$, $|\tau| = n$). Suppose $v_j = (v_{j1}, \ldots, v_{js})^T$, $j = 1, \ldots, s$. It follows from the above equation that

$$\frac{1}{\mu!}(v_{11}x_1 + \cdots + v_{1s}x_s)^{\mu_1} \cdots (v_{s1}x_1 + \cdots + v_{ss}x_s)^{\mu_s} = \sum_{|\tau|=n} c_{\mu\tau} \frac{x^\tau}{\tau!},$$

where $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$. Choose $x = (\lambda_1, \ldots, \lambda_s)^T$, where $\lambda_1, \ldots, \lambda_s \in \mathbb{N}_0$ and $\lambda_1 + \cdots + \lambda_s = n$. Then

$$\sum_{|\tau|=n} c_{\mu\tau} \frac{\lambda^\tau}{\tau!} = \frac{1}{\mu!}(v_1^T\lambda)^{\mu_1} \cdots (v_s^T\lambda)^{\mu_s}.$$

But $v_j^T\lambda = e_j^T(M^{-1}\lambda)$. Hence,

$$\sum_{|\tau|=n} c_{\mu\tau} \frac{\lambda^\tau}{\tau!} = \frac{(M^{-1}\lambda)^\mu}{\mu!}.$$

The matrix $(\lambda^\tau/\tau!)_{|\tau|=n,|\lambda|=n}$ is invertible. Let $(t_{\lambda\nu})_{|\lambda|=n,|\nu|=n}$ be its inverse. In other words,

$$\sum_{|\lambda|=n} \frac{\lambda^\tau}{\tau!} t_{\lambda\nu} = \delta_{\tau\nu}, \quad |\tau| = n, \ |\nu| = n.$$

Consequently,

$$c_{\mu\nu} = \sum_{|\lambda|=n} \frac{(M^{-1}\lambda)^\mu}{\mu!} t_{\lambda\nu}.$$

It follows that

$$\frac{(-iD_V)^\mu B(0)}{\mu!} = \sum_{|\tau|=n} \sum_{|\lambda|=n} \frac{(M^{-1}\lambda)^\mu}{\mu!} t_{\lambda\tau} \frac{(-iD)^\tau B(0)}{\tau!} = \sum_{|\tau|=n} \left[\sum_{|\lambda|=n} \frac{(M^{-1}\lambda)^\mu}{\mu!} t_{\lambda\tau} \right] B_\tau.$$

The foregoing discussion can be summarized as follows.

Theorem 5.3. *Let Φ be an $r \times 1$ vector of compactly supported distributions satisfying the refinement equation (4.1) with a general expansive matrix M and a finitely supported mask a. If there exist $1 \times r$ vectors $B_\mu \in \mathbb{C}^{1\times r}$, $\mu \in \Delta_k$, such that $B_0\hat{\Phi}(0) = 1$ and*

$$\sum_{\nu\le\mu} (-1)^{|\nu|} B_{\mu-\nu} J_{\nu,\gamma} = \sum_{|\tau|=|\mu|} \left[\sum_{|\lambda|=|\mu|} \frac{(M^{-1}\lambda)^\mu}{\mu!} t_{\lambda\tau} \right] B_\tau \quad \forall\, \mu \in \Delta_k \ \text{and} \ \gamma \in \Gamma, \qquad (5.12)$$

then Φ has accuracy k. Moreover,

$$\frac{x^\mu}{\mu!} = \sum_{\nu \le \mu} \sum_{\alpha \in \mathbb{Z}^s} \frac{\alpha^\nu}{\nu!} B_{\mu-\nu} \Phi(x-\alpha) \quad \forall\, \mu \in \Delta_k.$$

Conversely, if Φ has accuracy k, and if span$\{\hat{\Phi}(2\pi(M^T)^{-1}\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$ for all $\omega \in \Omega$, then there exist $1 \times r$ vectors $B_\mu \in \mathbb{C}^{1\times r}$, $\mu \in \Delta_k$, such that $B_0\hat{\Phi}(0) = 1$ and (5.12) holds true.

A mask a is said to satisfy the **sum rules** of order k if there exist $1 \times r$ vectors $B_\mu \in \mathbb{C}^{1\times r}$, $\mu \in \Delta_k$, such that $B_0 \ne 0$ and (5.12) holds true. A mask a is said to satisfy the **weak sum rules** of order k if there exist $1 \times r$ vectors $B_\mu \in \mathbb{C}^{1\times r}$, $\mu \in \Delta_k$, such that $B_0 \ne 0$ and (5.10) holds true. If a satisfies the sum rules of order k, and if Φ is a solution of the refinement equation associated with the mask a and $B_0\hat{\Phi}(0) = 1$, then Φ has accuracy k. For the vector case ($r > 1$), however, this conclusion is not valid if a only satisfies the *weak* sum rules of order k, as was demonstrated by Example 4.3.

The accuracy of Φ is related to the spectrum of the transition operator associated with the mask (see [11], [21], and [23]). The **transition operator** T_a is the linear operator on $(\ell_0(\mathbb{Z}^s))^{r\times 1}$ defined by

$$T_a v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(M\alpha - \beta) v(\beta), \quad \alpha \in \mathbb{Z}^s,\ v \in (\ell_0(\mathbb{Z}^s))^{r\times 1}.$$

For the scalar case ($r = 1$), it was shown in [12] that the transition operator T_a has only finitely many nonzero eigenvalues. The same proof works for the general case. The following is an outline of the proof. For a bounded subset H of $\mathbb{R}^s$, the set $\sum_{n=1}^\infty M^{-n}H$ is defined as

$$\Big\{\sum\nolimits_{n=1}^\infty M^{-n}h_n \,:\, h_n \in H \ \text{ for } \ n = 1, 2, \ldots\Big\}.$$

If H is a compact set, then $\sum_{n=1}^\infty M^{-n}H$ is also compact. By $\operatorname{supp} a$ we denote the set $\{\alpha \in \mathbb{Z}^s : a(\alpha) \ne 0\}$. Let

$$E := \Big(\sum\nolimits_{n=1}^\infty M^{-n}(\operatorname{supp} a)\Big) \cap \mathbb{Z}^s.$$

We use $\ell(E)$ to denote the linear space of all sequences supported in E. It is easily seen that $(\ell(E))^{r\times 1}$ is invariant under T_a. Moreover, if v is an eigenvector of T_a corresponding to a nonzero eigenvalue of T_a, then v must lie in $(\ell(E))^{r\times 1}$. Consequently, any nonzero eigenvalue of T_a must be an eigenvalue of the block matrix

$$\big(a(M\alpha - \beta)\big)_{\alpha,\beta \in E}.$$

In particular, T_a has only finitely many nonzero eigenvalues.

The following theorem extends [21, Theorem 2.1] and [23, Theorem 2.2]. Its proof is similar to those given in [21] and [23].

Theorem 5.4. *Let Φ be an $r \times 1$ vector of compactly supported distributions satisfying the refinement equation (4.1) with a general expansive matrix M and a finitely supported mask a. If Φ has accuracy k, then the spectrum of the block matrix $\big(a(M\alpha - \beta)\big)_{\alpha,\beta\in E}$ contains $\{\sigma^{-\mu} : |\mu| < k\}$, where $\sigma = (\sigma_1, \ldots, \sigma_s)$ is the s-tuple of the eigenvalues of M and $\sigma^{-\mu} := \sigma_1^{-\mu_1} \cdots \sigma_s^{-\mu_s}$ for a multi-index $\mu = (\mu_1, \ldots, \mu_s)$.*

§6. Examples

In this section we give two examples to show the reader how to apply Theorem 5.2 or Theorem 5.3 to concrete problems. To determine the accuracy of a refinable vector Φ of compactly supported distributions, we need to solve the system of linear equations in (5.8) or (5.12) for B_μ, $\mu \in \Delta_k$, subject to the condition $B_0\hat{\Phi}(0) = 1$. Note that the solution is unique if $\operatorname{span}\{\hat{\Phi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$.

Example 6.1. Let $r = 4$, $s = 2$, and

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Suppose the mask a is supported in $[-1,1]^2$, $a(-1,-1)$, $a(0,-1)$, $a(1,-1)$ are given by

$$\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1/4 & 0 & 1/2 & 1/4 \\ 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$a(-1,0)$, $a(0,0)$, $a(1,0)$ are given by

$$\begin{bmatrix} 1/4 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 1/4 & 0 & 1/4 & 0 \\ 1/2 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 \end{bmatrix},$$

and $a(-1,1)$, $a(0,1)$, $a(1,1)$ are given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 0 & 1/4 \end{bmatrix}, \quad \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 1/2 & 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

We observe that 1 is a simple eigenvalue of the matrix

$$A := \frac{1}{4} \sum_{\alpha_1=-1}^{1} \sum_{\alpha_2=-1}^{1} a(\alpha_1, \alpha_2) = \frac{1}{8} \begin{bmatrix} 5 & 3 & 3 & 3 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix},$$

and $(1/2, 1/6, 1/6, 1/6)^T$ is a right eigenvector of A corresponding to eigenvalue 1. In light of [25, Theorem 2.4] there exists a unique 4×1 vector Φ of compactly supported distributions such that

$$\Phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha)\Phi(2 \cdot - \alpha)$$

and $\hat{\Phi}(0) = (1/2, 1/6, 1/6, 1/6)^T$. It can be proved that Φ actually is continuous.

Clearly, $\Gamma := \{(0,0),(1,0),(0,1),(1,1)\}$ is a complete set of representatives of the cosets $\mathbb{Z}^2/2\mathbb{Z}^2$. For $\mu \in \mathbb{N}_0^s$ and $\gamma \in \Gamma$, $K_{\mu,\gamma}$ is computed as in (5.1). For $k = 3$, the equations in (5.4) can be written as follows:

$$
\begin{aligned}
B_{(0,0)} &= B_{(0,0)}K_{(0,0),\gamma},\\
B_{(1,0)} &= 2B_{(1,0)}K_{(0,0),\gamma} - B_{(0,0)}K_{(1,0),\gamma},\\
B_{(0,1)} &= 2B_{(0,1)}K_{(0,0),\gamma} - B_{(0,0)}K_{(0,1),\gamma},\\
B_{(2,0)} &= 4B_{(2,0)}K_{(0,0),\gamma} - 2B_{(1,0)}K_{(1,0),\gamma} + B_{(0,0)}K_{(2,0),\gamma},\\
B_{(1,1)} &= 4B_{(1,1)}K_{(0,0),\gamma} - 2B_{(1,0)}K_{(0,1),\gamma} - 2B_{(0,1)}K_{(1,0),\gamma} + B_{(0,0)}K_{(1,1),\gamma},\\
B_{(0,2)} &= 4B_{(0,2)}K_{(0,0),\gamma} - 2B_{(0,1)}K_{(0,1),\gamma} + B_{(0,0)}K_{(0,2),\gamma},
\end{aligned}
$$

where $\gamma \in \Gamma$. Solving the above system of linear equations subject to the condition $B_{(0,0)}(1/2, 1/6, 1/6, 1/6)^T = 1$, we obtain

$$
\begin{aligned}
&B_{(0,0)} = (1,1,1,1), \quad B_{(1,0)} = (0,1/2,0,1/2), \quad B_{(0,1)} = (0,0,1/2,1/2),\\
&B_{(2,0)} = (0,0,0,0), \quad B_{(1,1)} = (0,0,0,0), \quad B_{(0,2)} = (0,0,0,0).
\end{aligned}
$$

By Theorem 5.2 we conclude that Φ has accuracy 3. Moreover, we observe that 1/8 is not an eigenvalue of the block matrix $(a(2\alpha - \beta))_{\alpha,\beta\in[-1,1]^2\cap\mathbb{Z}^2}$. Hence, Φ does not have accuracy 4, by Theorem 5.4. Thus, the optimal accuracy of Φ is 3. $\square$

If the expansive matrix M is not diagonal, then Theorem 5.3 will be used. In this case the matrix $(\tau^\lambda/\tau!)_{|\tau|=n,|\lambda|=n}$ and its inverse $(t_{\lambda\nu})_{|\lambda|=n,|\nu|=n}$ need to be computed. For this purpose we fix an ordering in $\mathbb{N}_0^s$ as follows: For two multi-indices $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_s)$, if there exists some $j \in \{1, \ldots, s\}$ such that $\mu_1 = \nu_1, \ldots, \mu_{j-1} = \nu_{j-1}$, and $\mu_j > \nu_j$, then we write $\mu \prec \nu$. For the two-dimensional case ($s = 2$), $(\tau^\lambda/\tau!)_{|\tau|=0,|\lambda|=0}$ is (1), and $(\tau^\lambda/\tau!)_{|\tau|=1,|\lambda|=1}$ is the 2×2 identity matrix. Furthermore, $(\tau^\lambda/\tau!)_{|\tau|=2,|\lambda|=2}$ and its inverse are

$$
\begin{bmatrix} 2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1/2 & -1/4 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1/2 \end{bmatrix}.
$$

Example 6.2. Let $r = 2$, $s = 2$, and

$$
M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
$$

Suppose the mask a is supported in $[-1,1]^2$,

$$
a(-1,-1) = \frac{1}{32}\begin{bmatrix} 2 & 2 \\ 0 & -8 \end{bmatrix}, \quad a(-1,0) = \frac{1}{32}\begin{bmatrix} 3 & 2 \\ -12 & 16 \end{bmatrix}, \quad a(-1,1) = \frac{1}{32}\begin{bmatrix} 1 & 0 \\ -16 & -4 \end{bmatrix},
$$

$$
a(0,-1) = \frac{1}{32}\begin{bmatrix} 13 & -7 \\ -4 & 4 \end{bmatrix}, \quad a(0,0) = \frac{1}{32}\begin{bmatrix} 26 & 0 \\ 0 & -8 \end{bmatrix}, \quad a(0,1) = \frac{1}{32}\begin{bmatrix} 13 & 7 \\ 4 & 4 \end{bmatrix},
$$

and

$$
a(1,-1) = \frac{1}{32}\begin{bmatrix} 1 & 0 \\ 16 & -4 \end{bmatrix}, \quad a(1,0) = \frac{1}{32}\begin{bmatrix} 3 & -2 \\ 12 & 16 \end{bmatrix}, \quad a(1,1) = \frac{1}{32}\begin{bmatrix} 2 & -2 \\ 0 & -8 \end{bmatrix}.
$$

We observe that 1 is a simple eigenvalue of the matrix

$$A := \frac{1}{2} \sum_{\alpha_1=-1}^{1} \sum_{\alpha_2=-1}^{1} a(\alpha_1, \alpha_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1/8 \end{bmatrix},$$

and $(1,0)^T$ is a right eigenvector of A corresponding to eigenvalue 1. In light of [18] there exist unique compactly supported distributions ϕ_1 and ϕ_2 such that $\Phi = (\phi_1, \phi_2)^T$ satisfies

$$\Phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha)\Phi(M \cdot - \alpha)$$

and $\hat{\Phi}(0) = (1,0)^T$. It can be proved that Φ actually is continuous.

Clearly, $\Gamma := \{(0,0),(1,0)\}$ is a complete set of representatives of the cosets $\mathbb{Z}^2/M\mathbb{Z}^2$. For $\mu \in \mathbb{N}_0^s$ and $\gamma \in \Gamma$, $J_{\mu,\gamma}$ is computed as in (5.9). For $k = 3$, the equations in (5.12) can be written as follows:

$$\begin{aligned}
B_{(0,0)} &= B_{(0,0)} J_{(0,0),\gamma}, \\
\frac{1}{2}B_{(1,0)} + \frac{1}{2}B_{(0,1)} &= B_{(1,0)} J_{(0,0),\gamma} - B_{(0,0)} J_{(1,0),\gamma}, \\
\frac{1}{2}B_{(1,0)} - \frac{1}{2}B_{(0,1)} &= B_{(0,1)} J_{(0,0),\gamma} - B_{(0,0)} J_{(0,1),\gamma}, \\
\frac{1}{4}B_{(2,0)} + \frac{1}{4}B_{(1,1)} + \frac{1}{4}B_{(0,2)} &= B_{(2,0)} J_{(0,0),\gamma} - B_{(1,0)} J_{(1,0),\gamma} + B_{(0,0)} J_{(2,0),\gamma}, \\
\frac{1}{2}B_{(2,0)} - \frac{1}{2}B_{(0,2)} &= B_{(1,1)} J_{(0,0),\gamma} - B_{(1,0)} J_{(0,1),\gamma} - B_{(0,1)} J_{(1,0),\gamma} + B_{(0,0)} J_{(1,1),\gamma}, \\
\frac{1}{4}B_{(2,0)} - \frac{1}{4}B_{(1,1)} + \frac{1}{4}B_{(0,2)} &= B_{(0,2)} J_{(0,0),\gamma} - B_{(0,1)} J_{(0,1),\gamma} + B_{(0,0)} J_{(0,2),\gamma},
\end{aligned}$$

where $\gamma \in \Gamma$. Solving the above system of linear equations subject to the condition $B_{(0,0)}(1,0)^T = 1$, we obtain

$$\begin{aligned}
B_{(0,0)} = (1,0), \quad B_{(1,0)} = (0,1/8), \quad B_{(0,1)} = (0,-1/8), \\
B_{(2,0)} = (-11/96,0), \quad B_{(1,1)} = (11/48,0), \quad B_{(0,2)} = (-31/96,0).
\end{aligned}$$

By Theorem 5.3 we conclude that Φ has accuracy 3. Let us show that ϕ does not have accuracy 4. The matrix M has two eigenvalues $\sigma_1 = \sqrt{2}$ and $\sigma_2 = -\sqrt{2}$. Note that the mask a is supported in $[-1,1]^2$ and

$$\sum_{n=1}^{\infty} M^{-n}([-1,1]^2) = \bigl\{(x_1,x_2) \in \mathbb{R}^2 : |x_1| \le 3, |x_2| \le 3, |x_1 - x_2| \le 4, |x_1 + x_2| \le 4\bigr\}.$$

The set $E := \mathbb{Z}^2 \cap \bigl(\sum_{n=1}^{\infty} M^{-n}([-1,1]^2)\bigr)$ has exactly 37 points. Among the 37 eigenvalues of the matrix $(a(M\alpha - \beta))_{\alpha,\beta \in E}$ the following are of the form $\sigma_1^{-\mu_1}\sigma_2^{-\mu_2}$ for some double-index (μ_1, μ_2) with $\mu_1 + \mu_2 \le 3$:

$$1, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}.$$

If Φ had accuracy 4, then $\sqrt{2}/4 = \sigma_1^{-3} = \sigma_1^{-1}\sigma_2^{-2}$ would be a double eigenvalue of the matrix $(a(M\alpha - \beta))_{\alpha,\beta \in E}$, by Theorem 5.4. Therefore, the optimal accuracy of Φ is 3. $\square$

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