PARAMETERIZATIONS OF MASKS FOR TIGHT AFFINE FRAMES WITH TWO SYMMETRIC/ANTISYMMETRIC GENERATORS

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ABSTRACT. Parameterizations of FIR orthogonal systems are of fundamental importance to the design of filters with desired properties. By constructing paraunitary matrices, one can construct tight affine frames. In this paper we discuss parameterizations of paraunitary matrices which generate tight affine frames with two symmetric/antisymmetric generators (framelets). Based on the parameterizations, several symmetric/antisymmetric framelets are constructed.

1. INTRODUCTION

Let H be a (separable) Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and norm $\|\cdot\| := \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. A system $X \subset H$ is called a frame of H if there are two positive constants A and B such that

$$A\|f\|^2 \le \sum_{x\in X} |\langle f, x\rangle|^2 \le B\|f\|^2, \quad \forall f \in H.$$

The constants A and B are called bounds of the frame. If A, B can be chosen such that $A = B$, then X is called a tight frame. For $H = L^2(\mathbb{R})$, if X is the collection of the dilations of $2^j, j \in \mathbb{Z}$ and the integer shifts of a set of functions $\{\psi_1, \cdots, \psi_L\}$, i.e., $X = \{2^{j/2}\psi_\ell(2^j x - k) : 1 \le \ell \le L, j, k \in \mathbb{Z}\}$, then X is called an affine (wavelet) frame. In this case $\psi_1, \cdots, \psi_L$ are called the generators or framelets. Frames differ from (bi)orthogonal systems in that their elements may be linearly dependent. Or in other words, frames can be redundant. This redundancy may be useful in some applications such as signal analysis (noise reduction, feature detections) (see [8]). The reader is referred to [14], [7], [11], [8], [5], [1], [17] for discussions on frames and tight frames.

In [17] a simple sufficient condition is provided to construct tight affine frames with more than one generator (see [13] for the similar condition for affine bi-frames). More precisely, let $h(z), q_1(z), \cdots, q_L(z)$

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be FIR filters (Laurent polynomials of z^{-1}). Denote

$$M_{h,q_1,\cdots,q_L}(z) := \begin{bmatrix} h(z) & h(-z) \\ q_1(z) & q_1(-z) \\ \vdots & \vdots \\ q_L(z) & q_L(-z) \end{bmatrix}. \tag{1.1}$$

Assume that $\phi \in L^2(\mathbb{R})$ is the refinable function satisfying the refinement equation

$$\phi = 2\sum_k h_k \phi(2\cdot -k).$$

Let $\psi_\ell, 1 \le \ell \le L$, be the functions defined by

$$\hat{\psi}_\ell(\omega) = q_\ell(e^{\frac{i\omega}{2}})\hat{\phi}(\frac{\omega}{2}).$$

It was shown in [17], [13] that if $\phi \in L^2(\mathbb{R})$ satisfies that $\hat{\phi}(0) = 1$, $|\hat{\phi}(\omega)| \le c(1+|\omega|)^{-\frac{1}{2}-\epsilon}, \epsilon > 0$, and if $M_{h,q_1,\cdots,q_L}$ is paraunitary, then $\psi_\ell, 1 \le \ell \le L$ generate a tight affine (wavelet) frame of $L^2(\mathbb{R})$. Recall that a $j \times l$ $(j \ge l)$ matrix filter $P(z)$ with real coefficients P_k is called paraunitary if

$$P(z)^T P(z^{-1}) = I_l, \quad z \ne 0,$$

that is $P(e^{i\omega})$ is a matrix of orthonormal columns for all $\omega \in \mathbb{R}$. Throughout this paper we assume that the coefficients of filters discussed are real.

It was proved in [6], [16] that the decay assumption of $\hat{\phi}$ at infinity can be removed. (See [3] about the necessary and sufficient condition for the minimum-energy affine frames. The reader is referred to [15] and [2] for the similar results for the case $L = 1$.) Thus if we have filters $h, q_1, \cdots, q_L$ such that h generates a $L^2(\mathbb{R})$ function ϕ with $\hat{\phi}(0) = 1$ and if $M_{h,q_1,\cdots,q_L}$ is paraunitary, then we have a tight affine frame. The larger the number L of framelets, the more freedom or redundancy there is for the construction of tight affine frames (see [17, 18, 19]). However if too many framelets are used, then there would be the problem of complicated computations in applications. So in this paper we consider frames generated by two framelets. We discuss parameterizations of FIR filters for tight affine frames with two symmetric/antisymmetric generators. In the following for a refinable function ϕ with the refinement mask $h(z)$, we use $g(z) = \sum_{k\in\mathbb{Z}} g_k z^{-k}$ and $f(z) = \sum_{k\in\mathbb{Z}} f_k z^{-k}$ to denote the FIR filters for framelets, and let ψ_1, ψ_2 be the framelets defined by

$$\psi_1 = 2\sum_k g_k \phi(2\cdot -k), \quad \psi_2 = 2\sum_k f_k \phi(2\cdot -k).$$

Filters $g(z), f(z)$ are also called frame masks.

Parameterizations of FIR orthogonal systems are of fundamental importance to the design of filters with special properties. On the other hand, for some symmetric refinable functions, e.g., the cubic B-spline, one cannot construct the corresponding symmetric/antisymmetric framelets by this paraunitary matrix extension approach (the "Unitary Extension Principle", according to [17]). Even in the case that one can construct the corresponding symmetric/antisymmetric framelets from some symmetric refinable functions by this approach, the resulting framelets may not have high vanishing moments since the sum rule order of the refinement mask does not imply automatically the vanishing moment order of framelets. So to construct symmetric/antisymmetric framelets with high vanishing moments, one needs to construct appropriate refinable functions. The parameterization based construction method provides one way to this goal, and it produces framelets with high order vanishing moments and good smoothness by solving some equations related to the sum rules for the refinement mask and the vanishing moments of framelets.

We say a filter $p(z) = \sum_{k\in\mathbb{Z}} p_k z^{-k}$ is causal if $p_k = 0$ for any $k < 0$ and it is an FIR filter if only finite p_k are not zeros. For an FIR filter $p(z) = \sum_{k\in\mathbb{Z}} p_k z^{-k}$, write

$$
\begin{aligned}
p(z) &= \sum p_{2k} z^{-2k} + (\sum p_{2k+1} z^{-2k}) z^{-1} \\
&=: \frac{\sqrt{2}}{2} p_e(z^2) + \frac{\sqrt{2}}{2} p_o(z^2) z^{-1}.
\end{aligned}
\tag{1.2}
$$

For FIR filters h, g and f, let h_e, h_o, g_e, g_o and f_e, f_o be the filters defined by (1.2). Define

$$
P_{h,g,f}(z) := \begin{bmatrix} h_e(z) & h_o(z) \\ g_e(z) & g_o(z) \\ f_e(z) & f_o(z) \end{bmatrix}
$$

Let $M_{h,g,f}(z)$ be the matrix filter defined by (1.1). Then

$$
M_{h,g,f}(z) = \frac{\sqrt{2}}{2} P_{h,g,f}(z^2) \begin{bmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{bmatrix}. \tag{1.3}
$$

Thus $M_{h,g,f}$ is paraunitary if and only if so is $P_{h,g,f}$.

To construct symmetric/antisymmetric framelets, one starts with a symmetric refinement mask h and symmetric/antisymmetric frame masks g, f. For an FIR filter p, if there exits $N \in \mathbb{Z}$ such that

$$
z^{-N} p(z^{-1}) = s p(z), \quad s = \pm 1,
$$

then we say that p is symmetric (antisymmetric, respectively) if $s = 1$ ($s = -1$, respectively), and call $N/2$ the center of symmetry of $p(z)$. Suppose $h(z), g(z), f(z)$ are causal FIR filters. If they are symmetric/antisymmetric, and $M_{h,g,f}$ is paraunitary (or equivalently $P_{h,g,f}$ is paraunitary), then we can find symmetric/antisymmetric causal FIR filters $h_r(z), g_r(z), f_r(z)$ such that $[h_r(z), g_r(z), f_r(z)]^T$ is the paraunitary extension of $P_{h,g,f}(z)$, and furthermore, the extended square matrix filter is causal and has some symmetry. Therefore to obtain the parameterization of masks for tight affine frames, one needs only to consider that for the 3×3 symmetric paraunitary matrix filter.

After finishing this paper, the author became aware of recent papers of [4], [9]. In their papers, the "Oblique Extension Principle" is introduced and used for the construction of tight frames. In this paper, the author used the "Unitary Extension Principle". The advantage of the "Unitary Extension Principle" based construction is that the lengths of filters for the constructed framelets are not larger than that of the filter for the scaling function.

This paper is organized as follows. The symmetric extension of $P_{h,g,f}(z)$ is considered in Section 2. Parameterizations of masks are discussed in Sections 3 and 4. The parameterization based construction of symmetric/antisymmetric framelets is presented in Section 5.

In this paper we use $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of all natural numbers, nonnegative integers and integers, respectively. For an FIR filter $p(z) = \sum_{k=k_1}^{k_2} p_k z^{-k}$ with $p_{k_1} p_{k_2} \neq 0$, we use $\mathrm{leng}(p) := k_2 - k_1 + 1$ to denote the filter length of p. If $p(0) = 1, p^{(j)}(-1) = 0$, $0 \le j < J$, then we say that p has the sum rules of order J. For $n \in \mathbb{N}$, let I_n denote the $n \times n$ identity matrix, and let J_n denote the $n \times n$ exchange matrix with ones on the anti–diagonal.

For $s \ge 0$, we use $W^s(\mathbb{R})$ to denote the Sobolev space consisting of all functions f with $(1 + |\omega|^2)^{\frac{s}{2}} \widehat{f}(\omega) \in L^2(\mathbb{R})$. For a compactly supported function ψ, we say that ψ has the vanishing moments of order J if

$$\int \psi(x) x^j dx = 0, \quad 0 \le j < J.$$

If ψ is defined by $\hat{\psi}(\omega) = q(e^{i\frac{\omega}{2}}) \hat{\phi}(\frac{\omega}{2})$ for an FIR filter $q(z)$ and a compactly supported $\phi \in L^2(\mathbb{R})$, then the conditions $q^{(j)}(1) = 0$, $0 \le j < J$, imply that ψ has vanishing moments of order J.

2. Symmetric Extension

Suppose h, g, f are FIR filters. Let h_e, h_o, g_e, g_o and f_e, f_o be the filters defined by (1.2). For $L \in \mathbb{Z}$, define

$$\begin{aligned}
h_r(z) &:= z^{-L}\big(g_e(z^{-1}) f_o(z^{-1}) - f_e(z^{-1}) g_o(z^{-1})\big), \\
g_r(z) &:= -z^{-L}\big(h_e(z^{-1}) f_o(z^{-1}) - f_e(z^{-1}) h_o(z^{-1})\big), \\
f_r(z) &:= z^{-L}\big(h_e(z^{-1}) g_o(z^{-1}) - g_e(z^{-1}) h_o(z^{-1})\big).
\end{aligned} \tag{2.1}$$

By a direct calculation, one has the following proposition.

Proposition 2.1. *Suppose h, g, f are FIR filters. Let h_r, g_r, f_r be the filters defined by (2.1) with $L \in \mathbb{Z}$. Then $M_{h,g,f}(z)$ is paraunitary if and only if $A_L(z)$ defined by*

$$A_L(z) := \begin{bmatrix} h_e & h_o & h_r \\ g_e & g_o & g_r \\ f_e & f_o & f_r \end{bmatrix}, \tag{2.2}$$

is paraunitary.

From Proposition 2.1, any set $\{h, g, f\}$ of FIR filters with $M_{h,g,f}$ paraunitary can be constructed by 3×3 paraunitary matrices $A_L(z)$. If h, g, f are causal, then for a sufficiently large L, A_L is also causal. A causal and paraunitary FIR matrix can be factorized into the products of $U_0 U_1(z) \cdots U_{\gamma_1 - 1}(z) U_{\gamma_1}(z)$ (see e.g., [23, 22, 24, 21]), where $\gamma_1 \in \mathbb{N}_0$, U_0 is a 3×3 orthogonal matrix, and

$$U_j(z) := I_3 + (z^{-1} - 1) u_j u_j^T, \quad u_j \in \mathbb{R}^3, u_j^T u_j = 1. \tag{2.3}$$

Therefore we have that for causal FIR filters h, g, f, $M_{h,g,f}$ is paraunitary if and only if h, g, f can be factorized as

$$[h(z), g(z), f(z)]^T = \frac{\sqrt{2}}{2} U_0 U_1(z^2) \cdots U_{\gamma_1}(z^2) [1, z^{-1}, 0]^T$$

for some $\gamma_1 \in \mathbb{N}$.

The main purpose of this paper to give parameterizations of masks h, g, f which generate symmetric/antisymmetric framelets. We first have the following proposition.

Proposition 2.2. *Suppose h, g, f are symmetric/antisymmetric nonzero FIR filters. If $M_{h,g,f}$ is paraunitary, then leng(h), leng(g), leng(f) have the same parity.*

Proof. Without loss of generality, we assume that $\text{leng}(f) \le \text{leng}(g) \le \text{leng}(h)$. Suppose the centers of symmetry for h, g, f are $N/2, K/2, S/2$ respectively, where $N, K, S \in \mathbb{Z}$. Then $\text{leng}(h) = 2k_1 + N + 1$, $\text{leng}(g) = 2k_2 + K + 1$, $\text{leng}(f) = 2k_3 + S + 1$ for some $k_1, k_2, k_3 \in \mathbb{Z}$. By $|h(z)|^2 + |g(z)|^2 + |f(z)|^2 = 1$, $\text{leng}(g) = \text{leng}(h)$. Thus $K - N$ is even.

To complete the proof of Proposition 2.2, it suffices to show that $S - N$ is also even. For this, denote

$$H(z) := h(z)h(-z^{-1}), \quad G(z) := g(z)g(-z^{-1}), \quad F(z) := f(z)f(-z^{-1}).$$

Then $H(z^{-1}) = (-1)^N H(z), G(z^{-1}) = (-1)^K G(z), F(z^{-1}) = (-1)^S F(z)$. The paraunitariness of $M_{h,g,f}$ implies

$$H(z) + G(z) + F(z) = 0.$$

Thus

$$(-1)^N H(z) + (-1)^K G(z) + (-1)^S F(z) = 0.$$

Therefore

$$(1-(-1)^{K-N})G(z)+(1-(-1)^{S-N})F(z)=0.$$

That is $(1-(-1)^{S-N})F(z)=0$. Since $F(z)\neq 0$, $S-N$ is even. The proof of Proposition 2.2 is complete. $\square$

In the following we consider the case leng$(g)\leq$leng(h), leng$(f)\leq$leng(h). In this case if $M_{h,g,f}$ is paraunitary, leng$(g)=$leng(h) or leng$(f)=$leng(h). Without loss of any generality, we assume that leng$(g)=$leng(h), and h,g,f are causal.

Now let us discuss the symmetry of A_L when h,f,g are symmetric/antisymmetric. We first consider the case leng(h) is even. In this case by Proposition 2.2, leng(g), leng(f) are also even if $M_{h,g,f}$ is paraunitary. For $x\in\mathbb{R}$, denote

$$\lceil x\rceil := \inf\{k : x\leq k, k\in\mathbb{Z}\}.$$

Proposition 2.3. *Let $h(z)=\sum_{k=0}^{2\gamma-1}h_kz^{-k}$, $g(z)=\sum_{k=0}^{2\gamma-1}g_kz^{-k}$ and $f(z)=\sum_{k=0}^{2m-1}f_kz^{-k}$ be causal FIR filters with $h_0\neq 0$ and $m\leq\gamma,\gamma\in\mathbb{N}$. Suppose $h(z)$ is symmetric about $\gamma-1/2$ and*

$$g(z)=s_0z^{-(2\gamma-1)}g(z^{-1}),\quad f(z)=s_1z^{-(2m-1)}f(z^{-1}),\quad s_0=\pm 1,\quad s_1=\pm 1.$$

If $M_{h,g,f}$ is paraunitary, then

(i). $\gamma+m+leng(h_e),\gamma+m+leng(g_e)$ and $leng(f_e)$ are odd.

(ii). $(s_0+1)(s_1+1)=0$, $(s_0+1)(s_1+(-1)^{\gamma-m})=0$.

(iii). $A(z):=A_{L_0}(z)$ defined by (2.2) with $L_0:=\gamma+\lceil\frac{m-1}{2}\rceil-1$ is causal, paraunitary and satisfies

$$z^{-(\gamma-1)}diag(1,s_0,s_1z^{-(m-\gamma)})A(z^{-1})diag(J_2,-s_0s_1z^{m-1-2\lceil\frac{m-1}{2}\rceil})=A(z). \tag{2.4}$$

Proof. By the symmetry of h,g,f,

$$z^{-(\gamma-1)}h_e(z^{-1})=h_o(z),\quad z^{-(\gamma-1)}g_e(z^{-1})=s_0g_o(z),\quad z^{-(m-1)}f_e(z^{-1})=s_1f_o(z).$$

Let h_r,g_r,f_r be the filters defined by (2.1) for some integer L. Then

$$z^{-(2L-\gamma-m+2)}h_r(z^{-1})=-s_0s_1h_r(z),z^{-(2L-\gamma-m+2)}g_r(z^{-1})=-s_1g_r(z),z^{-(2L-2\gamma+2)}f_r(z^{-1})=-s_0f_r(z). \tag{2.5}$$

Thus

$$z^{-(\gamma-1)}\mathrm{diag}(1,s_0,s_1z^{-(m-\gamma)})A_L(z^{-1})\mathrm{diag}(J_2,-s_0s_1z^{-(2L-2\gamma-m+3)})=A_L(z).$$

By $|h_e(z)|^2+|h_o(z)|^2+|h_r(z)|^2=1$, leng$(h_r)$=leng$(h_e)$. Assume that

$$h_r(z)=a_kz^{-k}+a_{k+1}z^{-(k+1)}+\cdots-s_0s_1a_kz^{-(k+\mathrm{leng}(h_e)-1)},$$

for some $k \in \mathbb{Z}$ and $a_k \neq 0$. Then $z^{-(2k+\mathrm{leng}(h_e)-1)}h_r(z^{-1}) = -s_0 s_1 h_r(z)$, which together with (2.5) leads to

$$2k + \mathrm{leng}(h_e) - 1 = 2L - \gamma - m + 2.$$

Thus $\mathrm{leng}(h_e) + \gamma + m$ is odd. One can prove similarly that both $\mathrm{leng}(g_e) + \gamma + m$ and $\mathrm{leng}(f_e)$ are odd.

Let $L_0 = \gamma + \lceil \frac{m-1}{2} \rceil - 1$. Then h_r, g_r, f_r defined by (2.1) with $L = L_0$ satisfy

$$z^{-(\gamma - m + 2\lceil \frac{m-1}{2} \rceil)}[h_r(z^{-1}), g_r(z^{-1})] = [-s_0 s_1 h_r(z), -s_1 g_r(z)], \quad z^{-2\lceil \frac{m-1}{2} \rceil} f_r(z^{-1}) = -s_0 f_r(z). \tag{2.6}$$

Note that $\mathrm{leng}(h_r) = \mathrm{leng}(h_e) \le \gamma \le \gamma - m + 2\lceil \frac{m-1}{2} \rceil + 1$, $\mathrm{leng}(g_r) = \mathrm{leng}(g_e) \le \gamma - m + 2\lceil \frac{m-1}{2} \rceil + 1$ and $\mathrm{leng}(f_r) = \mathrm{leng}(f_e) \le m \le 2\lceil \frac{m-1}{2} \rceil + 1$. This fact and (2.6) yield that h_r, g_r, f_r are causal. Thus $A(z)$ is causal, paraunitary and satisfies (2.4).

Finally, we show (ii). By the paraunitariness of $A(z)$ and (2.4) for $z = 1, z = -1$, we have

$$\begin{aligned} &\mathrm{trace}(\mathrm{diag}(1, s_0, s_1)) = \mathrm{trace}(\mathrm{diag}(J_2, -s_0 s_1)), \\ &(-1)^{\gamma-1}\mathrm{trace}(\mathrm{diag}(1, s_0, s_1(-1)^{(m-\gamma)})) = \mathrm{trace}\big(\mathrm{diag}(J_2, -s_0 s_1 (-1)^{(m-1+2\lceil \frac{m-1}{2} \rceil)})\big), \end{aligned}$$

which imply (ii). □

Now let us consider the case $\mathrm{leng}(h)$ is odd. By Proposition 2.2, $\mathrm{leng}(g)$, $\mathrm{leng}(f)$ are also odd if $M_{h,g,f}$ is paraunitary and h, g, f are symmetric/antisymmetric.

Proposition 2.4. *Let $h(z) = \sum_{k=0}^{2\gamma} h_k z^{-k}$, $g(z) = \sum_{k=0}^{2\gamma} g_k z^{-k}$ and $f(z) = \sum_{k=0}^{2m} f_k z^{-k}$ be causal filters with $h_0 \neq 0$ and $m \le \gamma, \gamma \in \mathbb{N}$. Suppose $h(z)$ is symmetric about γ and*

$$g(z) = s_0 z^{-2\gamma} g(z^{-1}), \quad f(z) = s_1 z^{-2m} f(z^{-1}), \quad s_0 = \pm 1, \quad s_1 = \pm 1.$$

If $M_{h,g,f}$ is paraunitary, then

(i). m is odd.

(ii). $s_1 = -s_0$, $(1 + s_0)(1 + (-1)^\gamma) = 0$.

(iii). $A(z) := A_{L_0}(z)$ defined by (2.2) with $L_0 := \gamma + (m-1)/2$ is causal, paraunitary and satisfies

$$z^{-\gamma}\,\mathit{diag}(1, s_0, s_1 z^{-(m-\gamma)}) A(z^{-1})\,\mathit{diag}(1, z, -1) = A(z). \tag{2.7}$$

Proof. By the symmetry of h, g, f,

$$\begin{aligned} z^{-\gamma} h_e(z^{-1}) &= h_e(z), \quad z^{-(\gamma-1)} h_o(z^{-1}) = h_o(z), \\ z^{-\gamma} g_e(z^{-1}) &= s_0 g_e(z), \quad z^{-(\gamma-1)} g_o(z^{-1}) = s_0 g_o(z), \\ z^{-m} f_e(z^{-1}) &= s_1 f_e(z), \quad z^{-(m-1)} f_o(z^{-1}) = s_1 f_o(z). \end{aligned}$$

Let h_r, g_r, f_r be the filters defined by (2.1) for some integer L. Then

(2.8)
$$z^{-(2L-\gamma-m+1)}h_r(z^{-1}) = s_0s_1h_r(z), z^{-(2L-\gamma-m+1)}g_r(z^{-1}) = s_1g_r(z), z^{-(2L-2\gamma+1)}f_r(z^{-1}) = s_0f_r(z).$$

Thus

$$z^{-\gamma}\text{diag}(1, s_0, s_1z^{-(m-\gamma)})A_L(z^{-1})\text{diag}(1, z, s_0s_1z^{-(2L-2\gamma-m+1)}) = A_L(z).$$

Note that $\text{leng}(h_e) = \gamma+1$ and $\text{leng}(h_o) \le \gamma$. This together with $|h_e(z)|^2 + |h_o(z)|^2 + |h_r(z)|^2 = 1$ implies that $\text{leng}(h_r)$=$\text{leng}(h_e) = \gamma+1$, and h_r cannot be symmetric. Thus h_r is antisymmetric, i.e., $s_0s_1 = -1$.

Assume that

$$h_r(z) = a_kz^{-k} + a_{k+1}z^{-(k+1)} + \cdots - a_kz^{-(k+\gamma)}$$

for some $k \in \mathbb{Z}$ and $a_k \neq 0$. Then $z^{-(2k+\gamma)}h_r(z^{-1}) = -h_r(z)$. This and (2.8) yield

$$2k + \gamma = 2L - \gamma - m + 1.$$

Therefore m is odd.

Let $L_0 = \gamma + (m-1)/2$. Then h_r, g_r, f_r defined by (2.1) with $L = L_0$ satisfy

$$z^{-\gamma}h_r(z^{-1}) = -h_r(z), \quad z^{-\gamma}g_r(z^{-1}) = s_1g_r(z), \quad z^{-m}f_r(z^{-1}) = s_0f_r(z),$$

which, together with $\text{leng}(h_r) = \text{leng}(g_r) = \gamma+1$ and $\text{leng}(f_r) \le m+1$, yields that h_r, g_r, f_r are causal. Thus $A(z)$ is causal, paraunitary and satisfies (2.7).

The other statement in (ii) can be proved similarly to that for (ii) in Proposition 2.3, and details are omitted here. $\square$

By Propositions 2.2, 2.3 and 2.4, any causal symmetric/antisymmetric filters h, g, f with $M_{h,g,f}$ paraunitary and $\text{leng}(g)$, $\text{leng}(f) \le \text{leng}(h)$ are given by causal paraunitary filter $A(z)$ satisfying (2.4) or (2.7). Thus to give parameterizations of masks h, g, f for symmetric/antisymmetric framelets, one needs only to present those for $A(z)$ satisfying (2.4) or (2.7).

The parameterization of symmetric paraunitary M-channel filter banks are discussed in [12], [20]. In [12], [20], M is an even number, and the polyphase matrix of the filter bank can be factorized into the product of symmetric *linear* paraunitary factors. For $A(z)$ satisfying (2.4) or (2.7), since $A(z)$ is a 3×3 matrix, $A(z)$ cannot be factorized into the product of symmetric linear paraunitary factors. Instead, $A(z)$ will be factorized into the product of symmetric *quadratic* paraunitary factors.

In the following we discuss parameterizations for causal paraunitary filters $A(z)$ satisfying (2.4) or (2.7) with $m = \gamma$ if γ is odd and $m = \gamma - 1$ if γ is even. For $A(z)$ satisfying (2.4), we factorize it into the form of $P_0(z)V_2(z)\cdots V_{[\gamma/2]}(z)$, where $P_0(z)$ is the causal paraunitary filter satisfying (2.4) for $\gamma = 2$ or $\gamma = 3$, $V_j(z)$ are quadratic paraunitary factors (matrix polynomials of z^{-1} of degree 2). The factorization for $A(z)$ satisfying (2.7) is derived from that for $A(z)$ satisfying (2.4).

By the method used in this paper, one can also give the factorization of $A(z)$ for the general case, for example the factorization of such a causal symmetric paraunitary filter $A(z)$ that is derived from $h(z), g(z), f(z)$ with $h(z), f(z)$ having the forms in Proposition 2.3 and $g(z) = \sum_{k=1}^{2\gamma} g_k z^{-k}$. One will find below that the only difference between the factorization for the cases discussed in this paper and that for the general case is that one needs to calculate the different initial filters $P_0(z)$. Our policy here is that we consider such cases that for fixed filter lengths of masks $h(z), g(z)$ and $f(z)$, the expressions of $h(z), g(z), f(z)$ will have as many (but not redundant) free parameters as possible.

Parameterizations of causal paraunitary filters $A(z)$ satisfying (2.4) and (2.7) with $m = \gamma$ or $m = \gamma - 1$ are discussed in the following two sections, respectively.

3. Parameterizations for masks with even filter lengths

In this section we discuss parameterizations of symmetric/antisymmetric masks with the form: $h(z) = \sum_{k=0}^{2\gamma-1} h_k z^{-k}$, $g(z) = \sum_{k=0}^{2\gamma-1} g_k z^{-k}$ and $f(z) = \sum_{k=0}^{2m-1} f_k z^{-k}$, where $m = \gamma$ if γ is odd and $m = \gamma - 1$ if γ is even.

Let us first consider the case $\gamma = 2n+1$, $n \in \mathbb{N}_0$, and $m = \gamma$. Let $A(z) := A_{L_0}(z)$ be the causal filter defined by (2.2) with $L_0 = 3n$. By Proposition 2.3 (ii), $s_1 = -s_0$ or $s_1 = s_0 = -1$. If $s_1 = -s_0$, by interchanging g, f, we know that it is suffices to consider the parameterization for either the case $s_1 = -s_0 = 1$ or $s_1 = -s_0 = -1$. We consider the case $s_1 = -s_0 = 1$. Denote

$$E(z) := A(z)\mathrm{diag}(R_0, 1), \quad R_0 := \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then $E(z)$ is paraunitary. By Proposition 2.3 (iii),

$$E(z) = z^{-2n}\mathrm{diag}(1, -1, 1)E(z^{-1})\mathrm{diag}(1, -1, 1). \tag{3.1}$$

For the case $s_1 = -s_0 = 1$, define $\tilde{E}(z) := A(z)\mathrm{diag}(R_0, 1)$. Then

$$\tilde{E}(z) = z^{-2n}\mathrm{diag}(-1, 1, 1)\tilde{E}(z^{-1})\mathrm{diag}(-1, 1, 1). \tag{3.2}$$

Thus $\mathrm{diag}(J_2, 1)\tilde{E}(z)\mathrm{diag}(J_2, 1)$ satisfies (3.1). Therefore to find the parameterization for $\tilde{E}(z)$, we need only to find that for $E(z)$ satisfying (3.1).

For the case $\gamma = 2n$, $n \in \mathbb{N}$, we choose $m = \gamma - 1$. Let $A(z) := A_{L_0}(z)$ be the causal filter defined by (2.2) with $L_0 = 3n - 2$. In this case $s_0 = -1, s_1 = \pm 1$. If $s_1 = 1$, then $E(z) := A(z)\mathrm{diag}(R_0, 1)$

satisfies

$$E(z) = z^{-(2n-1)}\mathrm{diag}(1,-1,z)E(z^{-1})\mathrm{diag}(1,-1,1). \tag{3.3}$$

If $s_1 = -1$, then $\tilde{E}(z) := A(z)\mathrm{diag}(R_0, 1)$ satisfies

$$\tilde{E}(z) = z^{-(2n-1)}\mathrm{diag}(-1,1,z)\tilde{E}(z^{-1})\mathrm{diag}(-1,1,1). \tag{3.4}$$

Again, to find the parameterization for $\tilde{E}(z)$ satisfying (3.4), it suffices to find that for $E(z)$ satisfying (3.3).

We give parameterizations of $E(z)$ satisfying (3.1) (for $\gamma = 2n+1$) and (3.3) (for $\gamma = 2n$) in the following two subsections, respectively.

3.1. **The case $\gamma = 2n + 1$.** We first consider the case $n = 0$. By a direct calculation, one has that an orthogonal matrix P_0 satisfies (3.1) with $n = 0$ if and only if it can be written as

$$P_0 = \begin{bmatrix} \cos\theta & 0 & -\rho_1 \sin\theta \\ 0 & \rho_2 & 0 \\ \sin\theta & 0 & \rho_1 \cos\theta \end{bmatrix}, \quad \rho_1 = \pm 1, \quad \rho_2 = \pm 1, \quad \theta \in [-\pi, \pi). \tag{3.5}$$

Define a matrix filter $V(z)$ by

$$V(z) := \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}[\cos\alpha, 1, \sin\alpha] + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}[\sin\alpha, 0, -\cos\alpha]z^{-1} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}[\cos\alpha, -1, \sin\alpha]z^{-2}. \tag{3.6}$$

It is easy to check that $V(z)$ is paraunitary and satisfies (3.1) with $n = 1$.

Theorem 3.1. *A causal paraunitary filter $E(z)$ satisfies (3.1) for some $n \in \mathbb{N}_0$ if and only if it can be factorized in the form of*

$$E(z) = P_0 V_1(z) \cdots V_n(z), \tag{3.7}$$

where P_0 is defined by (3.5) with $\theta \in [-\pi, \pi)$ and $V_j(z), 1 \le j \le n$ are defined by (3.6) with $\alpha_j \in [-\pi, \pi)$.

Proof. Clearly if $E(z)$ is given by (3.7), then it is causal, paraunitary and satisfies (3.1). Conversely, suppose $E_n(z) = e_0 + \cdots + e_{2n}z^{-2n}$ with $n \in \mathbb{N}_0$ is paraunitary and satisfies (3.1). We show that it can be factorized in the form (3.7) by induction on n.

If $n = 0$, $E(z)$ is P_0 and the statement is true. Assume that n is a positive integer. Define

$$E_{n-1}(z) := E_n(z)V(z^{-1})^T,$$

where $V(z)$ is the filter defined by (3.6) with $\alpha \in [-\pi, \pi)$. Then $E_{n-1}(z)$ is paraunitary and satisfies (3.1) with $n-1$. We show that there exists α such that $E_{n-1}(z)$ defined above is causal. Note that

$$V(z^{-1})^T = \frac{1}{2}\begin{bmatrix} \cos\alpha \\ 1 \\ \sin\alpha \end{bmatrix}[1,1,0] + \begin{bmatrix} \sin\alpha \\ 0 \\ -\cos\alpha \end{bmatrix}[0,0,-1]z + \frac{1}{2}\begin{bmatrix} \cos\alpha \\ -1 \\ \sin\alpha \end{bmatrix}[1,-1,0]z^2.$$

Thus we need only to show that there exists α such that

$$e_0[\cos\alpha, -1, \sin\alpha]^T = 0, \quad e_0[\sin\alpha, 0, -\cos\alpha]^T = 0, \tag{3.8}$$

$$e_1[\cos\alpha, -1, \sin\alpha]^T = 0. \tag{3.9}$$

First we consider the case $e_0 = 0$. By the symmetry of $E_n(z)$, $e_j = \mathrm{diag}(1,-1,1)e_{2n-j}\mathrm{diag}(1,-1,1), 0 \leq j \leq 2n$. By the paraunitariness of $E_n(z)$, $e_1 e_{2n-1}^T = 0$. Thus $e_1\mathrm{diag}(1,-1,1)e_1^T = 0$. Therefore $\mathrm{rank}(e_1) = 1$ and $e_1 = u(\cos\theta, 1, \sin\theta)$ for some $u \in \mathbb{R}^3, \theta \in [-\pi, \pi)$. Let $\alpha = \theta$. Then (3.8) and (3.9) hold.

Then let us consider the case $e_0 \neq 0$. The facts $e_0 e_{2n}^T = 0$ and $e_{2n} = \mathrm{diag}(1,-1,1)e_0\mathrm{diag}(1,-1,1)$ yield that $e_0 = u[\cos\theta, 1, \sin\theta]$ for some $u \in \mathbb{R}^3\backslash\{0\}, \theta \in [-\pi, \pi)$. Thus (3.8) holds for $\alpha = \theta$.

To show that (3.9) holds for $\alpha = \theta$, denote $X := e_1[\cos\theta, -1, \sin\theta]^T$. By $e_0 e_{2n-1}^T + e_1 e_{2n}^T = 0$ (a consequence of the paraunitariness of $E_n(z)$),

$$uX^T + Xu^T = 0.$$

Since $u \neq 0$, the above equation yields $X = 0$. Thus (3.9) holds for $\alpha = \theta$.

Therefore for $\alpha = \theta$, $E_{n-1}(z)$ is causal. By our induction assumption, $E_{n-1}(z)$ can be factorized into the product of $P_0 V_1(z) \cdots V_{n-1}(z)$. Thus $E_n(z)$ can be factorized into the form of (3.7). The proof of Theorem 3.1 is complete. □

By Theorem 3.1 and the relationship between $E(z)$ and masks h, g, f, we have the following corollary.

Corollary 3.1. *Assume that $h(z) = \sum_{k=0}^{4n+1} h_k z^{-k}$, $g(z) = \sum_{k=0}^{4n+1} g_k z^{-k}$ and $f(z) = \sum_{k=0}^{4n+1} f_k z^{-k}$ are causal filters with $h_0 \neq 0$. Suppose $h(z)$ is symmetric about $2n + 1/2$ and*

$$g(z) = s_0 z^{-(4n+1)} g(z^{-1}), \quad f(z) = s_1 z^{-(4n+1)} f(z^{-1}), \quad s_0 = \pm 1, \quad s_1 = \pm 1.$$

Then $M_{h,g,f}$ is paraunitary if and only if $s_1 = -s_0 = \pm 1$ and h, g, f are factorized as

$$\begin{bmatrix} h(z) \\ J_{\pm} \begin{bmatrix} g(z) \\ f(z) \end{bmatrix} \end{bmatrix} = \frac{1}{2} P_0 V_1(z^2) \cdots V_n(z^2) \begin{bmatrix} z^{-1}+1 \\ z^{-1}-1 \\ 0 \end{bmatrix}, J_{\pm} = I_2 \text{ if } s_1 = 1, J_{\pm} = J_2 \text{ if } s_1 = -1, \tag{3.10}$$

or $s_1 = s_0 = -1$ and h, g, f are factorized as

$$\begin{bmatrix} g(z) \\ h(z) \\ f(z) \end{bmatrix} = \frac{1}{2} P_0 V_1(z^2) \cdots V_n(z^2) \begin{bmatrix} z^{-1}-1 \\ z^{-1}+1 \\ 0 \end{bmatrix}, \tag{3.11}$$

where P_0 is defined by (3.5) with $\theta \in [-\pi, \pi)$ and $V_j(z), 1 \le j \le n$ are defined by (3.6) with $\alpha_j \in [-\pi, \pi)$.

3.2. **The case $\gamma = 2n$.** In this subsection, we discuss the parameterization of $A(z)$ satisfying (3.3). First, we consider the case $n = 1$.

Proposition 3.1. *A causal filter $P_1(z)$ is paraunitary and satisfies (3.3) with $n = 1$ if and only if it can be written as*

$$P_1(z) = diag(\rho_1, \rho_2, \rho_3)\left(\frac{1}{2} \begin{bmatrix} \cos\theta & 1 & \sin\theta \\ \cos\theta & 1 & \sin\theta \\ -2\sin\theta & 0 & 2\cos\theta \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \cos\theta & -1 & \sin\theta \\ -\cos\theta & 1 & -\sin\theta \\ 0 & 0 & 0 \end{bmatrix} z^{-1}\right), \tag{3.12}$$

where $\theta \in [-\pi, \pi), \rho_1 = \pm 1, \rho_2 = \pm 1, \rho_3 = \pm 1$.

Proof. One can check directly that if $P_1(z)$ is given by (3.12), then it is paraunitary and satisfies (3.3) with $n = 1$. Conversely, assume that $P_1(z) = A + Bz^{-1}$. By the symmetry of P_1, the entries a_{ij} of A satisfy $a_{32} = 0$ and $B = \text{diag}(1, -1, 0)A \text{ diag}(1, -1, 1)$. By $AB^T = 0$,

$$[a_{ij}]_{1 \le i \le 2, 1 \le j \le 3} \text{diag}(1, -1, 1) \left([a_{ij}]_{1 \le i \le 2, 1 \le j \le 3}\right)^T = 0.$$

Thus the rank of matrix $[a_{ij}]_{1 \le i \le 2, 1 \le j \le 3}$ is 1 and it can be written as

$$[a_{ij}]_{1 \le i \le 2, 1 \le j \le 3} = v[\cos\theta, 1, \sin\theta], \quad v = (v_1, v_2)^T \in \mathbb{R}^2 \backslash \{0\}, \theta \in [-\pi, \pi).$$

On the other hand $A^T B = 0$ implies

$$[a_{31}, 0, a_{33}][\cos\theta, 1, \sin\theta]^T = 0.$$

Thus $[a_{31}, a_{33}] = t[\sin\theta, -\cos\theta]$ for some $t \in \mathbb{R}$. Finally $AA^T + BB^T = I_3$ yields $t = \pm 1, v_1 = \pm 1/2, v_2 = \pm 1/2$. □

Theorem 3.2. *A causal filter $E(z)$ satisfies (3.3) for some $n \in \mathbb{N}$ if and only if it can be factorized in the form of*

$$E(z) = P_1(z)V_2(z)\cdots V_n(z), \tag{3.13}$$

where $P_1(z)$ is defined by (3.12) with $\theta \in [-\pi, \pi)$ and $V_j(z), 2 \le j \le n$ are defined by (3.6) with $\alpha_j \in [-\pi, \pi)$.

Proof. One can check directly that if $E(z)$ is given by (3.13), then it is causal, paraunitary and satisfies (3.3). Conversely, suppose that $E_n(z) = e_0 + \cdots + e_{2n-1}z^{-(2n-1)}$, with $n \in \mathbb{N}$, is paraunitary and satisfies (3.3). We prove that it can be factorized in the form (3.13) by induction on n.

By Proposition 3.1, it is true for $n = 1$. Assume that $n \ge 2$. Define

$$E_{n-1}(z) := E_n(z)V(z^{-1})^T,$$

where $V(z)$ is the filter defined by (3.6) for some $\alpha \in [-\pi, \pi)$. Then $E_{n-1}(z)$ is paraunitary and it satisfies (3.1) with $n-1$. We show that there exists α such that $E_{n-1}(z)$ is causal, or precisely, there exists α such that

$$e_0[\cos\alpha, -1, \sin\alpha]^T = 0, \quad e_0[\sin\alpha, 0, -\cos\alpha]^T = 0, \tag{3.14}$$

$$e_1[\cos\alpha, -1, \sin\alpha]^T = 0. \tag{3.15}$$

By the symmetry of $E_n(z)$, $e_{2n-1} = \mathrm{diag}(1, -1, 0)e_0\mathrm{diag}(1, -1, 1)$ and

$$e_{2n-j} = \big(\mathrm{diag}(1, -1, 0)e_j + \mathrm{diag}(0, 0, 1)e_{j-1}\big)\mathrm{diag}(1, -1, 1), 2 \le j \le 2n.$$

If $e_0 = 0$, then $e_{2n-1} = 0$. Then $e_1e_{2n-2}^T = 0, e_1e_{2n-3}^T + e_2e_{2n-2}^T = 0$ (followed from the paraunitariness of $E(z)$) yield

$$e_1\mathrm{diag}(1, -1, 1)e_1^T = 0.$$

Thus $\mathrm{rank}(e_1) = 1$ and $e_1 = u(\cos\theta, 1, \sin\theta)$ for some $u \in \mathbb{R}^3, \theta \in [-\pi, \pi)$. Therefore (3.14) and (3.15) hold with $\alpha = \theta$.

Assume that $e_0 \ne 0$. Then $e_0e_{2n-1}^T = 0$ and $e_0e_{2n-2}^T + e_1e_{2n-1}^T = 0$ imply that

$$e_0\mathrm{diag}(1, -1, 1)e_0^T = 0.$$

Thus e_0 can be written as $e_0 = v(\cos\theta, 1, \sin\theta)$ for some $v \in \mathbb{R}^3\backslash\{0\}, \theta \in [-\pi, \pi)$. On the other hand $e_0e_{2n-2}^T + e_1e_{2n-1}^T = 0$ and $e_0e_{2n-3}^T + e_1e_{2n-2}^T + e_2e_{2n-1}^T = 0$ yield

$$e_0\mathrm{diag}(1, -1, 1)e_1^T + e_1\mathrm{diag}(1, -1, 1)e_0^T = 0.$$

Denote $Y := e_1[\cos\theta, -1, \sin\theta]^T$. Then the above equation is

$$vY^T + Yv^T = 0,$$

which implies $Y = 0$. Thus (3.14) and (3.15) hold for $\alpha = \theta$.

Thus for $\alpha = \theta$, $E_{n-1}(z)$ is causal. By our induction assumption, $E_{n-1}(z)$ can be factorized into the form of $P_1(z)V_2(z)\cdots V_{n-1}(z)$. Therefore $E_n(z)$ can be factorized into the form of (3.13). The proof of Theorem 3.2 is complete. $\square$

Corollary 3.2. *Let $h(z) = \sum_{k=0}^{4n-1} h_k z^{-k}$, $g(z) = \sum_{k=0}^{4n-1} g_k z^{-k}$, and $f(z) = \sum_{k=0}^{4n-3} f_k z^{-k}$ be causal filters with $h_0 \neq 0$. Suppose that $h(z)$ is symmetric about $2n - 1/2$ and*

$$g(z) = s_0 z^{-(4n-1)} g(z^{-1}), \quad f(z) = s_1 z^{-(4n-3)} f(z^{-1}), \quad s_0 = \pm 1, \quad s_1 = \pm 1.$$

Then $M_{h,g,f}$ is paraunitary if and only if $s_1 = -s_0 = 1$ and h, g, f are factorized as

$$\begin{bmatrix} h(z) \\ g(z) \\ f(z) \end{bmatrix} = \frac{1}{2} P_1(z^2)V_2(z^2)\cdots V_n(z^2)\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} z^{-1}\right), \tag{3.16}$$

or $s_1 = s_0 = -1$ and h, g, f are factorized as

$$\begin{bmatrix} g(z) \\ h(z) \\ f(z) \end{bmatrix} = \frac{1}{2} P_1(z^2)V_2(z^2)\cdots V_n(z^2)\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} z^{-1}\right), \tag{3.17}$$

where $P_1(z)$ is defined by (3.12) with $\theta \in [-\pi, \pi)$ and $V_j(z), 2 \leq j \leq n$ are defined by (3.6) with $\alpha_j \in [-\pi, \pi)$.

4. Parameterizations for masks with odd filter lengths

In this section we consider parameterizations of symmetric/antisymmetric masks h, g, f with the form of $h(z) = \sum_{k=0}^{2\gamma} h_k z^{-k}, g(z) = \sum_{k=0}^{2\gamma} g_k z^{-k}, f(z) = \sum_{k=0}^{2m} f_k z^{-k}$, where $m = \gamma$ if γ is odd and $m = \gamma - 1$ if γ is even.

Assume that $\gamma = m = 2n-1$ for some $n \in \mathbb{N}$. In this case $s_1 = -s_0 = \pm 1$ by Proposition 2.4 (ii). Let $A(z) := A_{L_0}(z)$ be the causal filter defined by (2.2) with $L_0 = 3n-2$. Denote $E(z) := A(z)\mathrm{diag}(1, J_2)$. Then $E(z)$ satisfies

$$E(z) = z^{-(2n-1)}\mathrm{diag}(1, s_0, s_1)E(z^{-1})\mathrm{diag}(1, -1, z). \tag{4.1}$$

Thus $E(z)^T$ satisfies (3.3) provided $s_1 = -s_0 = 1$. By Theorem 3.2, $E(z)^T$ can be factorized in the form of (3.13). Therefore we have the following corollary.

Corollary 4.1. *Let* $h(z) = \sum_{k=0}^{4n-2} h_k z^{-k}$, $g(z) = \sum_{k=0}^{4n-2} g_k z^{-k}$, *and* $f(z) = \sum_{k=0}^{4n-2} f_k z^{-k}$ *be causal filters with* $h_0 \neq 0$. *Suppose* $h(z)$ *is symmetric about* $2n-1$ *and*

$$g(z) = s_0 z^{-(4n-2)} g(z^{-1}), \quad f(z) = s_1 z^{-(4n-2)} f(z^{-1}), \quad s_0 = \pm 1, \quad s_1 = \pm 1.$$

Then $M_{h,g,f}$ *is paraunitary if and only if* $s_1 = -s_0 = \pm 1$ *and* h, g, f *are factorized as*

$$\begin{bmatrix} h(z) \\ J_\pm \begin{bmatrix} g(z) \\ f(z) \end{bmatrix} \end{bmatrix} = \frac{\sqrt{2}}{2} V_n(z^2)^T \cdots V_2(z^2)^T P_1(z^2)^T \begin{bmatrix} 1 \\ 0 \\ z^{-1} \end{bmatrix}, J_\pm = I_2 \text{ if } s_1 = 1,\ J_\pm = J_2 \text{ if } s_1 = -1, \tag{4.2}$$

where $P_1(z)$ *is defined by (3.12) with* $\theta \in [-\pi, \pi)$ *and* $V_j(z), 2 \le j \le n$ *are defined by (3.6) with* $\alpha_j \in [-\pi, \pi)$.

For the case $\gamma = 2n$, $n \in \mathbb{N}$, we choose $m = 2n-1$. In this case $s_1 = -s_0 = 1$. Let $A(z) := A_{L_0}(z)$ be the causal filter defined by (2.2) with $L_0 = 3n-1$. Denote $E(z) := A(z)\mathrm{diag}(1, J_2)$. Then $E(z)$ satisfies

$$E(z) = z^{-2n}\mathrm{diag}(1, -1, z) E(z^{-1}) \mathrm{diag}(1, -1, z). \tag{4.3}$$

To give the parameterization of $E(z)$ satisfying (4.3), we define

$$W_0(z) := \frac{1}{2}\begin{bmatrix} 1 & \rho & 0 \\ 1 & \rho & 0 \\ 0 & 0 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -\rho & 0 \\ -1 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix} z^{-1}, \quad \rho = \pm 1. \tag{4.4}$$

One can check that W_0 is paraunitary and satisfies

$$W_0(z) = z^{-1}\mathrm{diag}(1, -1, 1) W_0(z^{-1}) \mathrm{diag}(1, -1, z). \tag{4.5}$$

Theorem 4.1. *A causal filter* $E(z)$ *satisfies (4.3) for some* $n \in \mathbb{N}$ *if and only if it can be factorized in the form of*

$$E(z) = P_1(z) V_2(z) \cdots V_n(z) W_0(z), \tag{4.6}$$

where $W_0(z)$ *is defined by (4.4),* $P_1(z)$ *is defined by (3.12) with* $\theta \in [-\pi, \pi)$ *and* $V_j(z), 2 \le j \le n$ *are defined by (3.6) with* $\alpha_j \in [-\pi, \pi)$.

Proof. If $E(z)$ is given by (4.6), then it is easy to check that it is causal, paraunitary and satisfies (4.3). Conversely, suppose $E_n(z) = e_0 + \cdots + e_{2n} z^{-2n}$ with $n \in \mathbb{N}$ is paraunitary and satisfies (4.3). Define

$$\tilde{E}(z) := E_n(z) W_0(z^{-1})^T.$$

Then $\tilde{E}(z)$ is paraunitary and satisfies (3.3). We will show that for a suitable sign choice of ρ in $W_0(z)$, $\tilde{E}(z)$ is causal. That is

$$e_0[1,1,0]^T = 0 \text{ or } e_0[1,-1,0]^T = 0.$$

By the symmetry of $E(z)$, $e_{2n} = \mathrm{diag}(1,-1,0)e_0\mathrm{diag}(1,-1,0)$ and

$$e_{2n-1} = \mathrm{diag}(1,-1,0)\big(e_1\mathrm{diag}(1,-1,0) + e_0\mathrm{diag}(0,0,1)\big) + \mathrm{diag}(0,0,1)e_0\mathrm{diag}(1,-1,0).$$

By $e_0e_{2n}^T = 0, e_0e_{2n-1}^T + e_1e_{2n}^T = 0$ (followed from the paraunitariness of $E_n(z)$),

$$e_0\mathrm{diag}(1,-1,0)e_0^T\mathrm{diag}(1,-1,1) = 0.$$

Thus $\mathrm{rank}(e_0\mathrm{diag}(1,1,0)) = 1$ and the first two columns of e_0 can be written as $e_0\mathrm{diag}(1,1,0) = u\mathrm{diag}(1,\rho_1,0), \rho_1 = \pm 1, u \in \mathbb{R}^3$. Therefore $e_0[1,-\rho_1,0]^T = 0$. Hence if we choose $\rho = \rho_1$ in $W_0(z)$, then $\tilde{E}(z)$ is causal. By Theorem 3.2, $\tilde{E}(z)$ can be factorized into the product of $P_1(z)V_2(z)\cdots V_n(z)$. Therefore $E(z)$ can be factorized into the form of (4.6). □

Note that $W_0(z)[1,0,z^{-1}]^T = [1+z^{-2}, 1-z^{-2}, 2z^{-1}]^T/2$. Theorem 4.1 leads to the following corollary.

Corollary 4.2. *Assume that $h(z) = \sum_{k=0}^{4n} h_k z^{-k}$, $g(z) = \sum_{k=0}^{4n} g_k z^{-k}$, and $f(z) = \sum_{k=0}^{4n-2} f_k z^{-k}$ are causal filters with $h_0 \neq 0$. Suppose $h(z)$ is symmetric about $2n$ and*

$$g(z) = s_0 z^{-4n} g(z^{-1}), \quad f(z) = s_1 z^{-(4n-2)} f(z^{-1}), \quad s_0 = \pm 1, \quad s_1 = \pm 1.$$

Then $M_{h,g,f}$ is paraunitary if and only if $s_1 = -s_0 = 1$ and h, g, f are factorized as

$$\begin{bmatrix} h(z) \\ g(z) \\ f(z) \end{bmatrix} = \frac{\sqrt{2}}{4} P_1(z^2)V_2(z^2)\cdots V_n(z^2) \begin{bmatrix} 1+z^{-2} \\ 1-z^{-2} \\ 2z^{-1} \end{bmatrix}, \tag{4.7}$$

where $P_1(z)$ is defined by (3.12) with $\theta \in [-\pi,\pi)$ and $V_j(z), 2 \le j \le n$ are defined by (3.6) with $\alpha_j \in [-\pi,\pi)$.

5. Examples

In this section we construct framelets based on parameterizations of masks h, g, f provided in Corollaries 3.1, 3.2, 4.1 and 4.2. We use ϕ to denote the refinable function and ψ_1, ψ_2 to denote the symmetric/antisymmetric framelets corresponding to g, f, respectively.

Example 5.1. Let h, g, f be the filters given by (4.2) with $n = 1, s_1 = 1, \rho_1 = \rho_3 = 1$. Then

$$h(z) = \frac{\sqrt{2}}{4}(\cos\theta - 2\sin\theta z^{-1} + \cos\theta z^{-2}), g(z) = \frac{\sqrt{2}}{4}(1-z^{-2}), f(z) = \frac{\sqrt{2}}{4}(\sin\theta + 2\cos\theta z^{-1} + \sin\theta z^{-2}).$$

Clearly $h(1) = 1$ if and only if $\theta = -\frac{\pi}{4}$. For $\theta = -\frac{\pi}{4}$, $h(z) = (1+z^{-1})^2/4$ and the refinable function ϕ is the hat function. In this case the corresponding framelets ψ_1, ψ_2 are those constructed in [17].

Example 5.2. Let h, g, f be the filters given by (3.17) with $n = 1, \rho_1 = \rho_2 = \rho_3 = 1$. Then

$$\begin{aligned} h(z) &= (1+z^{-1})(1-\cos\theta + 2\cos\theta z^{-1} + (1-\cos\theta)z^{-2})/4, \\ g(z) &= (1-z^{-1})(1-\cos\theta + 2z^{-1} + (1-\cos\theta)z^{-2})/4, \\ f(z) &= \sin\theta(1-z^{-1})/2. \end{aligned}$$

For $\theta = \frac{\pi}{3}$, $h(z) = (1+z^{-1})^3/8$ and the refinable function is the quadratic B-spline. The framelets ψ_1, ψ_2 are those constructed in [3].

One notes that framelets ψ_1 in Example 5.1 and ψ_1, ψ_2 in Example 5.2 have the vanishing moments of order 1. In the following we construct symmetric/antisymmetric framelets with good smoothness and higher vanishing moment orders. Our method is, based on parameterizations for masks, to fix parameters by solving some equations related to the sum rules for the refinement mask and the vanishing moments for framelets such that the resulting framelets have good smoothness and higher vanishing moment orders. We will make the change of variables:

$$\sin\alpha = \frac{2t}{1+t^2}, \quad \cos\alpha = \frac{1-t^2}{1+t^2},$$

which makes it easy to solve the equations by Maple programs. The refinable functions ϕ constructed below are $L^2(\mathbb{R})$ stable, and we use the Sobolev smoothness estimate in [10] and [25] to calculate the smoothness of ϕ.

Example 5.3. Let $h(z) = \sum_{k=0}^{5} h_k z^{-k}, g(z) = \sum_{k=0}^{5} g_k z^{-k}, f(z) = \sum_{k=0}^{5} f_k z^{-k}$ be the filters given by (3.10) with $n = 2, s_1 = 1, \rho_1 = \rho_2 = 1$. Then

$$\begin{aligned} &h_0 = \frac{t_1^2(-1+t_0^2)}{2(1+t_0^2)(1+t_1^2)}, \quad h_1 = \frac{1-t_0^2}{2(1+t_0^2)(1+t_1^2)}, \quad h_2 = \frac{2t_0t_1}{(1+t_0^2)(1+t_1^2)}, \\ &g_0 = -\frac{t_1^2}{2(1+t_1^2)}, \quad g_1 = \frac{1}{2(1+t_1^2)}, \quad g_2 = 0, \\ &f_0 = -\frac{t_0t_1^2}{(1+t_0^2)(1+t_1^2)}, \quad f_1 = \frac{t_0}{(1+t_0^2)(1+t_1^2)}, \quad f_2 = \frac{t_1(-1+t_0^2)}{(1+t_0^2)(1+t_1^2)}, \end{aligned}$$

and $h_j = h_{5-j}, g_j = -g_{5-j}, f_j = f_{5-j}, 3 \le j \le 5$. For $t_1 = t_0$, $h(1) = 1, h(-1) = 0$ and $g(1) = 0$, $f(1) = f'(1) = 0$. For $t_1 = t_0 = \pm\sqrt{\frac{3}{5}}$, h has the sum rules of order 3, and the resulting $\phi \in W^{1.64688}(\mathbb{R})$, ψ_1

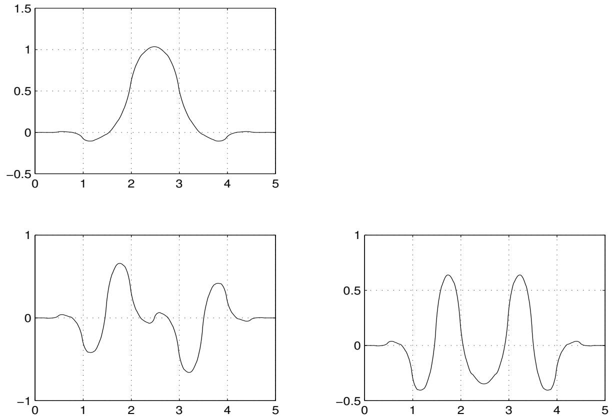

FIGURE 1. Symmetric refinable function $\phi \in W^{1.64688}$ and antisymmetric/symmetric framelets ψ_1, ψ_2 with vanishing moments of orders 3 and 2 respectively.

and ψ_2 have the vanishing moments of order 3 and 2, respectively. See Figure 1 for the graphs of ϕ, ψ_1 and ψ_2. In this case the masks are

$$\begin{aligned}
h(z) &= -\frac{1}{64}(1+z^{-1})^3(3-14z^{-1}+3z^{-2}),\\
g(z) &= -\frac{1}{16}(1-z^{-1})^3(3+4z^{-1}+3z^{-2}),\\
f(z) &= \mp\frac{\sqrt{15}}{64}(1-z^{-1})^2(1+z^{-1})(3-2z^{-1}+3z^{-2}).
\end{aligned}$$

Example 5.4. Let $h(z)=\sum_{k=0}^{6} h_k z^{-k}, g(z)=\sum_{k=0}^{6} g_k z^{-k}, f(z)=\sum_{k=0}^{6} f_k z^{-k}$ be the filters given by (4.2) with $n=2, s_1=1, \rho_1=\rho_3=1$. Then

$$\begin{aligned}
h_0 &= \frac{\sqrt{2}}{4}\frac{1-t_1^2}{(1+t_1^2)(1+t_0^2)}, \quad h_1 = \frac{\sqrt{2}}{2}\frac{t_0(-1+t_1^2)}{(1+t_1^2)(1+t_0^2)},\\
h_2 &= \frac{\sqrt{2}}{4}\frac{t_0(t_0t_1^2-t_0-4t_1)}{(1+t_1^2)(1+t_0^2)}, \quad h_3 = \frac{\sqrt{2}t_1(-1+t_0^2)}{(1+t_1^2)(1+t_0^2)},\\
g_0 &= \frac{\sqrt{2}}{4}\frac{1}{1+t_0^2}, \quad g_1 = -\frac{\sqrt{2}}{2}\frac{t_0}{1+t_0^2}, \quad g_2 = -\frac{\sqrt{2}}{4}\frac{t_0^2}{1+t_0^2}, \quad g_3 = 0,\\
f_0 &= \frac{\sqrt{2}}{2}\frac{t_1}{(1+t_1^2)(1+t_0^2)}, \quad f_1 = -\frac{\sqrt{2}t_0t_1}{(1+t_1^2)(1+t_0^2)},\\
f_2 &= -\frac{\sqrt{2}}{2}\frac{t_0(t_0t_1-1+t_1^2)}{(1+t_1^2)(1+t_0^2)}, \quad f_3 = \frac{\sqrt{2}}{2}\frac{(1-t_0^2)(1-t_1^2)}{(1+t_1^2)(1+t_0^2)},
\end{aligned}$$

and $h_j=h_{6-j}, g_j=-g_{6-j}, f_j=f_{6-j}, 4\le j\le 6$. For $t_1=((1+\sqrt{2})t_0+1)/(t_0-1-\sqrt{2})$, $h(1)=1, h(-1)=0$ and $g(1)=0$, $f(1)=f'(1)=0$. Furthermore, if $t_0=10.22065693302211$ (a root of the polynomial $x^4-8x^3-22x^2-8x+9$), h has the sum rules of order 4 and the resulting $\phi\in W^{3.37460}(\mathbb{R})$. In this case ψ_1 and ψ_2 have the vanishing moments of order 1 and 2 respectively. If $t_0=-2+\sqrt{7}$ ($t_1=-(1+2\sqrt{2})\sqrt{7}/7$ then), then h has the sum rules of order 2. In this case $\phi\in W^{1.22062}(\mathbb{R})$, ψ_1, ψ_2 have the vanishing moments of order 3 and 2, respectively, and the masks are

$$\begin{aligned}
h(z)&=-\frac{3+\sqrt{7}}{64}(1+z^{-1})^2\big(1+2(1-\sqrt{7})z^{-1}-6(5-2\sqrt{7})z^{-2}+2(1-\sqrt{7})z^{-3}+z^{-4}\big),\\
g(z)&=\frac{3\sqrt{2}+\sqrt{7}}{32}(1-z^{-1})^3(1+z^{-1})\big(1+(6-2\sqrt{7})z^{-1}+z^{-2}\big),\\
f(z)&=-\frac{3\sqrt{7}+7}{448}(1-z^{-1})^2\big(7+14(3-\sqrt{7})z^{-1}+2(7-2\sqrt{7})z^{-2}+14(3-\sqrt{7})z^{-3}+7z^{-4}\big).
\end{aligned}$$

Example 5.5. Let $h(z)=\sum_{k=0}^{9}h_kz^{-k}, g(z)=\sum_{k=0}^{9}g_kz^{-k}, f(z)=\sum_{k=0}^{9}f_kz^{-k}$ be the filters given by (3.10) with $n=3, s_1=1, \rho_1=\rho_2=1$. Here we do not give the parametric expression for them. If

$$t_2=(-16t_1\pm 2\sqrt{62t_1^2-t_1^4+63})/(t_1^2+9)/2,\quad t_0=(t_1+t_2)/(1-t_1t_2),$$

then h has the sum rules of order 3, $g(1)=g'(1)=g''(1)=0$, $f(1)=f'(1)=0$. If $t_1=\sqrt{32\sqrt{226}-481}$, $t_0=(59+4\sqrt{226})t_1/27$, and $t_2=27t_0/45$, then h has the sum rules of order 3, the resulting $\phi\in W^{1.82127}(\mathbb{R})$, ψ_1 and ψ_2 have the vanishing moments of order 3 and 4 respectively. See Figure 2 for the graphs. The corresponding masks are

$$\begin{aligned}
h(z)&=2^{-12}(1+z^{-1})^3\big((43+2c)-6(33+2c)z^{-1}+3(71+10c)z^{-2}+\\
&\qquad 4(99-10c)z^{-3}+3(71+10c)z^{-4}-6(33+2c)z^{-5}+(43+2c)z^{-6}\big),\\
g(z)&=2^{-10}(1-z^{-1})^3\big((76+5c)+4(16+c)z^{-1}-3(4-c)z^{-2}-\\
&\qquad 8(16-c)z^{-3}-3(4-c)z^{-4}+4(16+c)z^{-5}+(76+5c)z^{-6}\big),\\
f(z)&=-2^{-13}\sqrt{32c-481}(1-z^{-1})^4(1+z^{-1})\big(3(391+26c)+4(241+16c)z^{-1}+\\
&\qquad 2(751+50c)z^{-2}+4(241+16c)z^{-3}+3(391+26c)z^{-4}\big),
\end{aligned}$$

where $c=\sqrt{226}$.

Example 5.6. Let $h(z)=\sum_{k=0}^{11}h_kz^{-k}, g(z)=\sum_{k=0}^{11}g_kz^{-k}, f(z)=\sum_{k=0}^{9}f_kz^{-k}$ be the filters given by (3.17) with $n=3, s_1=1, \rho_1=\rho_2=\rho_3=1$. For the choices of

$$t_0=-\sqrt{-16345+2800\sqrt{37}}/35,\quad t_1=35(53+8\sqrt{37})t_0/441,\quad t_2=-\sqrt{1195+200\sqrt{37}}/\sqrt{77},$$

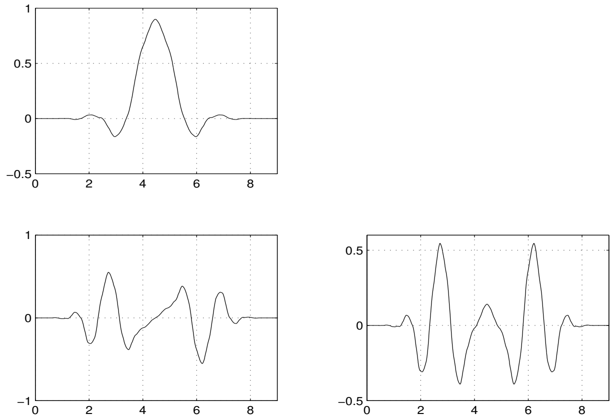

FIGURE 2. Symmetric refinable function $\phi \in W^{1.82127}$ and antisymmetric/symmetric framelets ψ_1, ψ_2 with vanishing moments of orders 3 and 4 respectively.

the resulting $\phi \in W^{3.27435}(\mathbb{R})$, and both ψ_1 and ψ_2 have vanishing moments of order 3. See Figure 3 for the graphs. In this case the masks are

$$\begin{aligned}
h(z) &= 2^{-17}(1+z^{-1})^5\big(25(13+5c) - 10(83+75c)z^{-1} - 5(433-375c)z^{-2} + \\
&\quad 4(2359-625c)z^{-3} - 5(433-375c)z^{-4} - 10(83+75c)z^{-5} + 25(13+5c)z^{-6}\big), \\
g(z) &= 2^{-17}(1-z^{-1})^3\big(25(13+5c) + 10(177+25c)z^{-1} - 1600(7-c)z^{-2} - 62(43+35c)z^{-3} - \\
&\quad (55818-2950c)z^{-4} - 62(43+35c)z^{-5} - 1600(7-c)z^{-6} + 10(177+25c)z^{-7} + 25(13+5c)z^{-8}\big), \\
f(z) &= \frac{2^{-16}\sqrt{2800c-16345}}{63}(1-z^{-1})^3\big(45(13+5c) + 18(177+25c)z^{-1} + 513(13+5c)z^{-2} + \\
&\quad 4(6503+959c)z^{-3} + 513(13+5c)z^{-4} + 18(177+25c)z^{-5} + 45(13+5c)z^{-6}\big),
\end{aligned}$$

where $c = \sqrt{37}$.

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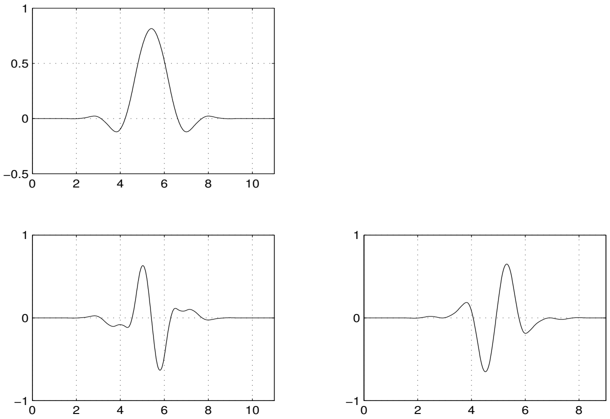

FIGURE 3. Symmetric refinable function $\phi \in W^{3.27435}$ and antisymmetric framelets ψ_1, ψ_2 with vanishing moments of order 3.

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