## TWO-SAMPLE TEST BASED ON CLASSIFICATION PROBABILITY

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ABSTRACT. Robust classification algorithms have been developed in recent years with great success. We take advantage of this development and recast the classical two-sample test problem in the framework of classification. Based on the estimates of classification probabilities from a classifier trained from the samples, a test statistic is proposed. We explain why such a test can be a powerful test and compare its performance in terms of the power and efficiency with those of some other recently proposed tests with simulation and real-life data. The test proposed is nonparametric and can be applied to complex and high dimensional data wherever there is a classifier that provides consistent estimate of the classification probability for such data.

arXiv:1909.07836v1 [math.ST] 17 Sep 2019

Key words and phrases. Two-sample test, classification.

#### 1. INTRODUCTION

A two-sample test detects if two sets of data have different underlying probability distributions. This is a classical problem in statistics with numerous applications in a range of scientific inquiries. The problem becomes challenging when data have higher dimensions relative to their sizes and are in complex form (texts, curves, images, graphs etc.). Classical methods with a focus on testing the difference in the first or second moments of the distributions have become inadequate for such data and remedies have been suggested [1, 3, 4, 5, 9]. Other and more general methods have also been proposed. For example, [2] investigates a test based on the  $L_1$  distance between the two empirical distributions, [12] proposes a test statistic (the maximum mean discrepancy) which is the largest of the differences in means over functions in the unit ball of a reproducing kernel Hilbert space and [6] proposes tests based on an ensemble of distances between analytic functions representing each of the distributions. How these tests relate to the most powerful test in the Nayman-Pearson framework is not clear however.

On the other hand, robust algorithms have been developed recently for classification problems. These algorithms are capable of training classifiers to discriminate complex and very high dimensional data from different distributions. A good classifier learned from training data can predict with high probability the correct class for a new/testing data point. It is therefore reasonable to ask if a good classifier can be adopted for testing the difference between two samples from different probability distributions [14, 18]. One can assign labels 1 or 0 to data points from the two samples. Then an intuitive approach is to utilize the prediction accuracy of a classifier trained from the labeled data for the test. Under the null hypothesis of identical distributions, the prediction accuracy (the probability of predicting class labels correctly) should not be too high, and therefore the null hypothesis should be rejected if otherwise. The versatility of such a test has been demonstrated in [14] with large and high dimensional datasets. There are some disadvantages in this approach however. The prediction accuracy has to be estimated through cross-validation which makes the test less efficient in data utilization and can slow down the computation. More importantly, as we will see in the next section, a more powerful test which is not based on prediction accuracy can be derived.

Following a different consideration, we propose a simple yet powerful method for the twosample test. By treating the two sample data as a set of sample feature points with labels of either 1 or 0, depending on which sample a feature point is from, we can re-formulate the original null and alternative hypotheses into an equivalent pair of hypotheses on the joint distribution of the feature points and their labels. In this setting, the concept of classification probability in classification problems, the probability that the label of a given feature point is 1, is naturally induced, and a classification algorithm, like logistic regression, random forest, k-nearest neighbor or support vector machine, can be used to estimate such a probability from the given data [11, 13, 15, 21]. The nice thing is that the likelihood ratio in the two-sample testing problem can be expressed exactly as the odds ratio of the classification probability multiplied by a known constant. This connection allows us to approximate the likelihood ratio with the odds ratio of the classification probability to derive a powerful test statistic. Unlike the test based on classification accuracy which in [18] is considered as an indirect approach to the two-sample problem, our classification probability test is basically an approximation of the likelihood ratio test. Therefore asymptotically there should be no loss of information like the classification accuracy test.

Our approach assumes that we have a consistent estimator of the classification probability. There are some results in the literature [11, 20, 21, 22], but more studies are needed in order to identify better theoretical sufficient conditions and this will not be the focus of the current paper. Through examples, we will compare our method with the maximum mean discrepancy (MMD) test [12] and a classifier test based on classification accuracies [14, 18] and show significant improvements in power and efficiency of our test. We will call our test Classification Probability Test (CP test or CPT for short).

In the following, we introduce our test in Section 2 and demonstrate its performance through examples in Section 3. Summary and discussions are given in Section 4.

### 2. Two-sample tests based on classification probability

Suppose we have independent samples from probability distributions F and G respectively on some feature space  $\mathcal{X}$ . For simplicity, we assume F and G have density functions f(x) and g(x) respectively on  $\mathcal{X}$  relative to some reference measure. Let

$$\mathcal{S}_f = \{X_{f,1}, ..., X_{f,n}\} \sim f^n \text{ and } \mathcal{S}_g = \{X_{g,1}, ..., X_{g,m}\} \sim g^m$$

be two independent samples. The problem is to test

$$(2.1) H_0: f = g \quad \text{vs.} \ H_1: f \neq g.$$

It is possible to reformulate this test within the framework of a classification problem, as it will become clear below. To this end, we first pool the samples together and let  $\mathcal{S} = \mathcal{S}_f \cup \mathcal{S}_g$ , N = n + m. To each  $X_i \in \mathcal{S}$ , i = 1, ..., N, we assign a class variable  $Y_i \in \{0, 1\}$  such that

$$Y_i = \begin{cases} 1 & \text{if } X_i \in \mathcal{S}_f \\ 0 & \text{if } X_i \in \mathcal{S}_g. \end{cases}$$

In this way we obtain an augmented dataset:  $(X_i, Y_i), i = 1, ..., N$ . The data in the new form can be viewed as an i.i.d. samples from the joint distribution of a pair of random variables (X, Y) for which  $P(Y = 1) = \pi = 1 - P(Y = 0)$  for some  $\pi \in (0, 1)$  and the conditional distribution of X is f given Y = 1, and g given Y = 0. Let p(x) = P(Y = 1|X = x) be the class probability for a given  $x \in \mathcal{X}$ . Then, by Bayes Theorem,

(2.2) 
$$p(x) = \frac{\pi f(x)}{\pi f(x) + (1 - \pi)g(x)}.$$

Note that p(x) depends on  $\pi$ . At first glance, it might seem that we are giving ourself one extra parameter  $\pi$  to deal with. But in reallity, we don't need to know its value, as long as we consider n/N as its estimate or simply assume n/N is  $\pi$ .

Next, let us consider the power of a test in testing (2.1) at a given significance level  $\alpha$ . Note that the underlying density functions f and g are unknown but fixed. It turns out that an upper bound for the power of any test in our two-sample problem can be expressed in terms of the class probability p(x). To see this, let

(2.3) 
$$U = \frac{1}{n} \sum_{i=1}^{n} \left( \log \frac{p(X_{f,i})}{1 - p(X_{f,i})} - \log \frac{\pi}{1 - \pi} \right).$$

Note that the functions p(x) and f(x) have the same support and  $X_{f,i}$  are sampled from f, therefore the terms  $p(X_{f,i})$  in (2.3) are always strictly positive. If  $p(X_{f,i}) = 1$  for some i = 1, ..., n, we set  $U = \infty$ . This happens only when some  $X_{f,i}$  assumes a value that is not in the support of the density function g(x). The following result motivates our test to be given later. It is essentially a form of the usual Neyman-Pearson Lemma.

**Proposition 1.** Let T be any test statistic based on samples from the probability densities f and g for testing (2.1). For every  $\alpha \in (0, 1)$ , let  $C_T \subset \mathbb{R}$  be the critical region of T such that  $P(T \in C_T) = \alpha$  under  $H_0$ . Then the power of T has the bound:

$$(2.4) P(T \in C_T) \le P(U > c_U),$$

where  $c_U \in \mathbb{R}$  is such that

$$P(U > c_U) = \alpha \ under \ H_0.$$

*Proof.* We can restate the hypotheses in (2.1) in a slightly different but equivalent way. First, let us us assume that sample  $S_g$  is from some unknown density g, and f is another unknown density which is different from g:

 $f \neq g$ .

Under this assumption, the two-sample problem can then be stated as: under  $H_0$ , the sample  $S_f$  is from the same density g as  $S_g$  and, under  $H_1$ ,  $S_f$  is from f, or

(2.5) 
$$H'_0: \mathcal{S}_f \sim g^n \ vs. \ H'_1: S_f \sim f^n.$$

Let's adopt this form of the two-sample test in the proof. Next, we consider the joint density of (X, Y). Under  $H'_1$ , this joint density has the form

$$h_1(x,y) = (yf(x) + (1-y)g(x))\pi^y(1-\pi)^{1-y}.$$

This  $h_1$  can be rewritten as

$$h_1(x,y) = [g(x) + y(f(x) - g(x))]\pi^y (1 - \pi)^{1-y}.$$

We see that the joint density of (X, Y) under  $H_0$  takes the form

$$h_0(x,y) = g(x)\pi^y(1-\pi)^{1-y}$$

Let h(x, y) denote the generic joint density of (X, Y). We can rewrite (2.5) as

(2.6) 
$$H_0'': h = h_0 \ vs. \ H_1'': h = h_1.$$

The log of the density ratio for testing (2.6) can be written as:

(2.7) 
$$\log \frac{h_0(x,y)}{h_1(x,y)} = y \log \frac{g(x)}{f(x)}, y \in \{0,1\}, x \in \mathcal{X}$$

From (2.2),

$$\frac{f(x)}{g(x)} = \frac{(1-\pi)p(x)}{\pi(1-p(x))}$$

and hence U given in (2.3) can also be written as

$$U = \frac{1}{n} \sum_{i=1}^{n} \log \frac{f(X_{f,i})}{g(X_{f,i})} = -\frac{1}{n} \sum_{i=1}^{N} Y_i \log \frac{g(X_i)}{f(X_i)}.$$

This, plus (2.7), shows that, up to a constant factor 1/n, U is actually the negative loglikelihood ratio of the data from the joint distribution of (X, Y). Now the standard arguments used in the proof of the Neyman-Pearson Lemma lead us to the inequality (2.4).  $\Box$  *Remark.* (1) Whenever the law of large numbers holds here, the quantity U converges in probability to the Kullback-Liebler divergence from g to f:

$$U \to E_f\left(\log \frac{f(X)}{g(X)}\right)$$
, as  $n, m \to \infty$  and  $n/N \to \pi \in (0, 1)$ .

Therefore U estimates the K-L distance from g to f. (2) We see in the proof that the quantities U and the critical value  $c_U$  in (2.4) can be replaced with

$$V = \frac{1}{m} \sum_{i=1}^{m} \left( \log \frac{1 - p(X_{g,i})}{p(X_{g,i})} - \log \frac{1 - \pi}{\pi} \right)$$

and the critical value  $c_V$ , defined in a similar way as  $c_U$ .

Unfortunately, U depends on unknown f and g. However, with the bound given in Proposition 1, if we can find a statistic W such that  $P(W \neq U) \rightarrow 0$  asymptotically, then we can use W to perform a test that is asymptotically most powerful. Of course, the sampling distribution of W under  $H_0$  would be generally unknown. But in many practical problems, one can circumvent the difficulty with a permutation test. An important feature of U is that we don't need to estimate f and g, or the ratio f/g, to calculate U, which we know can be increasingly impossible as the dimensionality of the feature space  $\mathcal{X}$  grows. On the other hand, there are many powerful classification algorithms (random forests, support vector machines, deep neural networks etc.) working on large, complex and very high dimensional data that estimate the class probabilities effectively. These observations motivate the following test.

Let  $\hat{p}(x)$  be a consistent estimate of the classification probability p(x). We propose the plug-in test statistic

(2.8) 
$$W_1 = \frac{1}{n} \sum_{i=1}^n \log \frac{\hat{p}(X_{1i})}{1 - \hat{p}(X_{1i})} - \log \frac{n}{m}$$

for a permutation test. In this test, the corresponding null hypothesis  $H_0^{\text{perm}}$  states that in the data, each  $(X_i, Y_i)$  is from a distribution with  $Y_i \sim B(1, \pi)$  and  $X_i \sim \pi f + (1 - \pi)g$ independently. Note that the original  $H_0$  implies  $H_0^{\text{perm}}$ , but not neccessaryly the other away around. This however will not affect the validity of the test. Here is the test.

# The Classification Probability Test 1. Given $\alpha \in (0, 1)$ .

(1) Generate a number of values of  $W_1$  using  $\hat{p}(x)$  and samples from the null hypothesis  $H_0^{\text{perm}}$ .

- (2) Find the critical value  $c_1$  satisfying (approximately)  $P_{H_0^{\text{perm}}}(W_1 > c_1) = \alpha$  based on the values of  $W_1$  generated in step (1).
- (3) Calculate  $W_1$  from the original data and reject  $H_0$  if  $W_1 > c_1$ .

More specifically, the steps (1) and (2) above is implemented as follows. First, following the simple random sampling principle, we randomly divide the pooled sample S into subsets  $S_f^{\text{perm}}$  and  $S_g^{\text{perm}}$  of sizes n and m respectively. We assign label 1 to  $X_i$ 's in  $S_f^{\text{perm}}$  and label 0 to  $X_i$ 's in  $S_g^{\text{perm}}$  and calculate  $\hat{p}(x)$  from a given classifier (a support vector machine classifier or a random forest classifier, for example). A value of  $W_1$  in (2.8) based on this shuffled data is obtained. This computation is repeated independently for a sufficiently large number of times. The critical value c is then the  $\alpha$ th sample percentile based on the values of  $W_1$ . Finally,  $W_1$  is culculated from the original data using the same  $\hat{p}(x)$  and we reject  $H_0^{\text{perm}}$  whenever  $W_1 > c$ .

In the next session we will demonstrate the actual power of this test with several examples. Gnerally, the performance of the test depends on the underlying distribution of the data and the method one uses to obtain  $\hat{p}(x)$ , but in our experiments, we were always able to find a classifier so the test based on it out performed all other tests. The efficiency of the test depends on the rate of convergence of the estimator  $\hat{p}$ . To have some idea of theoretical guarantee of our method, we state the following proposition under a uniform consistency condition.

**Proposition 2.** Assume when  $n, m \to \infty$  and  $n/(n+m) \to \pi \in (0, 1)$ ,

(2.9) 
$$\sup_{x \in \mathcal{X}} \left( \hat{p}(x) - p(x) \right) \to 0 \quad in \ probability.$$

Then the test given above is an asymptotically most powerful test.

*Proof.* Here is a sketch of the proof. The condition (2.9) implies that for any small  $\delta > 0$ 

$$\sup_{x:p(x)\in[\delta,1-\delta]} \left| \left( \log \frac{\hat{p}(x)}{1-\hat{p}(x)} - \log \frac{n}{m} \right) - \left( \log \frac{p(x)}{1-p(x)} - \log \frac{\pi}{1-\pi} \right) \right| \to 0 \text{ in probability,}$$

due to the uniform continuity of the function  $\log u/(1-u)$  on the interval  $[\delta, 1-\delta]$ . On the other hand, if p(x) = 0 or 1 then (2.9) implies that  $\log \hat{p}(x)/(1-\hat{p}(x)) \to -\infty$  or  $\infty$ respectively. From this it can be shown that

$$|W_1 - U| \to 0$$
 in probability as  $n, m \to \infty$  and  $n/(n+m) \to \pi \in (0, 1)$ .

or

(2.10) 
$$P(W_1 \neq U) \rightarrow 0 \text{ as } n, m \rightarrow \infty \text{ and } n/(n+m) \rightarrow \pi \in (0,1)$$

Hence

$$|P(W_1 > c) - P(U > c)| \le P(W_1 \ne U) \to 0.$$

Therefore the power of the test based on  $W_1$  asymptotically achieves the upper bound given in Proposition 1.

*Remark.* The uniform consistency condition (2.9) is somewhat strong and artificial. More studies are needed to identify weaker conditions for (2.10) to hold.

Next, we point out that it is possible to propose other tests using  $\hat{p}(x)$  based on more heuristic arguments. We give one here. The form of the conditional probability in (2.2) suggests that the two-sample test is also equivalent to determining if the mapping  $x \to p(x)$ is a constant function. For if p(x) is constant for all x, then  $p(x) = E_{X,Y}p(X)$  for all x. But  $E_{X,Y}p(X) = \pi$ . Thus  $p(x) = \pi$  for all x and this condition forces  $f \equiv g$  by (2.2). Now to determine if p(x) is a constant function, we can consider the variance

$$\theta = Var_{X,Y}(p(X)) = E_{X,Y}p(X)^2 - \pi^2.$$

The hypotheses in (2.1) can now be restated as

(2.11) 
$$H_0'': \theta = 0 \text{ vs. } H_1'': \theta > 0.$$

A natural test statistic for this test can then be

(2.12) 
$$W_2 = \frac{1}{N} \sum_{i=1}^{N} \left( \hat{p}(X_i) - \bar{p} \right)^2,$$

where  $\bar{p} = \frac{1}{N} \sum_{i=1}^{N} p(X_i)$ . Here is the permutation test based on  $W_2$ :

# The Classification Probability Test 2. Given $\alpha \in (0, 1)$ .

- (1) Independently, generate a number of values of  $W_2$  using  $\hat{p}(x)$  and the samples from the null hypothesis  $H_0^{\text{perm}}$ .
- (2) Find the critical value  $c_2$  satisfying (approximately)  $P_{H_0^{\text{perm}}}(W_2 > c_2) = \alpha$  based on the values of  $W_2$  generated in step (1).
- (3) Calculate  $W_2$  from the original data and reject  $H_0$  if  $W_2 > c_2$ .

In our experiments, the performance of this second test can be on a par with or, sometimes, even be slightly better than Test 1. In our examples, this is particularly the case when random forest is used as the classifier for estimating p(x).

#### 3. Examples

In this section we compare the performance of the classification probability test (CPT) against maximum mean discrepancy (MMD) test [12] and the test based on classification accuracy (ACC) [14, 18]. We choose the MMD test for comparison because the test is known to have better overall performance among many commonly used two-sample tests. We also include the ACC tests because of their similarities to our tests. In our tests, the classifiers we use to estimate the classification probabilities are random forests (RF) and support vector machines (SVM), because they performed best among several popular classifiers we tried in our experiments. The comparisons will be done on simulated data from four different types of probability distributions and on one real-life dataset. As a standard step in checking the validity of a testing procedure, we first ran our test under the null hypothesis  $H_0: f = g$  with the datasets and confirmed that the test had the correct type I error probability.

We use the receiver operating characteristic (ROC) curves and power versus sample-size plots to compare the performance of the tests. In the following, to obtain an approximation of the ROC curve for a test, we first perform the test 400 times. Each time a new pair of samples from the same pair of distributions is generated independently. We collect all 400 *p*-values from these tests. The estimated ROC curve is then the curve of the empirical distribution function of these *p*-values. The curve allows us to determine the power of the test at each given significance level. To obtain a plot of power versus sample-size for a test, we set the significance level of the test at  $\alpha = 0.05$  and then, for each of an increasing sequence of sample sizes, we run the test independently 250 times to get a *p*-value for that sample size. The power is then the proportion of these values which are less than 0.05. In the following examples, each scatter plot of the points of sample-size versus power has a smoothed curve added, which is obtained from local polynomial regression fitting to these points.

The same computations described above are applied to all examples below. In the figures showing these examples, we write CPT-rf and CPT-svm for the classification probability tests using RF and SVM classifiers respectively, MMD for the maximum mean discrepancy test, and ACC-rf and ACC-svm for the classification accuracy tests using RF and SVM classifiers. We use function ranger() from the R package ranger for RF classification, function svm() from the R package e1071 for SVM classification and function kmmd() from the R package kernlab for MMD test. The default values for the parameters in these functions are used. The ACC-rf and ACC-svm tests use 2-fold cross-validation. All five

tests (CPT-rf, CPT-svm, ACC-rf, ACC-svm, MMD) in our examples below are permutation tests. The null permutation distributions are all approximated with 200 independent runs.

3.1. A comparison with the minimax rate for normal data. First, we apply the tests in a standard setting to show how well it performs against the known minimax power of the two-sample problem [18]. The first sample is from distribution  $N(0, \sigma^2 I_{d \times d})$  and the second sample from  $N(\delta, \sigma^2 I_{d \times d})$  with  $\delta \in \mathbb{R}^d$ . Assume  $n_1 = n_2 = n$ . In this case the minimax power over all tests for testing  $\delta = 0$  vs  $\delta \neq 0$  is known to be [18]

$$\phi(\alpha) = \Phi\left(\frac{\sqrt{d}}{\sqrt{d+n||\delta||_2^2/\sigma^2}} z_\alpha + \frac{||\delta||^2/\sigma^2}{\sqrt{8d/n^2 + 8||\delta||_2^2/n\sigma^2}}\right) + o(1)$$

where  $\Phi(\cdot)$  is the standard normal CDF. In our example, we use d = 100, n = 100,  $\sigma = 2$ and try two different forms of  $\delta = (\delta_1, \delta_2, ..., \delta_{100})$ : a) a sparse case with  $\delta_1 = 1.6$  and  $\delta_i = 0$ for all i > 1 and b) a dense case with  $\delta_i = 0.16$  for all i = 1, ..., 100. Both of these forms of  $\delta$  give the same  $||\delta||_2 = 1.6$  and therefore the same minimax lower bound.

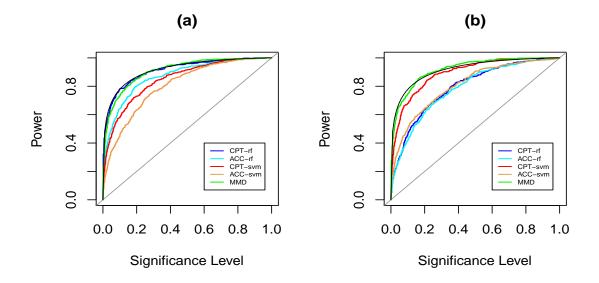


FIGURE 3.1. The smooth black curves are the minimax power function  $\phi(\alpha)$ . In (a)  $\delta_1 = 1.6$ ,  $\delta_2 = \cdots = \delta_{100} = 0$  and in (b)  $\delta_1 = \cdots = \delta_{100} = 0.16$ .

Figure 3.1 displays the ROC curves of the tests for both cases. The minimax power function  $\phi(\alpha)$  is plotted as black curves. We see that in (a) CPT-rf achieves the minimax bound and in (b) CPT-sym gives a good approximation to the bound. We also notice that the performance of MMD test are among the best in both cases and that the default kernel

used in MMD test is Gaussian. All ACC tests are less powerful than their CPT counter parts. In the next few examples, we will show how the CP tests outperform MMD in other distribution settings.

3.2. Distributions with the same means and variances but different covariances. In this example, samples are drawn from two d-dimensional multivariate normal distributions  $N_d(\mu, \Sigma_1)$  and  $N_d(\mu, \Sigma_2)$ , where the mean vector  $\mu \in \mathbb{R}^d$  is identical in both distributions,  $\Sigma_1$  and  $\Sigma_2$  are  $d \times d$  covariance matrices having the same diagonal elements but different off-diagonal elements. Specifically, in this example, we set d = 100,  $\operatorname{diag}(\Sigma_1) = \operatorname{diag}(\Sigma_2) = (1.0, 1.1, 1.2, ..., 10.9), \mu_1 = \mu_2 = 0_d$  and  $\Sigma_1$  has all the off-diagonal elements 0.01 and  $\Sigma_2$  has all the off-diagonal elements 0.21.

The performance of the tests on the data from these distributions are shown in Figure 3.2. The ROC curves in (a) are based on samples of size m = n = 100 and the powers in (b) are for the tests at the significance level 0.05. We see that CPT-svm test is the most powerful and efficient test among the five tests.

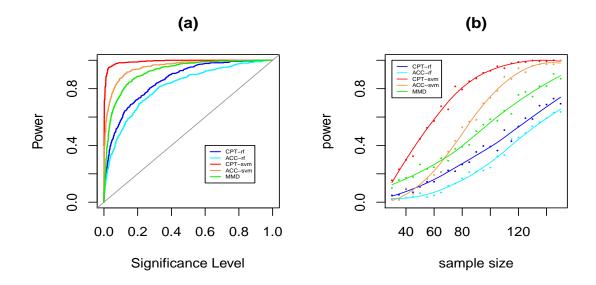


FIGURE 3.2. Tests for distributions with the same mean vector but different covariance matrices.

3.3. Gaussian graphical models with different sparsities. Here we have samples drawn from two Gaussian graphical models with the same mean vectors but different precision matrices  $Q_1$  and  $Q_2$  to reflect different degrees of connectivities of the underlying graphs. More specifically, let  $A_1$  be a weighted  $d \times d$  symmetric adjacency matrix of a simple graph with its off-diagonal entries randomly and independently taken from a uniform distribution U(0, 1). Let  $A_2$  be a more sparse adjacency matrix obtained from  $A_1$  by setting randomly and independently some of the entries in  $A_1$  to 0 according to a Bernoulli distribution  $B(1, \tau)$ . Let  $D_1$  and  $D_2$  be the diagonal degree matrix of  $A_1$  and  $A_2$  respectively so that their *i*-th diagonal element is the sum of the *i*-th row of their corresponding adjacency matrices. Let  $Q_1 = (D_1 - A_1) + \delta_1 I$  and  $Q_2 = (D_2 - A_2) + \delta_2 I$ , where  $\delta_1, \delta_2 > 0$ are added to diagonals to make  $Q_1$  and  $Q_2$  nonsingular. In our experiment, means are the zero vector  $0_{200}$ ,  $Q_1$  and  $Q_2$  are 200 × 200 matrices (the graphs have 200 nodes each) and  $Q_2$  is obtained from  $Q_1$  as described above with  $\tau = 0.65$ .

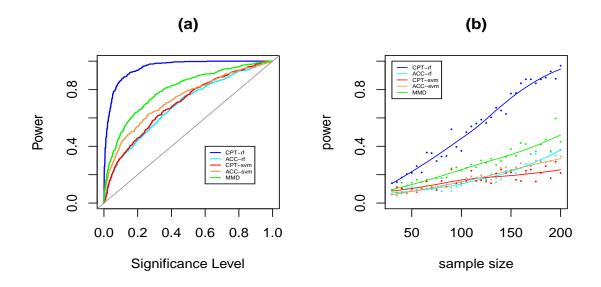


FIGURE 3.3. Tests for different Gaussian graphical models with the same mean vector but different rate of connectivities

The performance of the tests on data from these distributions are shown in Figure 3.3. The ROC curves in (a) are based on samples of sizes  $n_1 = n_2 = 150$  and the powers in (b) are for the tests at the significance level 0.05. In this example, the CPT-rf test is significantly more powerful and efficient than other tests.

3.4. Distributions with the same means and covariances but one different marginal distribution. In this fourth example, sample one is from a d dimensional distribution which is a product measure of a one dimensional exponential distribution with mean 1 and d-1 iid normal distributions of mean 1 and variance 1,  $Exp(1) \times N_{d-1}(1_{d-1}, I_{d-1})$ . The second sample is drawn from a product measure of d iid normal distributions of mean 1 and variance 1,  $N_d(1_d, I_d)$ . Therefore both samples have identical first and second moments and the only difference between the two distributions is that one component in the first distribution is exponential and the corresponding component in the second distribution is normal. In the experiment, we use d = 100.

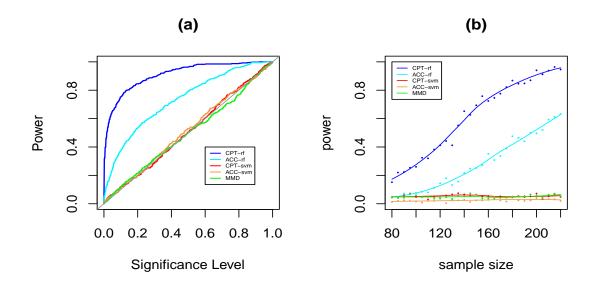


FIGURE 3.4. Tests for distributions with the same means and covariances but one different marginal distribution.

The performance of the tests on data from these distributions are shown in Figure 3.4. The ROC curves in (a) are based on samples of sizes m = n = 200 and the powers in (b) are for the tests at the significance level 0.05. We see that CAP-rf test outperforms other tests in terms of the power and efficiency. In contrast, MMD and SVM tests fail to detect any difference between the two distributions from the samples.

3.5. Movie review sentiment data. In this example, the samples of the same size m = n are drawn from two sets of movie reviews with 1000 reviews in each set. Of these two sets of reviews, one has all the reviews with positive sentiments and another all the reviews with negative sentiments. The null hypothesis in the test is that there is no difference in sentiments between two sets of reviews. All the reviews within each sample are encoded into a so-called document-term matrix in which each review is represented by a row of numbers 0 or 1 depending on wether a term which appears in at least 5% of all the reviews of the combined sample also appears in this review. To make the data more challenging, in our experiment, we purposely removed 50 highest influential words, including words like

"bad", "worst", "dull", "excellent", "perfect" etc. The dimension of the data depends on the samples we draw. It varies around 1000.

The performance of the tests on these data are shown in Figure 3.5 The ROC curves in (a) are based on samples of sizes m = n = 75 and the powers in (b) are for the tests at the significance level 0.05. It shows that CPT-rf test is best in terms of power and efficiency.

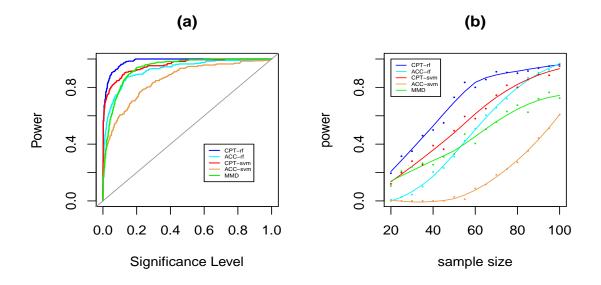


FIGURE 3.5. Tests for difference of text files with two sentiments.

3.6. Comparing Classification Probability Test 1 and Test 2. In this last example, we use the data from the first four examples above to show the difference between Test 1 and Test 2 based on classification probabilities. It shows that if SVM is used as the classifier, then Test 1 is consistently better than Test 2. But if the classifier is RF, then Test 2 can often be better.

### 4. Summary and discussion

We propose a test for two-sample problem based on estimates of classification probabilities obtained from a consistent classification algorithm. This test is effectively a likelihoodratio-based test in which the ratio is estimated not by maximizing the likelihoods involved - which would require knowledge on the density functions, but by estimating the odds ratio of classification probabilities. Our test is more powerful and efficient than many other tests.

Our test also has taken advantage of the recent advances in classification algorithms and computing software to derive its performance in computation. For any given dataset,

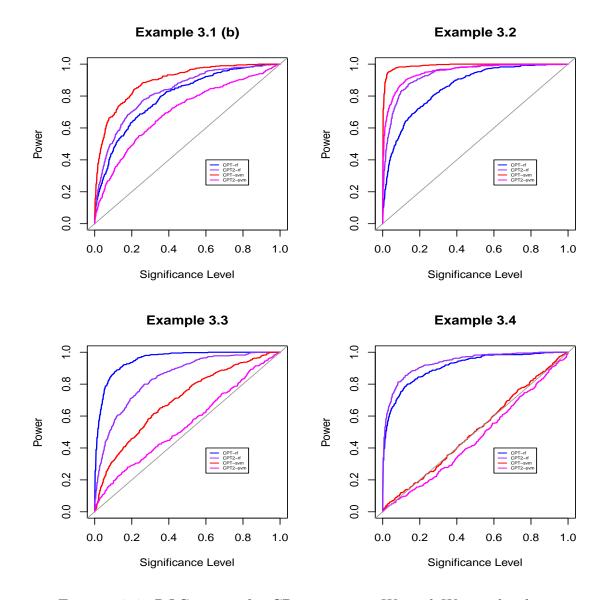


FIGURE 3.6. ROC curves for CP tests using  $W_1$  and  $W_2$  on the datasets from the first four examples.

tests using different classifiers deliver different performance, depending on the structure of underlying probability distributions. We conjecture that this phenomenon is due to the quality of estimates of the classification probability when using different classifiers on different types of data. More studies are needed to better understand the statistical properties of these estimates. In practice, one can choose a best classifier using K-fold cross-validation to ensure the best performance. Because these are permutation tests, they can be computationally intensive. To get some idea on CPU time usage in a single test, we provide the following table, in which each entry for time is the elapsed time in seconds returned from the function *system.time()* in R running on a MacBook Pro with 2.2 GHz Intel Core i7 processor and macOS Mojave version 10.14.5. In all these tests, every null permutation distribution was approximated using 200 independent runs.

TABLE 1. Elapsed Time of the Tests

	d	n	CPT-svm	ACC-svm	CPT-rf	ACC-rf	MMD
Example 1	100	100	12.143	8.221	15.532	14.293	2.408
Example 5	1000	75	63.772	48.820	30.772	43.555	4.373

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