REFINABLE BIVARIATE QUARTIC C^2-SPLINES FOR MULTI-LEVEL DATA REPRESENTATION AND SURFACE DISPLAY

CHARLES K. CHUI AND QINGTANG JIANG

ABSTRACT. In this paper, a second-order Hermite basis of the space of C^2-quartic splines on the six-directional mesh is constructed and the refinable mask of the basis functions is derived. In addition, the extra parameters of this basis are modified to extend the Hermite interpolating property at the integer lattices by including Lagrange interpolation at the half integers as well. We also formulate a compactly supported super function in terms of the basis functions to facilitate the construction of quasi-interpolants to achieve the highest (i.e. fifth) order of approximation in an efficient way. Due to the small (minimum) support of the basis functions, the refinable mask immediately yields (up to) four-point matrix-valued coefficient stencils of a vector subdivision scheme for efficient display of C^2-quartic spline surfaces. Finally, this vector subdivision approach is further modified to reduce the size of the coefficient stencils to two-point templates while maintaining the second-order Hermite interpolating property.

1. INTRODUCTION

Let $\triangle^1$ denote the triangulation of the x-y plane $\mathbb{R}^2$ by using the grid lines $x = i, y = j$, and $x - y = k$, $i, j, k \in \mathbb{Z}$, and $\triangle^3$ be its refinement by drawing in the addditional grid lines $x + y = \ell, x - 2y = m$, and $2x - y = n$, $\ell, m, n \in \mathbb{Z}$. Hence, $\triangle^3$, which may be considered as a Powell-Sabin split for each triangle of the triangulation $\triangle^1$, is called a six-directional mesh. A truncated portion of the triangulation $\triangle^3$ is shown in Fig.1. For integers d and r, with $0 \le r < d$, let $S_d^r(\triangle^3)$ be the collection of all (real-valued) functions in $C^r(\mathbb{R}^2)$ whose restrictions on each triangle of the triangulation $\triangle^3$ are bivariate polynomials of total degree $\le d$. Each function ϕ in $S_d^r(\triangle^3)$ is called a bivariate C^r-spline of degree d on $\triangle^3$. In addition, if the support of ϕ (denoted by *supp* ϕ) contains at most one point of the lattice $\mathbb{Z}^2$ in its interior, then ϕ is called a vertex spline (or more precisely, generalized vertex spline in [2, Chap. 6]). Also, if $\phi_1, \cdots, \phi_n$ are compactly supported functions in $S_d^r(\triangle^3)$ such that the column vector $\Phi := [\phi_1, \cdots, \phi_n]^T$ satisfies a 2-dilated refinement equation

$$(1.1) \qquad \Phi(\mathbf{x}) = \sum_{\mathbf{k}} P_{\mathbf{k}} \Phi(2\mathbf{x} - \mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^2,$$

Date: xx, 2003 and, in revised form, xx, 2003.
1991 *Mathematics Subject Classification.* Primary 65D07, 65D18; Secondary 41A15.
Key words and phrases. Multi-level data representation, Hermite interpolation, refinable quartic C^2-splines, vector subdivision, $\sqrt{3}$ topological rule, 2-point coefficient stencils.
The first author was supported in part by NSF Grants #CCR-9988289 and #CCR-0098331; and ARO Grant # DAAD 19-00-1-0512.
The second author was supported in part by Univ. of Missouri–St. Louis Research Award 03.

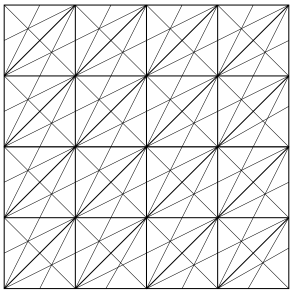

FIGURE 1. six-directional mesh $\triangle^3$

for some $n \times n$ matrices $P_{\mathbf{k}}$ with constant entries, Φ is called a refinable function vector with (refinement) mask $\{P_{\mathbf{k}}\}$.

Refinable function vectors of vertex splines $\phi_1, \cdots, \phi_n$ with finite masks $\{P_{\mathbf{k}}\}$ have important applications to computer-aided surface design as well as to various problem areas in data interpolation/approximation and visualization, particularly if $\phi_1, \cdots, \phi_n$ satisfy additional desirable properties such as polynomial reproduction (for high-order of L^2-approximation) and Lagrange/Hermite interpolating conditions. For applications to computer-aided surface design and interactive manipulation, the small support and interpolating property give rise to interpolating vector subdivision schemes, with matrix-valued coefficient stencils given by the mask $\{P_{\mathbf{k}}\}$, assuring the C^r-smoothness of the subdivided surfaces. For discrete data representations, vertex splines $\phi_1, \cdots, \phi_n$ with interpolating properties are readily implementable and the "super function" (of highest approximation order), formulated in terms of integer translates of $\phi_1, \cdots, \phi_n$, facilitates construction of quasi-interpolants, which are again readily decomposable in terms of the vertex splines, so that the refinement masks $\{P_{\mathbf{k}}\}$ can be applied to such schemes as multi-level approximations.

The C^1 problem is relatively simple. In fact, the vertex spline function vector $\Phi^a := [\phi_1^a, \phi_2^a, \phi_3^a]^T$, with $\phi_j^a \in S_2^1(\triangle^3)$, $j = 1, 2, 3$, formulated explicitly in terms of the Bézier coefficients (or Bézier-nets) of quadratic polynomial pieces in [4] (see

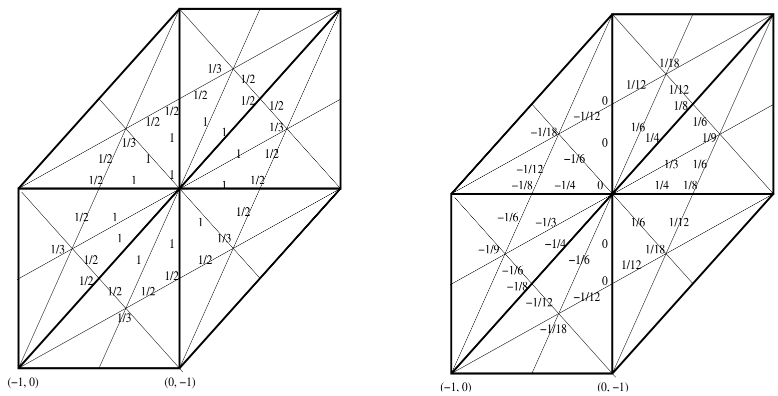

FIGURE 2. Support and Bézier coefficients of ϕ_1^a, ϕ_2^a, with $\phi_3^a(x,y) := \phi_2^a(y,x)$

also [3] for further elaboration) with Bézier coefficients shown in Fig.2 (with other obviously zero coefficients not shown), already satisfies the (first-order) Hermite interpolating condition

$$\left[\Phi^a,\ \frac{\partial}{\partial x}\Phi^a,\ \frac{\partial}{\partial y}\Phi^a\right](\mathbf{k}) = \delta_{\mathbf{k},\mathbf{0}} I_3, \quad \mathbf{k}\in\mathbb{Z}^2, \tag{1.2}$$

and generates a Hermite (and hence, Riesz) basis of $S_2^1(\Delta^3)$. Furthermore, it was shown in our earlier work [4] that the two-scale symbol of Φ^a with the refinement mask $\{P_{\mathbf{k}}\}$, where

$$\begin{aligned}
&P_{0,0} = \mathrm{diag}(1, \tfrac{1}{2}, \tfrac{1}{2}),\ P_{1,1} = \frac{1}{8}\begin{bmatrix} 4 & -4 & -4 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix},\ P_{1,0} = \frac{1}{8}\begin{bmatrix} 4 & -8 & 4 \\ 1 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix},\\
&P_{-1,0} = \frac{1}{8}\begin{bmatrix} 4 & 8 & -4 \\ -1 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix},\ P_{-1,-1} = \frac{1}{8}\begin{bmatrix} 4 & 4 & 4 \\ -1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix},\\
&P_{0,1} = \frac{1}{8}\begin{bmatrix} 4 & 4 & -8 \\ 0 & 2 & 0 \\ 1 & 2 & -2 \end{bmatrix},\ P_{0,-1} = \frac{1}{8}\begin{bmatrix} 4 & -4 & 8 \\ 0 & 2 & 0 \\ -1 & 2 & -2 \end{bmatrix},
\end{aligned} \tag{1.3}$$

also satisfies the sum rules of order 3 (so that Φ^a locally reproduces all bivariate quadratic polynomials and has the third order of approximation). When a spline series $s(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}}^0 \Phi^a(\mathbf{x}-\mathbf{k})$, where $\mathbf{v}_{\mathbf{k}}^0 = [\mathbf{v}_{1,\mathbf{k}}^0, \mathbf{v}_{2,\mathbf{k}}^0, \mathbf{v}_{3,\mathbf{k}}^0]$, $\mathbf{v}_{1,\mathbf{k}}^0, \mathbf{v}_{2,\mathbf{k}}^0, \mathbf{v}_{3,\mathbf{k}}^0 \in \mathbb{R}^3$, is considered, the (interpolating) vector subdivision scheme provides an efficient algorithm for displaying the surface $s(\mathbf{x})$. The matrix-valued coefficient stencils of this particular subdivision scheme is shown in Fig.3. Observe that these are 2-point coefficient templates.

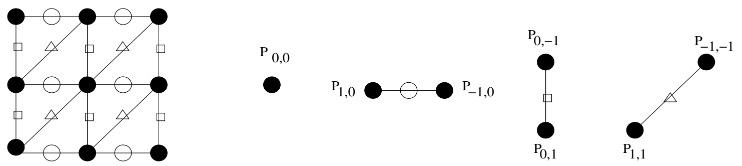

FIGURE 3. Coefficient stencils for C^1 local averaging rule

On the other hand, the C^2 problem is more complicated. In [5], we have shown that the space $S_3^2(\Delta^3)$ has only one vertex spline ϕ_1^b with the normalization condition $\phi_1^b(0) = 1$ and non-zero Bézier coefficients shown in Fig.4, and that the other spline function $\phi_2^b := \phi_1^b \circ A^{-1}$ in $S_3^2(\Delta^3)$, where

$$A := \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}, \tag{1.4}$$

with *supp* ϕ_2^b containing seven lattice points of $\mathbb{Z}^2$ in its interior, provides a second basis function in $S_3^2(\Delta^3)$ with "minimum support." By this, we mean that the space $S_3^2(\Delta^3)$ is the L^2-closure of the sum of the two linear algebraic spans V_1 and V_2, where

$$V_j := \langle \phi_j^b(\cdot - \mathbf{k}) : \ \mathbf{k}\in\mathbb{Z}^2\rangle, \quad j = 1, 2,$$

and that any $\phi \in S_3^2(\Delta^3)$ such that *supp* ϕ has a "reasonable" shape and contains less than seven lattice points of $\mathbb{Z}^2$ in its interior, is necessarily a spline function

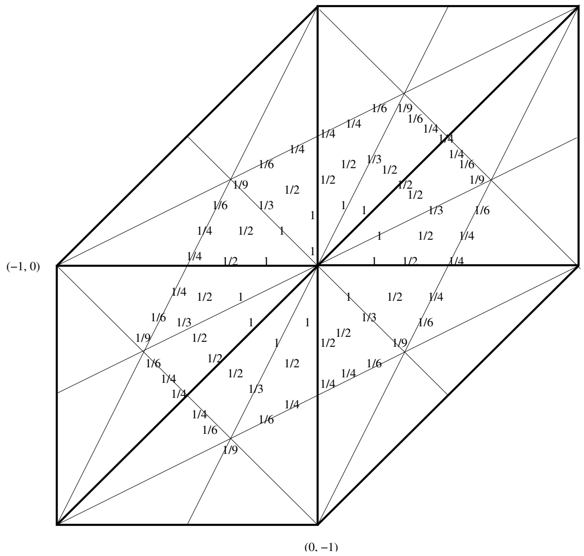

FIGURE 4. Support and Bézier coefficients of ϕ_1^b

in V_1. Furthermore, it was also shown in [4] that $V_1 \cap V_2 \neq \{0\}$, and in fact the integer-translates of ϕ_1^b and ϕ_2^b are governed precisely by two linear dependency identities.

The linear dependency of the basis functions of $S_3^2(\triangle^3)$ is an undesirable feature for various problems in approximation theory. Furthermore, in applications to surface subdivisions, the refinement mask of the refinable function vector $\Phi^b := [\phi_1^b, \phi_2^b]^T$ cannot be modified to derive suitable coefficient stencils for an interpolatory vector subdivision scheme, mainly because the support of ϕ_2^b is too large. What is more serious is that there is need of sufficient degrees of freedom for adjusting the basis functions, and hence the corresponding mask, near extraordinary points with valences different from six, to extend the subdivision scheme for the computer-aided design of surfaces with arbitrary topologies. For these and other reasons, we are willing to sacrifice the elegance of the space $S_3^2(\triangle^3)$ of cubic C^2-splines in order to acquire the important properties of linear independency, second-order Hermite interpolating condition, vertex spline basis functions, matrix-valued coefficient stencils for interpolating subdivisions, etc. by using quartic C^2-splines. Hermite quadratic C^1-splines were investigated in [15, 16], and splines of arbitrary smoothness on Powell-Sabin triangulations including the quartic C^2-splines were studied by using the macro-element method in [1]. However, since such basis functions as those discussed in [1, 15, 16] do not necessarily span the corresponding spline spaces, they are, in general, not refinable. The reader is also referred to the survey paper [14] on (non-refinable) interpolating bivariate splines.

The objective of this paper is to construct a refinable function vector $\Phi = [\phi_1, \cdots, \phi_n]^T$ of vertex splines $\phi_1, \cdots, \phi_n \in S_4^2(\triangle^3)$, such that its refinement mask $\{P_{\mathbf{k}}\}$ provides the coefficient stencils for an interpolating vector subdivision scheme, that its approximation order is maximum (meaning 5), that $\{\phi_1, \cdots, \phi_n\}$ is linearly independent, meaning that $\sum_{\ell=1}^n \sum_{\mathbf{k}} c_{\mathbf{k}}^\ell \phi_\ell(\mathbf{x} - \mathbf{k}) = 0, \mathbf{x} \in \mathbb{R}^2$, for any $c_{\mathbf{k}}^\ell \in \mathbb{R}$ implies $c_{\mathbf{k}}^\ell = 0$ (from which we will see that $n = 11$), and that Φ satisfies the second-order Hermite interpolating property:

$$\left[\Phi \quad \frac{\partial}{\partial x}\Phi \quad \frac{\partial}{\partial y}\Phi \quad \frac{\partial^2}{\partial x^2}\Phi \quad \frac{\partial^2}{\partial x\partial y}\Phi \quad \frac{\partial^2}{\partial y^2}\Phi\right](\mathbf{k}) = \delta_{\mathbf{k},\mathbf{0}}\left[\begin{array}{c} I_6 \\ \mathbf{0} \end{array}\right], \quad \mathbf{k}\in\mathbb{Z}^2. \tag{1.5}$$

The organization of this paper is as follows. The refinable function vector Φ of vertex splines in $S_4^2(\triangle^3)$ will be constructed in the next section, where the main result is stated and the matrix-valued coefficient stencils for the 1-to-4 subdivision scheme are also displayed. The proof of the main result will be given in Section 3. The fourth section is divided into two subsections, with Section 4.1 devoted to the construction of some super functions φ^a, φ^b, and φ (in terms) of Φ^a, Φ^b, and Φ, respectively. Here, taking Φ as an example, we say that

$$\varphi := \sum_{\mathbf{k}} \mathbf{t}_{\mathbf{k}}\Phi(\cdot - \mathbf{k}) \tag{1.6}$$

is a super function of Φ, if a finite sequence $\{\mathbf{t}_{\mathbf{k}}\}$ of row-vectors exists, such that the scalar-valued function φ satisfies the modified Strang-Fix condition:

$$D^\alpha\widehat{\varphi}(2\pi\mathbf{k}) = \delta_{\alpha,\mathbf{0}}\delta_{\mathbf{k},\mathbf{0}}, \qquad |\alpha| < m, \quad \mathbf{k}\in\mathbb{Z}^s, \tag{1.7}$$

with $m = 5$ (since Φ will be shown to have maximum approximation order, which is order $4+1$). In Section 4.2, the refinable function vector Φ is modified to satisfy certain combined canonical Hermite and Lagrange interpolating conditions, with second-order Hermite interpolating property at $\mathbb{Z}^2$ and Lagrange interpolating property at $\left(\mathbb{Z}^2 + \left(\frac{1}{2}, 0\right)\right) \cup \left(\mathbb{Z}^2 + \left(0, \frac{1}{2}\right)\right) \cup \left(\mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. For surface subdivision, the dilation matrix $2I_2$ in (1.1) corresponding to the so-called 1-to-4 split topological rule for triangular mesh refinement. In Section 5, the matrix A defined in (1.4) is used to modify the mathematical theory to adapt to the $\sqrt{3}$-split topological rule introduced recently in [11, 12]. This is possible since the six-directional mesh $\triangle^3$ satisfies the refinability property $\triangle^3 \subset A^{-1}\triangle^3$. In the final section, the symmetry/anti-symmetry property of basis functions $\phi_\ell \in S_4^2(\triangle^3), 1 \le \ell \le 6$, are followed to construct second order Hermite interpolating subdivisions with 2-point matrix coefficient stencils for the local averaging rule.

2. Second-order Hermite interpolating basis

Let M, N be arbitrary positive integers and $S_4^2(\triangle^3_{MN})$ denote the restriction of $S_4^2(\triangle^3)$ on $[0, M+1]\times[0, N+1]$. Then by applying the dimension formula in [2, Theorem 4.3], we have

$$\dim S_4^2(\triangle^3_{MN}) = 11MN + 20(M+N) + 35. \tag{2.1}$$

Since the coefficient of MN is 11, it is natural to investigate the existence of 11 compactly supported basis functions whose integer translates span all of $S_4^2(\triangle^3)$. In search of these functions, we first extend the first-order Hermite basis function vector $\Phi^a = [\phi_1^a, \phi_2^a, \phi_3^a]^T$ in $S_2^1(\triangle^3)$ to a second-order Hermite basis function vector $\Phi^c = [\phi_1, \cdots, \phi_6]^T$ in $S_4^2(\triangle^3)$, namely:

$$\left[\Phi^c \quad \frac{\partial}{\partial x}\Phi^c \quad \frac{\partial}{\partial y}\Phi^c \quad \frac{\partial^2}{\partial x^2}\Phi^c \quad \frac{\partial^2}{\partial x\partial y}\Phi^c \quad \frac{\partial^2}{\partial y^2}\Phi^c\right](\mathbf{k}) = \delta_{\mathbf{k},\mathbf{0}}I_6, \ \ \mathbf{k}\in\mathbb{Z}^2, \tag{2.2}$$

such that *supp* $\phi_j =$ *supp* ϕ_1^a, $j = 1, \cdots, 6$. Of course, there are quite a few of free parameters, which unfortunately cannot be adjusted to yield a refinable function vector Φ^c. So, instead, we temporarily shift our attention to acquire as much

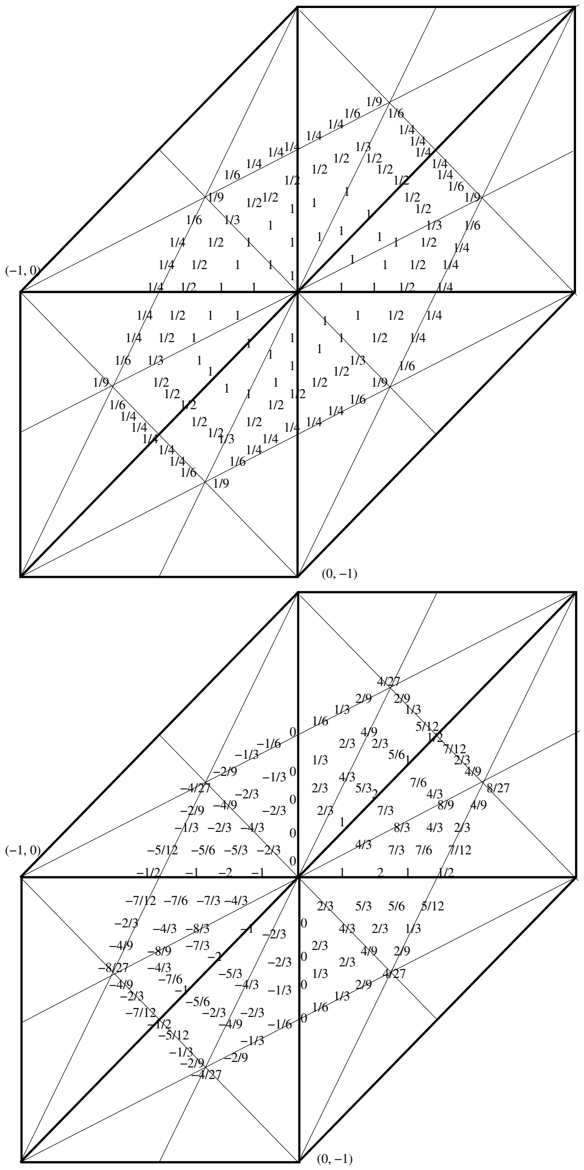

FIGURE 5. Support and Bézier coefficients of ϕ_1 (top), support and Bézier coefficients of $8\phi_2$ (bottom), with $\phi_3(x,y) := \phi_2(y,x)$

symmetry and/or anti-symmetry as possible. The Bézier coefficients of the six components $\phi_1, \cdots, \phi_6$ of Φ^c are shown in Fig.5–Fig.6, where those that are obviously equal to zero are not displayed. There are remaining 6 free parameters, and we are able to construct ϕ_7, ϕ_8, ϕ_9 with supports and Bézier coefficients shown in Fig.7 (top). Observe that all of the supports of $\phi_7, \phi_7(\cdot + (1,0)), \phi_8, \phi_8(\cdot + (1,1)), \phi_9$, and $\phi_9(\cdot + (0,1))$ are subsets of *supp* ϕ_1. Hence, all of the 6 free parameters have been taken care of, but we still need two more compactly supported basis

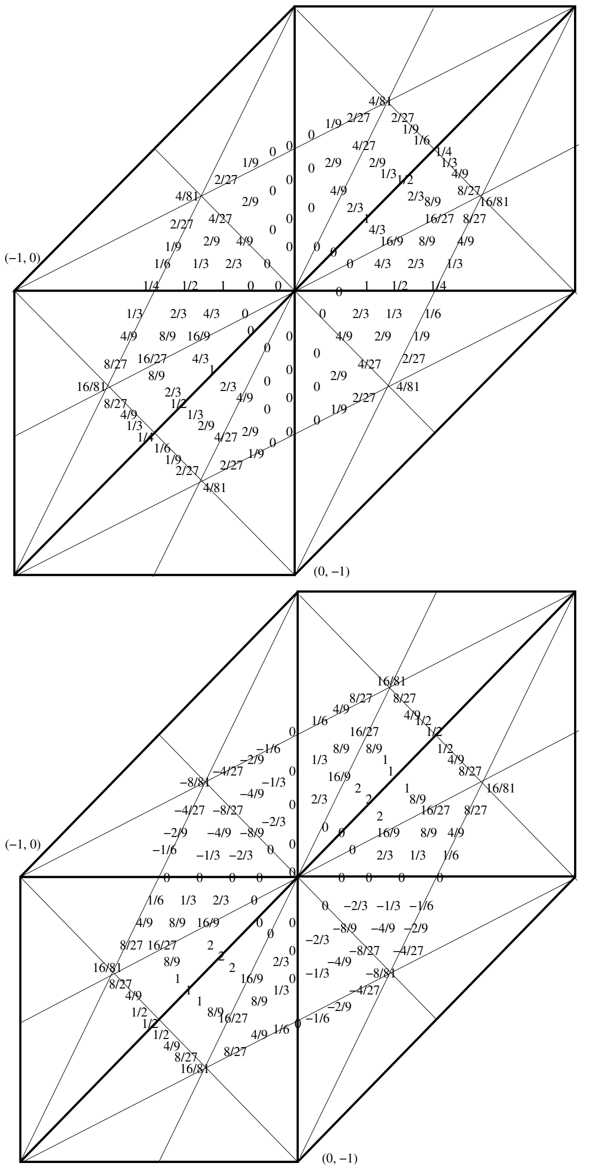

FIGURE 6. Support and Bézier coefficients of $48\phi_4$ (top), with $\phi_6(x,y) := \phi_4(y,x)$; support and Bézier coefficients of $48\phi_5$ (bottom)

functions, whose supports do not lie in *supp* ϕ_1. In Fig.7 (bottom), we show the support of ϕ_{10} and display its non-zero Bézier coefficients; and we define ϕ_{11} by $\phi_{11}(x,y) := \phi_{10}(y,x)$. Observe that the supports of $\phi_7, \cdots, \phi_{11}$ do not contain any lattice point of $\mathbb{Z}^2$ in their interiors, and hence, are called vertex splines also. Therefore, we have a total of 11 vertex splines that constitute the function vector

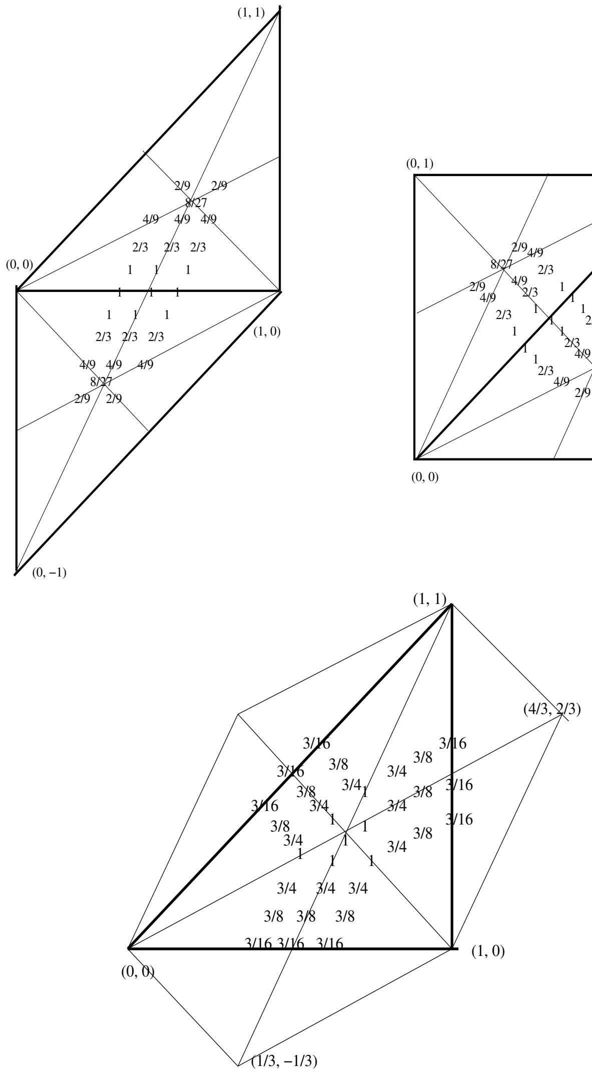

FIGURE 7. Supports and Bézier coefficients of ϕ_7 (top-left), ϕ_8 (top-right), with $\phi_9(x, y) := \phi_7(y, x)$; support and Bézier coefficients of ϕ_{10} (bottom), with $\phi_{11}(x, y) := \phi_{10}(y, x)$

$\Phi := [\phi_1, \cdots, \phi_{11}]^T$. It turns out that $\{\phi_\ell(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^2, 1 \leq \ell \leq 11\}$ is indeed a basis of $S_4^2(\triangle^3)$, as will be seen in Theorem 2.1 below.

Before stating our main result, we need to recall the concepts of sum rules and polynomial reproduction by Φ. If Φ is indeed refinable with the refinement mask $\{P_{\mathbf{k}}\}$ as described by the refinement (or two-scale) equation (1.1), then the two-scale symbol

$$P(\mathbf{z}) := \frac{1}{4} \sum_{\mathbf{k}} P_{\mathbf{k}} \, \mathbf{z}^{\mathbf{k}}$$

of Φ is said to satisfy the sum rules of order m, if there exist constant vectors $\mathbf{y}_\alpha,\ |\alpha| < m$, with $\mathbf{y}_0 \neq 0$, such that

$$\sum_{\beta \le \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} \mathbf{y}_{\alpha-\beta}\ \mathbf{J}_{\beta,\gamma} = 2^{-\alpha} \mathbf{y}_\alpha, \tag{2.3}$$

for all $|\alpha| < m$ and $\gamma = (0,0), (1,0), (0,1), (1,1)$, with

$$\mathbf{J}_{\beta,\gamma} := \sum_{\mathbf{k}} (\mathbf{k} + 2^{-1}\gamma)^\beta P_{2\mathbf{k}+\gamma}.$$

It is well-known (see, for example [8]) that if $P(\mathbf{z})$ satisfies the sum rules of order m, then Φ has the polynomial reproduction property of order m (or degree $m-1$), and in fact, the vectors $\mathbf{y}_\alpha$ can be used to give the following explicit polynomial reproduction formula:

$$\mathbf{x}^{\mathbf{j}} = \sum_{\mathbf{k}} \left\{ \sum_{\alpha \le \mathbf{j}} \binom{\mathbf{j}}{\alpha}\ \mathbf{k}^{\mathbf{j}-\alpha}\ \mathbf{y}_\alpha \right\} \Phi(\mathbf{x}-\mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^2,\ |\mathbf{j}| < m. \tag{2.4}$$

It is also known that under the assumption that the matrix $\sum_{\mathbf{k}\in\mathbb{Z}^2} \widehat{\Phi}(2\mathbf{k}\pi)\overline{\widehat{\Phi}(2\mathbf{k}\pi)^T}$ is positive definite, the function vector Φ has polynomial reproduction order m if and only if Φ has L^2-approximation order m (see [7]).

We are now ready to state the following main result of the paper.

Theorem 2.1. *For $\phi_\ell \in S_4^2(\Delta^3)$, $1 \le \ell \le 11$, with Bézier coefficients shown in Fig.5–Fig.7, the following statements hold.*

(i) *$\Phi = [\phi_1, \cdots, \phi_{11}]^T$ satisfies the second-order Hermite interpolating condition (1.5);*
(ii) *$\{\phi_1, \cdots, \phi_{11}\}$ is linearly independent;*
(iii) *$S_4^2(\Delta^3) = \mathrm{clos}_{L^2}\langle \phi_\ell(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^2, 1 \le \ell \le 11\rangle$;*
(iv) *The two-scale symbol of Φ satisfies the sum rules of order 5, and hence, Φ locally reproduces all bivariate quartic polynomials;*
(v) *Φ has L^2-approximation order 5;*
(vi) *Φ satisfies the refinement equation (1.1) with mask $\{P_{\mathbf{k}}\}$ given by*

$$P_{0,0} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{21}{64} & \frac{21}{64} & \frac{21}{64} & \frac{7}{24} & \frac{7}{24} \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{9}{128} & \frac{9}{128} & 0 & \frac{1}{12} & \frac{1}{24} \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{9}{128} & \frac{9}{128} & \frac{1}{24} & \frac{1}{12} \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{11}{2304} & \frac{11}{2304} & -\frac{1}{2304} & \frac{7}{864} & \frac{1}{864} \\
0 & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{2304} & \frac{23}{2304} & -\frac{1}{2304} & \frac{7}{864} & \frac{7}{864} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{2304} & \frac{11}{2304} & \frac{11}{2304} & \frac{1}{864} & \frac{7}{864} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}
\end{bmatrix},$$

$$P_{-1,-1}=\begin{bmatrix}
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{16} & \frac{21}{64} & \frac{3}{16} & \frac{7}{24} & \frac{7}{24}\\
-\frac{1}{16} & -\frac{1}{16} & -\frac{3}{16} & 0 & -\frac{3}{16} & -\frac{3}{8} & -\frac{1}{32} & -\frac{9}{128} & -\frac{1}{16} & -\frac{1}{24} & -\frac{1}{12}\\
-\frac{1}{16} & -\frac{3}{16} & -\frac{1}{16} & -\frac{3}{8} & -\frac{3}{16} & 0 & -\frac{1}{16} & -\frac{9}{128} & -\frac{1}{32} & -\frac{1}{12} & -\frac{1}{24}\\
\frac{1}{192} & 0 & \frac{1}{48} & 0 & 0 & \frac{1}{16} & \frac{1}{576} & \frac{11}{2304} & \frac{1}{144} & \frac{1}{864} & \frac{7}{864}\\
\frac{1}{96} & \frac{1}{48} & \frac{1}{48} & 0 & \frac{1}{16} & 0 & \frac{1}{144} & \frac{23}{2304} & \frac{1}{144} & \frac{7}{864} & \frac{7}{864}\\
\frac{1}{192} & \frac{1}{48} & 0 & \frac{1}{16} & 0 & 0 & \frac{1}{144} & \frac{11}{2304} & \frac{1}{576} & \frac{7}{864} & \frac{1}{864}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$P_{-1,0}=\begin{bmatrix}
\frac{1}{4} & 1 & -\frac{1}{2} & 3 & -\frac{3}{2} & \frac{3}{4} & \frac{21}{64} & \frac{3}{16} & 0 & \frac{7}{24} & 0\\
-\frac{1}{16} & -\frac{1}{4} & \frac{3}{16} & -\frac{3}{4} & \frac{9}{16} & -\frac{3}{8} & -\frac{9}{128} & -\frac{1}{32} & 0 & -\frac{1}{24} & 0\\
0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} & -\frac{3}{8} & 0 & \frac{1}{32} & 0 & \frac{1}{24} & 0\\
\frac{1}{192} & \frac{1}{48} & -\frac{1}{48} & \frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{11}{2304} & \frac{1}{576} & 0 & \frac{1}{864} & 0\\
0 & 0 & -\frac{1}{48} & 0 & -\frac{1}{16} & \frac{1}{8} & -\frac{1}{2304} & -\frac{1}{288} & 0 & -\frac{5}{864} & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{16} & -\frac{1}{2304} & \frac{1}{576} & 0 & \frac{1}{864} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & \frac{1}{6}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$P_{0,-1}=\begin{bmatrix}
\frac{1}{4} & -\frac{1}{2} & 1 & \frac{3}{4} & -\frac{3}{2} & 3 & 0 & \frac{3}{16} & \frac{21}{64} & 0 & \frac{7}{24}\\
0 & \frac{1}{8} & 0 & -\frac{3}{8} & \frac{3}{8} & 0 & 0 & \frac{1}{32} & 0 & 0 & \frac{1}{24}\\
-\frac{1}{16} & \frac{3}{16} & -\frac{1}{4} & -\frac{3}{8} & \frac{9}{16} & -\frac{3}{4} & 0 & -\frac{1}{32} & -\frac{9}{128} & 0 & -\frac{1}{24}\\
0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & \frac{1}{576} & -\frac{1}{2304} & 0 & \frac{1}{864}\\
0 & -\frac{1}{48} & 0 & \frac{1}{8} & -\frac{1}{16} & 0 & 0 & -\frac{1}{288} & -\frac{1}{2304} & 0 & -\frac{5}{864}\\
\frac{1}{192} & -\frac{1}{48} & \frac{1}{48} & \frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & \frac{1}{576} & \frac{11}{2304} & 0 & \frac{1}{864}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & \frac{1}{6} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$P_{0,1}=\begin{bmatrix}
\frac{1}{4} & \frac{1}{2} & -1 & \frac{3}{4} & -\frac{3}{2} & 3 & \frac{3}{16} & 0 & 0 & 0 & 0\\
0 & \frac{1}{8} & 0 & \frac{3}{8} & -\frac{3}{8} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0\\
\frac{1}{16} & \frac{3}{16} & -\frac{1}{4} & \frac{3}{8} & -\frac{9}{16} & \frac{3}{4} & \frac{1}{16} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{16} & 0 & 0 & \frac{1}{576} & 0 & 0 & 0 & 0\\
0 & \frac{1}{48} & 0 & \frac{1}{8} & -\frac{1}{16} & 0 & \frac{1}{144} & 0 & 0 & 0 & 0\\
\frac{1}{192} & \frac{1}{48} & -\frac{1}{48} & \frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{1}{144} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & \frac{1}{6} & 0\\
1 & 0 & 0 & -12 & 6 & -12 & \frac{1}{16} & \frac{1}{16} & \frac{1}{8} & \frac{1}{6} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{3}{16} & \frac{9}{8} & 0 & \frac{9}{2} & \frac{9}{8} & -\frac{9}{4} & \frac{45}{128} & \frac{45}{128} & 0 & \frac{7}{16} & \frac{1}{8}
\end{bmatrix},$$

$$P_{1,0} = \begin{bmatrix} \frac{1}{4} & -1 & \frac{1}{2} & 3 & -\frac{3}{2} & \frac{3}{4} & 0 & 0 & \frac{3}{16} & 0 & 0 \\ \frac{1}{16} & -\frac{1}{4} & \frac{3}{16} & \frac{3}{4} & -\frac{9}{16} & \frac{3}{8} & 0 & 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & -\frac{3}{8} & \frac{3}{8} & 0 & 0 & \frac{1}{32} & 0 & 0 \\ \frac{1}{192} & -\frac{1}{48} & \frac{1}{48} & \frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & \frac{1}{144} & 0 & 0 \\ 0 & 0 & \frac{1}{48} & 0 & -\frac{1}{16} & \frac{1}{8} & 0 & 0 & \frac{1}{144} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & \frac{1}{576} & 0 & 0 \\ 1 & 0 & 0 & -12 & 6 & -12 & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{16} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{16} & 0 & \frac{9}{8} & -\frac{9}{4} & \frac{9}{8} & \frac{9}{2} & 0 & \frac{45}{128} & \frac{45}{128} & \frac{1}{8} & \frac{7}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_{1,1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{16} & -\frac{1}{16} & -\frac{3}{16} & 0 & \frac{3}{16} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{16} & -\frac{3}{16} & -\frac{1}{16} & \frac{3}{8} & \frac{3}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{192} & 0 & -\frac{1}{48} & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{96} & -\frac{1}{48} & -\frac{1}{48} & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{192} & -\frac{1}{48} & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -12 & 6 & -12 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 \\ \frac{3}{16} & \frac{9}{8} & -\frac{9}{8} & \frac{9}{2} & -\frac{45}{8} & \frac{9}{2} & \frac{45}{128} & 0 & 0 & \frac{1}{8} & 0 \\ \frac{3}{16} & -\frac{9}{8} & \frac{9}{8} & \frac{9}{2} & -\frac{45}{8} & \frac{9}{2} & 0 & 0 & \frac{45}{128} & 0 & \frac{1}{8} \end{bmatrix},$$

$$P_{-1,1} = \frac{1}{16}[\delta_{i,9}\delta_{j,7}],\ P_{1,-1} = P^T_{-1,1},\ P_{1,2} = \begin{bmatrix} \mathbf{0}_{10\times 11} \\ \mathbf{u}_1 \end{bmatrix},\ P_{2,1} = \begin{bmatrix} \mathbf{0}_{9\times 11} \\ \mathbf{u}_2 \\ \mathbf{0}_{1\times 11} \end{bmatrix},$$

where

$$\mathbf{u}_1 = [\frac{3}{16},\ 0,\ -\frac{9}{8},\ -\frac{9}{4},\ \frac{9}{8},\ \frac{9}{2},\ 0, \cdots, 0],\ \mathbf{u}_2 = [\frac{3}{16},\ -\frac{9}{8},\ 0,\ \frac{9}{2},\ \frac{9}{8},\ -\frac{9}{4},\ 0, \cdots, 0].$$

The refinement equation (1.1) with refinement mask $\{P_{\mathbf{k}}\}$ given in (vi) above translates into the local averaging rule for the vector subdivision scheme as follows:

$$\mathbf{v}_{\mathbf{j}}^{m+1} = \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}}^m P_{\mathbf{j}-2\mathbf{k}}, \quad m = 0, 1, \cdots,$$

where $\mathbf{v}_{\mathbf{k}}^m := [\mathbf{v}_{1,\mathbf{k}}^m, \cdots, \mathbf{v}_{11,\mathbf{k}}^m]$ is a "row-vector" whose ℓ^{th} component $\mathbf{v}_{\ell,\mathbf{k}}^m$ is a "point" in the 3-D space, for $\ell = 1, \cdots, 11$. In particular, the first components $\mathbf{v}_{1,\mathbf{k}}^m$ are position vectors, meaning that they are the vertices of the triangular planes resulting from the m^{th} iterative step, with $\{\mathbf{v}_{1,\mathbf{k}}^0\}$ denoting the set of vertices of the initial triangular planes. Observe that this is an interpolating subdivision scheme, in that the old vertices $\mathbf{v}_{1,\mathbf{k}}^m$ are not changed in position (in the 3-D space), while the new vertices among $\{\mathbf{v}_{1,\mathbf{j}}^{m+1}\}$ are considered as "midpoints" of the triangular planes with vertices $\mathbf{v}_{1,\mathbf{k}}^m$ (though these so-called midpoints do not lie on the same triangular planes in the 3-D space). More precisely, we have

(2.5)

$$\mathbf{v}_{\mathbf{j}}^{m+1} = \left[\mathbf{v}_{1,\frac{1}{2}\mathbf{j}}^m,\ \frac{1}{2}\mathbf{v}_{2,\frac{1}{2}\mathbf{j}}^m,\ \frac{1}{2}\mathbf{v}_{3,\frac{1}{2}\mathbf{j}}^m,\ \frac{1}{4}\mathbf{v}_{4,\frac{1}{2}\mathbf{j}}^m,\ \frac{1}{4}\mathbf{v}_{5,\frac{1}{2}\mathbf{j}}^m,\ \frac{1}{4}\mathbf{v}_{6,\frac{1}{2}\mathbf{j}}^m,\ *,\ *,\ *,\ *,\ *\right],\ \mathbf{j} \in 2\mathbb{Z}^2.$$

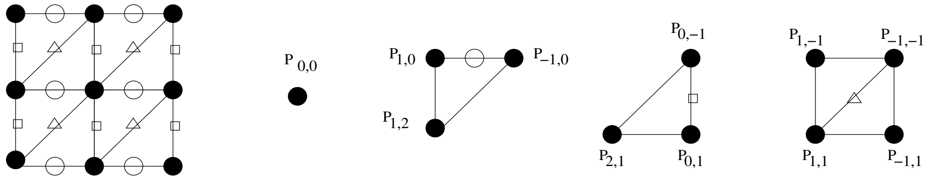

FIGURE 8. Coefficient stencils for C^2 local averaging rule

The three "midpoints" of each triangular plane are joined by three new edges, changing the triangular plane to four new triangular planes; and hence, it is called a 1-to-4 vector subdivision scheme. The matrix-valued coefficient stencils for determining $\mathbf{v}_{\mathbf{j}}^{m+1}$ from $\mathbf{v}_{\mathbf{k}}^{m}$ are given in Fig.8, where the solid circles denote the "old vertices" (meaning $\mathbf{v}_{\mathbf{k}}^{m}$, with the first components representing the actual positions of the vertices in the 3-D space), and the hollow circles, squares, and triangles denote the "new vertices," depending on their orientations as described in the 2-D parametric domain. Observe that while the first components of $\mathbf{v}_{\mathbf{k}}^{m}$ are unchanged, the second through the sixth components are simply scaled by $\frac{1}{2}$ or $\frac{1}{4}$ in (2.5).

Here we mention that if the basis functions in Theorem 2.1 (see Fig.5–Fig.7) are replaced by $\tilde{\phi}_j$, with

$$\tilde{\phi}_j = \phi_j,\ j = 1, \cdots, 6, 10, \quad \tilde{\phi}_7 = \phi_7(\cdot + (1,0)), \quad \tilde{\phi}_j = \phi_j(\cdot + (1,1)),\ j = 8, 9, 11,$$

then $\widetilde{\Phi} := [\tilde{\phi}_1, \cdots, \tilde{\phi}_{11}]$ remains refinable with refinement mask $\{\widetilde{P}_{\mathbf{k}}\}$, say. The importance of this transformation is that two of the four coefficient stencils in Fig.8 are reduced to 2-point coefficient stencils, as shown in Fig.9. In the final section of this paper, we will show that all coefficient stencils can be further reduced to 2-point templates when spline representation is less important.

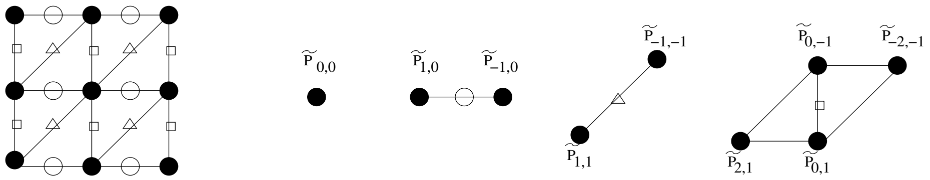

FIGURE 9. Coefficient stencils for C^2 local averaging rule

3. PROOF OF THEOREM 2.1

The statement (i), saying that Φ satisfies the second-order Hermite interpolating condition (1.5), can be easily verified by noting that the Bézier coefficients at the vertices are function values and by applying the formulas of partial derivatives in terms of Bézier coefficients in [2, p.94], using the Bézier coefficients shown in Fig.5–Fig.7.

To prove (ii), assume that there exist some real constants $c_{\mathbf{k}}^{\ell}$, $1 \le \ell \le 6$, $d_{\mathbf{k}}^{j}$, $1 \le j \le 3$, $e_{\mathbf{k}}^{s}$, $s = 1, 2$, $\mathbf{k} \in \mathbb{Z}^2$, such that for $\mathbf{x} \in \mathbb{R}^2$,

$$f(\mathbf{x}) := \sum_{\mathbf{n}} \{ \sum_{\ell=1}^{6} c_{\mathbf{n}}^{\ell} \phi_\ell(\mathbf{x} - \mathbf{n}) + \sum_{j=1}^{3} d_{\mathbf{n}}^{j} \phi_{6+j}(\mathbf{x} - \mathbf{n}) + \sum_{s=1}^{2} e_{\mathbf{n}}^{s} \phi_{s+9}(\mathbf{x} - \mathbf{n}) \} = 0. \tag{3.1}$$

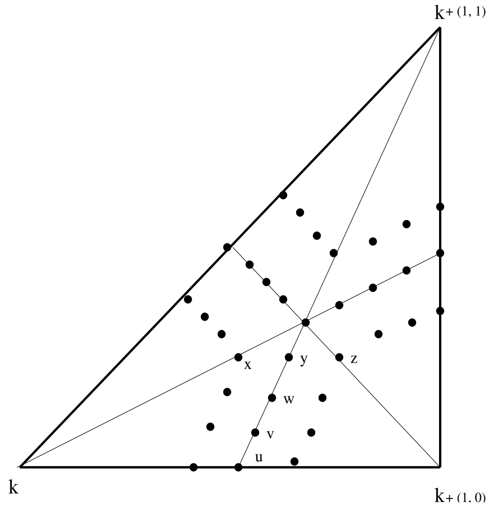

FIGURE 10. Bézier coefficients of bivariate spline f in (3.3) with u, v, w, x, y, z in (3.4).

By (i) with $\mathbf{x} = \mathbf{k} \in \mathbb{Z}^2$, we then have

$$[c^1_{\mathbf{k}}, \cdots, c^6_{\mathbf{k}}] = \left[f \quad \frac{\partial}{\partial x} f \quad \frac{\partial}{\partial y} f \quad \frac{\partial^2}{\partial x^2} f \quad \frac{\partial^2}{\partial x \partial y} f \quad \frac{\partial^2}{\partial y^2} f \right] (\mathbf{k}) = \mathbf{0}, \tag{3.2}$$

so that $f(\mathbf{x})$ in (3.1) reduces to

$$f(\mathbf{x}) = \sum_{\mathbf{n}} \{ \sum_{j=1}^{3} d^j_{\mathbf{n}} \phi_{6+j}(\mathbf{x} - \mathbf{n}) + \sum_{s=1}^{2} e^s_{\mathbf{n}} \phi_{s+9}(\mathbf{x} - \mathbf{n}) \}, \ \mathbf{x} \in \mathbb{R}^2. \tag{3.3}$$

The Bézier coefficients of f restricted to the triangle with vertices $\mathbf{k}, \mathbf{k} + (1,0), \mathbf{k} + (1,1)$, for any (fixed) $\mathbf{k} \in \mathbb{Z}^2$, is displayed in Fig.10 with

$$\begin{aligned} & u := d^1_{\mathbf{k}} + \frac{3}{16}(e^1_{\mathbf{k}} + e^2_{\mathbf{k}-(0,1)}), \quad v := d^1_{\mathbf{k}} + \frac{3}{8} e^1_{\mathbf{k}}, \quad w := \frac{2}{3} d^1_{\mathbf{k}} + \frac{3}{4} e^1_{\mathbf{k}}, \\ & x := e^1_{\mathbf{k}} + \frac{4}{9}(d^1_{\mathbf{k}} + d^2_{\mathbf{k}}), \quad y := e^1_{\mathbf{k}} + \frac{4}{9} d^1_{\mathbf{k}} + \frac{2}{9}(d^2_{\mathbf{k}} + d^3_{\mathbf{k}+(1,0)}), \\ & z := e^1_{\mathbf{k}} + \frac{4}{9}(d^1_{\mathbf{k}} + d^3_{\mathbf{k}+(1,0)}). \end{aligned} \tag{3.4}$$

From the assumption that $f \equiv 0$ in (3.1), we have

$$u = v = w = x = y = z = 0,$$

and it follows from (3.4) that

$$d^1_{\mathbf{k}} = 0, \ d^2_{\mathbf{k}} = 0, \ d^3_{\mathbf{k}+(1,0)} = 0, \ e^1_{\mathbf{k}} = 0, \ e^2_{\mathbf{k}-(0,1)} = 0. \tag{3.5}$$

Since (3.2) and (3.5) hold for an arbitrary $\mathbf{k} \in \mathbb{Z}^2$, we may conclude that $c^\ell_{\mathbf{k}}, d^j_{\mathbf{k}}, e^s_{\mathbf{k}}$ in (3.1) are all equal to 0. That is $\{\phi_1, \cdots, \phi_{11}\}$ is linearly independent.

To prove (iii), again let M, N be arbitrary positive integers and $S^2_4(\triangle^3_{MN})$ denote the restriction of $S^3_4(\triangle^3)$ on $[0, M+1] \times [0, N+1]$. Then the dimension of $S^2_4(\triangle^3_{MN})$ is given by (2.1). One can easily verify that for each $1 \le \ell \le 6$, the number of $\phi_\ell(\cdot - \mathbf{k})$ whose support overlaps with $[0, M+1] \times [0, N+1]$ is equal to $(M+2)(N+2)$, the numbers of $\phi_7(\cdot-\mathbf{k}), \phi_8(\cdot-\mathbf{k})$ and $\phi_9(\cdot-\mathbf{k})$ whose supports overlap with $[0, M+1]\times[0, N+1]$ are equal to $(M+1)(N+2)$, $(M+1)(N+1)$ and $(M+2)(N+1)$, respectively, while the numbers of $\phi_{10}(\cdot-\mathbf{k}), \phi_{11}(\cdot-\mathbf{k})$ whose supports overlap with $[0, M+1]\times[0, N+1]$ are both equal to $(M+2)(N+2)-1$. Hence, the total number of $\phi_\ell(\cdot-\mathbf{k}), 1\le\ell\le 11$, whose supports overlap with $[0, M+1]\times[0, N+1]$ is given by

$$\begin{aligned}&6(M+2)(N+2)+(M+1)(N+2)+(M+1)(N+1)+\\&\quad(M+2)(N+1)+2(M+2)(N+2)-2\\&=11MN+20M+20N+35,\end{aligned}$$

which is exactly the same as $\dim S_4^2(\triangle^3_{MN})$. This fact, along with the linear independency property (ii) and the assumption that M and N are arbitrary, assures the validity of the statement (iii).

Next, let us verify the correctness of (vi), before tackling the proof of (iv)–(v). To do so, we first observe that since $\triangle^3\subset\frac{1}{2}\triangle^3$, the spline space $S_4^2(\triangle^3)$ is a subspace of $S_4^2(\frac{1}{2}\triangle^3)$. Hence, in view of (iii), Φ is indeed refinable. To find the refinement mask $\{P_\mathbf{k}\}$ of Φ, we need to compute the Bézier representation of $\phi_\ell(\frac{\cdot}{2}), 1\le\ell\le 11$, by applying the C^4-smoothing formula in [2, Theorem 5.1], and then write down the linear equations of $\phi_\ell(\frac{\cdot}{2})$, formulated as (finite) linear combinations of $\phi_m(\cdot-\mathbf{k})$, $\mathbf{k}\in\mathbb{Z}^2$, at the Bézier points for $1\le\ell, m\le 11$. The (unique) solution, arranged in 11×11 matrix formulation, gives the mask $\{P_\mathbf{k}\}$ in (vi).

To prove (iv), it is not difficult to show that the two-scale symbol of Φ with the refinement mask $\{P_\mathbf{k}\}$ satisfies the sum rules of order 5, by solving equations (2.3) to find the following vectors $\mathbf{y}_\alpha, |\alpha|<5$:

$$\begin{aligned}\mathbf{y}_{0,0}&=\frac{1}{24}[24,\ 0,\ 0,\ 0,\ 0,\ 0,\ 9,\ 9,\ 9,\ 8,\ 8],\\ \mathbf{y}_{1,0}&=\frac{1}{144}[0,\ 18,\ 0,\ 0,\ 0\ ,0,\ 27,\ 27,\ 0,\ 32,\ 16],\\ \mathbf{y}_{0,1}&=\frac{1}{144}[0,\ 0,\ 18,\ 0,\ 0,\ 0,\ 0,\ 27,\ 27,\ 16,\ 32],\\ \mathbf{y}_{2,0}&=\frac{1}{432}[0,\ 0,\ 0,\ 18,\ 0,\ 0,\ 33,\ 33,\ -3,\ 56,\ 8],\\ \mathbf{y}_{1,1}&=\frac{1}{864}[0,\ 0,\ 0,\ 0,\ 18,\ 0,\ -3,\ 69,\ -3,\ 56,\ 56],\\ \mathbf{y}_{0,2}&=\frac{1}{432}[0,\ 0,\ 0,\ 0,\ 0,\ 18,\ -3,\ 33,\ 33,\ 8,\ 56],\\ \mathbf{y}_{3,0}&=\frac{1}{144}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 3,\ 3,\ 0,\ 8,\ 0],\\ \mathbf{y}_{2,1}&=\frac{1}{864}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -3,\ 21,\ -3,\ 24,\ 8],\\ \mathbf{y}_{1,2}&=\frac{1}{864}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -3,\ 21,\ -3,\ 8,\ 24],\\ \mathbf{y}_{0,3}&=\frac{1}{144}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 3,\ 3,\ 0,\ 8],\\ \mathbf{y}_{4,0}&=\mathbf{y}_{3,1}=\mathbf{y}_{2,2}=\mathbf{y}_{1,3}=\mathbf{y}_{0,4}=[0,\ \cdots,\ 0].\end{aligned}\tag{3.6}$$

This gives the polynomial reproduction formula (2.4) with $m=5$.

To prove (v), we first note that the linear independence of $\{\phi_1, \cdots, \phi_{11}\}$ in (ii) implies that $\sum_{\mathbf{k}\in\mathbb{Z}^2} \widehat{\Phi}(2\mathbf{k}\pi)\overline{\widehat{\Phi}(2\mathbf{k}\pi)^T}$ is positive definite (see [10]). Recall that under this condition, the order of local polynomial reproduction is equivalent to the L^2-approximation order. Therefore, (v) follows from (iv). This completes the proof of the theorem.

4. Applications to data representation

In this section, we will give two applications of the Hermite basis functions $\phi_1, \cdots, \phi_{11}$ of $S_4^2(\triangle^3)$ to discrete data representation. In §4.1, one single function φ, called a super function, is formulated as a finite linear combination of integer translates of $\phi_1, \cdots, \phi_{11}$ to achieve the full approximation order of $S_4^2(\triangle^3)$. In §4.2, we modify Φ to extend the second-order Hermite interpolating condition at $\mathbb{Z}^2$ to include Lagrange interpolation at the half integers as well.

4.1. **Super functions.** In this subsection, we compute a super function for $S_4^2(\triangle^3)$. For completeness, we also formulate certain super functions for $S_2^1(\triangle^3)$ and $S_3^2(\triangle^3)$ based on the refinable splines constructed in [4] and [5], respectively.

Suppose that the two-scale symbol $P(\mathbf{z})$ of a refinable function vector Φ satisfies the sum rules of order m, namely (2.3), for some vectors $\mathbf{y}_\alpha, |\alpha| < m$, with $\mathbf{y}_0\hat{\Phi}(0) = 1$. Let $\{\mathbf{t_k}\}$ be a finite sequence of row-vectors so chosen that the vector-valued trigonometric polynomial $\mathbf{t}(\omega) := \sum_{\mathbf{k}} \mathbf{t_k} e^{-i\mathbf{k}\omega}$ satisfies

$$(-iD)^\alpha \mathbf{t}(0) = \mathbf{y}_\alpha, \quad |\alpha| < m. \tag{4.1}$$

Then the function φ defined by (1.6) in terms of this sequence $\{\mathbf{t_k}\}$ is a super function, meaning that φ satisfies the modified Strang-Fix condition (1.7).

We first demonstrate the procedure by considering the simple example $\Phi^a = [\phi_1^a, \phi_2^a, \phi_3^a]$ in Fig.2 for $S_2^1(\triangle^3)$, where the two-scale symbol satisfies the sum rules of order 3, with vectors $\mathbf{y}_\alpha$ given in [5] by

$$\begin{aligned} \mathbf{y}_{0,0} = [1,\ 0,\ 0],\ \mathbf{y}_{1,0} = [0,\ 1,\ 0],\ \mathbf{y}_{0,1} = [0,\ 0,\ 1], \\ \mathbf{y}_{2,0} = \mathbf{y}_{1,1} = \mathbf{y}_{0,2} = [0,\ 0,\ 0]. \end{aligned} \tag{4.2}$$

Let $\mathbf{t}(\omega) = \sum_{\mathbf{k}} \mathbf{t_k} e^{-i\mathbf{k}\omega}$ be a vector-valued trigonometric polynomial satisfying (4.1) with $m = 3$. For the $\mathbf{y}_\alpha$ in (4.2), one can easily choose (among many other choices) non-zero $\mathbf{t_k}$, as follows:

$$\begin{aligned} &\mathbf{t}_{0,0} = [1,\ 0,\ 0],\ \mathbf{t}_{1,0} = -\frac{1}{2}[0,\ 1,\ 0],\ \mathbf{t}_{0,1} = -\frac{1}{2}[0,\ 0,\ 1], \\ &\mathbf{t}_{-1,0} = \frac{1}{2}[0,\ 1,\ 0],\ \mathbf{t}_{0,-1} = \frac{1}{2}[0,\ 0,\ 1]. \end{aligned}$$

Then the super function φ^a defined by

$$\varphi^a := \sum_{\mathbf{k}\in\{(0,0),(1,0),(0,1),(-1,0),(0,-1)\}} \mathbf{t_k}\Phi^a(\cdot - \mathbf{k})$$

satisfies

$$D^\alpha\widehat{\varphi^a}(2\pi\mathbf{k}) = \delta_{\alpha,\mathbf{0}}\delta_{\mathbf{k},\mathbf{0}} \qquad |\alpha| < 3, \quad \mathbf{k} \in \mathbb{Z}^2.$$

Here and in the following, we obtain $\mathbf{t_k}$ by solving the equations (4.1) for the vector coefficients $\mathbf{t_k}$.

For the basis functions ϕ_1^b, ϕ_2^b constructed in [5], one can verify (see [5]) that the two-scale symbol satisfies the sum rules of order 4, with vectors $\mathbf{y}_\alpha$ given by

$$\begin{aligned}
&\mathbf{y}_{0,0} = \frac{1}{6}[1,\ 3],\ \mathbf{y}_{1,0} = \mathbf{y}_{0,1} = [0,\ 0],\ \mathbf{y}_{2,0} = \mathbf{y}_{0,2} = \frac{1}{18}[1,\ -3],\\
&\mathbf{y}_{1,1} = \frac{1}{36}[1,\ -3],\ \mathbf{y}_{3,0} = \mathbf{y}_{2,1} = \mathbf{y}_{1,2} = \mathbf{y}_{0,3} = [0,\ 0],
\end{aligned}$$

and hence, $\Phi^b = [\phi_1^b,\ \phi_2^b]$ reproduces all cubic monomials $1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3$. Let $\mathbf{t}(\omega) = \sum_{\mathbf{k}} \mathbf{t}_{\mathbf{k}} e^{-i\mathbf{k}\omega}$ be a vector-valued trigonometric polynomial satisfying (4.1) with $|\alpha| < 4$. For the $\mathbf{y}_\alpha$ given above, one can choose $\mathbf{t}(\omega)$ with non-zero coefficients

$$\begin{aligned}
&\mathbf{t}_{0,0} = \frac{1}{36}[1,\ 33],\ \mathbf{t}_{1,0} = \mathbf{t}_{0,1} = \mathbf{t}_{-1,0} = \mathbf{t}_{0,-1} = \frac{1}{24}[1,\ -3],\\
&\mathbf{t}_{-1,1} = \mathbf{t}_{1,-1} = \frac{1}{72}[-1,\ 3].
\end{aligned}$$

The super function φ^b defined by

$$\varphi^b := \sum_{\mathbf{k}\in\{(0,0),(1,0),(0,1),(-1,0),(0,-1),(1,-1),(-1,1)\}} \mathbf{t}_{\mathbf{k}} \Phi^b(\cdot - \mathbf{k})$$

satisfies

$$D^\alpha \widehat{\varphi^b}((2\pi\mathbf{k}) = \delta_{\alpha,\mathbf{0}}\delta_{\mathbf{k},\mathbf{0}} \qquad |\alpha| < 4, \quad \mathbf{k} \in \mathbb{Z}^2.$$

Now let us return to the basis functions $\phi_\ell, 1 \le \ell \le 11$, of $S_4^2(\triangle^3)$ constructed in this paper. As shown in the above section, the two-scale symbol corresponding to $\Phi = [\phi_1, \cdots, \phi_{11}]^T$ satisfies the sum rules of order 5, with vectors $\mathbf{y}_\alpha, |\alpha| < 5$ given by (3.6). For these vectors, we can find $\mathbf{t}(\omega)$ by solving (4.1). In particular, we may choose $\mathbf{t}(\omega)$ with (non-zero) coefficients given by

$$\begin{aligned}
\mathbf{t}_{-1,-1} &= -\frac{1}{3456}[0,\ 36,\ 36,\ 6,\ 6,\ 6,\ 63,\ 99,\ 63,\ 88,\ 88],\\
\mathbf{t}_{-1,0} &= \frac{1}{1728}[0,\ 72,\ 72,\ 36,\ 12,\ 24,\ 177,\ 369,\ 111,\ 392,\ 264],\\
\mathbf{t}_{-1,1} &= \frac{1}{3456}[0,\ 0,\ -216,\ 0,\ -36,\ -54,\ 15,\ -561,\ -345,\ -392,\ -552],\\
\mathbf{t}_{-1,2} &= \frac{1}{1728}[0,\ 0,\ 72,\ 0,\ 12,\ 6,\ -3,\ 141,\ 117,\ 88,\ 152],\\
\mathbf{t}_{-1,3} &= \frac{1}{10368}[0,\ 0,\ -108,\ 0,\ -18,\ 0,\ 3,\ -177,\ -177,\ -104,\ -200],\\
\mathbf{t}_{0,-1} &= \frac{1}{1728}[0,\ 72,\ 72,\ 12,\ 36,\ 24,\ 111,\ 369,\ 177,\ 264,\ 392],\\
\mathbf{t}_{0,0} &= \frac{1}{864}[864,\ 54,\ 54,\ -36,\ -36,\ -18,\ 339,\ 267,\ 339,\ 224,\ 224],\\
\mathbf{t}_{0,1} &= \frac{1}{864}[0,\ 0,\ -108,\ 0,\ 18,\ 9,\ -6,\ -75,\ -123,\ -48,\ -80],\\
\mathbf{t}_{0,2} &= \frac{1}{5184}[0,\ 0,\ 108,\ 0,\ 0,\ -18,\ 3,\ 75,\ 147,\ 40,\ 88],
\end{aligned}$$

$$\begin{aligned}
\mathbf{t}_{1,-1} &= \frac{1}{3456}[0,\ -216,\ 0,\ -36,\ 0,\ -54,\ -345,\ -561,\ 15,\ -552,\ -392],\\
\mathbf{t}_{1,0} &= \frac{1}{864}[0,\ -108,\ 0,\ 18,\ 0,\ 9,\ -123,\ -75,\ -6,\ -80,\ -48],\\
\mathbf{t}_{1,1} &= \frac{1}{1152}[0,\ 0,\ 0,\ 0,\ 0,\ 6,\ 1,\ 9,\ 1,\ 8,\ 8],\\
t_{2,-1} &= \frac{1}{1728}[0,\ 72,\ 0,\ 12,\ 0,\ 6,\ 117,\ 141,\ -3,\ 152,\ 88],\\
\mathbf{t}_{2,0} &= \frac{1}{5184}[0,\ 108,\ 0,\ 0,\ 0,\ -18,\ 147,\ 75,\ 3,\ 88,\ 40],\\
\mathbf{t}_{3,-1} &= \frac{1}{10368}[0,\ -108,\ 0,\ -18,\ 0,\ 0,\ -177,\ -177,\ 3,\ -200,\ -104].
\end{aligned}$$

Again the super function φ^c defined by (1.6) with the above $\mathbf{t}_{\mathbf{k}}$ satisfies

$$D^\alpha\widehat{\varphi^c}((2\pi\mathbf{k}) = \delta_{\alpha,\mathbf{0}}\delta_{\mathbf{k},\mathbf{0}} \qquad |\alpha| < 5, \quad \mathbf{k}\in\mathbb{Z}^2.$$

4.2. **Combined Hermite-Lagrange interpolation.** In this subsection, we modify Φ to Φ^n so that in addition to satisfying the second-order Hermite interpolating condition (1.5), Φ^n satisfies the Lagrange interpolating condition at $(\mathbb{Z}^2+(\frac12,0))\cup(\mathbb{Z}^2+(0,\frac12))\cup(\mathbb{Z}^2+(\frac12,\frac12))$ as well. The modified basis functions are given by

$$\begin{aligned}
\phi_1^{\mathrm{n}} &:= \phi_1 - \frac14\big(\phi_7+\phi_8+\phi_9+\phi_7(\cdot+(1,0))+\phi_8(\cdot+(1,1))+\phi_9(\cdot+(0,1))\big),\\
\phi_2^{\mathrm{n}} &:= \phi_2 - \frac{1}{16}\big(\phi_7+\phi_8-\phi_7(\cdot+(1,0))-\phi_8(\cdot+(1,1))\big),\\
\phi_3^{\mathrm{n}} &:= \phi_3 - \frac{1}{16}\big(\phi_8+\phi_9-\phi_8(\cdot+(1,1))-\phi_9(\cdot+(0,1))\big),\\
\phi_4^{\mathrm{n}} &:= \phi_4 - \frac{1}{192}\big(\phi_7+\phi_8+\phi_7(\cdot+(1,0))+\phi_8(\cdot+(1,1))\big),\\
\phi_5^{\mathrm{n}} &:= \phi_5 - \frac{1}{96}\big(\phi_8+\phi_8(\cdot+(1,1))\big),\\
\phi_6^{\mathrm{n}} &:= \phi_6 - \frac{1}{192}\big(\phi_8+\phi_9+\phi_8(\cdot+(1,1))+\phi_9(\cdot+(0,1))\big),\\
\phi_7^{\mathrm{n}} &:= \phi_7,\ \phi_8^{\mathrm{n}} := \phi_8,\ \phi_9^{\mathrm{n}} := \phi_9,\\
\phi_{10}^{\mathrm{n}} &:= \phi_{10} - \frac{3}{16}\big(\phi_7+\phi_8+\phi_9(\cdot-(1,0))\big),\\
\phi_{11}^{\mathrm{n}} &:= \phi_{11} - \frac{3}{16}\big(\phi_7(\cdot-(0,1))+\phi_8+\phi_9\big).
\end{aligned}$$

It is clear that $\Phi^n := [\phi_1^n,\cdots,\phi_{11}^n]^T$ satisfies the same second-order Hermite interpolating property (1.5) as Φ. One can also easily verify that Φ^n satisfies:

$$\begin{aligned}
&\Phi^n(\mathbf{k}+(\frac12,0)) = \delta_{\mathbf{k},\mathbf{0}}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0]^T;\\
&\Phi^n(\mathbf{k}+(\frac12,\frac12)) = \delta_{\mathbf{k},\mathbf{0}}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0]^T,\\
&\Phi^{\mathrm{n}}(\mathbf{k}+(0,\frac12)) = \delta_{\mathbf{k},\mathbf{0}}[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0]^T, \quad \mathbf{k}\in\mathbb{Z}^2.
\end{aligned} \tag{4.3}$$

That is, Φ^n satisfies the Lagrange interpolating condition at the " half integers" as well. To relate Φ^n with Φ in the Fourier domain, we set

$$M(\mathbf{z}) :=$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}(1+\frac{1}{z_1}) & -\frac{1}{4}(1+\frac{1}{z_1 z_2}) & -\frac{1}{4}(1+\frac{1}{z_2}) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{16}(1-\frac{1}{z_1}) & -\frac{1}{16}(1-\frac{1}{z_1 z_2}) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{16}(1-\frac{1}{z_1 z_2}) & -\frac{1}{16}(1-\frac{1}{z_2}) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{192}(1+\frac{1}{z_1}) & -\frac{1}{192}(1+\frac{1}{z_1 z_2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{96}(1+\frac{1}{z_1 z_2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{192}(1+\frac{1}{z_1 z_2}) & -\frac{1}{192}(1+\frac{1}{z_2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{16} & -\frac{3}{16} & -\frac{3}{16}z_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{16}z_2 & -\frac{3}{16} & -\frac{3}{16} & 0 & 1 \end{bmatrix}.$$

Then we have

$$\hat{\Phi}^n(\omega) = M(e^{-i\omega})\hat{\Phi}(\omega).$$

Note that the inverse $M^{-1}(\mathbf{z})$ of $M(\mathbf{z})$ is still a matrix-valued Laurant polynomial, given by

$$M^{-1}(\mathbf{z}) = 2I_{11} - M(\mathbf{z}).$$

Hence, Φ^n is refinable with a finite mask $\{P^n_{\mathbf{k}}\}$, and the corresponding two-scale symbol $P^n(\mathbf{z})$ is given by the matrix-valued Laurent polynomial

$$M(\mathbf{z}^2)P(\mathbf{z})M^{-1}(\mathbf{z}),$$

where $P(\mathbf{z})$ is the two-scale symbol for Φ. Of course, Φ^n is still linearly independent, and has L^2-approximation of order equal to 5.

Let f be any C^2 function on $\mathbb{R}^2$. Set

$$S^j_f(x) := \sum_{\mathbf{k}}\{\sum_{\ell=1}^{6} c^j_{\mathbf{k},\ell}\phi^{\mathrm{n}}_\ell(2^j\cdot-\mathbf{k}) + \sum_{\ell=7}^{9} d^j_{\mathbf{k},\ell}\phi^{\mathrm{n}}_\ell(2^j\cdot-\mathbf{k}) + \sum_{\ell=10}^{11} e^j_{\mathbf{k},\ell}\phi^{\mathrm{n}}_\ell(2^j\cdot-\mathbf{k})\},$$

with

$$\begin{aligned} & c^j_{\mathbf{k},1} = f(2^{-j}\mathbf{k}),\ c^j_{\mathbf{k},2} = 2^{-j}\frac{\partial f}{\partial x}(2^{-j}\mathbf{k}),\ c^j_{\mathbf{k},3} = 2^{-j}\frac{\partial f}{\partial y}(2^{-j}\mathbf{k}), \\ & c^j_{\mathbf{k},4} = 2^{-2j}\frac{\partial^2 f}{\partial x^2}(2^{-j}\mathbf{k}),\ c^j_{\mathbf{k},5} = 2^{-2j}\frac{\partial^2 f}{\partial x\partial y}(2^{-j}\mathbf{k}), \\ & c^j_{\mathbf{k},6} = 2^{-2j}\frac{\partial^2 f}{\partial y^2}(2^{-j}\mathbf{k}),\ d^j_{\mathbf{k},7} = f(2^{-j}(\mathbf{k}+(\frac{1}{2},0))), \\ & d^j_{\mathbf{k},8} = f(2^{-j}(\mathbf{k}+(\frac{1}{2},\frac{1}{2}))),\ d^j_{\mathbf{k},9} = f(2^{-j}(\mathbf{k}+(0,\frac{1}{2}))), \end{aligned}$$

where $e^j_{\mathbf{k},10}, e^j_{\mathbf{k},11}$ are free parameters to be determined. Then S^j_f is a second-order Hermite interpolant of f at $2^{-j}\mathbb{Z}^2$ and Lagrange interpolant of f at $2^{-j-1}\mathbb{Z}^2\backslash(2^{-j}\mathbb{Z}^2)$. The free parameters can be used for shape control, or could be determined by certain best approximation criterion.

5. $\sqrt{3}$-SUBDIVISION

The multi-level structure discussed in the previous sections is governed by the refinement equation (1.1) with the dilation matrix $2I_2$. The corresponding subdivision for this matrix dilation employs the so-called 1-to-4 split topological rule as used in [13, 6], meaning that each triangle is subdivided into 4 triangles by joining the midpoint of each edge (see Section 2). More recently, another surface subdivision scheme, called $\sqrt{3}$-subdivision, was introduced in [11, 12]. To describe the topological rule of this newer scheme (that governs how new vertices are chosen and how they are connected to yield a finer triangular subdivided surface in $\mathbb{R}^3$), we use a two-dimensional regular triangulation $\triangle$ as a guideline. That is, each triangular plane of the subdivided surface in $\mathbb{R}^3$ is represented by a triangular cell of $\triangle$. For the $\sqrt{3}$-subdivision scheme, the new vertices are represented by the midpoints of the triangular cells of $\triangle$, while the new edges are obtained by following the topological rule of joining the midpoint of each triangular cell of $\triangle$ to its three (old) vertices as well as to the (new) vertices that are midpoints of the three adjacent triangular cells. To complete describing this topological rule, the old edges are to be removed. Hence, if the regular triangulation is the triangulation $\triangle^1$ of $\mathbb{R}^2$ generated by the three-directional mesh of grid lines $x = i, y = j, x - y = k$, where $i, j, k \in \mathbb{Z}$, as shown in Fig.11A, then before removing the old edges as dictated by the topological rule, we have the six-directional mesh $\triangle^3$ as shown in Fig.11B. This topological rule is shown in Fig.12A–Fig.12B. Observe that if the topological rule is applied for a second time, then we arrive at the 3-dilated triangulation shown in Fig.12C of the original triangulation in Fig.12A. That is why it is called $\sqrt{3}$-subdivision.

In our recent work [5], based on the basis function vectors Φ^a of $S_2^1(\triangle^3)$ and Φ^b of

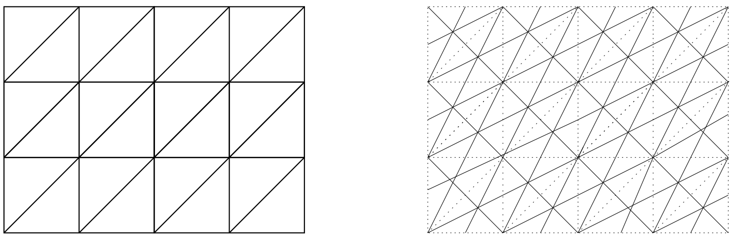

FIGURE 11. (Fig.11A, Fig.11B) Three-directional mesh $\triangle^1$, and six-directional mesh $\triangle^3$

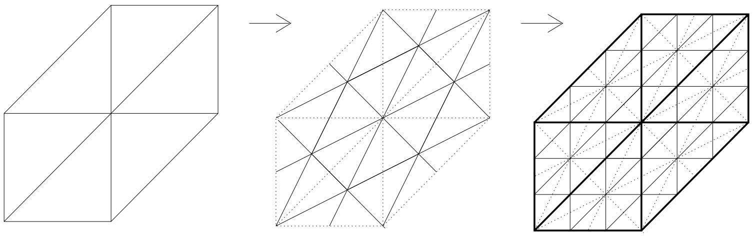

FIGURE 12. (Fig.12A, Fig.12B, Fig.12C) Topological rule of $\sqrt{3}$-subdivision scheme

$S_3^2(\triangle^3)$, respectively, we introduced the local averaging rules of a C^1-interpolating $\sqrt{3}$-subdivision scheme and a C^2-approximation (but non-interpolating) $\sqrt{3}$-subdivision scheme, by observing that the matrix A in (1.4) satisfies the "mesh refinability" property

$$\triangle^3 \subset A^{-1}\triangle^3, \tag{5.1}$$

and computing the corresponding masks $\{P^a_{\mathbf{k}}\}$ and $\{P^b_{\mathbf{k}}\}$. This is valid since $S_2^1(\triangle^3) \subset S_2^1(A^{-1}\triangle^3)$ and $S_3^2(\triangle^3) \subset S_3^2(A^{-1}\triangle^3)$ due to (5.1), and that Φ^a and Φ^b generate $S_2^1(\triangle^3)$ and $S_3^2(\triangle^3)$, respectively. Since the function vector Φ constructed in this paper generates a basis of $S_4^2(\triangle^3)$, it is also refinable with respect to the dilation matrix A, and hence its mask $\{\widehat{P}_{\mathbf{k}}\}$, say, provides the local averaging rule of a C^2-interpolating $\sqrt{3}$-subdivision scheme, namely

$$\mathbf{v}_{\mathbf{j}}^{m+1} = \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}}^m \widehat{P}_{\mathbf{j}-A\mathbf{k}}.$$

The details are not given here.

6. Two-point matrix-valued coefficient stencils

While the matrix-valued coefficient stencils in Fig.3 for C^1 surface display are 2-point templates, those for C^2 surface display introduced in the previous sections require at least one 4-point coefficient stencil, as shown in Fig.8 and Fig.9. In the following, we give a second-order Hermite interpolating scheme with 2-point templates as shown in Fig.13. A necessary condition is that all the refinement (6×6 matrix) coefficients, with the exception of $P^\#_{0,0}, P^\#_{1,0}, P^\#_{1,1}, P^\#_{0,1}, P^\#_{-1,0}, P^\#_{-1,-1}, P^\#_{0,-1}$, must be zero matrices. To compute these (possibly non-zero) matrices, we impose the sum rule (2.3) of order 4 to the two-scale symbol (denote by $P^\#(\mathbf{z}) = \frac{1}{4}\sum_{\mathbf{k}} P^\#_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}$) along with

$$\begin{bmatrix} \mathbf{y}^T_{0,0} & \mathbf{y}^T_{1,0} & \mathbf{y}^T_{0,1} & \frac{1}{2}\mathbf{y}^T_{2,0} & \mathbf{y}^T_{1,1} & \frac{1}{2}\mathbf{y}^T_{0,2} \end{bmatrix} = I_6$$

(which is a necessary condition for second-order Hermite interpolation), and

$$\mathbf{y}_{3,0} = \mathbf{y}_{2,1} = \mathbf{y}_{1,2} = \mathbf{y}_{0,3} = [0, \cdots, 0].$$

Hence, by following the symmetry properties of the C^2-quartic basis functions $\phi_\ell, \ell = 1, \cdots, 6$, namely, those of the Bézier coefficients of ϕ_1, ϕ_2 (in Fig.5) and ϕ_4, ϕ_5 (in Fig.6), as well as the properties of $\phi_3(x,y) = \phi_2(y,x)$ and $\phi_6(x,y) = \phi_4(y,x)$, the mask $\{P^\#_{\mathbf{k}}\}$ is reduced to a five-parameter family, given by

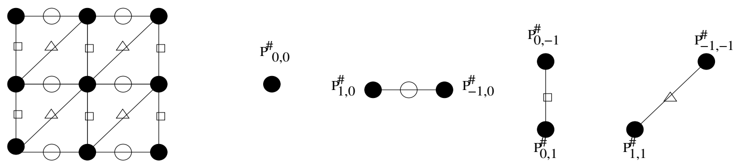

FIGURE 13. Coefficient stencils for C^2 local averaging rule

$$P^{\#}_{0,0} = \mathrm{diag}\,(1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4}),$$

$$P^{\#}_{1,0} = \begin{bmatrix} \frac{1}{2} & 6t_1 & -3t_1 & 0 & 0 & 0 \\ 2t_3+\frac{1}{8} & \frac{1}{4}+3t_1 & -\frac{3}{2}t_1 & 2t_2+4t_4 & -2t_4-2t_5 & 2t_5 \\ 0 & 0 & \frac{1}{4} & 0 & 2t_2-4t_5 & 4t_5-2t_2 \\ t_3 & \frac{1}{16}+\frac{1}{2}t_1 & -\frac{1}{4}t_1 & \frac{1}{8}+t_2+2t_4 & -t_4-t_5 & t_5 \\ 0 & 0 & \frac{1}{16} & 0 & \frac{1}{8}-2t_5+t_2 & 2t_5-t_2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \end{bmatrix},$$

$$P^{\#}_{1,1} = \begin{bmatrix} \frac{1}{2} & 3t_1 & 3t_1 & 0 & 0 & 0 \\ 2t_3+\frac{1}{8} & \frac{1}{4}+\frac{3}{2}t_1 & \frac{3}{2}t_1 & 2t_2-2t_5 & 2t_4 & 2t_5 \\ 2t_3+\frac{1}{8} & \frac{3}{2}t_1 & \frac{1}{4}+\frac{3}{2}t_1 & 2t_5 & 2t_4 & 2t_2-2t_5 \\ t_3 & \frac{1}{16}+\frac{1}{4}t_1 & \frac{1}{4}t_1 & \frac{1}{8}-t_5+t_2 & t_4 & t_5 \\ 2t_3 & \frac{1}{16}+\frac{1}{2}t_1 & \frac{1}{16}+\frac{1}{2}t_1 & t_2 & \frac{1}{8}+2t_4 & t_2 \\ t_3 & \frac{1}{4}t_1 & \frac{1}{16}+\frac{1}{4}t_1 & t_5 & t_4 & \frac{1}{8}-t_5+t_2 \end{bmatrix},$$

$$P^{\#}_{0,1} = \begin{bmatrix} \frac{1}{2} & -3t_1 & 6t_1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 4t_5-2t_2 & 2t_2-4t_5 & 0 \\ 2t_3+\frac{1}{8} & -\frac{3}{2}t_1 & \frac{1}{4}+3t_1 & 2t_5 & -2t_4-2t_5 & 2t_2+4t_4 \\ 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 2t_5-t_2 & \frac{1}{8}-2t_5+t_2 & 0 \\ t_3 & -\frac{1}{4}t_1 & \frac{1}{16}+\frac{1}{2}t_1 & t_5 & -t_4-t_5 & \frac{1}{8}+t_2+2t_4 \end{bmatrix},$$

$$P^{\#}_{-1,0} = \begin{bmatrix} \frac{1}{2} & -6t_1 & 3t_1 & 0 & 0 & 0 \\ -2t_3-\frac{1}{8} & \frac{1}{4}+3t_1 & -\frac{3}{2}t_1 & -2t_2-4t_4 & 2t_4+2t_5 & -2t_5 \\ 0 & 0 & \frac{1}{4} & 0 & 4t_5-2t_2 & 2t_2-4t_5 \\ t_3 & -\frac{1}{16}-\frac{1}{2}t_1 & \frac{1}{4}t_1 & \frac{1}{8}+t_2+2t_4 & -t_4-t_5 & t_5 \\ 0 & 0 & -\frac{1}{16} & 0 & \frac{1}{8}-2t_5+t_2 & 2t_5-t_2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \end{bmatrix},$$

$$P^{\#}_{-1,-1} = \begin{bmatrix} \frac{1}{2} & -3t_1 & -3t_1 & 0 & 0 & 0 \\ -2t_3-\frac{1}{8} & \frac{1}{4}+\frac{3}{2}t_1 & \frac{3}{2}t_1 & 2t_5-2t_2 & -2t_4 & -2t_5 \\ -2t_3-\frac{1}{8} & \frac{3}{2}t_1 & \frac{1}{4}+\frac{3}{2}t_1 & -2t_5 & -2t_4 & 2t_5-2t_2 \\ t_3 & -\frac{1}{16}-\frac{1}{4}t_1 & -\frac{1}{4}t_1 & \frac{1}{8}-t_5+t_2 & t_4 & t_5 \\ 2t_3 & -\frac{1}{16}-\frac{1}{2}t_1 & -\frac{1}{16}-\frac{1}{2}t_1 & t_2 & \frac{1}{8}+2t_4 & t_2 \\ t_3 & -\frac{1}{4}t_1 & -\frac{1}{16}-\frac{1}{4}t_1 & t_5 & t_4 & \frac{1}{8}-t_5+t_2 \end{bmatrix},$$

$$P^{\#}_{0,-1} = \begin{bmatrix} \frac{1}{2} & 3t_1 & -6t_1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 2t_2-4t_5 & 4t_5-2t_2 & 0 \\ -2t_3-\frac{1}{8} & -\frac{3}{2}t_1 & \frac{1}{4}+3t_1 & -2t_5 & 2t_4+2t_5 & -2t_2-4t_4 \\ 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & -\frac{1}{16} & 0 & 2t_5-t_2 & \frac{1}{8}-2t_5+t_2 & 0 \\ t_3 & \frac{1}{4}t_1 & -\frac{1}{16}-\frac{1}{2}t_1 & t_5 & -t_4-t_5 & \frac{1}{8}+t_2+2t_4 \end{bmatrix}.$$

The free parameters $t_1, \cdots, t_5$ can be adjusted to achieve certain desirable properties. For example, one may choose

$$t_1 = -0.199332,\ t_2 = -0.186915,\ t_3 = 0.022376,\ t_4 = -0.041126,\ t_5 = 0.087064,$$

to assure that the corresponding refinable function vector $\Phi^{\#}$ is in the Sobolev space $W^{2.9092}(\mathbb{R}^2)$. For fix-point computer implementation, one may choose

$$t_1 = -3/16,\ t_2 = -3/16,\ t_3 = 23/1024,\ t_4 = -21/512,\ t_5 = 11/128,$$

for which $\Phi^{\#}$ is in $W^{2.8588}(\mathbb{R}^2)$. These smoothness exponents can be calculated by following the formula in [9].

Acknowledgment. We are very grateful to the referee for making many valuable suggestions that significantly improve the presentation of the paper.

References

1. P. Alfeld, L. L. Schumaker, *Smooth macro-elements based on Powell-Sabin triangle splits*, Adv. Comput. Math. **16** (2002), 29–46.
2. C. K. Chui, *Multivariate Splines*, NSF-CBMS Series, vol. 54, SIAM Publ., Philadelphia, 1988.
3. C. K. Chui, *Vertex splines and their applications to interpolation of discrete data*, In *Computation of Curves and Surfaces*, 137–181, W. Dahmen, M. Gasca and C.A. Micchelli (eds.), Kluwer Academic, 1990.
4. C. K. Chui, H. C. Chui, T. X. He, *Shape-preserving interpolation by bivariate C^1 quadratic splines*, In *Workshop on Computational Geometry*, 21–75, A. Conte, V. Demichelis, F. Fontanella, and I. Galligani (eds.), World Sci. Publ. Co., Singapore, 1992.
5. C. K. Chui, Q. T. Jiang, *Surface subdivision schemes generated by refinable bivariate spline function vectors*, Appl. Comput. Harmon. Anal. **15** (2003), 147–162.
6. N. Dyn, D. Levin, J. A. Gregory, *A butterfly subdivision scheme for surface interpolation with tension control*, ACM Trans. Graphics **2** (1990), 160–169.
7. R. Q. Jia, *Shift-invariant spaces and linear operator equations*, Israel J. Math. **103** (1998), 259–288.
8. R. Q. Jia, Q. T. Jiang, *Approximation power of refinable vectors of functions*, In *Wavelet analysis and applications*, 155–178, Studies Adv. Math., vol. 25, Amer. Math. Soc., Providence, RI, 2002.
9. R. Q. Jia, Q. T. Jiang, *Spectral analysis of transition operators and its applications to smoothness analysis of wavelets*, SIAM J. Matrix Anal. Appl. **24** (2003), 1071–1109.
10. R. Q. Jia, C. A. Micchelli, *On linear independence of integer translates of a finite number of functions*, Proc. Edinburgh Math. Soc. **36** (1992), 69–85.
11. L. Kobbelt, *$\sqrt{3}$-subdivision*, In *Computer Graphics Proceedings*, Annual Conference Series, 2000, pp. 103–112.
12. U. Labsik, G. Greiner, *Interpolatory $\sqrt{3}$-subdivision*, *Proceedings of Eurographics 2000, Computer Graphics Forum*, vol. 19, 2000, pp. 131–138.
13. C. Loop, *Smooth subdivision surfaces based on triangles*, Master's thesis, University of Utah, Department of Mathematics, Salt Lake City, 1987.
14. G. Nürnberger, F. Zeilfelder, *Developments in bivariate spline interpolation*, J. Comput. Appl. Math. **121** (2000), 125–152.
15. M. J. D. Powell, M. A. Sabin, *Piecewise quadratic approximations on triangles*, ACM Trans. Math. Software **3** (1977), 316–325.
16. P. Sablonniére, *Error bounds for Hermite interpolation by quadratic splines on an α-triangulation*, IMA J. Numer. Anal. **7** (1987), 495–508.

Department of Mathematics & Computer Science, University of Missouri–St. Louis, St. Louis, MO 63121 and Department of Statistics, Stanford University, Stanford, CA 94305

E-mail address: `ckchui@stanford.edu`

Department of Mathematics & Computer Science, University of Missouri–St. Louis, St. Louis, MO 63121

E-mail address: `jiang@math.umsl.edu`