

Analysis of Adaptive Short-time Fourier Transform-based Synchrosqueezing Transform

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Abstract

Recently the study of modeling a non-stationary signal as a superposition of amplitude and frequency-modulated Fourier-like oscillatory modes has been a very active research area. The synchrosqueezing transform (SST) is a powerful method for instantaneous frequency estimation and component separation of non-stationary multicomponent signals. The short-time Fourier transform-based SST (FSST for short) reassigns the frequency variable to sharpen the time-frequency representation and to separate the components of a multicomponent non-stationary signal. Very recently the FSST with a time-varying parameter, called the adaptive FSST, was introduced. The simulation experiments show that the adaptive FSST is very promising in instantaneous frequency estimation of the component of a multicomponent signal, and in accurate component recovery. However the theoretical analysis of the adaptive FSST has not been carried out. In this paper, we study the theoretical analysis of the adaptive FSST and obtain the error bounds for the instantaneous frequency estimation and component recovery with the adaptive FSST and the 2nd-order adaptive FSST.

Keywords: Adaptive short-time Fourier transform; adaptive synchrosqueezing transform; instantaneous frequency estimation; multicomponent signal separation

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1 Introduction

To model a non-stationary oscillatory signal $x(t)$ as

$$x(t) = \sum_{k=1}^K x_k(t), \quad x_k(t) = A_k(t)e^{i2\pi\phi_k(t)}, \quad (1)$$

with $A_k(t), \phi'_k(t) > 0$ is important to extract information, such as the underlying dynamics, hidden in $x(t)$.

The empirical mode decomposition (EMD) [10] is a widely used method for the representation of (1). Recently the continuous wavelet transform (CWT)-based synchrosqueezing transform (WSST) developed in [8] provides a mathematically sound alternative to EMD. The short-time Fourier transform (STFT)-based SST (FSST) was proposed in [24] and further studied in [27, 19]. Both WSST and FSST are a special type of the reassignment method [1]. WSST reassigns the scale variable of the CWT to the frequency variable and FSST reassigns the frequency variable, both aiming to sharpen the time-frequency representation and to separate the components of a multicomponent non-stationary signal. SST was proved to be robust to noise and small perturbations [23, 11, 17]. However SST does not provide sharp representations for signals with significant changes of instantaneous frequency. In this regard, the 2nd-order FSST and the 2nd-order WSST were introduced in [20] and [18] respectively, and theoretical analysis of the 2nd-order FSST was carried out in [2]. The 2nd-order SST improves the concentration of the time-frequency representation. Other SST related methods include the generalized WSST [13], a hybrid EMT-SST computational scheme [7], the synchrosqueezed wave packet transform [28], WSST with vanishing moment wavelets [5], the multitapered WSST [9], the demodulation-transform based SST [25, 12, 26], higher-order FSST [21], signal separation operator [6] and empirical signal separation algorithm [16]. The statistical analysis of synchrosqueezing transforms has been studied in [29].

Most of the WSST or FSST algorithms available in the literature are based on a continuous (admissible) wavelet or a window function with a fixed window, which means high time resolution and frequency resolution cannot be obtained simultaneously. For broadband signals, a narrow window is suitable for the high-frequency parts while a wide window is suitable for the low-frequency parts. Recently the authors in [22] introduced a method to select the time-varying window width for sharp SST representation by minimizing the Rényi entropy. The window width of the signal separation operator algorithm in [6] is also time-varying. More recently the authors of [4, 3] study the adaptive FSST with the window function containing time and frequency parameters. Very recently the authors of [14, 15] proposed the adaptive WSST and adaptive FSST with a time-varying adaptive Gaussian window. They obtain the well-separated condition for multicomponent signals using linear frequency modulation signals to approximate a non-stationary signal during any local time, along with a new definition of bandwidth of Gaussian window. The experiments with synthetic and real data show that the adaptive FSST is very promising in instantaneous frequency estimation of the component of a multicomponent signal, and in accurate component recovery. However the theoretical analysis of the adaptive FSST has not been carried out. The goal of this paper is to study the theoretical analysis of such an FSST. We obtain the error bounds for the instantaneous frequency estimation and component recovery with the adaptive FSST and

the 2nd-order adaptive FSST.

The rest of this paper is organized as follows. In Section 2 we briefly review FSST, the 2nd-order FSST, the adaptive FSST and the 2nd-order adaptive FSST. In Section 3, we obtain the theoretical analysis of the (1st-order) adaptive FSST. We establish the error bounds for the IF estimation and component recovery. In Section 4, we consider the theoretical analysis of the 2nd-order adaptive FSST. The error bounds for the IF estimation and component recovery for the 2nd-order adaptive FSST are obtained. The proofs of two lemmas are presented in the appendix.

2 Short-time Fourier transform-based synchrosqueezing transform

In this section we briefly review the short-time Fourier transform (STFT)-based synchrosqueezing transform (FSST) and the adaptive FSST. The (modified) STFT of $x(t) \in L_2(\mathbb{R})$ with a window function $g(t) \in L_2(\mathbb{R})$ is defined by

$$V_x(t, \eta) := \int_{\mathbb{R}} x(\tau)g(\tau - t)e^{-i2\pi\eta(\tau-t)} d\tau,$$

where t and η are the time variable and the frequency variable respectively.

The original signal $x(t)$ can be recovered back from its STFT:

$$x(t) = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}} \int_{\mathbb{R}} V_x(\tau, \eta) \overline{g(t - \tau)} e^{i2\pi\eta(t-\tau)} d\tau d\eta.$$

If $g(0) \neq 0$, then one can show that $x(t)$ can also be recovered back from its STFT $V_x(t, \eta)$ with integrals involving only η :

$$x(t) = \frac{1}{g(0)} \int_{\mathbb{R}} V_x(t, \eta) d\eta. \quad (2)$$

In addition, if the window function $g(t) \in L_2(\mathbb{R})$ is real, then for a real-valued $x(t) \in L_2(\mathbb{R})$, we have

$$x(t) = \frac{2}{g(0)} \operatorname{Re} \left(\int_0^\infty V_x(t, \eta) d\eta \right). \quad (3)$$

Here we remark that if the window function $g(t)$ has certain smoothness and certain decaying order as $t \rightarrow \infty$, then STFT $V_x(t, \eta)$ of a slowly growing $x(t)$ with $g(t)$ is well defined. Furthermore, the above formulas still hold. In this following, unless otherwise stated, we always assume a window function $g(t)$ has certain smoothness and decaying properties, and a signal $x(t)$ is a slowly growing function. In addition, in this paper we assume $g(0) \neq 0$. For a signal $x(t)$, its Fourier transform $\widehat{x}(\xi)$ (maybe in the distribution sense) is defined by

$$\widehat{x}(\xi) := \int_{\mathbb{R}} x(t) e^{-i2\pi\xi t} dt.$$

2.1 STFT-based synchrosqueezing transform

The STFT-based synchrosqueezing transform (FSST) was first studied in [24]. For a signal $x(t)$, at (t, η) for which $V_x(t, \eta) \neq 0$, denote

$$\omega_x(t, \eta) := \operatorname{Re}\left(\frac{\partial_t V_x(t, \eta)}{2\pi i V_x(t, \eta)}\right).$$

The quantity $\omega_x(t, \eta)$ is called the ‘‘phase transformation’’ [8] or ‘‘instantaneous frequency information’’ in [24]. FSST is to reassign the frequency variable η by transforming STFT $V_x(t, \eta)$ of $x(t)$ to a quantity, denoted by $R_{x,\gamma}^\lambda(t, \xi)$, on the time-frequency plane:

$$R_{x,\gamma}^\lambda(t, \xi) := \int_{|V_x(t,\eta)| > \gamma} V_x(t, \eta) \frac{1}{\lambda} h\left(\frac{\xi - \omega_x(t, \eta)}{\lambda}\right) d\eta. \quad (4)$$

where throughout this paper $h(t)$ is a compactly supported function with certain smoothness and $\int_{\mathbb{R}} h(t) dt = 1$. Throughout this paper $\int_{|V_x(t,\eta)| > \gamma}$ means the integral $\int_{\{\eta: |V_x(t,\eta)| > \gamma\}}$ with η over the set $\{\eta: |V_x(t, \eta)| > \gamma\}$.

We consider multicomponent signals $x(t)$ given by (1) with $A_k(t), \phi_k(t)$ satisfying

$$A_k(t) \in C^1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \phi_k(t) \in C^2(\mathbb{R}), \quad (5)$$

$$A_k(t) > 0, \inf_{t \in \mathbb{R}} \phi'_k(t) > 0, \sup_{t \in \mathbb{R}} \phi'_k(t) < \infty. \quad (6)$$

Let $\varepsilon > 0$ and $\Delta > 0$, and let $\mathcal{B}_{\varepsilon, \Delta}$ denote the set of multicomponent signals of (1) satisfying (5), (6), and the following conditions:

$$|A'_k(t)| \leq \varepsilon \phi'_k(t), |\phi''_k(t)| \leq \varepsilon \phi'_k(t), t \in \mathbb{R}; M''_k := \sup_{t \in \mathbb{R}} |\phi''_k(t)| < \infty, \quad (7)$$

$$\phi'_k(t) - \phi'_{k-1}(t) \geq 2\Delta, 2 \leq k \leq K, t \in \mathbb{R}. \quad (8)$$

The condition (8) is called the well-separated condition with resolution Δ . For the well-separated condition, [24] uses a stronger condition than that in (8):

$$\inf_{t \in \mathbb{R}} \phi'_k(t) - \sup_{t \in \mathbb{R}} \phi'_{k-1}(t) \geq 2\Delta, 2 \leq k \leq K. \quad (9)$$

The condition (7), which was considered in [27], means that $A_k(t)$ and IF $\phi'_k(t)$ change slowly compared with $\phi_k(t)$. [19] uses another condition for the change of $A_k(t)$ and IF $\phi'_k(t)$:

$$|A'_k(t)| \leq \varepsilon, |\phi''_k(t)| \leq \varepsilon, t \in \mathbb{R}. \quad (10)$$

We let $B_{\varepsilon, \Delta}$ denote the set of multicomponent signals of (1) satisfying (5), (6), (10) and well-separated condition (8).

Let

$$\mathcal{Z}_k := \{(t, \eta) : |\eta - \phi'_k(t)| < \Delta, t \in \mathbb{R}\}. \quad (11)$$

Then the well-separated condition (8) implies that $\mathcal{Z}_k, 1 \leq k \leq K$ are not overlapping.

Denote

$$\mu(t) := \min_{1 \leq k \leq K} A_k(t), \quad M(t) := \sum_{k=1}^K A_k(t) \quad (12)$$

and

$$\Gamma_0(t) := KI_1 + \pi I_2 M(t), \quad \tilde{\Gamma}_0(t) := K\tilde{I}_1 + \pi\tilde{I}_2 M(t), \quad (13)$$

where

$$I_n := \int_{\mathbb{R}} |\tau^n g(\tau)| d\tau, \quad \tilde{I}_n := \int_{\mathbb{R}} |\tau^n g'(\tau)| d\tau, \quad n = 1, 2, \dots \quad (14)$$

Theorem A. *Let $x(t) \in B_{\varepsilon, \Delta}$ and g be a window function. Let $\mu(t), \Gamma_0(t), \tilde{\Gamma}_0(t)$ be defined by (12) and (13). Suppose $2\varepsilon\Gamma_0(t) \leq \mu(t)$ and $\tilde{\varepsilon}$ satisfies*

$$\varepsilon\Gamma_0(t) \leq \tilde{\varepsilon} \leq \mu(t) - \varepsilon\Gamma_0(t). \quad (15)$$

Then we have the following statements.

(a) *The set $\{\eta : |V_x(t, \eta)| > \tilde{\varepsilon}\}$ can be represented as the union of disjoint non-empty sets $\{\eta : |V_x(t, \eta)| > \tilde{\varepsilon}\} \cap \{(t, \eta) \in \mathcal{Z}_k\}, 1 \leq k \leq K$.*

(b) *Suppose (t, η) satisfies $|V_x(t, \eta)| > \tilde{\varepsilon}$ and $(t, \eta) \in \mathcal{Z}_k$. Then*

$$|\omega_x(t, \eta) - \phi'_k(t)| < \frac{\varepsilon}{\tilde{\varepsilon}} (\Gamma_0(t)\Delta + \frac{1}{2\pi}\tilde{\Gamma}_0(t)). \quad (16)$$

(c) *Suppose that $\tilde{\varepsilon}$ satisfies $(\Gamma_0(t)\Delta + \frac{1}{2\pi}\tilde{\Gamma}_0(t))\varepsilon/\tilde{\varepsilon} \leq \Delta$. Then, for any $k \in \{1, \dots, K\}$ and any $\tilde{\varepsilon}_3$ satisfying $(\Gamma_0(t)\Delta + \frac{1}{2\pi}\tilde{\Gamma}_0(t))\varepsilon/\tilde{\varepsilon} \leq \tilde{\varepsilon}_3 \leq \Delta$, we have*

$$\left| \lim_{\lambda \rightarrow 0} \frac{1}{g(0)} \int_{|\xi - \phi'_k(t)| < \tilde{\varepsilon}_3} R_{x, \tilde{\varepsilon}}^\lambda(t, \xi) d\xi - x_k(t) \right| \leq \frac{2\Delta(\varepsilon\Gamma_0(t) + \tilde{\varepsilon})}{|g(0)|}. \quad (17)$$

(d) *If $x(t) \in \mathcal{B}_{\varepsilon, \Delta}$, then the above statements (a)-(c) hold with $\Gamma_0(t)$ and $\tilde{\Gamma}_0(t)$ in (13) replaced by*

$$\Gamma_0(t) := \sum_{k=1}^K \left\{ \phi'_k(t)I_1 + \frac{1}{2}M''_k I_2 + \pi A_k(t)(\phi'_k(t)I_2 + \frac{1}{3}M''_k I_3) \right\}, \quad (18)$$

$$\tilde{\Gamma}_0(t) := \sum_{k=1}^K \left\{ \phi'_k(t)\tilde{I}_1 + \frac{1}{2}M''_k \tilde{I}_2 + \pi A_k(t)(\phi'_k(t)\tilde{I}_2 + \frac{1}{3}M''_k \tilde{I}_3) \right\}. \quad (19)$$

Here we remark that $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_3$ in Theorem A could be a function of t .

If we choose $\tilde{\varepsilon} = \varepsilon^{1/3}$ and if ε is small enough such that

$$\tilde{\varepsilon} \leq \min \left\{ \Delta, \frac{1}{2}\|\mu(t)\|_\infty, \left\| \frac{1}{\Gamma_0(t)} \right\|_\infty^{1/2}, \left\| \frac{1}{\Gamma_0(t)\Delta + \frac{1}{2\pi}\tilde{\Gamma}_0(t)} \right\|_\infty \right\}, \quad (20)$$

then (15) holds. In addition, $\tilde{\varepsilon}(\Gamma_0(t)\Delta + \frac{1}{2\pi}\tilde{\Gamma}_0(t)) \leq 1$. Hence,

$$(\Gamma_0(t)\Delta + \frac{1}{2\pi}\tilde{\Gamma}_0(t))\varepsilon/\tilde{\varepsilon} \leq \tilde{\varepsilon} \leq \Delta.$$

Thus, the conditions in Theorem A are satisfied, and Theorem A (with $\tilde{\varepsilon}_3 = \tilde{\varepsilon}$) can be stated in the following theorem.

Theorem B. [27, 19] *Let $x(t) \in \mathcal{B}_{\varepsilon, \Delta}$ or $B_{\varepsilon, \Delta}$, and $\tilde{\varepsilon} = \varepsilon^{1/3}$. Let g be a window function with $\text{supp}(\hat{g}) \subseteq [-\Delta, \Delta]$. If ε is small enough, then the following statements hold.*

(a) *For (t, η) satisfying $|V_x(t, \eta)| > \tilde{\varepsilon}$, there exists a unique $k \in \{1, 2, \dots, K\}$ such that $(t, \eta) \in \mathcal{Z}_k$.*

(b) *Suppose (t, η) satisfies $|V_x(t, \eta)| > \tilde{\varepsilon}$ and $(t, \eta) \in \mathcal{Z}_k$. Then*

$$|\omega_x(t, \eta) - \phi'_k(t)| < \tilde{\varepsilon}.$$

(c) *For any $k \in \{1, \dots, K\}$,*

$$\left| \lim_{\lambda \rightarrow 0} \frac{1}{g(0)} \int_{|\xi - \phi'_k(t)| < \tilde{\varepsilon}} R_{x, \tilde{\varepsilon}}^\lambda(t, \xi) d\xi - x_k(t) \right| \leq \frac{4\Delta}{|g(0)|} \tilde{\varepsilon}.$$

The meaning of “ ε is small enough” in Theorem B is that $\tilde{\varepsilon}$ defined by $\tilde{\varepsilon} = \varepsilon^{1/3}$ satisfies some inequalities like (20). Most theorems on the WSST and FSST analysis are stated in the form of Theorem B, see e.g. [8, 24, 27, 19, 2]. Actually the statements of part (b) and part (c) in Theorem A give us more direct bounds of the estimates. We call the quantity on the left-hand side (LHS) of (16) the IF estimate error, and call that on LHS of (17) the error of component recovery (or component separation). The statements in Theorem A can be found in [27, 19, 2] but with some different IF estimate errors. For example, [19, 2] gave IF estimate error $\frac{\varepsilon}{\tilde{\varepsilon}}(\Gamma_0(t)(\Delta + 2\phi'_k(t)) + \frac{1}{2\pi}\tilde{\Gamma}_0(t))$ instead of $\frac{\varepsilon}{\tilde{\varepsilon}}(\Gamma_0(t)\Delta + \frac{1}{2\pi}\tilde{\Gamma}_0(t))$ in (16). One can also find that Theorem A is a special case of Theorem 1 in Section 3 below (see Remark 3).

Observe that the condition (7) or (10) requires the slow change of the IF $\phi'_k(t)$ of each component $x_k(t)$. There is no mathematical guarantee for the IF estimate and the component separation for a multicomponent signal $x(t)$ with a component $x_k(t)$ having a fast-changing frequency (e.g. $\phi''_k(t)$, the changing rate of IF of $x_k(t)$, is not very small). To this regard, the 2nd-order FSST was introduced in [20] and later the 2nd-order WSST was proposed in [18] with the theoretical analysis of the 2nd-order FSST established in [2].

Suppose $V_x(t, \eta) \neq 0$ and $\partial_t \left(\frac{\partial_\eta V_x(t, \eta)}{V_x(t, \eta)} \right) \neq i2\pi$. Denote

$$\tilde{q}(t, \eta) := \frac{\partial_t \left(\frac{\partial_\eta V_x(t, \eta)}{V_x(t, \eta)} \right)}{\partial_t \left(i2\pi t - \frac{\partial_\eta V_x(t, \eta)}{V_x(t, \eta)} \right)}.$$

The 2nd-order FSST in [2] is defined as

$$R_{x,\gamma}^{2nd,\lambda}(t, \xi) := \int_{\{|V_x(t,\eta)| > \gamma\}} V_x(t, \eta) \frac{1}{\lambda} h\left(\frac{\xi - \omega_x^{2nd}(t, \eta)}{\lambda}\right) d\eta,$$

where $\omega_x^{2nd}(t, \eta)$ is the phase transformation for the 2nd-order FSST: for (t, η) with $V_x(t, \eta) \neq 0$,

$$\omega_x^{2nd}(t, \eta) := \begin{cases} \operatorname{Re}\left\{\frac{\partial_t V_x(t, \eta)}{2\pi i V_x(t, \eta)}\right\} + \operatorname{Re}\left\{\tilde{q}(t, \eta) \frac{\partial_\eta V_x(t, \eta)}{i2\pi V_x(t, \eta)}\right\}, & \text{if } \partial_t\left(\frac{\partial_\eta V_x(t, \eta)}{V_x(t, \eta)}\right) \neq i2\pi, \\ \operatorname{Re}\left\{\frac{\partial_t V_x(t, \eta)}{2\pi i V_x(t, \eta)}\right\}, & \text{elsewhere.} \end{cases}$$

Let $\varepsilon > 0$ and $\Delta > 0$. $B_{\varepsilon, \Delta}^{(2)}$ denote the set of multicomponent signals of (1) satisfying (6), the well-separated condition (8), and the following conditions:

$$A_k(t) \in C^2(\mathbb{R}) \cap L_\infty(\mathbb{R}), \phi_k(t) \in C^3(\mathbb{R}), \phi_k''(t) \in L_\infty(\mathbb{R}), \quad (21)$$

$$|A_k'(t)| \leq \varepsilon, |A_k''(t)| \leq \varepsilon, |\phi_k^{(3)}(t)| \leq \varepsilon, t \in \mathbb{R}. \quad (22)$$

Then for $x(t) \in B_{\varepsilon, \Delta}^{(2)}$, statements for the 2nd-order FSST similar to those in Theorem B hold, under some conditions which are more complicated than (20). See [2] for the details. Observe that there is no direct boundedness restriction on $\phi_k''(t)$ in the definition of $B_{\varepsilon, \Delta}^{(2)}$.

2.2 Adaptive FSST with a time-varying parameter

We consider the window function given by

$$g_\sigma(t) := \frac{1}{\sigma} g\left(\frac{t}{\sigma}\right), \quad (23)$$

where $\sigma > 0$ is a parameter, $g(t)$ is a function with $g(0) \neq 0$ and having certain smoothness and decaying order as $t \rightarrow \infty$. If

$$g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, \quad (24)$$

then $g_\sigma(t)$ is the Gaussian window function. The parameter σ is also called the window width in the time-domain of the window function $g_\sigma(t)$ since the time duration Δ_{g_σ} of g_σ is σ (up to a constant): $\Delta_{g_\sigma} = \sigma \Delta_g$, where Δ_g is the time duration of g , which is defined as

$$\Delta_g := \left(\frac{\int_{\mathbb{R}} t^2 |g(t)|^2 dt}{\int_{\mathbb{R}} |g(t)|^2 dt} \right)^{1/2}.$$

For a signal $x(t)$, the STFT of $x(t)$ with a time-varying parameter is defined in [14] as

$$\tilde{V}_x(t, \eta) = \int_{\mathbb{R}} x(\tau) g_{\sigma(t)}(\tau - t) e^{-i2\pi\eta(\tau - t)} d\tau = \int_{\mathbb{R}} x(t + \tau) \frac{1}{\sigma(t)} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau, \quad (25)$$

where $\sigma = \sigma(t) > 0$ is a differentiable function of t . $\tilde{V}_x(t, \eta)$ is called the adaptive STFT of $x(t)$ with g_σ .

Before we move on to review the SST associated with the adaptive STFT, we introduce some notations used in this and next sections. Denote

$$g_1(\tau) := \tau g(\tau), \quad g_2(\tau) := \tau^2 g(\tau), \quad g_3(\tau) := \tau g'(\tau).$$

Thus

$$g_{1,\sigma}(\tau) = \frac{\tau}{\sigma^2} g\left(\frac{\tau}{\sigma}\right), \quad g_{2,\sigma}(\tau) = \frac{\tau^2}{\sigma^3} g\left(\frac{\tau}{\sigma}\right), \quad g_{3,\sigma}(\tau) = \frac{\tau}{\sigma^2} g'\left(\frac{\tau}{\sigma}\right).$$

We use $\tilde{V}_x^{g_j}(t, \eta)$ and $\tilde{V}_x^{g'_j}(t, \eta)$ to denote the adaptive STFT defined by (25) with g_σ replaced by $g_{j,\sigma}$ and $g'_\sigma(\tau) = \frac{1}{\sigma} g'\left(\frac{\tau}{\sigma}\right)$ respectively, where $1 \leq j \leq 3$.

For $x(t) = Ae^{i2\pi\xi_0 t}$, one can show that (see [14]) if $\tilde{V}_x(t, \eta) \neq 0$, then $\omega_x^{\text{adp},c}(t, \eta)$ defined by

$$\omega_x^{\text{adp},c}(t, \eta) := \frac{\partial_t \tilde{V}_x(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} + \frac{\sigma'(t)}{i2\pi\sigma(t)} + \frac{\sigma'(t)}{\sigma(t)} \frac{\tilde{V}_x^{g_3}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)}, \quad (26)$$

is ξ_0 , the IF of $x(t)$. Hence, for a general $x(t)$, at (t, η) for which $\tilde{V}_x(t, \eta) \neq 0$, [14] defines the real part of the quantity $\omega_x^{\text{adp},c}(t, \eta)$ in the above equation, denoted by $\omega_x^{\text{adp}}(t, \eta)$, as the phase transformation of the adaptive FSST:

$$\omega_x^{\text{adp}}(t, \eta) := \text{Re} \left\{ \frac{\partial_t \tilde{V}_x(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} \right\} + \frac{\sigma'(t)}{\sigma(t)} \text{Re} \left\{ \frac{\tilde{V}_x^{g_3}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} \right\}, \quad \text{for } \tilde{V}_x(t, \eta) \neq 0.$$

Then the (1st-order) adaptive FSST, denoted by $R_{x,\gamma}^{\text{adp},\lambda}$, is defined by

$$R_{x,\gamma}^{\text{adp},\lambda}(t, \xi) := \int_{|\tilde{V}_x(t,\eta)| > \gamma} \tilde{V}_x(t, \eta) \frac{1}{\lambda} h\left(\frac{\xi - \omega_x^{\text{adp}}(t, \eta)}{\lambda}\right) d\eta, \quad (27)$$

where $\gamma > 0, \lambda > 0$ and $h(t)$ is a compactly supported function as described in §2.1. Note that here and below, the letter c in $\omega_x^{\text{adp},c}(t, \eta)$ denotes the complex-valued version of the phase transformation.

Next we consider the 2nd-order adaptive FSST. For a linear chirp signal,

$$x(t) = Ae^{i2\pi\phi(t)} = Ae^{i2\pi(ct + \frac{1}{2}rt^2)}, \quad (28)$$

it was shown in [14] that $\omega_x^{\text{adp},2\text{nd},c}$ defined below is $c + rt$, the IF of $x(t)$:

$$\omega_x^{\text{adp},2\text{nd},c} := \frac{\sigma'(t)}{i2\pi\sigma(t)} + \frac{\partial_t \tilde{V}_x(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} - \frac{\tilde{V}_x^{g_1}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} P_0(t, \eta) + \frac{\sigma'(t)}{\sigma(t)} \frac{\tilde{V}_x^{g_3}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)}, \quad (29)$$

for (t, η) satisfying $\partial_\eta \left(\frac{\tilde{V}_x^{g_1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \neq 0$ and $\tilde{V}_x(t, \eta) \neq 0$, where

$$P_0(t, \eta) := \frac{1}{\partial_\eta \left(\frac{\tilde{V}_x^{g_1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right)} \left\{ \partial_\eta \left(\frac{\partial_t \tilde{V}_x(t, \eta)}{\tilde{V}_x(t, \eta)} \right) + \frac{\sigma'(t)}{\sigma(t)} \partial_\eta \left(\frac{\tilde{V}_x^{g_3}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \right\}. \quad (30)$$

Thus the authors of [14] define the real part of $\omega_x^{\text{adp},2\text{nd},\text{c}}$ as the phase transformation for the 2nd-order adaptive FSST. Namely, the phase transformation $\omega_x^{\text{adp},2\text{nd}}$ is defined by

$$\omega_x^{\text{adp},2\text{nd}}(t, \eta) := \begin{cases} \text{Re}\left\{\frac{\partial_t \tilde{V}_x(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)}\right\} - \text{Re}\left\{\frac{\tilde{V}_x^{g1}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} P_0(t, \eta)\right\} + \frac{\sigma'(t)}{\sigma(t)} \text{Re}\left\{\frac{\tilde{V}_x^{g3}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)}\right\}, \\ \quad \text{if } \frac{\partial}{\partial \eta} \left(\frac{\tilde{V}_x^{g1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \neq 0 \text{ and } \tilde{V}_x(t, \eta) \neq 0; \\ \text{Re}\left\{\frac{\partial_t \tilde{V}_x(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)}\right\} + \frac{\sigma'(t)}{\sigma(t)} \text{Re}\left\{\frac{\tilde{V}_x^{g3}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)}\right\}, \text{ if } \frac{\partial}{\partial \eta} \left(\frac{\tilde{V}_x^{g1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) = 0, \tilde{V}_x(t, \eta) \neq 0, \end{cases}$$

Finally we define the 2nd-order adaptive FSST with $V_x(t, \eta) \neq 0$ and $\frac{\partial}{\partial \eta} \left(\frac{\tilde{V}_x^{g1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \neq 0$ described by thresholds $\gamma_1 > 0, \gamma_2 > 0$. More precisely, define

$$\omega_{x, \gamma_1, \gamma_2}^{\text{adp},2\text{nd},\text{c}}(t, \eta) := \begin{cases} \text{quantity in (29),} & \text{if } |V_x(t, \eta)| > \gamma_1 \text{ and } \left| \frac{\partial}{\partial \eta} \left(\frac{\tilde{V}_x^{g1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \right| > \gamma_2, \\ \text{quantity in (26),} & \text{if } |V_x(t, \eta)| > \gamma_1 \text{ and } \left| \frac{\partial}{\partial \eta} \left(\frac{\tilde{V}_x^{g1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \right| \leq \gamma_2. \end{cases}$$

Let $\omega_{x, \gamma_1, \gamma_2}^{\text{adp},2\text{nd}}(t, \eta) := \text{Re}(\omega_{x, \gamma_1, \gamma_2}^{\text{adp},2\text{nd},\text{c}}(t, \eta))$.

Again, let $h(t)$ be a compactly supported function with certain smoothness and $\int_{\mathbb{R}} h(t) dt = 1$.

We define the 2nd-order adaptive FSST $R_{x, \gamma_1, \gamma_2}^{\text{adp},2\text{nd},\lambda}$ by

$$R_{x, \gamma_1, \gamma_2}^{\text{adp},2\text{nd},\lambda}(t, \xi) := \int_{\left\{ \eta: |\tilde{V}_x(t, \eta)| > \gamma_1, \left| \frac{\partial}{\partial \eta} \left(\frac{\tilde{V}_x^{g1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \right| > \gamma_2 \right\}} \tilde{V}_x(t, \eta) \frac{1}{\lambda} h\left(\frac{\xi - \omega_{x, \gamma_1, \gamma_2}^{\text{adp},2\text{nd}}(t, \eta)}{\lambda}\right) d\eta. \quad (31)$$

3 Analysis of adaptive FSST

We assume

$$d' := \min_{k \in \{2, \dots, K\}} \min_{t \in \mathbb{R}} (\phi'_k(t) - \phi'_{k-1}(t)) > 0. \quad (32)$$

Thus $x(t)$ satisfies the well-separated condition (8) with resolution $\Delta = d'/2$. However, the value d' may be very small. In this case, we cannot apply Theorem A directly. The reason is that to guarantee the results in Theorem A to hold, the window function g needs to satisfy $\text{supp}(\hat{g}) \subseteq [-\frac{d'}{2}, \frac{d'}{2}]$. If d' is quite small, then g has a very good frequency resolution, which implies by the uncertainty principle that g has a very poor time resolution, or equivalently g has a very large time duration, which results in large errors in the IF estimate and component recovery (see Remark 4). We use the adaptive STFT and FSST to adjust the time-varying window width $\sigma(t)$ at certain local time t where the frequencies of two components are close.

In this section we consider the case that each component $x_k(t) = A_k(t)e^{i2\pi\phi_k(t)}$ is approximated locally by a sinusoidal signal. More precisely, we assume $A'_k(t)$ and $\phi''_k(t)$ are small:

$$|A'_k(t)| \leq \varepsilon_1, |\phi''_k(t)| \leq \varepsilon_2, t \in \mathbb{R}, 1 \leq k \leq K, \quad (33)$$

for some positive number $\varepsilon_1, \varepsilon_2$. Let $\mathcal{D}_{\varepsilon_1, \varepsilon_2}$ denote the set of multicomponent signals of (1) satisfying (5), (6), (32) and (33).

Let $x(t) \in \mathcal{D}_{\varepsilon_1, \varepsilon_2}$. Write $x_k(t + \tau)$ as

$$\begin{aligned} x_k(t + \tau) &= x_k(t)e^{i2\pi\phi'_k(t)\tau} + (A_k(t + \tau) - A_k(t))e^{i2\pi\phi_k(t+\tau)} \\ &\quad + x_k(t)e^{i2\pi\phi'_k(t)\tau} (e^{i2\pi(\phi_k(t+\tau) - \phi_k(t) - \phi'_k(t)\tau)} - 1). \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{V}_x(t, \eta) &= \sum_{k=1}^K \int_{\mathbb{R}} x_k(t + \tau) \frac{1}{\sigma(t)} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau \\ &= \sum_{k=1}^K \int_{\mathbb{R}} x_k(t) e^{i2\pi\phi'_k(t)\tau} \frac{1}{\sigma(t)} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau + \text{rem}_0, \end{aligned}$$

or

$$\tilde{V}_x(t, \eta) = \sum_{k=1}^K x_k(t) \hat{g}(\sigma(t)(\eta - \phi'_k(t))) + \text{rem}_0, \quad (34)$$

where rem_0 is the remainder for the expansion of $\tilde{V}_x(t, \eta)$ in (34) given by

$$\begin{aligned} \text{rem}_0 &:= \sum_{k=1}^K \int_{\mathbb{R}} \left\{ (A_k(t + \tau) - A_k(t)) e^{i2\pi\phi_k(t+\tau)} \right. \\ &\quad \left. + x_k(t) e^{i2\pi\phi'_k(t)\tau} (e^{i2\pi(\phi_k(t+\tau) - \phi_k(t) - \phi'_k(t)\tau)} - 1) \right\} \frac{1}{\sigma(t)} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau. \end{aligned} \quad (35)$$

With $|A_k(t + \tau) - A_k(t)| \leq \varepsilon_1 |\tau|$ and

$$|e^{i2\pi(\phi_k(t+\tau) - \phi_k(t) - \phi'_k(t)\tau)} - 1| \leq 2\pi |\phi_k(t + \tau) - \phi_k(t) - \phi'_k(t)\tau| \leq \pi\varepsilon_2 |\tau|^2,$$

we have

$$\begin{aligned} |\text{rem}_0| &\leq \sum_{k=1}^K \int_{\mathbb{R}} \varepsilon_1 |\tau| \frac{1}{\sigma(t)} |g\left(\frac{\tau}{\sigma(t)}\right)| d\tau + M(t) \int_{\mathbb{R}} \pi\varepsilon_2 |\tau|^2 \frac{1}{\sigma(t)} |g\left(\frac{\tau}{\sigma(t)}\right)| d\tau \\ &= K\varepsilon_1 I_1 \sigma(t) + \pi\varepsilon_2 I_2 \sigma^2(t) M(t), \end{aligned}$$

where I_n and $M(t)$ are defined by (14) and (12) respectively. Hence we have

$$|\text{rem}_0| \leq \sigma(t) \Lambda_0(t), \quad (36)$$

where

$$\Lambda_0(t) := K\varepsilon_1 I_1 + \pi\varepsilon_2 I_2 \sigma(t) M(t). \quad (37)$$

$\tilde{V}_x^{g'}(t, \eta)$ can be expanded as (34) with remainder rem'_0 , defined as rem_0 in (35) with $g(\tau)$ replaced by $g'(\tau)$. Then we have the estimate for the remainder similar to (36). More precisely, we have

$$|\text{rem}'_0| \leq \sigma(t) \tilde{\Lambda}_0(t),$$

where

$$\tilde{\Lambda}_0(t) := K\varepsilon_1\tilde{I}_1 + \pi\varepsilon_2\tilde{I}_2\sigma(t)M(t), \quad (38)$$

with \tilde{I}_n defined in (14).

Remark 1. Condition (33) is essentially the condition (10). If $A_k(t), \phi_k(t)$ satisfy (7), then we have a similar error bound for the expansion of $\tilde{V}_x(t, \eta)$. More precisely, suppose $A_k(t), \phi_k(t)$ satisfy

$$|A'_k(t)| \leq \varepsilon_1\phi'_k(t), \quad |\phi''_k(t)| \leq \varepsilon_2\phi'_k(t), \quad t \in \mathbb{R}; \quad M''_k := \sup_{t \in \mathbb{R}} |\phi''_k(t)| < \infty. \quad (39)$$

Then (see [8])

$$\begin{aligned} |A_k(t + \tau) - A_k(t)| &\leq \varepsilon_1|\tau|(\phi'_k(t) + \frac{1}{2}M''_k|\tau|), \\ |\phi_k(t + \tau) - \phi_k(t) - \phi'_k(t)\tau| &\leq \varepsilon_2\tau^2(\frac{1}{2}\phi'_k(t) + \frac{1}{6}M''_k|\tau|). \end{aligned}$$

Thus, we can expand $\tilde{V}_x(t, \eta)$ as (34) with $|\text{rem}_0| \leq \sigma(t)\Lambda_0(t)$, where in this case $\Lambda_0(t)$ is

$$\Lambda_0(t) := \varepsilon_1 \sum_{k=1}^K (I_1\phi'_k(t) + \frac{1}{2}M''_k I_2\sigma(t)) + \pi\varepsilon_2\sigma(t) \sum_{k=1}^K A_k(t)(I_2\phi'_k(t) + \frac{1}{3}M''_k I_3\sigma(t)). \quad (40)$$

With the condition of (39), we have an estimate $\sigma(t)\tilde{\Lambda}_0(t)$ for rem'_0 with $\tilde{\Lambda}_0(t)$ defined by (40) with I_j replaced by \tilde{I}_j . In this paper we consider the condition (33). The statements for theoretical analysis of the adaptive FSST with condition (39) instead of (33) are still valid as long as $\Lambda_0(t)$ in (37), $\Lambda'_0(t)$ in (38) and so on are replaced respectively by that in (40) and similar terms. This also applies to the 2nd-order adaptive FSST in Section 4, where we will not repeat again this discussion on the condition like (39). \blacksquare

If the remainder rem_0 in (34) is small, then the term $x_k(t)\hat{g}(\sigma(t)(\eta - \phi'_k(t)))$ in (34) governs the time-frequency zone of the STFT \tilde{V}_{x_k} of the k th component $x_k(t)$ of $x(t)$. If in addition, g is band-limited, that is \hat{g} is compactly supported, to say $\text{supp}(\hat{g}) \subset [-\Delta, \Delta]$, then $x_k(t)\hat{g}(\sigma(t)(\eta - \phi'_k(t)))$ lies within the zone:

$$\{(t, \eta) : |\eta - \phi'_k(t)| < \frac{\Delta}{\sigma(t)}, t \in \mathbb{R}\}.$$

Thus the multicomponent signal $x(t)$ is well-separated (that is $Z_k \cap Z_\ell = \emptyset, k \neq \ell$), provided that $\sigma(t)$ satisfies

$$\sigma(t) \geq \frac{2\Delta}{\phi'_k(t) - \phi'_{k-1}(t)}, \quad t \in \mathbb{R}, k = 2, \dots, K. \quad (41)$$

Observe that our well-separated condition (41) is different from that in (8) considered in [27] and [19].

If \widehat{g} is not compactly supported, we need to define the *essential support* of \widehat{g} outside which $\widehat{g}(\xi) \approx 0$. More precisely, for a given threshold $0 < \tau_0 < 1$, if $|\widehat{g}(\xi)| \leq \tau_0$ for $|\xi| \geq \alpha$, then we say $\widehat{g}(\xi)$ is *essentially supported* in $[-\alpha, \alpha]$. When $|\widehat{g}(\xi)|$ is even and decreasing for $\xi \geq 0$, then α can be obtained by solving

$$|\widehat{g}(\alpha)| = \tau_0. \quad (42)$$

For example, when g is the Gaussian function given by (24), then, with $\widehat{g}(\xi) = e^{-2\pi^2\xi^2}$, the corresponding α is given by

$$\alpha = \frac{1}{2\pi} \sqrt{2 \ln\left(\frac{1}{\tau_0}\right)}. \quad (43)$$

For g with $\widehat{g}(\xi)$ *essentially supported* in $[-\alpha, \alpha]$, we then define the time-frequency zone Z_k of the k th-component $x_k(t)$ of $x(t)$ by

$$Z_k := \{(t, \eta) : |\widehat{g}(\sigma(t)(\eta - \phi'_k(t)))| > \tau_0, t \in \mathbb{R}\} = \{(t, \eta) : |\eta - \phi'_k(t)| < \frac{\alpha}{\sigma(t)}, t \in \mathbb{R}\}. \quad (44)$$

Thus the multicomponent signal $x(t)$ is well-separated, if $\sigma(t)$ satisfies

$$\sigma(t) \geq \frac{2\alpha}{\phi'_k(t) - \phi'_{k-1}(t)}, \quad t \in \mathbb{R}, k = 2, \dots, K. \quad (45)$$

In this case $Z_k \cap Z_\ell = \emptyset, k \neq \ell$. In this section we assume that (45) holds for some $\sigma(t)$. Due to (32), there always exists $C^1(\mathbb{R})$ and bounded $\sigma(t)$ such that (45) holds. For the sinusoidal function-based adaptive FSST, the authors in [14] suggest to choose $\sigma(t)$ as

$$\sigma_1(t) := \max\left\{\frac{2\alpha}{\phi'_k(t) - \phi'_{k-1}(t)}, k = 2, \dots, K\right\}.$$

In practice, $\phi'_k(t)$ is in general unknown and one needs to develop an algorithm to estimate it or to provide an approximation to $\sigma_1(t)$.

Observe that for $\sigma(t)$ satisfying (45), since $\phi'_k(t)$ is bounded, we know

$$\left\|\frac{1}{\sigma(t)}\right\|_\infty < \infty.$$

In addition, in this case

$$\sigma(t)|\phi'_k(t) - \phi'_\ell(t)| \geq 2\alpha|k - \ell|. \quad (46)$$

Next we will present our analysis results on the adaptive FSST in Theorem 1 below, where α is defined by (42), and $\sum_{\ell \neq k}$ denotes $\sum_{\ell \in \{1, \dots, K\} \setminus \{k\}}$. $\widetilde{V}_x(t, \eta)$ is the adaptive STFT of $x(t)$ with such a window function g that $|\widehat{g}(\xi)|$ is even and decreasing for $\xi \geq 0$.

Theorem 1. *Let $x(t) \in \mathcal{D}_{\varepsilon_1, \varepsilon_2}$ for some small $\varepsilon_1, \varepsilon_2 > 0$ and g be a window function. Let $\mu(t), M(t), \Lambda_0(t), \widetilde{\Lambda}_0(t)$ be defined by (12), (37) and (38). Suppose $2\sigma(t)\Lambda_0(t) + 2\tau_0M(t) \leq \mu(t)$ and $\widetilde{\varepsilon}_1$ satisfies*

$$\sigma(t)\Lambda_0(t) + \tau_0M(t) \leq \widetilde{\varepsilon}_1 \leq \mu(t) - \sigma(t)\Lambda_0(t) - \tau_0M(t). \quad (47)$$

Then the following statements hold.

(a) The set $H_t := \{\eta : |\tilde{V}_x(t, \eta)| > \tilde{\epsilon}_1\}$ can be represented as the union of disjoint non-empty sets $H_{t,k} := H_t \cap \{\eta : (t, \eta) \in Z_k\}$, $1 \leq k \leq K$.

(b) For (t, η) with $|\tilde{V}_x(t, \eta)| \neq 0$, we have

$$\omega_x^{\text{adp,c}}(t, \eta) - \phi'_k(t) = \frac{\text{Rem}_1}{i2\pi\tilde{V}_x(t, \eta)}, \quad (48)$$

where

$$\text{Rem}_1 := i2\pi(\eta - \phi'_k(t))\text{rem}_0 - \frac{\text{rem}'_0}{\sigma(t)} + i2\pi \sum_{\ell \neq k} x_\ell(t) (\phi'_\ell(t) - \phi'_k(t)) \widehat{g}(\sigma(t)(\eta - \phi'_\ell(t))). \quad (49)$$

Hence, for (t, η) satisfying $|\tilde{V}_x(t, \eta)| > \tilde{\epsilon}_1$ and $(t, \eta) \in Z_k$, we have

$$|\omega_x^{\text{adp}}(t, \eta) - \phi'_k(t)| < \text{bd}_1, \quad (50)$$

where

$$\begin{aligned} \text{bd}_1 := & \frac{1}{\tilde{\epsilon}_1} (\alpha\Lambda_0(t) + \frac{1}{2\pi}\tilde{\Lambda}_0(t)) \\ & + \frac{1}{\tilde{\epsilon}_1} \max_{k \in \{1, \dots, K\}} \left\{ \sum_{\ell \neq k} A_\ell(t) |\phi'_\ell(t) - \phi'_k(t)| \sup_{\{u: |u| < \alpha\}} |\widehat{g}(u + \sigma(t)(\phi'_k(t) - \phi'_\ell(t)))| \right\}. \end{aligned} \quad (51)$$

(c) Suppose that $\tilde{\epsilon}_1$ satisfies the condition in part (a) and that bd_1 in part (b) satisfies $\text{bd}_1 \leq \frac{\alpha}{\sigma(t)}$. Then for $\tilde{\epsilon}_3$ satisfying $\text{bd}_1 \leq \tilde{\epsilon}_3 \leq \frac{\alpha}{\sigma(t)}$, we have

$$\left| \lim_{\lambda \rightarrow 0} \frac{\sigma(t)}{g(0)} \int_{|\xi - \phi'_k(t)| < \tilde{\epsilon}_3} R_{x, \tilde{\epsilon}_1}^{\text{adp}, \lambda}(t, \xi) d\xi - x_k(t) \right| \leq \text{bd}_2, \quad (52)$$

where

$$\text{bd}_2 := \frac{1}{|g(0)|} \left\{ 2\alpha(\sigma(t)\Lambda_0(t) + \tilde{\epsilon}_1) + A_k(t) \left| \int_{|u| \geq \alpha} \widehat{g}(u) du \right| + \sum_{\ell \neq k} A_\ell(t) m_{\ell, k}(t) \right\}, \quad (53)$$

with

$$m_{\ell, k}(t) := \left| \int_{|u| \leq \alpha} \widehat{g}(u + \sigma(t)(\phi'_k(t) - \phi'_\ell(t))) du \right|.$$

Remark 2. When $\widehat{g}(\xi)$ is supported in $[-\alpha, \alpha]$, we can set τ_0 in Theorem 1 part (a) to be zero, and thus, the condition (47) is

$$\sigma(t)\Lambda_0(t) \leq \tilde{\epsilon}_1 \leq \mu(t) - \sigma(t)\Lambda_0(t). \quad (54)$$

In addition, in this case the 2nd term in (51) is zero, and $m_{\ell,k}(t)$ and $\int_{|u| \geq \alpha} \widehat{g}(u) du$ in (53) are also zero. Hence in this case, (51) and (52) are respectively reduced to

$$|\omega_x^{\text{adp}}(t, \eta) - \phi'_k(t)| \leq \frac{1}{\tilde{\varepsilon}_1} (\alpha \Lambda_0(t) + \frac{1}{2\pi} \tilde{\Lambda}_0(t)), \quad (55)$$

and

$$\left| \lim_{\lambda \rightarrow 0} \frac{\sigma(t)}{g(0)} \int_{|\xi - \phi'_k(t)| < \tilde{\varepsilon}_3} R_{x, \tilde{\varepsilon}_1}^{\text{adp}, \lambda}(t, \xi) d\xi - x_k(t) \right| \leq \frac{2\alpha(\sigma(t)\Lambda_0(t) + \tilde{\varepsilon}_1)}{|g(0)|}, \quad (56)$$

for any $\tilde{\varepsilon}_3$ with $\frac{1}{\tilde{\varepsilon}_1} (\alpha \Lambda_0(t) + \frac{1}{2\pi} \tilde{\Lambda}_0(t)) \leq \tilde{\varepsilon}_3 \leq \frac{\alpha}{\sigma(t)}$.

Furthermore, the statement of Theorem 1 can be written in the form of Theorem B. For simplicity, we just consider the case $\varepsilon_1 = \varepsilon_2$. Write $\Lambda_0(t), \tilde{\Lambda}_0(t)$ defined by (37) and (38) respectively as

$$\Lambda_0(t) = \varepsilon_1 \lambda_0(t), \quad \tilde{\Lambda}_0(t) = \varepsilon_1 \tilde{\lambda}_0(t),$$

with

$$\lambda_0(t) := KI_1 + \pi I_2 \sigma(t) M(t), \quad \tilde{\lambda}_0(t) := K\tilde{I}_1 + \pi \tilde{I}_2 \sigma(t) M(t).$$

Let $\tilde{\varepsilon}_1 = \varepsilon_1^3$. If ε_1 is small enough such that

$$\tilde{\varepsilon}_1 \leq \min \left\{ \alpha \left\| \frac{1}{\sigma(t)} \right\|_\infty, \frac{1}{2} \|\mu(t)\|_\infty, \left\| \frac{1}{\sigma(t)\lambda_0(t)} \right\|_\infty^{1/2}, \left\| \frac{1}{\alpha\lambda_0(t) + \frac{1}{2\pi}\tilde{\lambda}_0(t)} \right\|_\infty \right\}, \quad (57)$$

then (54) holds and

$$\frac{1}{\tilde{\varepsilon}_1} (\alpha \Lambda_0(t) + \frac{1}{2\pi} \tilde{\Lambda}_0(t)) \leq \tilde{\varepsilon}_1, \quad \text{and } \tilde{\varepsilon}_1 \leq \frac{\alpha}{\sigma(t)}.$$

Thus we have the following corollary. ■

Corollary 1. Suppose $x(t) \in \mathcal{D}_{\varepsilon_1, \varepsilon_1}$ for some small $\varepsilon_1 > 0$ and $\text{supp}(\widehat{g}) \subseteq [-\alpha, \alpha]$. Let $\tilde{\varepsilon}_1 = \varepsilon_1^3$. If ε_1 is small enough such that (57) holds, then we have the following.

(a) For (t, η) satisfying $|\tilde{V}_x(t, \eta)| > \tilde{\varepsilon}_1$, there exists a unique $k \in \{1, 2, \dots, K\}$ such that $(t, \eta) \in Z_k$.

(b) Suppose (t, η) satisfies $|\tilde{V}_x(t, \eta)| > \tilde{\varepsilon}_1$ and $(t, \eta) \in Z_k$. Then

$$|\omega_x^{\text{adp}}(t, \eta) - \phi'_k(t)| < \tilde{\varepsilon}_1.$$

(c) For any $k, 1 \leq k \leq K$,

$$\left| \lim_{\lambda \rightarrow 0} \frac{\sigma(t)}{g(0)} \int_{|\xi - \phi'_k(t)| < \tilde{\varepsilon}_1} R_{x, \tilde{\varepsilon}_1}^{\text{adp}, \lambda}(t, \xi) d\xi - x_k(t) \right| \leq \frac{2\alpha(\sigma(t)\Lambda_0(t) + \tilde{\varepsilon}_1)}{|g(0)|}.$$

Remark 3. When $\sigma(t) \equiv 1$, $R_{x, \tilde{\varepsilon}_1}^{\text{adp}, \lambda}(t, \xi)$ is the regular FSST $R_{x, \tilde{\varepsilon}_1}^\lambda(t, \xi)$ defined by (4). Suppose $\text{supp}(\widehat{g}) \subseteq [-\alpha, \alpha]$. Then (50) and (52) (which, by Remark 2, are (55) and (56)) with $\varepsilon_1 = \varepsilon_2$ are respectively (16) and (17) with $\varepsilon = \varepsilon_1, \Delta = \alpha$. Thus in the case $\sigma(t) \equiv 1$, Theorem 1 is reduced to Theorem A, and Corollary 1 is Theorem B. ■

Remark 4. Observe that $\Lambda_0(t)$ and $\tilde{\Lambda}_0(t)$ defined by (37) and (38) respectively depend on $\sigma(t)$. Smaller $\sigma(t)$ results in smaller $\Lambda_0(t)$ and $\tilde{\Lambda}_0(t)$, hence the corresponding $\omega_x^{\text{adp}}(t, \eta)$ provides a more accurate estimate for $\phi'_k(t)$ as implied by (55) and the adaptive FSST in (52) gives a recovery of $x_k(t)$ with a smaller error as shown in (56).

Remark 5. When $\hat{g}(\xi)$ is not supported on $[-\alpha, \alpha]$, but $|\hat{g}(\xi)|$ decays fast as $|\xi| \rightarrow \infty$, then the terms in the summation for bd_1 in (51) will be small as long as τ_0 is small. Recall that we assume $|\hat{g}(\xi)|$ is even and decreasing. Then for $\ell = k - 1$, since $\sigma(t)(\phi'_k(t) - \phi'_{k-1}(t)) > 2\alpha$, we have for $|u| \leq \alpha$,

$$|\hat{g}(u + \sigma(t)(\phi'_k(t) - \phi'_{k-1}(t)))| \leq |\hat{g}(u + 2\alpha)| \leq |\hat{g}(\alpha)| = \tau_0$$

Similarly, we have $\sup_{\{|u| \leq \alpha\}} |\hat{g}(u + \sigma(t)(\phi'_k(t) - \phi'_{k+1}(t)))| \leq \tau_0$. The quantities $\sup_{\{|u| \leq \alpha\}} |\hat{g}(u + \sigma(t)(\phi'_k(t) - \phi'_\ell(t)))|$ for other $\ell \neq k - 1, k, k + 1$ are much smaller than τ_0 since $\sigma(t)(\phi'_k(t) - \phi'_\ell(t))$ is large (see (46)) and \hat{g} is rapidly decreasing. Thus the summation in bd_1 is dominated by $\tau_0 \max_{k=2, \dots, K-1} \{A_{k+1}(t)(\phi'_{k+1}(t) - \phi'_k(t)), A_{k-1}(t)(\phi'_k(t) - \phi'_{k-1}(t))\}$.

The functions $m_{\ell,k}(t)$ in (53) could be small if τ_0 is small. More precisely, for $\ell = k - 1$, we have

$$\begin{aligned} m_{k-1,k}(t) &\leq \int_{|u| \leq \alpha} |\hat{g}(u + \sigma(t)(\phi'_k(t) - \phi'_{k-1}(t)))| du \\ &\leq \int_{|u| \leq \alpha} |\hat{g}(u + 2\alpha)| du \leq 2\alpha |\hat{g}(\alpha)| = 2\alpha\tau_0. \end{aligned}$$

We can show similarly that $m_{k+1,k}(t) \leq 2\alpha\tau_0$. For other ℓ , $m_{\ell,k}(t)$ will be much smaller than $2\alpha\tau_0$.

Finally, let us look at the term $|\int_{|u| \geq \alpha} \hat{g}(u) du|$. It will be small if α is large. Here we give an estimate of this term when $g(t)$ is the Gaussian function defined by (24). In this case, one can obtain

$$\int_{|u| \geq \alpha} e^{-2\pi^2 u^2} du \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-2\pi^2 \alpha^2}}{1 + \sqrt{1 - e^{-2\pi^2 \alpha^2}}} = \frac{1}{\sqrt{2\pi}} \frac{\tau_0}{1 + \sqrt{1 - \tau_0}}. \quad (58)$$

To summarize, for g with $\hat{g}(\xi)$ decaying rapidly as $\xi \rightarrow \infty$, the statements in Corollary 1 hold if the same conditions are assumed and that τ_0 is small enough (and hence α is large enough). ■

In the rest of this section, we give the proof of Theorem 1.

Proof of Theorem 1 Part (a). Clearly $\cup_{k=1}^K H_{t,k} \subseteq H_t$. To show $H_t \subseteq \cup_{k=1}^K H_{t,k}$, let $\eta \in H_t$. We need to show (t, η) is in some Z_k . Assume $(t, \eta) \notin \cup_{k=1}^K Z_k$. Then for any k , by the definition

of Z_k in (44), we have $|\widehat{g}(\sigma(t)(\eta - \phi'_k(t)))| \leq \tau_0$. Hence, by (34) and (36), we have

$$\begin{aligned} |\widetilde{V}_x(t, \eta)| &\leq \sum_{k=1}^K |x_k(t)\widehat{g}(\sigma(t)(\eta - \phi'_k(t)))| + |\text{rem}_0| \\ &\leq \tau_0 M(t) + \sigma(t)\Lambda_0(t) \leq \widetilde{\epsilon}_1, \end{aligned}$$

a contradiction to the assumption $|\widetilde{V}_x(t, \eta)| > \widetilde{\epsilon}_1$. Thus (t, η) is in some Z_ℓ . Hence $\eta \in H_{t,\ell}$. This shows $H_t = \cup_{k=1}^K H_{t,k}$.

Clearly $H_{t,k}, 1 \leq k \leq K$ are disjoint since $Z_k, 1 \leq k \leq K$ are disjoint. Next we prove that each $H_{t,k}$ is non-empty by showing $\phi'_k(t) \in H_{t,k}$. It is enough to show $\phi'_k(t) \in H_t$ since $(t, \phi'_k(t)) \in Z_k$. By (34) and (36) again, we have

$$\begin{aligned} |\widetilde{V}_x(t, \eta) - x_k(t)\widehat{g}(\sigma(t)(\eta - \phi'_k(t)))| &\leq \sum_{\ell \neq k} |x_\ell(t)\widehat{g}(\sigma(t)(\eta - \phi'_\ell(t)))| + |\text{rem}_0| \\ &\leq \sum_{\ell \neq k} A_\ell(t)\tau_0 + \sigma(t)\Lambda_0(t) < \tau_0 M(t) + \sigma(t)\Lambda_0(t). \end{aligned}$$

This, together with the fact $\widehat{g}(0) = 1$, leads to

$$|\widetilde{V}_x(t, \phi'_k(t))| > |x_k(t)\widehat{g}(0)| - (\tau_0 M(t) + \sigma(t)\Lambda_0(t)) \geq \mu(t) - (\tau_0 M(t) + \sigma(t)\Lambda_0(t)) \geq \widetilde{\epsilon}_1.$$

This shows $\phi'_k(t) \in H_t$, and hence, $H_{t,k}$ is non-empty. ■

Proof of Theorem 1 Part (b). By a direct calculation, we have

$$\partial_t \widetilde{V}_x(t, \eta) = (i2\pi\eta - \frac{\sigma'(t)}{\sigma(t)})\widetilde{V}_x(t, \eta) - \frac{\sigma'(t)}{\sigma(t)}\widetilde{V}_x^{g_3}(t, \eta) - \frac{1}{\sigma(t)}\widetilde{V}_x^{g'}(t, \eta). \quad (59)$$

By (34) with g replaced by g' ,

$$\begin{aligned} \widetilde{V}_x^{g'}(t, \eta) &= \sum_{\ell=1}^K x_\ell(t)(g')^\wedge(\sigma(t)(\eta - \phi'_\ell(t))) + \text{rem}'_0 \\ &= i2\pi\sigma(t) \sum_{\ell=1}^K x_\ell(t)(\eta - \phi'_\ell(t))\widehat{g}(\sigma(t)(\eta - \phi'_\ell(t))) + \text{rem}'_0. \end{aligned}$$

This and (59) imply that

$$\begin{aligned} &(\omega_x^{\text{adp,c}}(t, \eta) - \phi'_k(t))i2\pi\widetilde{V}_x(t, \eta) \\ &= \partial_t \widetilde{V}_x(t, \eta) + \frac{\sigma'(t)}{\sigma(t)}(\widetilde{V}_x(t, \eta) + \widetilde{V}_x^{g_3}(t, \eta)) - i2\pi\phi'_k(t)\widetilde{V}_x(t, \eta) \\ &= i2\pi\eta\widetilde{V}_x(t, \eta) - \frac{1}{\sigma(t)}\widetilde{V}_x^{g'}(t, \eta) - i2\pi\phi'_k(t)\widetilde{V}_x(t, \eta) \\ &= i2\pi(\eta - \phi'_k(t)) \left(\sum_{\ell=1}^K x_\ell(t)\widehat{g}(\sigma(t)(\eta - \phi'_\ell(t))) + \text{rem}_0 \right) \\ &\quad - \frac{1}{\sigma(t)} \left(i2\pi\sigma(t) \sum_{\ell=1}^K x_\ell(t)(\eta - \phi'_\ell(t))\widehat{g}(\sigma(t)(\eta - \phi'_\ell(t))) + \text{rem}'_0 \right) \end{aligned}$$

$$\begin{aligned}
&= i2\pi(\eta - \phi'_k(t))\text{rem}_0 - \frac{\text{rem}'_0}{\sigma(t)} + i2\pi \sum_{\ell \neq k} x_\ell(t)(\phi'_\ell(t) - \phi'_k(t))\widehat{g}(\sigma(t)(\eta - \phi'_\ell(t))) \\
&= \text{Rem}_1.
\end{aligned}$$

This shows (48).

For (50), with the assumption $|\widetilde{V}_x(t, \eta)| > \widetilde{\epsilon}_1$, we have

$$\begin{aligned}
&|\omega_x^{\text{adp}}(t, \eta) - \phi'_k(t)| \leq |\omega_x^{\text{adp},c}(t, \eta) - \phi'_k(t)| \\
&\leq \left| \frac{\text{Rem}_1}{i2\pi\widetilde{V}_x(t, \eta)} \right| < \frac{|\text{Rem}_1|}{2\pi\widetilde{\epsilon}_1} \\
&\leq \frac{1}{\widetilde{\epsilon}_1} \left\{ |(\eta - \phi'_k(t))\sigma(t)\Lambda_0(t) + \frac{1}{2\pi}\widetilde{\Lambda}_0(t) + \sum_{\ell \neq k} A_\ell(t)|\phi'_\ell(t) - \phi'_k(t)| |\widehat{g}(\sigma(t)(\eta - \phi'_\ell(t)))| \right\} \\
&\leq \frac{1}{\widetilde{\epsilon}_1} \left\{ \alpha\Lambda_0(t) + \frac{1}{2\pi}\widetilde{\Lambda}_0(t) + \sum_{\ell \neq k} A_\ell(t)|\phi'_\ell(t) - \phi'_k(t)| |\widehat{g}(\sigma(t)(\eta - \phi'_\ell(t)))| \right\} \\
&\leq \text{bd}_1,
\end{aligned}$$

as desired, where the second last inequality follows from $|\eta - \phi'_k(t)| < \frac{\alpha}{\sigma(t)}$ since $(t, \eta) \in Z_k$. \blacksquare

Proof of Theorem 1 Part (c). First we have the following result which can be derived as that on p.254 in [8]:

$$\lim_{\lambda \rightarrow 0} \int_{|\xi - \phi'_k(t)| < \widetilde{\epsilon}_3} R_{x, \widetilde{\epsilon}_1}^{\text{adp}, \lambda}(t, \xi) d\xi = \int_{X_t} \widetilde{V}_x(t, \eta) d\eta, \quad (60)$$

where

$$X_t := \{ \eta : |\widetilde{V}_x(t, \eta)| > \widetilde{\epsilon}_1 \text{ and } |\phi'_k(t) - \omega_x^{\text{adp}}(t, \eta)| < \widetilde{\epsilon}_3 \}.$$

Let

$$Y_t := \{ \eta : |\widetilde{V}_x(t, \eta)| > \widetilde{\epsilon}_1 \text{ and } (t, \eta) \in Z_k \}.$$

By Theorem 1 part (b), if $\eta \in Y_t$, then $|\phi'_k(t) - \omega_x^{\text{adp}}(t, \eta)| < \text{bd}_1 \leq \widetilde{\epsilon}_3$. Thus $\eta \in X_t$. Hence $Y_t \subseteq X_t$.

On the other hand, suppose $\eta \in X_t$. Since $|\widetilde{V}_x(t, \eta)| > \widetilde{\epsilon}_1$, by Theorem 1 part (a), $(t, \eta) \in Z_\ell$ for an ℓ in $\{1, 2, \dots, K\}$. If $\ell \neq k$, then by Theorem Theorem 1 part (b),

$$\begin{aligned}
|\phi'_k(t) - \omega_x^{\text{adp}}(t, \eta)| &\geq |\phi'_k(t) - \phi'_\ell(t)| - |\phi'_\ell(t) - \omega_x^{\text{adp}}(t, \eta)| \\
&> \frac{2\alpha}{\sigma(t)} - \text{bd}_1 \geq \frac{2\alpha}{\sigma(t)} - \widetilde{\epsilon}_3 \geq \widetilde{\epsilon}_3,
\end{aligned}$$

since $\text{bd}_1 \leq \widetilde{\epsilon}_3 \leq \frac{\alpha}{\sigma(t)}$. This contradicts to the assumption $|\phi'_k(t) - \omega_x^{\text{adp}}(t, \eta)| < \widetilde{\epsilon}_3$ since $\eta \in X_t$. Hence $\ell = k$ and $\eta \in Y_t$. Thus we get $X_t = Y_t$. Therefore, from (60), we have

$$\lim_{\lambda \rightarrow 0} \int_{|\xi - \phi'_k(t)| < \widetilde{\epsilon}_3} R_{x, \widetilde{\epsilon}_1}^{\text{adp}, \lambda}(t, \xi) d\xi = \int_{\{|\widetilde{V}_x(t, \eta)| > \widetilde{\epsilon}_1\} \cap \{(t, \eta) \in Z_k\}} \widetilde{V}_x(t, \eta) d\eta. \quad (61)$$

Furthermore,

$$\begin{aligned}
& \left| \int_{\{|\tilde{V}_x(t,\eta)| > \tilde{\epsilon}_1\} \cap \{\eta:(t,\eta) \in Z_k\}} \tilde{V}_x(t,\eta) d\eta - \frac{g(0)}{\sigma(t)} x_k(t) \right| \\
&= \left| \int_{\{\eta:(t,\eta) \in Z_k\}} \tilde{V}_x(t,\eta) d\eta - \frac{g(0)}{\sigma(t)} x_k(t) - \int_{\{|\tilde{V}_x(t,\eta)| \leq \tilde{\epsilon}_1\} \cap \{\eta:(t,\eta) \in Z_k\}} \tilde{V}_x(t,\eta) d\eta \right| \\
&\leq \tilde{\epsilon}_1 \frac{2\alpha}{\sigma(t)} + \left| \int_{\{\eta:(t,\eta) \in Z_k\}} \left(\sum_{\ell=1}^K x_\ell(t) \hat{g}(\sigma(t)(\eta - \phi'_\ell(t))) + \text{rem}_0 \right) d\eta - \frac{g(0)}{\sigma(t)} x_k(t) \right| \\
&\leq \tilde{\epsilon}_1 \frac{2\alpha}{\sigma(t)} + |\text{rem}_0| \frac{2\alpha}{\sigma(t)} + \left| \int_{|\eta - \phi'_k(t)| < \frac{\alpha}{\sigma(t)}} x_k(t) \hat{g}(\sigma(t)(\eta - \phi'_k(t))) d\eta - \frac{g(0)}{\sigma(t)} x_k(t) \right| \\
&\quad + \sum_{\ell \neq k} A_\ell(t) \left| \int_{|\eta - \phi'_k(t)| < \frac{\alpha}{\sigma(t)}} \hat{g}(\sigma(t)(\eta - \phi'_k(t))) d\eta \right| \\
&= (|\text{rem}_0| + \tilde{\epsilon}_1) \frac{2\alpha}{\sigma(t)} + \left| \frac{x_k(t)}{\sigma(t)} \int_{|u| < \alpha} \hat{g}(u) du - \frac{g(0)}{\sigma(t)} x_k(t) \right| \\
&\quad + \sum_{\ell \neq k} \frac{A_\ell(t)}{\sigma(t)} \left| \int_{|u| < \alpha} \hat{g}(u + \sigma(t)(\phi'_k(t) - \phi'_\ell(t))) du \right| \\
&= (|\text{rem}_0| + \tilde{\epsilon}_1) \frac{2\alpha}{\sigma(t)} \left| \frac{x_k(t)}{\sigma(t)} \int_{\mathbb{R}} \hat{g}(u) du - \frac{g(0)}{\sigma(t)} x_k(t) - \frac{x_k(t)}{\sigma(t)} \int_{|u| \geq \alpha} \hat{g}(u) du \right| + \sum_{\ell \neq k} \frac{A_\ell(t)}{\sigma(t)} m_{\ell,k}(t) \\
&= (|\text{rem}_0| + \tilde{\epsilon}_1) \frac{2\alpha}{\sigma(t)} + \left| x_k(t) \frac{g(0)}{\sigma(t)} - \frac{g(0)}{\sigma(t)} x_k(t) - \frac{x_k(t)}{\sigma(t)} \int_{|u| \geq \alpha} \hat{g}(u) du \right| + \frac{1}{\sigma(t)} \sum_{\ell \neq k} A_\ell(t) m_{\ell,k}(t) \\
&\leq (\sigma(t) \Lambda_0(t) + \tilde{\epsilon}_1) \frac{2\alpha}{\sigma(t)} + \frac{A_k(t)}{\sigma(t)} \left| \int_{|u| \geq \alpha} \hat{g}(u) du \right| + \frac{1}{\sigma(t)} \sum_{\ell \neq k} A_\ell(t) m_{\ell,k}(t).
\end{aligned}$$

The above estimate, together with (61), leads to (52). This completes the proof of Theorem 1 Part (c). \blacksquare

4 Analysis of 2nd-order adaptive FSST

We consider multicomponent signals $x(t)$ of (1) satisfying (21) and being well approximated locally by a linear chirp signal of (28) with $A'_k(t)$ and $\phi_k^{(3)}(t)$ are small:

$$|A'_k(t)| \leq \varepsilon_1, \quad |\phi_k^{(3)}(t)| \leq \varepsilon_3, \quad t \in \mathbb{R}, \quad 1 \leq k \leq K, \quad (62)$$

for some positive number $\varepsilon_1, \varepsilon_3$,

For a given t , we use $G_k(\xi)$ to denote the Fourier transform of $e^{i\pi\sigma(t)\phi_k''(t)\tau^2} g(\tau)$, namely,

$$G_k(\xi) := \mathcal{F} \left(e^{i\pi\sigma^2(t)\phi_k''(t)\tau^2} g(\tau) \right) (\xi) = \int_{\mathbb{R}} e^{i\pi\sigma^2(t)\phi_k''(t)\tau^2} g(\tau) e^{-i2\pi\xi\tau} d\tau,$$

where \mathcal{F} denotes the Fourier transform. Note that $G_k(\xi)$ depends on t also if $\phi_k''(t) \neq 0$. We drop t in G_ℓ for simplicity.

For each component $x_k(t) = A_k(t)e^{i2\pi\phi_k(t)}$, we write $x_k(t + \tau)$ as

$$\begin{aligned} x_k(t + \tau) &= x_k(t)e^{i2\pi(\phi'_k(t)\tau + \frac{1}{2}\phi''_k(t)\tau^2)} + (A_k(t + \tau) - A_k(t))e^{i2\pi\phi_k(t+\tau)} \\ &\quad + x_k(t)e^{i2\pi(\phi'_k(t)\tau + \frac{1}{2}\phi''_k(t)\tau^2)} (e^{i2\pi(\phi_k(t+\tau) - \phi_k(t) - \phi'_k(t)\tau - \frac{1}{2}\phi''_k(t)\tau^2)} - 1). \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{V}_x(t, \eta) &= \sum_{k=1}^K \int_{\mathbb{R}} x_k(t + \tau) \frac{1}{\sigma(t)} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau \\ &= \sum_{k=1}^K \int_{\mathbb{R}} x_k(t) e^{i2\pi(\phi'_k(t)\tau + \frac{1}{2}\phi''_k(t)\tau^2)} \frac{1}{\sigma(t)} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau + \text{res}_0 \end{aligned} \quad (63)$$

$$= \sum_{k=1}^K x_k(t) G_k(\sigma(t)(\eta - \phi'_k(t))) + \text{res}_0, \quad (64)$$

where

$$\begin{aligned} \text{res}_0 &:= \sum_{k=1}^K \int_{\mathbb{R}} \left\{ (A_k(t + \tau) - A_k(t)) e^{i2\pi\phi_k(t+\tau)} \right. \\ &\quad \left. + x_k(t) e^{i2\pi(\phi'_k(t)\tau + \frac{1}{2}\phi''_k(t)\tau^2)} (e^{i2\pi(\phi_k(t+\tau) - \phi_k(t) - \phi'_k(t)\tau - \frac{1}{2}\phi''_k(t)\tau^2)} - 1) \right\} \frac{1}{\sigma(t)} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau. \end{aligned} \quad (65)$$

To distinguish the different types of the remainders for the expansion of $\tilde{V}_x(t, \eta)$ resulted from different local approximations for $x_k(t + \tau)$, in this section we use ‘‘res’’, which means residual, to denote the remainder for the expansion of $\tilde{V}_x(t, \eta)$ in (63).

With $|A_k(t + \tau) - A_k(t)| \leq \varepsilon_1|\tau|$ and

$$|e^{i2\pi(\phi_k(t+\tau) - \phi_k(t) - \phi'_k(t)\tau - \frac{1}{2}\phi''_k(t)\tau^2)} - 1| \leq 2\pi \frac{1}{6} \sup_{\eta \in \mathbb{R}} |\phi_k^{(3)}(\eta)\tau^3| \leq \frac{\pi}{3} \varepsilon_2 |\tau|^3,$$

we have

$$\begin{aligned} |\text{res}_0| &\leq \sum_{k=1}^K \int_{\mathbb{R}} \varepsilon_1 |\tau| \frac{1}{\sigma(t)} |g\left(\frac{\tau}{\sigma(t)}\right)| d\tau + \sum_{k=1}^K A_k(t) \int_{\mathbb{R}} \frac{\pi}{3} \varepsilon_3 |\tau|^3 \frac{1}{\sigma(t)} |g\left(\frac{\tau}{\sigma(t)}\right)| d\tau \\ &= K\varepsilon_1 I_1 \sigma(t) + \frac{\pi}{3} \varepsilon_3 I_3 \sigma^3(t) M(t), \end{aligned}$$

where I_n and $M(t)$ are defined in (14) and (12) respectively. Hence we have

$$|\text{res}_0| \leq \sigma(t) \Pi_0(t), \quad (66)$$

where

$$\Pi_0(t) := K\varepsilon_1 I_1 + \frac{\pi}{3} \varepsilon_3 I_3 \sigma^2(t) M(t). \quad (67)$$

Thus if $\varepsilon_1, \varepsilon_2$ are small enough, then $|\text{res}_0|$ is small and hence, $G_k(\sigma(t)(\eta - \phi'_k(t)))$ provides the time-frequency zone for $\tilde{V}_{x_k}(t, \eta)$. In the following we describe those time-frequency zones mathematically. Let $0 < \tau_0 < 1$ be a given small number as the threshold. Denote

$$O_k := \{(t, \eta) : |G_k(\sigma(t)(\eta - \phi'_k(t)))| > \tau_0, t \in \mathbb{R}\}. \quad (68)$$

We assume again $|G_k(\xi)|$ is even and decreasing for $\xi \geq 0$. Then O_k can be written as

$$O_k = \{(t, \eta) : |\eta - \phi'_k(t)| < \frac{\alpha_k}{\sigma(t)}, t \in \mathbb{R}\}. \quad (69)$$

where $\alpha_k = \alpha_k(t)$ is obtained by solving $|G_k(\xi)| = \tau_0$. We will assume the multicomponent signal $x(t)$ is well-separated, that is there is $\sigma(t)$ such that

$$O_k \cap O_\ell = \emptyset, \quad k \neq \ell. \quad (70)$$

As an example, let us look at what are O_k and $\alpha_k(t)$ look like when g is the Gaussian function defined by (24). One can obtain for this g (see [14]),

$$G_k(u) = \frac{1}{\sqrt{1 - i2\pi\phi''_k(t)\sigma^2(t)}} e^{-\frac{2\pi^2 u^2}{1 + (2\pi\phi''_k(t)\sigma^2(t))^2} (1 + i2\pi\phi''_k(t)\sigma^2(t))}. \quad (71)$$

Thus

$$|G_k(u)| = \frac{1}{(1 + (2\pi\phi''_k(t)\sigma^2(t))^2)^{\frac{1}{4}}} e^{-\frac{2\pi^2}{1 + (2\pi\phi''_k(t)\sigma^2(t))^2} u^2}. \quad (72)$$

Therefore, in this case, assuming $\tau_0(1 + (2\pi\phi''_k(t)\sigma^2(t))^2)^{\frac{1}{4}} \leq 1$,

$$\alpha_k = \sqrt{1 + (2\pi\phi''_k(t)\sigma^2(t))^2} \frac{1}{2\pi} \sqrt{2\ln\left(\frac{1}{\tau_0}\right) - \frac{1}{2}\ln(1 + (2\pi\phi''_k(t)\sigma^2(t))^2)}. \quad (73)$$

Authors of [14] consider a larger zone O'_k in the time-frequency plane:

$$O'_k := \left\{ (t, \eta) : |\eta - \phi'_k(t)| < \frac{\alpha}{\sigma(t)} \left(1 + 2\pi|\phi''_k(t)|\sigma^2(t) \right), t \in \mathbb{R} \right\}. \quad (74)$$

where α is defined by (43). They obtain that if for $k = 2, \dots, K$,

$$4\alpha\sqrt{\pi} \sqrt{|\phi''_k(t)| + |\phi''_{k-1}(t)|} \leq \phi'_k(t) - \phi'_{k-1}(t), \text{ and} \quad (75)$$

$$\max_{2 \leq k \leq K} \left\{ \frac{4\alpha}{b_k(t) + \sqrt{b_k(t)^2 - 8\alpha a_k(t)}} \right\} \leq \min_{2 \leq k \leq K} \left\{ \frac{4\alpha}{b_k(t) - \sqrt{b_k(t)^2 - 8\alpha a_k(t)}} \right\}, \quad (76)$$

then the components $x_k(t), 1 \leq k \leq K$ of $x(t)$ are well-separated in the time-frequency plane in the sense that $O'_k \cap O'_\ell = \emptyset, k \neq \ell$, where

$$a_k(t) := 2\pi\alpha(|\phi''_{k-1}(t)| + |\phi''_k(t)|), \quad b_k(t) := \phi'_k(t) - \phi'_{k-1}(t).$$

[14] shows that any $\sigma(t)$ between the two quantities in (76) separates the components $x_k(t)$ of $x(t)$ in the time-frequency plane, and it suggests to choose $\sigma(t)$ to be

$$\sigma_2(t) := \max_{2 \leq k \leq K} \left\{ \frac{4\alpha}{b_k(t) + \sqrt{b_k(t)^2 - 8\alpha a_k(t)}} \right\}. \quad (77)$$

Let g be a window function with $|\widehat{g}(\xi)|$ even and decreasing for $\xi \geq 0$. Let $\mathcal{D}_{\varepsilon_1, \varepsilon_2}^{(2)}$ denote the set of multicomponent signals of (1) satisfying (21), (32), (62) and that $x(t)$ is well-separated with g , that is there is $\sigma(t)$ such that (70) holds.

We introduce more notations to describe our main theorem on the 2nd-order adaptive FSST. For $j \geq 0$, denote

$$\begin{aligned} G_{j,k}(t, \eta) &:= \int_{\mathbb{R}} e^{i2\pi(\phi'_k(t)\tau + \frac{1}{2}\phi''_k(t)\tau^2)} \frac{\tau^j}{\sigma(t)^{j+1}} g\left(\frac{\tau}{\sigma(t)}\right) e^{-i2\pi\eta\tau} d\tau \\ &= \mathcal{F}\left(e^{i\pi\sigma^2(t)\phi''_k(t)\tau^2} \tau^j g(\tau)\right) (\sigma(t)(\eta - \phi'_k(t))). \end{aligned} \quad (78)$$

Clearly,

$$G_{0,k}(t, \eta) = G_k(\sigma(t)(\eta - \phi'_k(t))),$$

and one can obtain for $j \geq 1$,

$$G_{j,k}(t, \eta) = \frac{1}{(-i2\pi)^j} G_k^{(j)}(\sigma(t)(\eta - \phi'_k(t))). \quad (79)$$

Let $\text{res}_1, \text{res}_2, \text{res}'_0$, and res'_1 be the residuals defined as res_0 in (65) with $g(\tau)$ replaced respectively by $g_1(\tau), g_2(\tau), g'(\tau)$, and $g_3(\tau) = \tau g'(\tau)$. Then we have the estimates for these residuals similar to (66). More precisely, we have

$$|\text{res}_1| \leq \sigma(t)\Pi_1(t), \quad |\text{res}_2| \leq \sigma(t)\Pi_2(t), \quad |\text{res}'_0| \leq \sigma(t)\widetilde{\Pi}_0(t), \quad |\text{res}'_1| \leq \sigma(t)\widetilde{\Pi}_1(t),$$

where

$$\begin{aligned} \Pi_1(t) &:= K\varepsilon_1 I_2 + \frac{\pi}{3}\varepsilon_3 I_4 \sigma^2(t) M(t), \quad \Pi_2(t) := K\varepsilon_1 I_3 + \frac{\pi}{3}\varepsilon_3 I_5 \sigma^2(t) M(t), \\ \widetilde{\Pi}_0(t) &:= K\varepsilon_1 \widetilde{I}_1 + \frac{\pi}{3}\varepsilon_3 \widetilde{I}_3 \sigma^2(t) M(t), \quad \widetilde{\Pi}_1(t) := K\varepsilon_1 \widetilde{I}_2 + \frac{\pi}{3}\varepsilon_3 \widetilde{I}_4 \sigma^2(t) M(t), \end{aligned}$$

with I_n, \widetilde{I}_n defined in (14).

Denote

$$\begin{aligned} B_k(t, \eta) &:= \sum_{\ell \neq k} x_\ell(t) (\phi'_\ell(t) - \phi'_k(t)) G_{0,\ell}(t, \eta), & D_k(t, \eta) &:= \sum_{\ell \neq k} x_\ell(t) (\phi''_\ell(t) - \phi''_k(t)) G_{1,\ell}(t, \eta), \\ E_k(t, \eta) &:= \sum_{\ell \neq k} x_\ell(t) (\phi'_\ell(t) - \phi'_k(t)) G_{1,\ell}(t, \eta), & F_k(t, \eta) &:= \sum_{\ell \neq k} x_\ell(t) (\phi''_\ell(t) - \phi''_k(t)) G_{2,\ell}(t, \eta), \end{aligned}$$

and

$$\text{Res}_1 := i2\pi B_k(t, \eta) + i2\pi\sigma(t)D_k(t, \eta) + i2\pi(\eta - \phi'_k(t))\text{res}_0 - \frac{\text{res}'_0}{\sigma(t)} - i2\pi\phi''_k(t)\sigma(t)\text{res}_1, \quad (80)$$

$$\begin{aligned} \text{Res}_2 := & 4\pi^2\sigma(t)E_k(t, \eta) + 4\pi^2\sigma^2(t)F_k(t, \eta) \\ & + i2\pi\text{res}_0 + 4\pi^2(\eta - \phi'_k(t))\sigma(t)\text{res}_1 + i2\pi\text{res}'_1 - 4\pi^2\phi''_k(t)\sigma^2(t)\text{res}_2. \end{aligned} \quad (81)$$

Next we will provide Theorem 2 on the 2nd-order adaptive FSST, which also consists of parts (a)-(c). The proof of part (b) is based on the following two lemmas whose proof is postponed to Appendix.

Lemma 1. *Let Res_1 be the quantity defined by (80). Then*

$$\partial_t \tilde{V}_x(t, \eta) = \left(i2\pi\phi'_k(t) - \frac{\sigma'(t)}{\sigma(t)} \right) \tilde{V}_x(t, \eta) + i2\pi\phi''_k(t)\sigma(t)\tilde{V}_x^{g_1}(t, \eta) - \frac{\sigma'(t)}{\sigma(t)}\tilde{V}_x^{g_3}(t, \eta) + \text{Res}_1. \quad (82)$$

Lemma 2. *Let $P_0(t, \eta)$ be the quantity defined by (30). Then for (t, η) satisfying $V_x(t, \eta) \neq 0$ and $\frac{\partial}{\partial \eta} \left(\frac{\tilde{V}_x^{g_1}(t, \eta)}{\tilde{V}_x(t, \eta)} \right) \neq 0$, we have*

$$P_0(t, \eta) = i2\pi\sigma(t)\phi''_k(t) + \text{Res}_3, \quad (83)$$

where

$$\text{Res}_3 := \frac{\tilde{V}_x(t, \eta)\text{Res}_2 - \partial_\eta \tilde{V}_x(t, \eta)\text{Res}_1}{\tilde{V}_x(t, \eta)\partial_\eta \tilde{V}_x^{g_1}(t, \eta) - \tilde{V}_x^{g_1}(t, \eta)\partial_\eta \tilde{V}_x(t, \eta)}, \quad (84)$$

with Res_1 and Res_2 defined by (80) and (81) respectively.

Denote

$$M_{\ell, k}(t) := \sigma(t) \int_{\{\eta: (t, \eta) \in O_k\}} |G_{0, \ell}(t, \eta)| d\eta = \int_{|u| < \alpha_k} |G_\ell(u + \sigma(t)(\phi'_k(t) - \phi'_\ell(t)))| du. \quad (85)$$

Theorem 2. *Suppose $x(t) \in \mathcal{D}_{\varepsilon_1, \varepsilon_3}^{(2)}$ with a window function $g(t)$ for some small $\varepsilon_1, \varepsilon_3 > 0$. Let $\mu(t), M(t), \Pi_0(t)$ be defined by (12) and (67). Suppose $2\sigma(t)\Pi_0(t) + 2\tau_0 M(t) \leq \mu(t)$ and $\tilde{\varepsilon}_1$ satisfies*

$$\sigma(t)\Pi_0(t) + \tau_0 M(t) \leq \tilde{\varepsilon}_1 \leq \mu(t) - \sigma(t)\Pi_0(t) - \tau_0 M(t).$$

Then the following statements hold.

(a) *The set $\mathcal{H}_t := \{\eta : |\tilde{V}_x(t, \eta)| > \tilde{\varepsilon}_1\}$ can be represented as the union of disjoint non-empty sets $\mathcal{H}_{t, k} := \mathcal{H}_t \cap \{\eta : (t, \eta) \in O_k\}$, $1 \leq k \leq K$.*

(b) *Suppose (t, η) satisfies $|\tilde{V}_x(t, \eta)| > \tilde{\varepsilon}_1$, $|\partial_\eta(\tilde{V}_x^{g_1}(t, \eta)/\tilde{V}_x(t, \eta))| > \tilde{\varepsilon}_2$, and $(t, \eta) \in O_k$. Then*

$$\omega_x^{\text{adp}, 2\text{nd}, \text{c}}(t, \eta) - \phi'_k(t) = \text{Res}_4, \quad (86)$$

where

$$\text{Res}_4 := \frac{\text{Res}_1}{i2\pi\tilde{V}_x(t, \eta)} - \frac{\tilde{V}_x^{g_1}(t, \eta)\text{Res}_3}{i2\pi\tilde{V}_x(t, \eta)}.$$

Furthermore,

$$|\omega_x^{\text{adp,2nd}}(t, \eta) - \phi'_k(t)| < \text{Bd}_1, \quad (87)$$

where

$$\text{Bd}_1 := \max_{1 \leq k \leq K} \sup_{\eta \in O_k} \left\{ \frac{|\text{Res}_1|}{2\pi\tilde{\varepsilon}_1} + \frac{1}{2\pi\tilde{\varepsilon}_1^3\tilde{\varepsilon}_2} |\tilde{V}_x^{g_1}(t, \eta)| (|\partial_\eta \tilde{V}_x(t, \eta)| |\text{Res}_1| + \tilde{\varepsilon}_1 |\text{Res}_2|) \right\}. \quad (88)$$

(c) Suppose that $\tilde{\varepsilon}_1$ satisfies the condition in part (a) and $\text{Bd}_1 \leq \frac{1}{2}L_k(t)$, where

$$L_k(t) := \frac{1}{\sigma(t)} \min\{\alpha_k + \alpha_{k-1}, \alpha_k + \alpha_{k+1}\}. \quad (89)$$

Then for any $\tilde{\varepsilon}_3 = \tilde{\varepsilon}_3(t) > 0$ satisfying $\text{Bd}_1 \leq \tilde{\varepsilon}_3 \leq \frac{1}{2}L_k(t)$,

$$\left| \lim_{\lambda \rightarrow 0} \frac{\sigma(t)}{g(0)} \int_{|\xi - \phi'_k(t)| < \tilde{\varepsilon}_3} R_{x, \tilde{\varepsilon}_1, \tilde{\varepsilon}_2}^{\text{adp,2nd}, \lambda}(t, \xi) d\xi - x_k(t) \right| \leq \text{Bd}_2, \quad (90)$$

where $\text{Bd}_2 = \text{Bd}'_2 + \text{Bd}''_2$ with

$$\begin{aligned} \text{Bd}'_2 &:= \frac{1}{|g(0)|} \left\{ 2\alpha_k (\tilde{\varepsilon}_1 + \sigma(t)\Pi_0(t)) + A_k(t) \left| \int_{|u| \geq \alpha_k} G_k(u) du \right| + \sum_{\ell \neq k} A_\ell(t) M_{\ell, k}(t) \right\}, \\ \text{Bd}''_2 &:= \frac{1}{|g(0)|} \left\{ 2\alpha_k \sigma(t)\Pi_0(t) + \sigma(t)A_k(t) \|g\|_1 |U_t| + \sum_{\ell \neq k} A_\ell(t) M_{\ell, k}(t) \right\} \end{aligned} \quad (91)$$

and $|U_t|$ denoting the Lebesgue measure of the set U_t :

$$U_t := \{\eta : (t, \eta) \in O_k, |V_x(t, \eta)| > \tilde{\varepsilon}_1, |\partial_\eta (\tilde{V}_x^{g_1}(t, \eta) / \tilde{V}_x(t, \eta))| \leq \tilde{\varepsilon}_2\}. \quad (92)$$

We postpone the proof of Theorem 2 to the end of this section.

Remark 6. With the decay conditions of $G_k(u)$ and $G_{j,\ell}(t, \eta)$, Theorem 2 can be stated in the formulation in Corollary 1. Here instead of giving such a statement for the 2nd-order adaptive FSST as in Corollary 1, we look at the estimate bounds when $g(t)$ is the Gaussian function given by (24).

First we look at the bounds for $\text{Res}_1, \text{Res}_2$. From (80) and (81), we have

$$\begin{aligned} |\text{Res}_1| &\leq 2\pi |B_k(t, \eta)| + 2\pi\sigma(t) |D_k(t, \eta)| + 2\pi\alpha_k \Pi_0(t) + \Pi_2(t) + 2\pi |\phi''_k(t)| \sigma^2(t) \Pi_1(t), \\ |\text{Res}_2| &\leq 4\pi^2 \sigma(t) |E_k(t, \eta)| + 4\pi^2 \sigma^2(t) |F_k(t, \eta)| \\ &\quad + 2\pi\sigma(t) \Pi_0(t) + 4\pi^2 \alpha_k \sigma(t) \Pi_1(t) + 2\pi\sigma(t) \tilde{\Pi}_1(t) + 4\pi^2 |\phi''_k(t)| \sigma^3(t) \Pi_2(t). \end{aligned}$$

We need to look at the estimates for $B_k(t, \eta), D_k(t, \eta), E_k(t, \eta), F_k(t, \eta)$, which are determined by $G_{j,\ell}(t, \eta)$, for $(t, \eta) \in O_k$.

When $g(t)$ is given by (24), then $G_k(u)$ is given by (71), and α_k by solving $|G_k(u)| = \tau_0$ for u is given by (73). For $(t, \eta) \in O_k$, we have

$$\eta - \phi'_{k-1}(t) \geq \phi'_k(t) - \frac{\alpha_k}{\sigma(t)} - \phi'_{k-1}(t) \geq \left(\frac{\alpha_{k-1}}{\sigma(t)} + \frac{\alpha_k}{\sigma(t)}\right) - \frac{\alpha_k}{\sigma(t)} = \frac{\alpha_{k-1}}{\sigma(t)}.$$

Thus

$$|G_{0,k-1}(t, \eta)| = |G_{k-1}(\sigma(\eta - \phi'_{k-1}(t)))| \leq |G_{k-1}(\alpha_{k-1})| = \tau_0.$$

Similarly, we can obtain for $(t, \eta) \in O_k$

$$|G_{0,k+1}(t, \eta)| \leq \tau_0.$$

For other ℓ , $|G_{0,\ell}(t, \eta)|$ is much smaller than τ_0 .

$G_{j,\ell}(t, \eta)$ can be estimated similarly. For example, for $G_{1,k-1}(t, \eta)$, with (79) and

$$G'_{k-1}(u) = G_{k-1}(u)(-2u) \frac{2\pi^2(1 + i2\pi\phi''_{k-1}(t)\sigma^2(t))}{1 + (2\pi\phi''_{k-1}(t)\sigma^2(t))^2},$$

we have

$$\begin{aligned} |G_{1,k-1}(t, \eta)| &= \frac{1}{2\pi} |G'_{k-1}(\sigma(t)(\eta - \phi'_{k-1}(t)))| \\ &= |G_{k-1}(\sigma(t)(\eta - \phi'_{k-1}(t)))| \frac{2\pi|\sigma(t)(\eta - \phi'_{k-1}(t))|}{\sqrt{1 + (2\pi\phi''_{k-1}(t)\sigma^2(t))^2}} \\ &\leq |G_{k-1}(\alpha_{k-1})| 2\pi(\phi'_k(t) - \phi'_{k-1}(t) + \alpha_k) = 2\pi\tau_0(\phi'_k(t) - \phi'_{k-1}(t) + \alpha_k). \end{aligned}$$

since for $(t, \eta) \in O_k$, $\alpha_{k-1} \leq \sigma(t)(\eta - \phi'_{k-1}(t)) \leq \phi'_k(t) - \phi'_{k-1}(t) + \alpha_k$. We also can get

$$|G_{1,k+1}(t, \eta)| \leq 2\pi\tau_0(\phi'_{k+1}(t) - \phi'_k(t) + \alpha_k).$$

For other ℓ , $|G_{1,\ell}(t, \eta)|$ are much smaller than $2\pi\tau_0(\phi'_k(t) - \phi'_{k-1}(t) + \alpha_k)$ or $2\pi\tau_0(\phi'_{k+1}(t) - \phi'_k(t) + \alpha_k)$.

For other j , one can show that $|G_{j,k\pm 1}(t, \eta)|$ are bounded by $C_k\tau_0$, where C_k is a polynomial of $(|\phi'_{k\pm 1}(t) - \phi'_k(t)| + \alpha_k)$, and $|G_{j,\ell}(t, \eta)|$ for other ℓ with $|\ell - k| \geq 2$ are much smaller.

By the above discussion, we can conclude that for $(t, \eta) \in O_k$, $B_k(t, \eta)$, $D_k(t, \eta)$, $E_k(t, \eta)$, $F_k(t, \eta)$ are dominated by $C'_k\tau_0$, where C'_k is a polynomial of $(\phi'_k(t) - \phi'_{k-1}(t) + \alpha_k)$ and $(\phi'_{k+1}(t) - \phi'_k(t) + \alpha_k)$ with degree ≤ 2 . Therefore, if $\varepsilon_1, \varepsilon_3, \tau_0$ are small, then Bd_1 in (88) is small.

Next we look at Bd_2 in (91). First we consider $M_{\ell,k}(t)$ defined by (85). For $\ell = k - 1$, since $\sigma(t)(\phi'_k(t) - \phi'_{k-1}(t)) > \alpha_{k-1} + \alpha_k$ and $|G_{k-1}(u)|$ is a decreasing and even function of u , we have

$$\begin{aligned} M_{k-1,k}(t) &= \int_{|u| < \alpha_k} |G_{k-1}(u + \sigma(t)(\phi'_k(t) - \phi'_{k-1}(t)))| du \\ &\leq \int_{|u| < \alpha_k} |G_{k-1}(u + \alpha_{k-1} + \alpha_k)| du \\ &\leq 2\alpha_k |G_{k-1}(\alpha_{k-1})| = 2\alpha_k\tau_0. \end{aligned}$$

We can show similarly $M_{k+1,k}(t) \leq 2\alpha_k\tau_0$. For other ℓ , $M_{\ell,k}(t)$ will be much smaller than $2\alpha_k\tau_0$.

Now let us look at $\int_{|u|\geq\alpha_k} G_k(u)du$:

$$\begin{aligned} \left| \int_{|u|\geq\alpha_k} G_k(u)du \right| &\leq \int_{|u|\geq\alpha_k} |G_k(u)|du = \frac{1}{(1+(2\pi\phi_k''(t)\sigma^2(t))^2)^{\frac{1}{4}}} \int_{|u|\geq\alpha_k} e^{-\frac{2\pi^2}{1+(2\pi\phi_k''(t)\sigma^2(t))^2}u^2} du \\ &= (1+(2\pi\phi_k''(t)\sigma^2(t))^2)^{\frac{1}{4}} \int_{|u|\geq\frac{\alpha_k}{\sqrt{1+(2\pi\phi_k''(t)\sigma^2(t))^2}}} e^{-2\pi^2u^2} du \\ &\leq (1+(2\pi\phi_k''(t)\sigma^2(t))^2)^{\frac{1}{4}} \int_{|u|\geq\alpha} e^{-2\pi^2u^2} du \leq (1+(2\pi\phi_k''(t)\sigma^2(t))^2)^{\frac{1}{4}} \frac{1}{\sqrt{2\pi}} \frac{\tau_0}{1+\sqrt{1-\tau_0}}, \end{aligned}$$

where the last inequality follows from (58).

Finally we present the proof of Theorem 2. The proof of Theorem 2 Part (a) is similar to that for Theorem 1 Part (a) and we skip the details.

Proof of Theorem 2 Part (b). Plugging $\partial_t \tilde{V}_x(t, \eta)$ in (82) to $\omega_x^{\text{adp,2nd,c}}$ in (29), we have

$$\begin{aligned} \omega_x^{\text{adp,2nd,c}} &= \frac{\partial_t \tilde{V}_x(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} + \frac{\sigma'(t)}{i2\pi\sigma(t)} - \frac{\tilde{V}_x^{g_1}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} P_0(t, \eta) + \frac{\sigma'(t)}{\sigma(t)} \frac{\tilde{V}_x^{g_3}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} \\ &= \frac{1}{i2\pi \tilde{V}_x(t, \eta)} \left\{ (i2\pi\phi_k'(t) - \frac{\sigma'(t)}{\sigma(t)}) \tilde{V}_x(t, \eta) + i2\pi\phi_k''(t)\sigma(t)\tilde{V}_x^{g_1}(t, \eta) - \frac{\sigma'(t)}{\sigma(t)} \tilde{V}_x^{g_3}(t, \eta) + \text{Res}_1 \right\} \\ &\quad + \frac{\sigma'(t)}{i2\pi\sigma(t)} - \frac{\tilde{V}_x^{g_1}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} P_0(t, \eta) + \frac{\sigma'(t)}{\sigma(t)} \frac{\tilde{V}_x^{g_3}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} \\ &= \phi_k'(t) + \phi_k''(t)\sigma(t) \frac{\tilde{V}_x^{g_1}(t, \eta)}{\tilde{V}_x(t, \eta)} + \frac{\text{Res}_1}{i2\pi \tilde{V}_x(t, \eta)} - \frac{\tilde{V}_x^{g_1}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} P_0(t, \eta) \\ &= \phi_k'(t) + \phi_k''(t)\sigma(t) \frac{\tilde{V}_x^{g_1}(t, \eta)}{\tilde{V}_x(t, \eta)} + \frac{\text{Res}_1}{i2\pi \tilde{V}_x(t, \eta)} - \frac{\tilde{V}_x^{g_1}(t, \eta)}{i2\pi \tilde{V}_x(t, \eta)} (i2\pi\sigma(t)\phi_k''(t) + \text{Res}_3) \\ &= \phi_k'(t) + \frac{\text{Res}_1}{i2\pi \tilde{V}_x(t, \eta)} - \frac{\tilde{V}_x^{g_1}(t, \eta)\text{Res}_3}{i2\pi \tilde{V}_x(t, \eta)} \\ &= \phi_k'(t) + \text{Res}_4, \end{aligned}$$

where the last third equation follows from (83). This shows (86).

For (87), with the assumptions $|\tilde{V}_x(t, \eta)| > \tilde{\varepsilon}_1$ and

$$|\partial_\eta(\tilde{V}_x^{g_1}(t, \eta)/\tilde{V}_x(t, \eta))| = \left| \tilde{V}_x(t, \eta)\partial_\eta \tilde{V}_x^{g_1}(t, \eta) - \partial_\eta \tilde{V}_x(t, \eta)\tilde{V}_x^{g_1}(t, \eta) \right| / |\tilde{V}_x(t, \eta)|^2 > \tilde{\varepsilon}_2,$$

we have

$$\begin{aligned}
|\text{Res}_4| &= \left| \frac{\text{Res}_1}{i2\pi\tilde{V}_x(t,\eta)} - \frac{\tilde{V}_x^{g_1}(t,\eta)\text{Res}_3}{i2\pi\tilde{V}_x(t,\eta)} \right| \\
&= \left| \frac{\text{Res}_1}{i2\pi\tilde{V}_x(t,\eta)} - \frac{\tilde{V}_x^{g_1}(t,\eta)}{i2\pi\tilde{V}_x(t,\eta)} \frac{\tilde{V}_x(t,\eta)\text{Res}_2 - \partial_\eta\tilde{V}_x(t,\eta)\text{Res}_1}{\tilde{V}_x(t,\eta)\partial_\eta\tilde{V}_x^{g_1}(t,\eta) - \tilde{V}_x^{g_1}(t,\eta)\partial_\eta\tilde{V}_x(t,\eta)} \right| \\
&\leq \frac{|\text{Res}_1|}{2\pi|\tilde{V}_x(t,\eta)|} + \frac{|\tilde{V}_x^{g_1}(t,\eta)|}{2\pi|\tilde{V}_x(t,\eta)|} \frac{\left(|\partial_\eta\tilde{V}_x(t,\eta)| |\text{Res}_1| + |\tilde{V}_x(t,\eta)| |\text{Res}_2| \right) / |\tilde{V}_x(t,\eta)|^2}{\left| \tilde{V}_x(t,\eta)\partial_\eta\tilde{V}_x^{g_1}(t,\eta) - \tilde{V}_x^{g_1}(t,\eta)\partial_\eta\tilde{V}_x(t,\eta) \right| / |\tilde{V}_x(t,\eta)|^2} \\
&< \frac{|\text{Res}_1|}{2\pi\tilde{\varepsilon}_1} + \frac{1}{2\pi\tilde{\varepsilon}_1^3\tilde{\varepsilon}_2} |\tilde{V}_x^{g_1}(t,\eta)| \left(|\partial_\eta\tilde{V}_x(t,\eta)| |\text{Res}_1| + \tilde{\varepsilon}_1 |\text{Res}_2| \right). \tag{93}
\end{aligned}$$

Thus $|\text{Res}_4| < \text{Bd}_1$, as desired. \blacksquare

Proof of Theorem 2 Part (c). First we have the following result which can be derived as that on p.254 in [8]:

$$\lim_{\lambda \rightarrow 0} \int_{|\xi - \phi'_k(t)| < \tilde{\varepsilon}_3} R_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd},\lambda}(t,\xi) d\xi = \int_{Z_t} \tilde{V}_x(t,\eta) d\eta, \tag{94}$$

where

$$Z_t := \{ \eta : |\tilde{V}_x(t,\eta)| > \tilde{\varepsilon}_1, |\partial_\eta(\tilde{V}_x^{g_1}(t,\eta)/\tilde{V}_x(t,\eta))| > \tilde{\varepsilon}_2 \text{ and } |\phi'_k(t) - \omega_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd}}(t,\eta)| < \tilde{\varepsilon}_3 \}.$$

Denote

$$W_t := \{ \eta : |\tilde{V}_x(t,\eta)| > \tilde{\varepsilon}_1, |\partial_\eta(\tilde{V}_x^{g_1}(t,\eta)/\tilde{V}_x(t,\eta))| > \tilde{\varepsilon}_2 \text{ and } (t,\eta) \in O_k \}.$$

Then we have $W_t = Z_t$. Indeed, by Theorem 2 part (b), if $\eta \in W_t$, then $|\phi'_k(t) - \omega_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd}}(t,\eta)| < \text{Bd}_1 \leq \tilde{\varepsilon}_3$. Thus $\eta \in Z_t$. Hence $W_t \subseteq Z_t$.

On the other hand, suppose $\eta \in Z_t$. Since $|\tilde{V}_x(t,\eta)| > \tilde{\varepsilon}_1$, by Theorem 2 part (a), $(t,\eta) \in O_\ell$ for an ℓ in $\{1, 2, \dots, K\}$. If $\ell \neq k$, then

$$\begin{aligned}
|\phi'_k(t) - \omega_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd}}(t,\eta)| &\geq |\phi'_k(t) - \phi'_\ell(t)| - |\phi'_\ell(t) - \omega_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd}}(t,\eta)| \\
&> L_k(t) - \tilde{\varepsilon}_3 \geq \tilde{\varepsilon}_3,
\end{aligned}$$

and this contradicts to the assumption $\eta \in Z_t$ with $|\phi'_k(t) - \omega_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd}}(t,\eta)| < \tilde{\varepsilon}_3$, where we have used the fact $|\phi'_k(t) - \phi'_\ell(t)| \geq L_k(t)$ and $|\phi'_\ell(t) - \omega_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd}}(t,\eta)| < \text{Bd}_1 \leq \tilde{\varepsilon}_3$ by Theorem 2 part (b). Hence $\ell = k$ and $\eta \in W_t$. Thus we know $Z_t = W_t$.

The facts $Z_t = W_t$ and $W_t \cap U_t = \emptyset$, together with (94), imply that

$$\begin{aligned}
\lim_{\lambda \rightarrow 0} \int_{|\xi - \phi'_k(t)| < \tilde{\varepsilon}_3} R_{x,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}^{\text{adp},2\text{nd},\lambda}(t,\xi) d\xi &= \int_{W_t} \tilde{V}_x(t,\eta) d\eta = \int_{W_t \cup U_t} \tilde{V}_x(t,\eta) d\eta - \int_{U_t} \tilde{V}_x(t,\eta) d\eta \\
&= \int_{\{|\tilde{V}_x(t,\eta)| > \tilde{\varepsilon}_1\} \cap \{(t,\eta) \in O_k\}} \tilde{V}_x(t,\eta) d\eta - \int_{U_t} \tilde{V}_x(t,\eta) d\eta. \tag{95}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \left| \sigma(t) \int_{\{|\tilde{V}_x(t,\eta)| > \tilde{\varepsilon}_1\} \cap \{\eta:(t,\eta) \in O_k\}} \tilde{V}_x(t,\eta) d\eta - g(0)x_k(t) \right| \\
&= \left| \sigma(t) \int_{\{\eta:(t,\eta) \in O_k\}} \tilde{V}_x(t,\eta) d\eta - g(0)x_k(t) - \sigma(t) \int_{\{|\tilde{V}_x(t,\eta)| \leq \tilde{\varepsilon}_1\} \cap \{\eta:(t,\eta) \in O_k\}} \tilde{V}_x(t,\eta) d\eta \right| \\
&\leq \sigma(t) \tilde{\varepsilon}_1 \frac{2\alpha_k}{\sigma(t)} + \left| \sigma(t) \int_{\{\eta:(t,\eta) \in O_k\}} \left(\sum_{\ell=1}^K x_\ell(t) G_{0,\ell}(t,\eta) + \text{res}_0 \right) d\eta - g(0)x_k(t) \right| \\
&\leq 2\tilde{\varepsilon}_1 \alpha_k + \sigma(t) \sigma(t) \Pi_0(t) \frac{2\alpha_k}{\sigma(t)} + |x_k(t)| \int_{|u| < \alpha_k} G_k(u) du - g(0)x_k(t) \\
&\quad + \sum_{\ell \neq k} A_\ell(t) \left| \sigma(t) \int_{\{\eta:(t,\eta) \in O_k\}} G_{0,\ell}(t,\eta) d\eta \right| \\
&\leq 2\alpha_k (\tilde{\varepsilon}_1 + \sigma(t) \Pi_0(t)) + |x_k(t)| \int_{\mathbb{R}} G_k(u) du - g(0)x_k(t) - x_k(t) \int_{|u| \geq \alpha_k} G_k(u) du + \sum_{\ell \neq k} A_\ell(t) M_{\ell,k}(t) \\
&= 2\alpha_k (\tilde{\varepsilon}_1 + \sigma(t) \Pi_0(t)) + |x_k(t)g(0) - g(0)x_k(t) - x_k(t) \int_{|u| \geq \alpha_k} G_k(u) du| + \sum_{\ell \neq k} A_\ell(t) M_{\ell,k}(t) \\
&= 2\alpha_k (\tilde{\varepsilon}_1 + \sigma(t) \Pi_0(t)) + A_k(t) \left| \int_{|u| \geq \alpha_k} G_k(u) du \right| + \sum_{\ell \neq k} A_\ell(t) M_{\ell,k}(t),
\end{aligned}$$

where we have used the fact:

$$\int_{\mathbb{R}} G_k(u) du = \int_{\mathbb{R}} \mathcal{F}\{e^{i\pi\phi_k''(t)\tau^2} g(\tau)\}(u) du = \left(e^{i\pi\phi_k''(t)\tau^2} g(\tau) \right) \Big|_{\tau=0} = g(0).$$

Hence, we have

$$\left| \frac{\sigma(t)}{g(0)} \int_{\{|\tilde{V}_x(t,\eta)| > \tilde{\varepsilon}_1\} \cap \{\eta:(t,\eta) \in O_k\}} \tilde{V}_x(t,\eta) d\eta - x_k(t) \right| \leq \text{Bd}'_2. \quad (96)$$

In addition,

$$\begin{aligned}
& \left| \sigma(t) \int_{U_t} \tilde{V}_x(t,\eta) d\eta \right| = \sigma(t) \left| \int_{U_t} \left(\sum_{\ell=1}^K x_\ell(t) G_{0,\ell}(t,\eta) + \text{res}_0 \right) d\eta \right| \\
&\leq \sigma(t) \sigma(t) \Pi_0(t) \frac{2\alpha_k}{\sigma(t)} + \sigma(t) A_k(t) \sup_{\eta \in U_t} |G_k(\sigma(t)(\eta - \phi_k'(t)))| |U_t| + \sum_{\ell \neq k} A_\ell(t) \left| \sigma(t) \int_{\{\eta:(t,\eta) \in O_k\}} G_{0,\ell}(t,\eta) d\eta \right| \\
&\leq 2\alpha_k \sigma(t) \Pi_0(t) + \sigma(t) A_k(t) \|g\|_1 |U_t| + \sum_{\ell \neq k} A_\ell(t) M_{\ell,k}(t) \leq \text{Bd}''_2 |g(0)|.
\end{aligned}$$

The above estimates, together with (95), lead to (90). This completes the proof of Theorem 2 part (c). \blacksquare

Appendix

In this appendix, we provide the proofs of Lemmas 1 and 2. For simplicity of presentation, we drop x, t, η in $\tilde{V}_x(t, \eta)$.

Proof of Lemma 1. By (63) with g replaced by g' ,

$$\begin{aligned}
\tilde{V}g' &= \sum_{\ell=1}^K \int_{\mathbb{R}} x_{\ell}(t) e^{i2\pi(\phi'_{\ell}(t)\tau + \frac{1}{2}\phi''_{\ell}(t)\tau^2)} \frac{1}{\sigma(t)} g' \left(\frac{\tau}{\sigma(t)} \right) e^{-i2\pi\eta\tau} d\tau + \text{res}'_0 \\
&= \sum_{\ell=1}^K \int_{\mathbb{R}} x_{\ell}(t) e^{-i2\pi(\eta - \phi'_{\ell}(t))\tau + i\pi\phi''_{\ell}(t)\tau^2} \frac{\partial}{\partial\tau} \left(g \left(\frac{\tau}{\sigma(t)} \right) \right) d\tau + \text{res}'_0 \\
&= - \sum_{\ell=1}^K \int_{\mathbb{R}} \frac{\partial}{\partial\tau} \left(x_{\ell}(t) e^{-i2\pi(\eta - \phi'_{\ell}(t))\tau + i\pi\phi''_{\ell}(t)\tau^2} \right) g \left(\frac{\tau}{\sigma(t)} \right) d\tau + \text{res}'_0 \\
&= i2\pi \sum_{\ell=1}^K x_{\ell}(t) (\eta - \phi'_{\ell}(t)) \int_{\mathbb{R}} e^{-i2\pi(\eta - \phi'_{\ell}(t))\tau + i\pi\phi''_{\ell}(t)\tau^2} g \left(\frac{\tau}{\sigma(t)} \right) d\tau \\
&\quad - i2\pi \sum_{\ell=1}^K x_{\ell}(t) \phi''_{\ell}(t) \int_{\mathbb{R}} e^{-i2\pi(\eta - \phi'_{\ell}(t))\tau + i\pi\phi''_{\ell}(t)\tau^2} \tau g \left(\frac{\tau}{\sigma(t)} \right) d\tau + \text{res}'_0 \\
&= i2\pi\sigma(t) \sum_{\ell=1}^K x_{\ell}(t) (\eta - \phi'_{\ell}(t)) G_{0,\ell}(t, \eta) - i2\pi\sigma^2(t) \sum_{\ell=1}^K x_{\ell}(t) \phi''_{\ell}(t) G_{1,\ell}(t, \eta) + \text{res}'_0.
\end{aligned}$$

This and (59) imply that

$$\begin{aligned}
&\partial_t \tilde{V} + \frac{\sigma'(t)}{\sigma(t)} (\tilde{V} + \tilde{V}^{g_3}) - i2\pi\phi'_k(t)\tilde{V} - i2\pi\phi''_k(t)\sigma(t)\tilde{V}^{g_1} \\
&= i2\pi\eta\tilde{V} - \frac{1}{\sigma(t)} \tilde{V}^{g'} - i2\pi\phi'_k(t)\tilde{V} - i2\pi\phi''_k(t)\sigma(t)\tilde{V}^{g_1} \\
&= i2\pi(\eta - \phi'_k(t)) \left(\sum_{\ell=1}^K x_{\ell}(t) G_{0,\ell}(t, \eta) + \text{res}_0 \right) \\
&\quad - \frac{1}{\sigma(t)} \left(i2\pi\sigma(t) \sum_{\ell=1}^K x_{\ell}(t) (\eta - \phi'_{\ell}(t)) G_{0,\ell}(t, \eta) - i2\pi\sigma^2(t) \sum_{\ell=1}^K x_{\ell}(t) \phi''_{\ell}(t) G_{1,\ell}(t, \eta) + \text{res}'_0 \right) \\
&\quad - i2\pi\phi''_k(t)\sigma(t) \left(\sum_{\ell=1}^K x_{\ell}(t) G_{1,\ell}(t, \eta) + \text{res}_1 \right) \\
&= i2\pi \sum_{\ell \neq k} x_{\ell}(t) (\phi'_{\ell}(t) - \phi'_k(t)) G_{0,\ell}(t, \eta) + i2\pi\sigma(t) \sum_{\ell \neq k} x_{\ell}(t) (\phi''_{\ell}(t) - \phi''_k(t)) G_{1,\ell}(t, \eta) \\
&\quad + i2\pi(\eta - \phi'_k(t)) \text{res}_0 - \frac{\text{res}'_0}{\sigma(t)} - i2\pi\phi''_k(t)\sigma(t) \text{res}_1 \\
&= i2\pi B_k(t, \eta) + i2\pi\sigma(t) D_k(t, \eta) + i2\pi(\eta - \phi'_k(t)) \text{res}_0 - \frac{\text{res}'_0}{\sigma(t)} - i2\pi\phi''_k(t)\sigma(t) \text{res}_1 \\
&= \text{Res}_1.
\end{aligned}$$

This completes the proof of Lemma 1. ■

Proof of Lemma 2. One can obtain for $j \geq 1$,

$$\frac{\partial}{\partial\eta} G_{j,k}(t, \eta) = -i2\pi\sigma(t) G_{j+1,k}(t, \eta).$$

Thus we have

$$\partial_\eta B_k(t, \eta) = -i2\pi\sigma(t)E_k(t, \eta), \quad \partial_\eta D_k(t, \eta) = -i2\pi\sigma(t)F_k(t, \eta).$$

In addition, it is straightforward to verify that

$$\partial_\eta \text{res}_j = -i2\pi\sigma(t) \text{res}_{j+1}, \quad \partial_\eta \text{res}'_j = -i2\pi\sigma(t) \text{res}'_{j+1}.$$

Hence we have

$$\partial_\eta \text{Res}_1 = \text{Res}_2. \quad (97)$$

Taking the partial derivate with respect to η to both sides of (82) and using (97), we have

$$\partial_\eta \partial_t \tilde{V} = \left(i2\pi\phi'_k(t) - \frac{\sigma'(t)}{\sigma(t)} \right) \partial_\eta \tilde{V} + i2\pi\phi''_k(t)\sigma(t)\partial_\eta \tilde{V}^{g_1} - \frac{\sigma'(t)}{\sigma(t)} \partial_\eta \tilde{V}^{g_3} + \text{Res}_2. \quad (98)$$

Note that

$$P_0(t, \eta) = \frac{1}{\tilde{V}\partial_\eta \tilde{V}^{g_1} - \tilde{V}^{g_1}\partial_\eta \tilde{V}} \left(\tilde{V}\partial_\eta \partial_t \tilde{V} - \partial_\eta \tilde{V}\partial_t \tilde{V} + \frac{\sigma'(t)}{\sigma(t)} (\tilde{V}\partial_\eta \tilde{V}^{g_3} - \tilde{V}^{g_3}\partial_\eta \tilde{V}) \right).$$

Thus, by (82) and (98),

$$\begin{aligned} & (P_0(t, \eta) - i2\pi\sigma(t)\phi''_k(t)) (\tilde{V}\partial_\eta \tilde{V}^{g_1} - \tilde{V}^{g_1}\partial_\eta \tilde{V}) \\ &= \tilde{V}\partial_\eta \partial_t \tilde{V} - \partial_\eta \tilde{V}\partial_t \tilde{V} + \frac{\sigma'(t)}{\sigma(t)} (\tilde{V}\partial_\eta \tilde{V}^{g_3} - \tilde{V}^{g_3}\partial_\eta \tilde{V}) - i2\pi\sigma(t)\phi''_k(t) (\tilde{V}\partial_\eta \tilde{V}^{g_1} - \tilde{V}^{g_1}\partial_\eta \tilde{V}) \\ &= \tilde{V} \left(\left(i2\pi\phi'_k(t) - \frac{\sigma'(t)}{\sigma(t)} \right) \partial_\eta \tilde{V} + i2\pi\phi''_k(t)\sigma(t)\partial_\eta \tilde{V}^{g_1} - \frac{\sigma'(t)}{\sigma(t)} \partial_\eta \tilde{V}^{g_3} + \text{Res}_2 \right) \\ & \quad - \partial_\eta \tilde{V} \left(\left(i2\pi\phi'_k(t) - \frac{\sigma'(t)}{\sigma(t)} \right) \tilde{V} + i2\pi\phi''_k(t)\sigma(t)\tilde{V}^{g_1} - \frac{\sigma'(t)}{\sigma(t)} \tilde{V}^{g_3} + \text{Res}_1 \right) \\ & \quad + \frac{\sigma'(t)}{\sigma(t)} (\tilde{V}\partial_\eta \tilde{V}^{g_3} - \tilde{V}^{g_3}\partial_\eta \tilde{V}) - i2\pi\sigma(t)\phi''_k(t) (\tilde{V}\partial_\eta \tilde{V}^{g_1} - \tilde{V}^{g_1}\partial_\eta \tilde{V}) \\ &= \tilde{V} \text{Res}_2 - \partial_\eta \tilde{V} \text{Res}_1. \end{aligned}$$

Therefore, we have $P_0(t, \eta) - i2\pi\sigma(t)\phi''_k(t) = \frac{\tilde{V} \text{Res}_2 - \partial_\eta \tilde{V} \text{Res}_1}{\tilde{V}\partial_\eta \tilde{V}^{g_1} - \tilde{V}^{g_1}\partial_\eta \tilde{V}} = \text{Res}_3$, as desired. This completes the proof of Lemma 2. \blacksquare

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