A New Sufficient Condition for the Orthonormality of Refinable Functions

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Abstract. A sufficient condition for the orthonormality of refinable vector valued functions is given. We shall use this sufficient condition to check the orthonormality of several well-known examples of multiscaling functions.

§1. Introduction

In recent years, multiscaling functions and multiwavelets have been studied extensively. In [6], a characterization of multiscaling functions and multiwavelets with semi-orthogonality was established. In particular, some examples of spline-wavelets with multiple knots were given. Examples of cubic and quintic finite elements and their corresponding multiwavelets were also studied in [7]. In [2], Chui and Lian introduced a general scheme for constructing symmetric and antisymmetric compactly supported refinable functions and multiwavelets. Geronimo, Hardin and Massopust in [5] used fractal interpolation to construct orthonormal multiscaling functions, and their corresponding multiwavelets were also given in [8].

One of the main difficulties in the construction of compactly supported multiscaling function and the corresponding multi-wavelets with arbitrarily high regularity is the orthonormal conditions. Even we know that the multivariate and multi-scale generalization of the Lawton condition is a sufficient condition for the orthogonality of multi-scaling functions, it is very difficult to be applied. It is interesting to find a feasible necessary and sufficient condition for the orthonormality of the multi-scaling functions. In [9], Wang gave a necessary and sufficient condition which is based on the infinite product of the matrix mask. In this paper, we shall give a new sufficient condition for the orthonormality which is sometimes easier to use, especially when the multiplicity $r = 2$. The sufficient condition is similar to the Cohen condition in [3] and is based on the structure of the zeros of the determinant of the matrix symbol. We shall use our sufficient condition to check the orthonormality

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of the well-known example of compactly supported Chui and Lian's refinable functions with support on [0, 2] and [0, 3] (cf. [2]) and Geronimo, Hardin and Massopust's multiscaling function (cf. [5]). We should point out that the integer translates of Geronimo, Hardin and Massopust's multiscaling function are indeed orthonormal by their construction.

§2. A New Orthonomality Condition

Let P_k be a real valued matrix of size $r \times r$ and let $\phi := (\phi_1, ..., \phi_r)^T$ with $\phi_1, ..., \phi_r \in L_2(\mathbf{R})$ satisfy a refinable equation:

$$\phi(x) = \sum_{k=0}^{n} P_k \phi(2x - k), x \in \mathbf{R}.$$

Such a ϕ is called a refinable vector valued function. Its Fourier transform is

$$\hat{\phi}(\omega) = \mathcal{P}(\omega/2)\hat{\phi}(\omega/2) \tag{1}$$

with $\mathcal{P}(\omega) = \frac{1}{2}\sum_{k=0}^{n} P_k e^{-ik\omega}$. We can easily see that symbol $\mathcal{P}(\omega)$ is an $r \times r$ matrix with trigonometric polynomial entries. By repeated applications of (1), we formly have

$$\hat{\phi}(\omega) = \left(\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)\right) \hat{\phi}(0).$$

If the infinite product above converges, the $\hat{\phi}$ is well-defined and we will say ϕ is generated by $\mathcal{P}$. The following lemma ensures the convergence of the above infinite product. (See [1] for a proof).

Lemma 1. *The infinite matrix product*

$$\prod_{j=1}^{\infty} \mathcal{P}(\omega/2^j)$$

converges uniformly on compact sets to a continuous matrix-valued function if and only if $\mathcal{P}(0)$ *has eigenvalues* $\lambda_1 = ... = \lambda_s = 1$ *and* $|\lambda_{s+1}|, ..., |\lambda_r| < 1$, *with the eigenvalue* 1 *non-degenerate for* $s \geq 1$.

For the continuous orthonormal refinable function $\phi(x)$, the eigenvalue 1 must be simple (see [9]). That is, we have

Lemma 2. *Suppose that* ϕ *generated by* $\mathcal{P}$ *is continuous. Suppose that the integer translates of the refinable function* ϕ *form an orthogonal set, i.e.,*

$$\int\int_{\mathbf{R}} \phi(x - \alpha)\phi^*(x - \beta)dx = \delta_{\alpha,\beta}\mathbf{I}_r, \quad \forall \alpha, \beta \in \mathbf{Z},$$

where $\mathbf{I}_r$ *is the identity matrix of size* $r \times r$. *Then* 1 *must be a simple eigenvalue of the matrix* $\mathcal{P}(0)$, *and all other eigenvalues* λ *of* $\mathcal{P}(0)$ *must have* $|\lambda| < 1$.

To study the orthonormality, we define the Gramian matrix

$$\mathbf{\Lambda}(\omega) := [\sum_{n \in Z} \hat{\phi}_l(\omega + 2\pi n)\overline{\hat{\phi}_m(\omega + 2\pi n)}]_{1 \leq l,m \leq r}.$$

The following lemmas are well-known.

Lemma 3. *$\{\phi_1(x), ..., \phi_r(x)\}$ is an orthonormal refinable function if and only if $\mathbf{\Lambda}(\omega)$ is an identity matrix $\mathbf{I}_r$.*

Lemma 4. *If $\phi(x)$ is an orthonormal refinable function, then $\mathcal{P}(\omega)$ satisfies the following identity:*

$$\mathcal{P}\mathcal{P}^*(\omega) + \mathcal{P}\mathcal{P}^*(\omega+\pi) = \mathbf{I}_r. \tag{2}$$

We thus define an operator on $r \times r$ matrix $\mathbf{L}(\omega)$ as

$$(\mathbf{P}_0\mathbf{L})(\omega) := \mathcal{P}\mathbf{L}\mathcal{P}^*(\omega/2) + \mathcal{P}\mathbf{L}\mathcal{P}^*(\omega/2+\pi) \tag{3}$$

It is easy to see that $\mathbf{\Lambda}(\omega)$ is a fixed point of the operator $\mathbf{P}_0$. The main result in this paper is as follows. We refer to [4] for a proof.

Theorem 5. *Let $\mathcal{P}(\omega)$ be a matrix symbol of a vector valued function ϕ and satisfy the condition in Lemma 2. Suppose that $\mathcal{P}$ satisfies (2). Then ϕ is a multiscaling function if the refinable matrix symbol $\mathcal{P}$ satisfies the following conditions.*

1° *The trigonometric polynomial $\det(\mathcal{P}(\omega+\pi))$ does not have zeros which are in a nontrivial finite cycle invariant under the operator τ, where $\tau\omega = 2\omega$ mod 2π;*

2° *Any nonzero invariant subspace of $\mathcal{P}^*(0)$ is not contained in Null($\mathcal{P}^*(\pi)$) or the dimension of Null($\mathcal{P}^*(\pi)$) is at most one;*

3° *For the multiplicity $r = 2$, the two operators $\mathbf{V}_i, i = 1, 2$ defined by*

$$(\mathbf{V}_0 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix})(\omega) := \frac{1}{2}\left((\mathcal{P}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix})(\omega/2) + (\mathcal{P}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix})(\omega/2+\pi)\right)$$
$$(\mathbf{V}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix})(\omega) := \frac{e^{i\omega/2}}{2}((\mathcal{P}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix})(\omega/2) - (\mathcal{P}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix})(\omega/2+\pi))$$

have no common eigenvectors; For $r \geq 2$, for any nonzero Hermitian nonnegative definite matrix Θ which is a fixed point of $\mathbf{P}_0$, $\det(\Theta(\omega))$ is not identically zero.

Remark 6. *When $r = 2$, the dimension of Null($\mathcal{P}^*(\pi)$) is at most one. Otherwise, we have $\mathcal{P}^*(\pi) = 0$. Thus, (2) implies $\mathcal{P}^*(0)\mathcal{P}(0) = \mathbf{I}_2$. That is, $\mathcal{P}(0)$ is a unitary matrix which contradicts to the condition in Lemma 2 on $\mathcal{P}(0)$.*
□

§3. Verifications of the Orthonormality of Some Refinable Functions

We shall verify that the integer translates of Chui and Lian's refinable functions and that of Geronimo, Hardin and Massopust's refinable function are orthonormal.

Example 7. *Chui and Lian's refinable function with support* $[0,2]$ *(cf. [2]).*

The matrix symbol $\mathcal{P}(\omega)$ is given by

$$\mathcal{P}(\omega)=\frac{1}{2}\begin{bmatrix}\frac{1}{2}+z+\frac{1}{2}z^2 & \frac{1}{2}-\frac{1}{2}z^2\\ -\frac{\sqrt{7}}{4}+\frac{\sqrt{7}}{4}z^2 & -\frac{\sqrt{7}}{4}+\frac{1}{2}z-\frac{\sqrt{7}}{4}z^2\end{bmatrix}=\frac{1}{2}(P_0+P_1z+P_2z^2)$$

where $z=e^{-i\omega}$, $P_0=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{\sqrt{7}}{4} & -\frac{\sqrt{7}}{4}\end{bmatrix}$, $P_1=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}$ and $P_2=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{\sqrt{7}}{4} & -\frac{\sqrt{7}}{4}\end{bmatrix}$. Note that $\mathcal{P}(0)=\begin{bmatrix}1 & 0\\ 0 & \frac{1-\sqrt{7}}{4}\end{bmatrix}$ satisfies the condition in Lemmas 1 and 2. And $\det(\mathcal{P}(\omega))=\frac{1-\sqrt{7}}{8}z(1+z)^2$. The only zero of $\det(\mathcal{P}(\omega+\pi))$ is $\omega=0$, thus, it satisfies 1° of Theorem 5.

Now we look at the operators $\mathbf{V}_0$ and $\mathbf{V}_1$ in Theorem 5. They act on trigonometric polynomial vectors $[f_1,f_2]^T(\omega)=\mathbf{x}_0+\mathbf{x}_1z+\mathbf{x}_2z^2$ with $z=e^{-i\omega}$ and $\mathbf{x}_i=[a_i,b_i]^T\in\mathbf{R}^2$ for $i=0,1,2$. The coefficients of the following polynomials with $C(\omega)=c_1+c_2z+c_3z^2$ and $D(\omega)=d_1+d_2z+d_3z^2$

$$\mathbf{V}_0\left(\begin{bmatrix}f_1\\ f_2\end{bmatrix}\right),\quad C(\omega)\begin{bmatrix}f_1\\ f_2\end{bmatrix},\quad \mathbf{V}_1\left(\begin{bmatrix}f_1\\ f_2\end{bmatrix}\right),\quad D(\omega)\begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

are given by

$$\begin{bmatrix}P_0 & 0 & 0\\ P_2 & P_1 & P_0\\ 0 & 0 & P_2\end{bmatrix}\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix},\quad c_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\\ 0\\ 0\end{bmatrix}+c_2\begin{bmatrix}0\\ \mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\\ 0\end{bmatrix}+c_3\begin{bmatrix}0\\ 0\\ \mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix},$$

$$\begin{bmatrix}P_1 & P_0 & 0\\ 0 & P_2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix},\quad d_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\\ 0\\ 0\end{bmatrix}+d_2\begin{bmatrix}0\\ \mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\\ 0\end{bmatrix}+d_3\begin{bmatrix}0\\ 0\\ \mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}.$$

For simplicity, let us denote the two matrices with entries P_i's by V_0 and V_1.

I : Suppose that $\mathbf{x}_2\neq 0$. We have to have $c_2=c_3=d_2=d_3=0$. Then

$$V_0\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=c_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix},\quad V_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=d_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}.$$

However, we can immediately see that $d_1=0$ which is a contradiction.

II : Suppose that $\mathbf{x}_2=0$, but $\mathbf{x}_1\neq 0$. We have to have $c_3=d_3=0$. Then

$$V_0\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ 0\end{bmatrix}=c_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ 0\end{bmatrix}+c_2\begin{bmatrix}0\\ \mathbf{x}_0\\ \mathbf{x}_1\end{bmatrix},V_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ 0\end{bmatrix}=d_1\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ 0\end{bmatrix}+d_2\begin{bmatrix}0\\ \mathbf{x}_0\\ \mathbf{x}_1\end{bmatrix}$$

By $\mathbf{x}_1 \neq 0$, we can easily see that $c_2 = d_2 = 0$. Also $P_0\mathbf{x}_0 = c_1\mathbf{x}_0, P_2\mathbf{x}_1 = d_1\mathbf{x}_1$ and $P_1\mathbf{x}_0 + P_0\mathbf{x}_1 = d_1\mathbf{x}_0$. From the third equation, we can see that $\mathbf{x}_0 \neq 0$. Since $c_1 \neq 0$, from the first equation, we get $\mathbf{x}_0 = [2, -\sqrt{7}], c_1 = 1/2 - \sqrt{7}/4$. Since $d_1 \neq 0$, from the second equation, we get $\mathbf{x}_1 = [2, \sqrt{7}], d_1 = 1/2 - \sqrt{7}/4$. One can check that the third equation is not satisfied.

III : Suppose that $\mathbf{x}_2 = \mathbf{x}_1 = 0$, but $\mathbf{x}_0 \neq 0$. Then we have

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1\mathbf{x}_0 \\ c_2\mathbf{x}_0 \\ c_3\mathbf{x}_0 \end{bmatrix}, V_1 \begin{bmatrix} \mathbf{x}_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1\mathbf{x}_0 \\ d_2\mathbf{x}_0 \\ d_3\mathbf{x}_0 \end{bmatrix}$$

for some real numbers c_1, c_2, c_3, d_1, d_2 and d_3. Similarly, we can see that $c_3 = d_2 = d_3 = 0$. And $\mathbf{x}_0$ is an common eigenvector of matrices P_0, P_1 and P_2, this is impossible.

Therefore, $\mathbf{V}_0$ and $\mathbf{V}_1$ do not have common eigenvectors and the condition 3° of Theorem 5 holds. Since $r = 2$, we do not need to check condition 2°. Hence, the integer translates of the Chui and Lian's refinable function are orthonormal. □

Example 8. *The Chui and Lian's refinable function with support [0,3] (cf. [2]).*

Recall the matrix symbol associated with Chui and Lian's refinable function with support [0, 3] is $\mathcal{P}(\omega) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix}$, where

$$\begin{aligned}
P_{11}(z) &= \frac{10 - 3\sqrt{10}}{80}(1+z)(1 + (38 + 12\sqrt{10})z + z^2) \\
P_{12}(z) &= \frac{5\sqrt{6} - 2\sqrt{15}}{80}(1-z)(1+z)^2 \\
P_{21}(z) &= \frac{5\sqrt{6} - 3\sqrt{15}}{80}(1-z)(1 - 10(3 + \sqrt{10})z + z^2) \\
P_{22}(z) &= \frac{5 - 3\sqrt{10}}{1040}(1+z)(13 - (10 + 6\sqrt{10})z + 13z^2)
\end{aligned}$$

And

$$\begin{aligned}
\det(\mathcal{P}(z)) = -\frac{1}{640}(1+z)^2(-158 + 51\sqrt{10} + 2(-22 + 39\sqrt{10})z \\
+ 6(-106 + 9\sqrt{10})z^2 + 2(-22 + 39\sqrt{10})z^3 + (-158 + 51\sqrt{10})z^4)
\end{aligned}$$

Thus, $\det(\mathcal{P}(\omega + \pi))$ has only one zero at $\omega = 0$, which satisfies 1° of Theorem 5. We can also write $\mathcal{P}(\omega) = \frac{1}{2}(P_0 + P_1 z + P_2 z^2 + P_3 z^3)$ where

$$P_0 = \begin{bmatrix} \frac{10 - 3\sqrt{10}}{40} & \frac{5\sqrt{6} - 2\sqrt{15}}{40} \\ \frac{5\sqrt{6} - 3\sqrt{15}}{40} & \frac{5 - 3\sqrt{10}}{40} \end{bmatrix}, P_1 = \begin{bmatrix} \frac{30 + 3\sqrt{10}}{40} & \frac{5\sqrt{6} - 2\sqrt{15}}{40} \\ -\frac{5\sqrt{6} + 7\sqrt{15}}{40} & \frac{15 - 3\sqrt{10}}{40} \end{bmatrix}$$

$$P_2 = \begin{bmatrix} \frac{30+3\sqrt{10}}{40} & -\frac{5\sqrt{6}-2\sqrt{15}}{40} \\ \frac{5\sqrt{6}+7\sqrt{15}}{40} & \frac{15-3\sqrt{10}}{40} \end{bmatrix}, P_3 = \begin{bmatrix} \frac{10-3\sqrt{10}}{40} & -\frac{5\sqrt{6}-2\sqrt{15}}{40} \\ -\frac{5\sqrt{6}-3\sqrt{15}}{40} & \frac{5-3\sqrt{10}}{40} \end{bmatrix}$$

Let $[f_1, f_2]^T(\omega) = \mathbf{x}_0 + \mathbf{x}_1 z + \mathbf{x}_2 z^2 + \mathbf{x}_3 z^3$ and $\mathbf{x}_i = [a_i, b_i]^T \in \mathbf{R}^2$ for $i = 0, 1, 2, 3$. We define the operators $\mathbf{V}_0$ and $\mathbf{V}_1$ as follows.

$$\mathbf{V}_0\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right) : \quad V_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \qquad \mathbf{V}_1\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right) : \quad V_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix},$$

where $V_0 = \begin{bmatrix} P_0 & 0 & 0 & 0 \\ P_2 & P_1 & P_0 & 0 \\ 0 & P_3 & P_2 & P_1 \\ 0 & 0 & 0 & P_3 \end{bmatrix}$ and $V_1 = \begin{bmatrix} P_1 & P_0 & 0 & 0 \\ P_3 & P_2 & P_1 & P_0 \\ 0 & 0 & P_3 & P_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

I : $\mathbf{x}_3 \neq 0$

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}, \quad V_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = d_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}$$

It is easy to see that $d_1 = 0$, which is a contradiction.

II : $\mathbf{x}_3 = 0, \mathbf{x}_2 \neq 0$.

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}, V_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Since $\mathbf{x}_2 \neq 0$, we must have $c_2 = d_2 = 0$. Also $P_0\mathbf{x}_0 = c_1\mathbf{x}_0, P_3\mathbf{x}_2 = d_1\mathbf{x}_2, P_3\mathbf{x}_1 + P_2\mathbf{x}_2 = c_1\mathbf{x}_2$ and $P_1\mathbf{x}_0 + P_0\mathbf{x}_1 = d_1\mathbf{x}_0$. But these equations do not hold for $\mathbf{x}_2 \neq 0$ simultaneously.

III : $\mathbf{x}_3 = \mathbf{x}_2 = 0, \mathbf{x}_1 \neq 0$.

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1\mathbf{x}_0 \\ c_1\mathbf{x}_1 + c_2\mathbf{x}_0 \\ c_2\mathbf{x}_1 + c_3\mathbf{x}_0 \\ c_3\mathbf{x}_1 \end{bmatrix}, \quad V_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1\mathbf{x}_0 \\ d_1\mathbf{x}_1 + d_2\mathbf{x}_0 \\ d_2\mathbf{x}_1 + d_3\mathbf{x}_0 \\ d_3\mathbf{x}_1 \end{bmatrix}.$$

Since $\mathbf{x}_1 \neq 0$, we must have $c_3 = d_3 = d_2 = 0$. Also $P_0\mathbf{x}_0 = c_1\mathbf{x}_0, P_3\mathbf{x}_1 = c_2\mathbf{x}_1$ and $P_2\mathbf{x}_0 + P_1\mathbf{x}_1 = c_1\mathbf{x}_1 + c_2\mathbf{x}_0$. One can check that this is impossible for $\mathbf{x}_1 \neq 0$.

IV : $\mathbf{x}_3 = \mathbf{x}_2 = \mathbf{x}_1 = 0, \mathbf{x}_0 \neq 0$.

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1\mathbf{x}_0 \\ c_2\mathbf{x}_0 \\ c_3\mathbf{x}_0 \\ c_4\mathbf{x}_0 \end{bmatrix}, \quad V_1 \begin{bmatrix} \mathbf{x}_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1\mathbf{x}_0 \\ d_2\mathbf{x}_0 \\ d_3\mathbf{x}_0 \\ d_4\mathbf{x}_0 \end{bmatrix}.$$

Since $\mathbf{x}_0 \neq 0$, we must have $c_4 = d_4 = c_3 = d_3 = 0$. Then we have that $\mathbf{x}_0$ is a common eigenvector of the four matrices P_0, P_1, P_2, and P_3, which is impossible. □

Example 9. *the Geronimo, Hardin and Massopust refinable function.*

The matrix symbol associated with the Geronimo, Hardin and Massopust refinable function is

$$\mathcal{P}(\omega) = \frac{1}{40}\begin{bmatrix} 12(1+z) & 16\sqrt{2} \\ -\sqrt{2}(z+1)(z^2-10z+1) & -6+20z-6z^2 \end{bmatrix}$$
$$= \frac{1}{2}(P_0 + P_1 z + P_2 z^2 + P_3 z^3)$$

where

$$P_0 = \begin{bmatrix} \frac{3}{5} & \frac{4\sqrt{2}}{5} \\ -\frac{\sqrt{2}}{20} & -\frac{3}{10} \end{bmatrix}, P_1 = \begin{bmatrix} \frac{3}{5} & 0 \\ \frac{9\sqrt{2}}{20} & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 \\ \frac{9\sqrt{2}}{20} & -\frac{3}{10} \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{20} & 0 \end{bmatrix}$$

Note that $\det(\mathcal{P}(\omega)) = -\frac{1}{40}(1+z)^3$. The only zero of $\det(\mathcal{P}(\omega+\pi))$ is $\omega = 0$. 1° of Theorem 5 is satisfied. Suppose the common eigenvector of $\mathbf{V}_0$ and $\mathbf{V}_1$ is $[\, f_1 \quad f_2\,]^T(\omega) = \mathbf{x}_0 + \mathbf{x}_1 z + \mathbf{x}_2 z^2$. As in Example 7, letting $V_0 = \begin{bmatrix} P_0 & 0 & 0 \\ P_2 & P_1 & P_0 \\ 0 & P_3 & P_2 \end{bmatrix}$ and $V_1 = \begin{bmatrix} P_1 & P_0 & 0 \\ P_3 & P_2 & P_1 \\ 0 & 0 & P_3 \end{bmatrix}$, we have

$$\mathbf{V}_0(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}) : V_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}, \text{ and } \mathbf{V}_1(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}) : V_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.$$

I : Suppose that $\mathbf{x}_2 \neq 0$. We have

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \quad V_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = d_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.$$

It follows from $P_3\mathbf{x}_2 = d_1\mathbf{x}_2$ that d_1 must be zero which is a contradiction.

II : Suppose that $\mathbf{x}_2 = 0, \mathbf{x}_1 \neq 0$. We have

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix}, V_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix}$$

We can see that $d_2 = 0$ and $c_2 = 0$ from $0 = d_2\mathbf{x}_1$ and $P_3\mathbf{x}_1 = c_2\mathbf{x}_1$. We find that when $\mathbf{x}_0 = [\sqrt{2}/2, -1/2]^T, \mathbf{x}_1 = [0, -1/2]^T$ and $c_1 = d_1 = -1/5$, the above two equations hold. This implies that the operators $\mathbf{V}_0, \mathbf{V}_1$ have a common eigenvector, which is $[\sqrt{2}/2, -1/2(1+z)]^T$. In this situation, we need to consider $\Theta(2\omega) = \mathcal{P}\Theta\mathcal{P}^*(\omega) + \mathcal{P}\Theta\mathcal{P}^*(\omega+\pi)$. If $\det(\Theta(\omega)) \equiv 0$, then $\Theta(\omega) = \begin{bmatrix} \sqrt{2}/2 \\ -1/2(1+e^{-i\omega}) \end{bmatrix} [\sqrt{2}/2, -1/2(1+e^{i\omega})]$ (up to a constant multiple). And

$$\mathcal{P}(\omega) \begin{bmatrix} \sqrt{2}/2 \\ -1/2(1+e^{-i\omega}) \end{bmatrix} = -\left(\frac{1+e^{-i\omega}}{10}\right) \begin{bmatrix} \sqrt{2}/2 \\ -1/2(1+e^{-2i\omega}) \end{bmatrix}.$$

Therefore, $\det(\Theta(\omega)) \not\equiv 0$ since

$$\mathcal{P}\Theta\mathcal{P}^*(\omega) + \mathcal{P}\Theta\mathcal{P}^*(\omega+\pi) = \frac{1}{25}\Theta(2\omega) \neq \Theta(2\omega).$$

III : Suppose that $\mathbf{x}_2 = \mathbf{x}_1 = 0$, but $\mathbf{x}_0 \neq 0$. We have

$$V_0 \begin{bmatrix} \mathbf{x}_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1\mathbf{x}_0 \\ c_2\mathbf{x}_0 \\ c_3\mathbf{x}_0 \end{bmatrix}, V_1 \begin{bmatrix} \mathbf{x}_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1\mathbf{x}_0 \\ d_2\mathbf{x}_0 \\ d_3\mathbf{x}_0 \end{bmatrix}.$$

We can see $c_3 = d_3 = 0$. Also we can see that $\mathbf{x}_0$ is a common eigenvector of P_0, P_1, P_2 and P_3. But P_0 and P_1 do not have any common eigenvectors. Thus, the condition 3° of Theorem 5 holds. Therefore, the integer translates of the Geronimo, Hardin, and Massopust refinable function are orthonormal. □

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