Nonstationary Tight Wavelet Frames, II: Unbounded Intervals[1)]

by

Charles K. Chui[2)], Wenjie He

Department of Mathematics and Computer Science
University of Missouri–St. Louis
St. Louis, MO 63121-4499, USA

Joachim Stöckler

Universität Dortmund
Fachbereich Mathematik
44221 Dortmund, Germany

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Corresponding author:
Joachim Stöckler
Universität Dortmund
Institut für Angewandte Mathematik
44221 Dortmund, Germany
phone: +49-231-7553100
fax: +49-231-7555923
email: joachim.stoeckler@math.uni-dortmund.de

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[2)] This author is also with the Department of Statistics, Stanford University, Stanford, CA 94305

Abstract. From the definition of tight frames, normalized with frame bound constant equal to one, a tight frame of wavelets can be considered as a natural generalization of an orthonormal wavelet basis, with the only exception that the wavelets are not required to have norm equal to one. However, without the orthogonality property, the tight-frame wavelets do not necessarily have vanishing moments of order higher than one, although the associated multiresolution spaces may contain higher order polynomials locally. This observation motivated a relatively recent parallel development of the general theory of affine (i.e. stationary) tight frames by Daubechies-Han-Ron-Shen and the authors, with both papers published in this journal in 2002-2003. In the second issue of this volume of Special Issues, we introduced a general theory of nonstationary wavelet frames on a bounded interval, and emphasized, with illustrative examples, that in general such tight frames cannot be easily constructed by adopting the above-mentioned stationary wavelets as "interior" frame elements, even for the "uniform" setting. Hence, the results on nonstationary tight frames on a bounded interval obtained in our previous paper are definitely not follow-up of the present paper, in which we will introduce a general mathematical theory of nonstationary tight frames on unbounded intervals. While the "Fourier" and "matrix culculus" approaches were used in the above-mentioned works on stationary and nonstationary frames, respectively, we will engage a "kernel operator" approach to the development of the theory of nonstationary tight frames on unbounded intervals, and observe that this somewhat new approach could be considered as a unification of the previous considerations. The nonstationary notion discussed in this paper is very general, with (polynomial) splines of any (fixed) order on arbitrary but dense nested knot vectors as canonical examples, and in particular, eliminates the rigid assumptions of invariance in translations and scalings among different levels. In addition to the development of approximate duals and construction of compactly supported tight-frame wavelets with desirable order of vanishing moments, a unified formulation of the degree of approximation in Sobolev spaces of negative exponent, of order up to twice of that

of the corresponding approximate dual, is established in this paper. A thorough development for the general spline setting is a major focus of our study, with examples of tight frames of splines with multiple knots included to illustrate our constructive approach.

1. Introduction

This paper is devoted to the development of a general theory, along with constructive proofs, of nonstationary wavelet tight frames on both types of unbounded intervals: the (one-sided) infinite interval $I = [0, \infty)$ and the bi-infinite interval $I = \mathbb{R}=(-\infty, \infty)$. With (polynomial) splines of arbitrary (but fixed) order and on arbitrary nested knot vectors (that do not have finite accumulation points and whose union is dense on the unbounded interval under consideration) as canonical examples, this general theory of nonstationary tight frames avoids any rigid assumptions of invariance in translations and scalings among different levels. Hence, even for the bi-infinite interval, the usual "Fourier" approach for the study of tight frames of affine (i.e. stationary) wavelets has to be abandoned. Instead, we will adopt the "kernel operator" approach in this paper, and point out that this somewhat new approach can be viewed as a generalization and unification of the Fourier approach for the stationary setting and the "matrix calculus" approach, followed in the main body of our work on nonstationary tight frames of wavelets on a bounded interval in the companion paper [4] that appeared in the previous issue of this volume of Special Issues of this journal.

As already pointed out with illustrative examples in our previous paper [4], the study of tight frames of wavelets on a bounded interval in [4] cannot be altered to be a follow-up work of the present paper in a direct way, since the compactly supported tight-frame wavelets introduced in this paper cannot be easily used, in general, as "interior" wavelets of the tight frames for the bounded interval. On the other hand, the results obtained in [4] will be applied to facilitate our development of the theory of nonstationary tight wavelet frames for the (one-sided) infinite interval. In fact, although different proofs are needed, the results for the (one-sided) infinite interval setting to be derived in this paper can be formulated in precisely the same way as those for the bounded interval setting in [4]. Our development of nonstationary tight wavelet frames for the bi-infinite interval, however, requires a more elaborate theoretical setting, with various basic assumptions, which are superfluous for the (one-sided) infinite and bounded interval considerations. These assumptions are necessary for the bi-infinite interval for introducing the more elaborate notion of approximate duals.

To give a somewhat unified treatment, a "kernel operator" approach will be adopted in this paper. This approach could be viewed as some generalization of the Fourier approach for the study of stationary affine frames on the bi-infinite interval and the "matrix calculus" approach for the bulk of the derivations in our study [4] of nonstationary tight frames on bounded intervals. The kernel operator approach also facilitates our discussion and derivation of the degree of approximation in Sobolev spaces of negative exponent, of order up to twice of that of the corresponding approximate duals. Other technical aspects developed in this paper that are perhaps of independent interest include certain (one-sided) infinite non-Toeplitz Cholesky matrix factorization and an explicit method for the construction of a symmetric factorization of positive semi-definite (spsd) bi-infinite matrices for the derivation of compactly supported tight frames of spline-wavelets with maximum order of vanishing moments, on arbitrary nested knot vectors.

Although the paper can be read independently by those who are somewhat familiar with the subjects of wavelet frames and spline functions, certain results from our earlier paper [4] on tight frames on bounded intervals are needed for the discussions in this paper. In addition, to save space in our presentation, the preliminary materials that have been presented in [4] are not repeated in this paper. The reader is therefore recommended to refer to the companion paper [4] while reading this paper. For this reason, we assume that the reader has some expectations of what the main results of this paper could be, and therefore the statements of such results as those on approximate duals, tight frame characterizations that include necessary conditions and sufficient conditions, approximation orders, more in-depth results on spline-wavelet tight frames, etc. are not highlighted in this introduction section, but rather delayed to the main body of the paper, after the necessary elaborate theoretical setting has been described.

This paper is organized as follows. In Section 2, the general notion of nonstationary multiresolution approximation/analysis (NMRA) is described, with precise statements of three assumptions, mainly for taking care of the study of nonstationary tight frames on the bi-infinite interval. In Section 3, the notion of approximate duals of Riesz bases for unbounded intervals is introduced, and the corresponding main results, Theorem 1 and Theorem 2, along with remarks and examples for clarifying certain points of view, are discussed, with proofs presented by using the kernel operator approach. The main results on tight NMRA frames for the unbounded intervals are derived in Section 4. The content of this section includes a general characterization result formulated as Theorem 3, an explicit but general formulation of tight NMRA frames with vanishing moments stated in Theorem 4, as well as an outline of some procedure for constructing such frame elements, for which Theorem 5 and Theorem 6 are relevant. The next section, Section 5, is devoted to an in-depth study of nonstationary tight frames of splines on arbitrary nested knot vectors, with the main result in this section, namely Theorem 9, along with its elaborate proof, given in Subsection 5.3. The necessary matrix calculus for approximate duals on infinite and bi-infinite intervals is derived beforehand in Subsection 5.2, where Theorem 7 formulates certain characterization of spline approximation duals, and another result of independent interest, namely Theorem 8, gives a Taylor-type expansion formula for symmetric matrices. For the proof of Theorem 9 in Subsection 5.3, an explicit factorization of spsd infinite and bi-infinite matrices is included in Theorem 10, and the "knot insertion" argument used in [4] is extended to a new "knot removal" argument in Theorem 11. Furthermore, the uniqueness, uniform boundedness, and convergence properties of the explicit approximate duals of B-splines are derived in Theorems 12–14. Finally, Section 6 presents examples of tight NMRA frames of linear and cubic splines.

2. Nonstationary multiresolution analysis (NMRA)

We begin with the specification of the generic setting of a nonstationary multiresolution analysis (NMRA) on unbounded intervals $I = [0, \infty)$ or $I = \mathbb{R}$. The consideration of any other unbounded interval can easily be transformed into one of these two cases. Specifically, the NMRA is given by a sequence of closed subspaces $V_j \subset L_2(I)$, $j \in \mathbf{Z}$, such that

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L_2(I), \tag{2.1}$$

and

$$\operatorname{clos}_{L_2}\Big(\bigcup_{j=-\infty}^{\infty} V_j\Big) = L_2(I), \qquad \bigcap_{j=-\infty}^{\infty} V_j = \{0\}. \tag{2.2}$$

The spaces V_j are infinite dimensional vector spaces, whose elements $f \in V_j$ have L_2-convergent representations

$$f = \sum_{k \in \mathbb{M}_j} c_k \phi_{j,k}$$

with coefficients $(c_k)_{k \in \mathbb{M}_j} \in \ell_2$ and $\mathbb{M}_j$ an appropriate index set (typically $\mathbb{N}$ or $\mathbf{Z}$). We assume throughout that the family

$$\Phi_j := \{\phi_{j,k};\ k \in \mathbb{M}_j\} \subset V_j, \tag{2.3}$$

is a Bessel family; *i.e.*, there exists a constant B_j such that

$$\sum_{k \in \mathbb{M}_j} |\langle f, \phi_{j,k} \rangle|^2 \le B_j \|f\|^2 \quad \text{for all} \quad f \in L_2(I). \tag{2.4}$$

For the theory to be valid for most cases of practical interest, we specify three assumptions (A1–A3) on the family $\{\Phi_j\}$.

Assumption A1. *We assume that $\{\Phi_j\}_{j\in\mathbf{Z}}$ satisfies the following conditions:*

(a) Each Φ_j constitutes a Riesz basis of V_j. In particular, there exist constants $A_j, B_j > 0$ such that

$$A_j \sum_{k\in\mathbb{M}_j} |c_k|^2 \leq \left\| \sum_{k\in\mathbb{M}_j} c_k \phi_{j,k} \right\|_{L_2(I)}^2 \leq B_j \sum_{k\in\mathbb{M}_j} |c_k|^2, \qquad (c_k) \in \ell_2(\mathbb{M}_j). \tag{2.5}$$

(b) Each Φ_j is uniformly bounded; i.e., $\sup_{k\in\mathbb{M}_j} \|\phi_{j,k}\|_\infty < \infty$,

(c) Each Φ_j is strictly local*; i.e.,*

(i) all functions $\phi_{j,k}$ have compact support

$$\operatorname{supp} \phi_{j,k} \subset [a_{j,k}, b_{j,k}], \qquad \text{and} \quad h_j := \sup_k (b_{j,k} - a_{j,k}) < \infty;$$

(ii) there exists $m_j \in \mathbb{N}$ such that at most m_j of these intervals overlap; in other words, the functions $[\phi_{j,k}]_{k\in\mathbb{M}_j}$ can be rearranged such that $a_{j,k+m_j} \geq b_{j,k}$.

(d) The maximal length of the support, h_j in (c), converges to 0 as j tends to infinity.

Note that by the condition (c) the family Φ_j is locally finite, *i.e.*, for every compact interval $[a, b] \subset I$ we have $\phi_{j,k}|_{[a,b]} = 0$ except for finitely many indices $k \in \mathbb{M}_j$. Conditions (a)–(c) combined require that the quotient of the length of the largest and smallest support interval, for each j, be bounded, but the bound need not be uniform with respect to j.

Without causing any confusion, we also use Φ_j to denote the row vector of functions $\phi_{j,k}$, associated with a natural ordering of $\mathbb{M}_j$. The refinement relation

$$\Phi_j = \Phi_{j+1} P_j \tag{2.6}$$

is assumed to hold, where P_j is an infinite or bi-infinite matrix with row indices in $\mathbb{M}_{j+1}$ and column indices in $\mathbb{M}_j$. We further assume that the columns of P_j contain only finitely many nonzero terms.

Remark. Some of our results developed in this paper are valid in a more general setting, where linear independence or Riesz stability (2.5) of the families Φ_j is not required. Note, however, that we do not require any conditions of "uniform" refinement, as usually assumed in the wavelet literature. In particular, we do not assume the spaces V_j to be shift-invariant, nor do we assume dilation invariance.

Assumption A1(a) implies that the Gramian matrix

$$\Gamma_j = [\langle \phi_{j,k}, \phi_{j,\ell} \rangle]_{k,\ell \in \mathbb{M}_j}$$

defines a bounded positive operator on $\ell_2(\mathbb{M}_j)$ with bounded inverse. Moreover, the condition (c) implies that Γ_j is banded, with bandwidth m_j. The main result in Demko [8] assures that the entries of Γ_j^{-1} decay exponentially; more precisely, there exist constants $c > 0$ and $0 < \lambda < 1$ such that the entries $\gamma_j^{k,\ell}$ of Γ_j^{-1} satisfy

$$|\gamma_j^{k,\ell}| \leq c\lambda^{|k-\ell|}. \tag{2.7}$$

Since the dual basis $\tilde{\Phi}_j$ of the Riesz basis Φ_j is given by

$$\tilde{\Phi}_j = [\tilde{\phi}_{j,k}]_{k \in \mathbb{M}_j} = \Phi_j \Gamma_j^{-1},$$

this also implies that the functions $\tilde{\phi}_{j,k}$ decay exponentially. The orthogonal projection onto V_j is given by the kernel operator

$$\mathcal{K}_j f = \int_I f(y) K_j(\cdot, y)\, dy, \qquad f \in L_2(I),$$

where the kernel K_j is given by

$$K_j(x,y) = \Phi_j(x) \Gamma_j^{-1} \Phi_j(y)^T = \sum_{k \in \mathbb{M}_j} \phi_{j,k}(x) \tilde{\phi}_{j,k}(y).$$

Again we can conclude from the property of exponential decay of the coefficients of Γ_j^{-1} that the kernel K_j decays exponentially, that is

$$|K_j(x,y)| \leq c e^{-\alpha|x-y|}, \qquad x, y \in I,$$

with some constants $c, \alpha > 0$.

Another way to describe the orthogonal projection $\mathcal{K}_j$ is by means of an orthonormal basis of V_j. Instead of applying the Gram-Schmidt orthogonalization procedure, an orthonormal basis of V_j can be defined by

$$\Phi_j^{\perp} = [\phi_{j,k}^{\perp}]_{k \in \mathbb{M}_j} := \Phi_j \Gamma_j^{-1/2},$$

where $\Gamma_j^{-1/2}$ is the square root of Γ_j^{-1} (in the sense of symmetric positive operators). Again, the entries of $\Gamma_j^{-1/2}$ decay exponentially, and so do the functions $\phi_{j,k}^{\perp}$. The kernel K_j of the orthogonal projection onto V_j has the equivalent representation

$$K_j(x,y) = \Phi_j^{\perp}(x) \Phi_j^{\perp}(y)^T.$$

Our second assumption on the family $\{\Phi_j\}$ is often used in connection with the study of the approximation order of the spaces V_j, see e.g. [2], [9].

Assumption A2. *All the kernels K_j, $j \in \mathbf{Z}$, reproduce polynomials of order $m \geq 1$ (or degree $m-1$); i.e.*

$$\int_I y^{\nu} K_j(x,y)\, dy = x^{\nu}, \qquad x \in I, \quad 0 \leq \nu \leq m-1.$$

The above assumption is equivalent to the identities

$$x^{\nu} = \sum_{k \in \mathbb{M}_j} g_{j,k}^{(\nu)} \phi_{j,k}(x), \qquad x \in I, \quad 0 \leq \nu \leq m-1, \tag{2.8}$$

where

$$g_{j,k}^{(\nu)} = \langle x^{\nu}, \tilde{\phi}_{j,k} \rangle, \qquad k \in \mathbb{M}_j, \ 0 \leq \nu \leq m-1.$$

Since the functions $\tilde{\phi}_{j,k}$, $k \in \mathbb{M}_j$, are uniformly bounded and decay exponentially, the coefficients $[g_{j,k}^{(\nu)}]_{k \in \mathbb{M}_j}$ grow at most polynomially as $|k|$ tends to infinity. Moreover, if

$$x^{\nu} = \sum_{k \in \mathbb{M}_j} a_k \phi_{j,k}$$

holds, where $[a_k]_{k\in\mathbb{M}_j}$ grows at most polynomially, we can interchange the order of integration and summation in

$$\langle x^\nu, \tilde{\phi}_{j,\ell}\rangle = \sum_{k\in\mathbb{M}_j} a_k \langle \phi_{j,k}, \tilde{\phi}_{j,\ell}\rangle = a_\ell,$$

to give $a_k = g_{j,k}^{(\nu)}$ for all $k \in \mathbb{M}_j$. An application of the refinement relation (2.6) then yields

$$x^\nu = [g_{j,k}^{(\nu)}]\Phi_j^T(x) = [g_{j,k}^{(\nu)}](P_j^T\Phi_{j+1}^T(x)) = ([g_{j,k}^{(\nu)}]P_j^T)\Phi_{j+1}^T(x), \tag{2.9}$$

and this shows that

$$[g_{j+1,k}^{(\nu)}]_{k\in\mathbb{M}_{j+1}} = [g_{j,k}^{(\nu)}]_{k\in\mathbb{M}_j} P_j^T. \tag{2.10}$$

(Note that the associative law can be applied in (2.9), because Φ_j is locally finite and P_j has only finitely many nonzero terms in each row and column.)

As usual, we let $H^m(I)$ denote the Sobolev space of functions f with distributional derivatives $f^{(\nu)} \in L_2(I)$, $0 \le \nu \le m$; in particular, every $f \in H^m(I)$ is $(m-1)$-times differentiable, and $f^{(m-1)}$ is absolutely continuous. The usual Sobolev norm is defined as

$$\|f\|_m := \left(\sum_{k=0}^m \|f^{(k)}\|_{L_2(I)}^2\right)^{1/2}.$$

Our third assumption on the family $\{\Phi_j\}$ is concerned with the characterization of vanishing moments of elements of V_j. This consideration is closely related to the concept of "commutation" in the study of irregular subdivision as described, e.g., in [6].

Assumption A3. *For some given integer $L \ge 1$, there exists a nonstationary NMRA*

$$\cdots \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \cdots \subset L_2(I),$$

and families

$$\Xi_j = \{\xi_{j,k};\, k \in \tilde{\mathbb{M}}_j\} \subset \tilde{V}_j$$

which satisfy Assumption A1 (with V_j replaced by $\tilde{V}_j$) and the following two conditions:

(a) *All the functions* $\xi_{j,k}$ *are elements of* $H^L(I)$ *and have* $L-1$ *vanishing derivatives at any finite endpoint of* I*.*

(b) $f \in V_j$ *has vanishing moments of order* L*, if and only if there exists* $v \in \ell_2(\tilde{\mathbb{M}}_j)$ *such that* $f(x) = \frac{d^L}{dx^L}\Xi_j(x)\, v$*. Moreover,* v *decays exponentially if* f *does.*

Of particular importance to our investigation are certain bounded self-adjoint operators whose integral kernel is an element of the tensor product space $V_j \otimes V_j$. These operators were defined by means of symmetric positive semi-definite (spsd) matrices S_j in the case of finite dimensional spaces V_j in [4].

Definition 1. *Let* $\Phi = [\phi_k]_{k\in\mathbb{M}}$ *be a locally finite Bessel family in* $L_2(I)$ *and* $S = [s_{k,l}]_{k,\ell\in\mathbb{M}}$ *a symmetric matrix which defines a bounded linear operator on* $\ell_2(\mathbb{M})$*. Then we define the kernel*

$$K_S(x,y) := \Phi(x)\ S\ \Phi^T(y) = \sum_{k,\ell\in\mathbb{M}} s_{k,\ell}\phi_k(x)\phi_\ell(y), \tag{2.11}$$

the associated symmetric operator

$$\mathcal{K}_S f := \int_I f(y) K_S(\cdot,y)\,dy, \qquad f \in L_2(I), \tag{2.12}$$

and the quadratic form

$$T_S f := \langle f, \mathcal{K}_S f\rangle = \Big[\langle f,\phi_k\rangle\Big]_{k\in\mathbb{M}}\ S\ \Big[\langle f,\phi_k\rangle\Big]_{k\in\mathbb{M}}^T, \qquad f \in L_2(I). \tag{2.13}$$

Note that the function values $K_S(x,y)$ of the kernel are well defined for an arbitrary matrix S, since Φ is assumed to be locally finite: for fixed x and y, the sum in (2.11) has only finitely many nonzero summands. Moreover, for symmetric S, the kernel K_S is symmetric, i.e., $K_S(x,y) = K_S(y,x)$. Since S is assumed to define a bounded operator on $\ell_2(\mathbb{M})$, the following argument shows that the operator $\mathcal{K}_S$ maps $L_2(I)$ into itself. Indeed, let B be an upper bound in (2.4) of the Bessel family Φ and $f \in L_2(I)$. We may conclude from (2.11)–(2.12) that

$$\mathcal{K}_S f(x) = \sum_{k\in\mathbb{M}} \left(\sum_{\ell\in\mathbb{M}} s_{k,\ell}\langle f,\phi_\ell\rangle\right)\phi_k(x), \qquad x \in I,$$

and this gives

$$\|\mathcal{K}_S f\|_2^2 \le B \sum_{k\in\mathbb{M}} \left| \sum_{\ell\in\mathbb{M}} s_{k,\ell}\langle f,\phi_\ell\rangle \right|^2 \le B\|S\|^2_{\ell_2\to\ell_2} \sum_{\ell\in\mathbb{M}} |\langle f,\phi_\ell\rangle|^2 \le B^2\|S\|^2_{\ell_2\to\ell_2}\|f\|^2_{L_2}.$$

Therefore, $\mathcal{K}_S$ is a bounded linear operator on $L_2(I)$, with $\|\mathcal{K}_S\| \le B\|S\|_{\ell_2\to\ell_2}$. The last expression also defines an upper bound for the quadratic form T_S in (2.13). If Φ defines a Riesz basis, then the moment sequences $[\langle f,\phi_k\rangle]_{k\in\mathbb{M}}$, where $f \in L_2(I)$, are known to be all of $\ell_2(\mathbb{M})$. Hence, there is a one-to-one correspondence between the bounded quadratic forms T_S and the symmetric matrices S that act on $\ell_2(\mathbb{M})$.

3. Approximate Duals

With rare exceptions, the dual Riesz basis $\tilde{\Phi}_j$ of a given Riesz basis is not compactly supported. In this section, we define the notion of approximate duals of Riesz bases and show how they are related to quasi-projection operators, that is, operators of the form $\mathcal{K}_S$ which reproduce polynomials of a certain degree. The approximation order of such operators is related to their polynomial accuracy. In Section 5, we will give examples of approximate duals of B-splines of order $m \ge 2$ which have compact support. Since our present discussion is only for one single space

$$V := \mathrm{clos}_{L_2}\left(\mathrm{span}\,\{\phi_k;\ k\in\mathbb{M}\}\right),$$

we can, and will, omit the first index j in this section.

For the Sobolev space $H^m(I)$ introduced earlier, we allow, for completeness, the interval $I = (a,b)$ to be bounded or unbounded; that is, a may be real or $-\infty$ and b may be real or $+\infty$. Then, the subspace $H_0^m(I)$ of $H^m(I)$ consists of all functions in $H^m(I)$ which satisfy the following boundary conditions: if c is a finite endpoint of I, then $f^{(\nu)}(c) = 0$ for $0 \le \nu \le m-1$. If c is an infinite "endpoint" of I (so $c = -\infty$ or $c = \infty$), then

$$\lim_{x\to c} x^\nu f^{(\nu)}(x) = 0, \quad 0 \le \nu \le m-1.$$

Hence, for every $f \in H_0^m(I)$, integration by parts leads to the identity

$$f(x) = \int_a^x \frac{(x-t)^{m-1}}{(m-1)!} f^{(m)}(t)\,dt = -\int_x^b \frac{(x-t)^{m-1}}{(m-1)!} f^{(m)}(t)\,dt, \qquad f \in H_0^m(I), \quad x \in I. \tag{3.1}$$

The concept under consideration in this section is valid under less restrictive assumptions than those described in Section 2. In particular, instead of requiring the family Φ to be strictly local as in Assumption A1(c), we only require that, for every $k \in \mathbb{M}$, positive constants c_k, r exist for which

$$|\phi_k(x)|,\ |\tilde{\phi}_k(x)| \leq c_k(1+|x|)^{-r}, \qquad x \in I, \tag{3.2}$$

where $r > 1$ will be chosen later. The same decay condition is assumed to hold for the functions ξ_k and their duals $\tilde{\xi}_k$ in Assumption A3. We also assume that $\xi_k, \tilde{\xi}_k \in H_0^L(I)$, and this is the proper extension of Assumption A3 to functions with unbounded support.

Without any further assumptions on the space V, the orthogonal projection

$$\mathcal{K} : L_2(I) \to V$$

admits a representation of the form

$$\mathcal{K}f(x) = \int_I f(y)K(x,y)\,dy, \qquad \text{a.e. } x \in I,$$

where the *orthoprojection kernel* (or kernel for orthogonal projection)

$$K(x,y) = \sum_\ell \eta_\ell(x)\eta_\ell(y)$$

is defined by choosing any orthonormal basis $\{\eta_\ell\}$ for V. We assume that this kernel (and all kernels defined later) satisfies the "localization property"

$$|K(x,y)| \leq \frac{C}{(1+|x-y|)^r}, \qquad x, y \in I, \tag{3.3}$$

where $r > 0$ is a parameter which will be chosen later.

Definition 2. *Let Φ be a Bessel family in $L_2(I)$ and $S = [s_{k,\ell}]_{k,\ell\in\mathbb{M}}$ some matrix that defines a bounded operator on $\ell_2(\mathbb{M})$. Denote by $\Phi^S := \Phi S = [\phi_k^S]_{k\in\mathbb{M}}$ a corresponding Bessel family. Then for an integer $L \geq 1$, Φ^S is said to constitute an* approximate dual of order *L relative to Φ, if the kernel*

$$K_S = \Phi(x)S\Phi(y)^T = \sum_{k,\ell\in\mathbb{M}} s_{k,\ell}\phi_k(x)\phi_\ell(y) \tag{3.4}$$

satisfies the decay condition (3.3), where $r > 2L+1$, and the following two conditions of vanishing moments in the x- and y-directions hold:

$$\int_I x^\nu (K - K_S)(x,y)\,dx = 0 \qquad \text{for a.e. } y \in I,\ 0 \leq \nu \leq L-1, \tag{3.5}$$

$$\int_I y^\nu \int_a^x \frac{(x-t)^{L-1}}{(L-1)!}(K-K_S)(t,y)\,dt\,dy = 0 \qquad \text{for a.e. } x \in I,\ 0 \leq \nu \leq L-1. \tag{3.6}$$

Remark. The notion of "approximate dual" for the family Φ^S reflects to the approximation properties of the operator

$$\mathcal{K}_S : L_2(I) \to V, \quad \mathcal{K}_S f(x) = \int_I f(y)K_S(x,y)\,dy, \qquad x \in I,$$

which will be presented in Theorem 2 later. We will also call the operator $\mathcal{K}_S$ a *quasi-projection* operator, in view of the polynomial reproduction property (3.5). The reason for the introduction of this notion is that approximate duals have the advantage over the canonical duals in that both families Φ and Φ^S may consist of compactly supported functions of arbitrary smoothness.

The next proposition shows that (3.6) is implied by (3.5) if $I \neq \mathbb{R}$. We also give an example where the implication fails for $I = \mathbb{R}$.

Proposition 1. *If $I = [a,b]$ or $[a,\infty)$ or $(-\infty,b]$, where a,b are real numbers, then (3.6) follows from (3.5) for every pair K and K_S of symmetric kernels that satisfy the decay condition (3.3) for some $r > L$.*

Proof: The result is established by a simple application of Fubini's theorem. Let $K_1, K_2 : I \times I \to \mathbb{C}$ be symmetric kernels, with

$$K_n(x,y) = \overline{K_n(y,x)}, \qquad x, y \in I, \ n = 1, 2,$$

which satisfy the decay condition (3.3). It is sufficient to consider the case $I = [0, b)$ where b is either real or ∞. Then the decay condition implies that

$$C_\nu := \sup_{x \in I} \int_I (1 + |x-y|)^\nu |(K_1 - K_2)(x,y)| \, dy < \infty, \qquad 0 \le \nu \le L-1. \tag{3.7}$$

We make use of the relation

$$|y|^\nu \le (|t-y| + |t|)^\nu \le 2^\nu (|t-y|^\nu + |t|^\nu).$$

Hence, for all $x \in I$, we obtain

$$\int_0^b \int_0^x |y|^\nu \frac{(x-t)^{L-1}}{(L-1)!} |(K_1 - K_2)(t,y)| \, dt \, dy \le \int_0^x 2^\nu (C_\nu + C_0 t^\nu) \frac{(x-t)^{L-1}}{(L-1)!} \, dt < \infty.$$

An application of Fubini's theorem leads to

$$\int_0^b \int_0^x y^\nu \frac{(x-t)^{L-1}}{(L-1)!} (K_1 - K_2)(t,y) \, dt \, dy = \int_0^x \frac{(x-t)^{L-1}}{(L-1)!} \left(\int_0^b y^\nu (K_1 - K_2)(t,y) \, dy \right) dt.$$

Now, the symmetry of the kernels and (3.5) imply that

$$\int_0^b y^\nu (K_1 - K_2)(t,y) \, dy = \int_0^b y^\nu \overline{(K_1 - K_2)(y,t)} \, dy = 0. \quad \blacksquare$$

The next example shows that the second condition (3.6) is indispensable in the case where $I = \mathbb{R}$. This arises from the fact that the application of Fubini's theorem in the proof of Proposition 1 cannot be extended to the bi-infinite setting.

Example 1. Let N_m denote the cardinal B-spline of order m (or degree $m-1$) with knots $0, 1, \ldots, m$. We consider $L = 2$ and define the piecewise bilinear kernel

$$K(x,y) = \sum_{k \in \mathbf{Z}} N_2(y-k)(N_2(x-k+1) - 2N_2(x-k) + N_2(x-k-1)), \qquad x, y \in \mathbb{R}.$$

By re-ordering the (locally finite) sum, we see that K is symmetric. The well-known formula for the derivatives of cardinal B-splines can be applied to give

$$N_4''(x-k+1) = N_3'(x-k+1) - N_3'(x-k) = N_2(x-k+1) - 2N_2(x-k) + N_2(x-k-1).$$

Hence, for $\nu = 0, 1$ in (3.5), we obtain

$$\int_{\mathbb{R}} x^\nu K(x,y)\,dx = 0, \qquad y \in \mathbb{R}\,.$$

On the other hand, the function

$$G_2(x,y) := \int_{-\infty}^{x} (x-t)K(t,y)\,dt = \sum_{k\in\mathbf{Z}} N_2(y-k)N_4(x-k+1)$$

does not have any vanishing moment in the $y-$direction at all. Hence, (3.6) is not satisfied for the kernel K and $\nu = 0$ or 1. On the other hand, it is worthwhile to observe, however, that K satisfies both conditions (3.5) and (3.6) of order $L = 1$. Indeed, the function

$$G_1(x,y) := \int_{-\infty}^{x} K(t,y)\,dt = \sum_{k\in\mathbf{Z}} N_2(y-k)(N_3(x-k+1) - N_3(x-k))$$

satisfies

$$\int_{\mathbb{R}} G_1(x,y)\,dy = \sum_{k\in\mathbf{Z}} (N_3(x-k+1) - N_3(x-k)) = 0, \qquad x \in \mathbb{R}\,.$$

We have thus seen that, although K annihilates polynomials of order 2 (or degree 1), its order, relative to the conditions of approximate duals, is only 1. ∎

Remark. The previous example is typical for the case where V is a shift-invariant subspace of $L_2(\mathbb{R})$. For certain approximate duals of order $L \leq m$ of the cardinal B-spline basis $\Phi = [N_m(\cdot - k)]_{k\in\mathbf{Z}}$, it is shown in [7] that K_S reproduces polynomials of degree up to $\min(m-1, 2L-1)$. Therefore, the order of polynomial reproduction (and the approximation order) of the kernel K_S can exceed its order as an approximate dual. Proposition 1 shows, however, that both of these orders agree, if Φ is the B-spline basis on a bounded interval $I = [a, b]$ with m-fold stacked knots at both endpoints or on $I = [0, \infty)$ with an m-fold knot at 0. For B-splines on non-uniform nested knot vectors on the bi-infinite interval, the situation is much more complicated. Here, although we may conclude that both orders agree for most cases, yet the order of polynomial reproduction of the kernel K_S could exceed its order as an approximate dual only in very special occasions.

There is a strong connection between approximate duals and vanishing moments of the kernel difference $K - K_S$, where K, as before, denotes the orthoprojection kernel of the subspace V. In order to formulate the next result in a general setting, we replace the property in Assumption A1(c) of being strictly local by the weaker decay assumptions (3.2).

Theorem 1. *Assume that Φ satisfies Assumptions A1(a) and A3, and that the decay conditions (3.2) hold. Then a symmetric matrix S defines an approximate dual Φ^S of order L, if and only if there exists a symmetric matrix $A = [a_{k,\ell}]_{k,\ell\in\tilde{\mathbb{M}}}$, which defines a bounded operator on $\ell_2(\tilde{\mathbb{M}})$, such that*

$$(K - K_S)(x, y) = \frac{\partial^{2L}}{\partial x^L \partial y^L} \sum_{k,\ell\in\tilde{\mathbb{M}}} a_{k,\ell}\xi_k(x)\xi_\ell(y), \qquad x, y \in I. \tag{3.8}$$

Proof: For the necessity condition, we see that, based on Assumption A1, the orthoprojection kernel K for the subspace V is given by

$$K(x, y) = \sum_{k\in\mathbb{M}} \phi_k(x)\tilde{\phi}_k(y) = \sum_{k,\ell\in\mathbb{M}} \gamma^{k,\ell}\phi_k(x)\phi_\ell(y),$$

where $\gamma^{k,\ell}$ denote the entries of the inverse Gramian of the Riesz basis Φ. Therefore, the kernel $\tilde{K} := K - K_S$ is given by

$$\tilde{K}(x, y) = \sum_{\ell\in\mathbb{M}} d_\ell(x)\phi_\ell(y), \qquad x, y \in I, \tag{3.9}$$

where

$$d_\ell(x) = \sum_{k\in\mathbb{M}} (\gamma^{k,\ell} - s_{k,\ell})\phi_k(x) = \langle (K - K_S)(x, \cdot), \tilde{\phi}_\ell\rangle. \tag{3.10}$$

Clearly, each function d_ℓ is a function in V.

Our first step of the proof is to show that the function d_ℓ has vanishing moments of order L, and then apply part (b) of Assumption A3. We infer from (3.3), that

$$C_\nu := \sup_{x\in I} \int_I (1+|x-y|)^\nu |\tilde{K}(x,y)|\, dy < \infty, \qquad 0 \le \nu \le L-1. \tag{3.11}$$

In order to apply Fubini's theorem, we make use of the relation

$$|x|^\nu \le (|y| + |x-y|)^\nu \le 2^\nu (|y|^\nu + |x-y|^\nu).$$

For all $0 \le \nu \le L-1$ and $\ell \in \mathbb{M}$, this leads to

$$\int_{I\times I} |x|^\nu |\tilde{\phi}_\ell(y)\tilde{K}(x,y)|\, dx\, dy \le 2^\nu \left(C_0 \int_I |y^\nu \tilde{\phi}_\ell(y)|\, dy + C_\nu \int_I |\tilde{\phi}_\ell(y)|\, dy \right).$$

The right-hand side is finite due to (3.2). An application of Fubini's theorem gives

$$\int_I x^\nu d_\ell(x)\, dx = \int_I x^\nu \left(\int_I \tilde{K}(x,y)\tilde{\phi}_\ell(y)\, dy \right) dx = \int_I \tilde{\phi}_\ell(y) \left(\int_I x^\nu \tilde{K}(x,y)\, dx \right) dy,$$

and the condition (3.5) implies

$$\int_I x^\nu d_\ell(x)\, dx = 0.$$

Therefore, by Assumption A3(b), we have

$$d_\ell(x) = \frac{d^L}{dx^L} \sum_{k\in\tilde{\mathbb{M}}} v_{k,\ell} \xi_k(x), \tag{3.12}$$

where $[v_{k,\ell}]_{k\in\tilde{\mathbb{M}}}$ is a sequence in $\ell_2(\tilde{\mathbb{M}})$.

In the second step of the proof, we introduce the kernel

$$G(x,y) := \int_a^x \frac{(x-t)^{L-1}}{(L-1)!} \tilde{K}(t,y)\, dt$$

and proceed in a similar way as before. First, we develop a decay condition for G which replaces (3.3). Note that, by (3.5), we have

$$G(x,y) = \int_a^x \frac{(x-t)^{L-1}}{(L-1)!} \tilde{K}(t,y)\, dt = -\int_x^b \frac{(x-t)^{L-1}}{(L-1)!} \tilde{K}(t,y)\, dt, \tag{3.13}$$

which includes the cases $a = -\infty$ and $b = \infty$. Therefore, for any $x, y \in I$ with $y > x$, we infer from (3.3) that

$$\begin{aligned} |G(x,y)| &\le \frac{C}{(L-1)!} \int_a^x \frac{(x-t)^{L-1}}{(1+(y-x)+(x-t))^r}\, dt \\ &\le \frac{\tilde{C}}{(1+|y-x|)^{r-L}}, \end{aligned} \tag{3.14}$$

where the constant $\tilde{C}$ does not depend on x and y. Likewise, for $y < x$, we use the second integral in (3.13) and obtain the same upper bound for $|G(x,y)|$. This establishes a similar decay condition as in (3.3), with the exception that r must be replaced by $r - L$. If we insert (3.12) into (3.9) and make use of (3.1), we obtain

$$\begin{aligned} G(x,y) &= \sum_{\ell \in \mathbb{M}} \sum_{k \in \tilde{\mathbb{M}}} v_{k,\ell} \xi_k(x) \phi_\ell(y) \\ &= \sum_{k \in \tilde{\mathbb{M}}} e_k(y) \xi_k(x) \end{aligned}$$

where

$$e_k(y) = \sum_{\ell \in \mathbb{M}} v_{k,\ell} \phi_\ell(y) = \langle G(\cdot, y), \tilde{\xi}_k \rangle, \qquad k \in \tilde{\mathbb{M}}.$$

Analogous to the first step of the proof, we see that e_k is an element of V, which, by (3.6), has vanishing moments of order L. Once more, we can apply Assumption A3(b) and obtain

$$e_k(y) = \frac{d^L}{dy^L} \sum_{\ell \in \tilde{\mathbb{M}}} a_{k,\ell} \xi_\ell(y), \qquad y \in I, \quad k \in \tilde{\mathbb{M}},$$

where $[a_{k,\ell}]_{\ell \in \tilde{\mathbb{M}}}$ is a sequence in $\ell_2(\tilde{\mathbb{M}})$.

Finally, as a consequence of the first two steps, we have already shown that

$$(K - K_S)(x,y) = \tilde{K}(x,y) = \frac{d^L}{dx^L} G(x,y) = \frac{\partial^{2L}}{\partial x^L \partial y^L} H(x,y),$$

where we let

$$H(x,y) := \int_a^y \frac{(y-s)^{L-1}}{(L-1)!} G(x,s)\, ds = \sum_{k,\ell \in \tilde{\mathbb{M}}} a_{k,\ell} \xi_k(x) \xi_\ell(y), \qquad x, y \in I.$$

As in (3.14), we find that

$$|H(x,y)| \le \frac{\tilde{C}}{(1+|y-x|)^{r-2L}}, \tag{3.15}$$

for almost all $x, y \in I$. We conclude that

$$\int_I |H(x,y)|\, dy \le C, \qquad x \in I,$$

where the constant C does not depend on x. Therefore, the kernel H defines a bounded operator on $L_2(I)$ and, likewise, the matrix A defines a bounded operator on $\ell_2(\tilde{\mathbb{M}})$.

The sufficiency condition is obvious from the definition of approximate duals. This completes the proof of the theorem. ■

The result of Theorem 1 can be extended by introducing a localization parameter $h > 0$ into the decay conditions (3.2) and (3.3); for the remainder of this section, we assume that

$$|\phi_k(x)|,\ |\tilde{\phi}_k(x)|,\ |\xi_k(x)|,\ |\tilde{\xi}_k(x)| \le \frac{\tilde{C}}{\sqrt{h}(1+|x-t_k|/h)^r}, \qquad x \in I, \tag{3.16}$$

$$|K(x,y)|,\ |K_S(x,y)| \le \frac{C}{h(1+|x-y|/h)^r}, \qquad x, y \in I, \tag{3.17}$$

where $C, r > 0$ and $(t_k)_{k\in\mathbb{M}}$ is a real sequence which satisfies

$$|t_k - t_\ell| \ge dh|k-\ell|, \qquad k, \ell \in \mathbb{M},$$

for some positive constant d.

Proposition 2. *Let $L \ge 1$ be an integer and assume that the functions $\phi_k, \tilde{\phi}_k$ in Assumption A1 and $\xi_k, \tilde{\xi}_k$ in Assumption A3 satisfy the decay condition (3.16), where $r > 2L+1$. If S defines an approximate dual of order L, and if K and K_S satisfy the decay condition (3.17), then the coefficients $a_{k,\ell}$ in equation (3.8) of Theorem 1 satisfy*

$$|a_{k,\ell}| \le Ch^{2L}(1+|k-\ell|)^{-(r-2L)}.$$

Moreover, the kernel

$$H(x,y) = \sum_{k,\ell \in \tilde{\mathbb{M}}} a_{k,\ell}\xi_k(x)\xi_\ell(y)$$

satisfies

$$|H(x,y)| \le \frac{\tilde{C}h^{2L-1}}{(1+|x-y|/h)^{r-2L}}, \qquad x,y \in I. \tag{3.18}$$

Here the constants C and $\tilde{C}$ depend only on r, L, and the constants in (3.16), (3.17), but not on h.

Proof: The kernel H, as in the proof of Theorem 1, can be constructed in two steps, as follows. First, let us consider

$$G(x,y) = \int_a^x \frac{(x-t)^{L-1}}{(L-1)!}(K-K_S)(t,y)\,dt.$$

Then by the condition (3.17), an analogous estimate as in (3.14) leads, for all $y \ge x$, to

$$\begin{aligned}
|G(x,y)| &\le \frac{C}{h(L-1)!}\int_a^x \frac{(x-t)^{L-1}}{(1+(y-x)/h+(x-t)/h)^r}\,dt \\
&= \frac{Ch^{L-1}}{(L-1)!}\int_0^{(x-a)/h} \frac{t^{L-1}}{(1+(y-x)/h+t)^r}\,dt \\
&\le \frac{Ch^{L-1}}{(L-1)!}\int_0^{\infty} \frac{t^{L-1}}{(1+(y-x)/h+t)^r}\,dt \\
&= \frac{Ch^{L-1}}{(L-1)!\binom{r-1}{L}(1+(y-x)/h)^{r-L}}.
\end{aligned}$$

The case $y < x$ is treated analogously. In the same manner, we further obtain that

$$\begin{aligned}
|H(x,y)| &= \left|\int_a^y \frac{(y-s)^{L-1}}{(L-1)!}G(x,s)\,ds\right| \\
&\le \frac{Ch^{2L-1}}{((L-1)!)^2}\frac{1}{\binom{r-1}{L}\binom{r-L-1}{L}(1+|x-y|/h)^{r-2L}}
\end{aligned}$$

for all $x,y \in I$. This proves the second assertion of the proposition.

The coefficients $a_{k,\ell}$ can be computed by using the formula

$$a_{k,\ell} = \int_I\int_I H(x,y)\tilde{\xi}_k(x)\tilde{\xi}_\ell(y)\,dx\,dy.$$

Therefore, from the estimate (3.16) for the functions $\tilde{\xi}_k$, we have

$$
\begin{aligned}
|a_{k,\ell}| &\leq Ch^{2L-2} \int_I \int_I \frac{1}{(1+|x-y|/h)^{r-2L}(1+|x-t_k|/h)^r(1+|y-t_\ell|/h)^r}\, dx\, dy \\
&\leq Ch^{2L-2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1+|x-y+t_k-t_\ell|/h)^{r-2L}(1+|x|/h)^r(1+|y|/h)^r}\, dx\, dy \\
&= Ch^{2L} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1+|x-y+(t_k-t_\ell)/h|)^{r-2L}(1+|x|)^r(1+|y|)^r}\, dx\, dy \\
&\leq Ch^{2L} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1+|y-(t_k-t_\ell)/h|)^{r-2L}(1+|x|)^{2L}(1+|y|)^r}\, dx\, dy \\
&\leq Ch^{2L} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{(1+\underbrace{|(t_k-t_\ell)/h|}_{\geq d|k-\ell|})^{r-2L}(1+|x|)^{2L}(1+|y|)^{2L}}\, dx\, dy \\
&\leq \tilde{C}h^{2L} \frac{1}{(1+|k-\ell|)^{r-2L}}.
\end{aligned}
$$

For the above third and fourth inequalities, we have made use of the fact that $(1+|x|)(1+|x-a|) \geq 1+|a|$ for all reals x and a. This completes the proof of Proposition 2. ■

We will now establish the following estimate of the approximation error

$$
\|f - \mathcal{K}_S f\|_{H^\beta} \leq h^{\alpha-\beta}\|f\|_{H^\alpha}, \qquad f \in H^\alpha,
$$

where $\beta < \alpha$ are real numbers which are used to define the order of the corresponding Sobolev spaces. Hence, for $\beta = 0$ we obtain error estimates in $L_2(I)$, and for $\beta > 0$ we have estimates of simultaneous approximation. On the other hand, for $\beta < 0$, $H^\beta(I)$ is defined to be the dual space of $H^{|\beta|}(I)$. Error estimates of this form have been developed for the approximation by kernel operators that map into shift-invariant subspaces of $L_2(I)$, see *e.g.*[11, 12].

The following result specifies the approximation order of the operator $\mathcal{K}_S$ in relation to the order of the approximate dual. The effect of "doubling" the order, as observed in the shift-invariant setting by [7], occurs here in a weaker form.

Theorem 2. *Let Φ^S be an approximate dual of order L. Furthermore, let the assumptions of Proposition 2 be satisfied. Then there exists a constant $c_1 > 0$, which depends only on*

r, L and the constant in (3.18), such that

$$\|f - \mathcal{K}_S f\|_{-L} \le \inf_{g\in V} \|f - g\|_{L_2(I)} + c_1 h^{2L} \|f\|_L, \qquad f \in H^L(I). \tag{3.19}$$

Proof: We split the error estimate into two parts, namely

$$\|f - \mathcal{K}_S f\|_{-L} \le \|f - \mathcal{K} f\|_{-L} + \|\mathcal{K} f - \mathcal{K}_S f\|_{-L}.$$

For the first error term, it follows from the definition of Sobolev spaces with negative exponent that

$$\|f - \mathcal{K} f\|_{-L} \le \|f - \mathcal{K} f\|_{L_2(I)},$$

and this is the first term on the right-hand side in (3.19). Therefore, it suffices to prove that

$$\|\mathcal{K} f - \mathcal{K}_S f\|_{-L} \le c_1 h^{2L} \|f\|_L, \qquad f \in H^L(I). \tag{3.20}$$

This is done by applying the duality relation

$$\|\mathcal{K} f - \mathcal{K}_S f\|_{-L} = \sup_{g\in H^L(I)} \frac{1}{\|g\|_L} \left| \int_I (\mathcal{K} f(x) - \mathcal{K}_S f(x)) g(x)\, dx \right|.$$

The integral on the right-hand side of this identity is given by

$$\begin{aligned}
\int_I (\mathcal{K} f(x) - \mathcal{K}_S f(x)) g(x)\, dx &= \int_I g(x) \int_I f(y)(K(x,y) - K_S(x,y))\, dy\, dx \\
&= \int_{I\times I} f(y) g(x) \frac{\partial^{2L}}{\partial x^L \partial y^L} H(x,y)\, dx\, dy \\
&= \int_I g^{(L)}(x) \int_I f^{(L)}(y) H(x,y)\, dy\, dx.
\end{aligned}$$

Here, the integration by parts does not introduce any boundary terms since H has the representation in Proposition 2 with $\xi_k \in H_0^L$, by Assumption 3. Furthermore, the upper bound for H in Proposition 2 implies that

$$\sup_{x\in I} \int_I |H(x,y)|\, dy \le \tilde{C} h^{2L-1} \int_I \left(1 + \frac{|x-y|}{h}\right)^{2L-r} dy \le \tilde{C} h^{2L} \int_{\mathbb{R}} \left(1 + |\frac{x}{h} - y|\right)^{2L-r} dy \le c_1 h^{2L}.$$

Hence, standard estimates for kernel operators give

$$\left\| \int_I f^{(L)}(y) H(\cdot, y)\, dy \right\|_{L_2(I)} \le c_1 h^{2L} \|f^{(L)}\|_{L_2(I)}.$$

Therefore, the Cauchy-Schwarz inequality can be applied to give

$$\left| \int_I (\mathcal{K}f(x) - \mathcal{K}_S f(x)) g(x)\, dx \right| \le c_1 h^{2L} \|f^{(L)}\|_{L_2(I)} \|g^{(L)}\|_{L_2(I)},$$

and this leads to the desired error estimate. $\blacksquare$

Remark. It was observed in [7] that for the shift-invariant setting, where $I = \mathbb{R}$ and $\phi_k(x) = h^{-1/2}\phi(x/h - k)$ with $\phi \in L_2(\mathbb{R})$ and $k \in \mathbf{Z}$, an estimate for the L_2-norm of the error

$$\|f - \mathcal{K}_S f\|_{L_2(\mathbb{R})} \le c h^{\kappa} \|f\|_{\kappa}, \qquad f \in H^{\kappa}(I),$$

can be derived with $\kappa = \min\{2L, m\}$. Here, m denotes the approximation order of the space V in the L_2-norm, so that κ represents the smaller of the two exponents on the right-hand side of (3.19). This result is closely related to the phenomenon of excess in the order of polynomial reproduction as mentioned in a previous remark. In this regard, Kyriazis [12] has developed several results which allow us to study error estimates of the form (3.19) to be "shifted" along a certain scale of the Triebel-Lizorkin spaces. However, application of these results to our consideration of approximate duals is not straightforward, and we leave this study to future research.

4. Theory of tight NMRA wavelet frames

The construction of wavelets and frames from a nonstationary MRA (or NMRA) is based on the definition of the derived function families

$$\Psi_j = [\psi_{j,k}]_{k \in \mathbb{N}_j} := \Phi_{j+1}\, Q_j, \qquad j \in \mathbf{Z}, \tag{4.1}$$

where Q_j is an infinite or bi-infinite matrix with row indices in $\mathbb{M}_{j+1}$ and column indices in some new index set $\mathbb{N}_j$. A natural ordering of $\mathbb{N}_j$ will be assumed throughout. Note that the columns of Q_j define the coefficient sequences of the functions $\psi_{j,k}$. In order to have localization of this family, we assume that $\{\Phi_j\}$ satisfies Assumption A1 and the columns of Q_j decay at least exponentially.

In this section, we present the characterization of tight NMRA frames in a general setting and describe a generic method for their construction, which is motivated by the "oblique matrix extension" method for the shift-invariant case, see [3,7]. More elaborate results will be derived in Section 5 for the NMRA of B-splines on nonuniform knot vectors.

Observe that the infinite matrix Q_j in (4.1) has no well-defined diagonal. In order to describe frames with compact support, the sparsity of this matrix is defined by the *upper* and *lower profile*

$$u_k(Q_j) \leq \ell_k(Q_j), \qquad k \in \mathbb{N}_j,$$

such that the column entries $q_{i,k}^{(j)}$ vanish if $i < u_k(Q_j)$ and $i > \ell_k(Q_j)$; moreover, both sequences are chosen to be nondecreasing. If, in addition, there exist positive integers n_j and $\tilde{n}_j$, such that

$$\ell_k(Q_j) - u_k(Q_j) \leq n_j - 1 \quad \text{and} \quad u_k(Q_j) < u_{k+\tilde{n}_j}(Q_j),$$

we say that the family Ψ_j is local with respect to Φ_{j+1}. Note that the totality of all these conditions on Q_j implies that Ψ_j is locally finite.

Our aim in this section is to give a definition as well as some characterization of NMRA tight frames of $L_2(I)$ in the given NMRA setting.

Definition 3. *Assume that $(V_j)_{j\in\mathbf{Z}}$ constitute an NMRA of $L_2(I)$ and the associated Riesz bases $\{\Phi_j\}_{j\in\mathbf{Z}}$ satisfy Assumption A1. Let $\Psi_j = \Phi_{j+1}Q_j$, $j \in \mathbf{Z}$, where Q_j is a real or complex matrix of dimension $\mathbb{M}_{j+1} \times \mathbb{N}_j$, whose columns either decay exponentially or which is local with respect to Φ_{j+1}. Then the family $\{\Psi_j\}_{j\in\mathbf{Z}}$ constitutes a (normalized)* NMRA tight frame *of $L_2(I)$, if*

$$\sum_{j\in\mathbf{Z}} \sum_{k\in\mathbb{N}_j} |\langle f, \psi_{j,k}\rangle|^2 = \|f\|^2, \quad \text{for all} \quad f \in L_2(I). \tag{4.2}$$

Our next result gives a characterization of such tight frames. This result is an extension of Theorem 1 in [4] to unbounded intervals.

Theorem 3. *Under the same assumptions as in Definition 3, the families* $\{\Psi_j\}_{j\in\mathbf{Z}} = \{\Phi_{j+1}Q_j\}_{j\in\mathbf{Z}}$ *constitute an NMRA tight frame of* $L_2(I)$*, if and only if there exist spsd matrices* S_j, $j \in \mathbf{Z}$*, such that the following conditions hold:*

(i) for each j*, the quadratic form* T_j *in (2.13) is bounded on* $L_2(I)$*,*

(ii) for every function $f \in L_2(I)$*,*

$$\lim_{j\to\infty} T_j f = \|f\|^2, \quad \textit{and} \quad \lim_{j\to-\infty} T_j f = 0, \tag{4.3}$$

(iii) for each j*, the following identity holds:*

$$S_{j+1} - P_j S_j P_j^T = Q_j Q_j^T, \qquad j \in \mathbf{Z}. \tag{4.4}$$

Proof: We first assume that $\psi_{j,k}$, $j \in \mathbf{Z}$, $k \in \mathbb{N}_j$, constitute an NMRA tight frame, and each family Ψ_j is defined by a matrix Q_j in (4.1). For $j \in \mathbf{Z}$, we define the sequence of matrices

$$S_{j,n} := Q_{j-1}Q_{j-1}^T + \sum_{\ell=1}^{n} \left(P_{j-1}\cdots P_{j-\ell}Q_{j-\ell-1}Q_{j-\ell-1}^T P_{j-\ell}^T \cdots P_{j-1}^T\right). \tag{4.5}$$

In order to take the limit of this sequence for $n \to \infty$, we observe that the quadratic form

$$T_{j,n}f := [\langle f, \Phi_j\rangle] S_{j,n} [\langle f, \Phi_j\rangle]^T = \sum_{\ell=1}^{n+1} \sum_{k\in\mathbb{N}_{j-\ell}} |\langle f, \psi_{j-\ell,k}\rangle|^2, \qquad f \in L_2(I),$$

satisfies

$$T_{j,n}f \le T_{j,n+1}f \le \|f\|^2, \qquad f \in L_2(I).$$

Our assumption that Φ_j defines a Riesz basis of V_j implies that the matrices $S_{j,n}$, $n \ge 1$, define a monotonic sequence of bounded symmetric operators on $\ell_2(\mathbb{M}_j)$. By [14], p. 263, the sequence converges in the strong operator topology to a symmetric operator S_j on $\ell_2(\mathbb{M}_j)$, which is the spsd matrix of the theorem. The corresponding quadratic form

$$T_j f = [\langle f, \Phi_j \rangle] S_j [\langle f, \Phi_j \rangle]^T, \qquad f \in L_2(I),$$

satisfies properties (i) and (ii) of the theorem. Moreover, (4.5) shows that

$$S_{j+1,n+1} - P_j S_{j,n} P_j^T = Q_j Q_j^T, \qquad n \geq 1.$$

Since this identity remains true for the limits S_{j+1} and S_j, we have also proved property (iii) of the theorem.

To establish the converse direction, we assume that spsd matrices S_j, $j \in \mathbf{Z}$, that satisfy (i)–(iii) are given. Then the identity

$$T_{J_2} f = T_{J_1} f + \sum_{j=J_1}^{J_2-1} \sum_{k \in \mathbb{N}_j} |\langle f, \psi_{j,k} \rangle|^2, \qquad J_2 > J_1, \tag{4.6}$$

is a direct consequence of condition (iii), and (i) implies that taking the limit for $J_2 \to \infty$ and $J_1 \to -\infty$ on both sides of (4.6) leads to the tight frame condition (4.2). ■

Remark. The following three conditions, which are also mentioned in [4], are sufficient for the validity of the property (i) in Theorem 3, namely:

$$\int_I |K_{S_j}(x,y)|\, dy \leq C \qquad \text{a.e. } x \in I, \quad j \geq 0, \tag{4.7}$$

for some constant $C > 0$;

$$\int_I K_{S_j}(x,y)\, dy = 1, \qquad \text{a.e. } x \in I, \quad j \geq 0; \tag{4.8}$$

and

$$\lim_{j \to \infty} \int_{|x-y|>\epsilon} |K_{S_j}(x,y)|\, dy = 0, \qquad j \geq 0, \tag{4.9}$$

for any $\epsilon > 0$. We remark that condition (4.9), by itself, is satisfied, if the matrices S_j have a fixed maximal bandwidth $r > 0$ and $\{\Phi_j\}_{j \in \mathbf{Z}}$ is locally supported, since the integral in

(4.9) is zero for sufficiently large values of j. We will return to the construction of kernels K_{S_j} of this type in section 5.

For practical applications, all of the wavelets $\psi_{j,k}$ must have vanishing moments of some order $L \geq 1$, meaning that

$$\int_I x^\nu \psi_{j,k}(x)\, dx = 0, \qquad 0 \leq \nu \leq L-1.$$

We assume from now on that the NMRA also satisfies Assumption A2; that is, the orthoprojection kernels K_j reproduce polynomials of degree $m-1$. A first construction of a nonstationary tight frame of $L_2(I)$ with m vanishing moments is given next. This construction will lead to wavelets with unbounded support. Nevertheless, this frame is of importance for our subsequent analysis of vanishing moments of nonstationary frames with compact support.

As for the construction of orthonormal wavelets, we define the space

$$W_j := V_{j+1} \cap V_j^\perp.$$

Note that the kernel $K_{j+1} - K_j$ is the orthoprojection kernel of W_j. The next result provides us with a tight frame for W_j, and thus with a representation of the form

$$(K_{j+1} - K_j)(x,y) = \sum_{k \in \mathbb{M}_{j+1}} \psi_{j,k}(x)\psi_{j,k}(y), \tag{4.10}$$

where all the functions $\psi_{j,k}$ have m vanishing moments.

Theorem 4. *Assume that Φ_j and Φ_{j+1} are strictly local Riesz bases as specified by Assumption A1(a)–(c), and that $\Phi_j = \Phi_{j+1}P_j$ as in (2.6). Furthermore, assume that the associated kernels K_j and K_{j+1} reproduce all polynomials of degree $m-1$. Then the orthonormal bases $\Phi_j^\perp := \Phi_j \Gamma_j^{-1/2}$ and $\Phi_{j+1}^\perp := \Phi_{j+1}\Gamma_{j+1}^{-1/2}$ satisfy the refinement relation*

$$\Phi_j^\perp = \Phi_{j+1}^\perp P_j^\perp, \tag{4.11}$$

where $P_j^\perp = \Gamma_{j+1}^{1/2} P_j \Gamma_j^{-1/2}$. *Moreover, the family*

$$\Theta_j = [\theta_{j,k}]_{k \in \mathbb{M}_{j+1}} := \Phi_{j+1}^\perp (I - P_j^\perp (P_j^\perp)^T) \tag{4.12}$$

constitutes a normalized tight frame of W_j. *All the functions* $\theta_{j,k}$ *of this frame decay exponentially and have* m *vanishing moments.*

Proof: The refinement relation (4.11) immediately follows from the definitions. By the fact that $\Phi_{j+1}^\perp$ is an orthonormal basis of V_{j+1}, the (k,ℓ)-entry of the matrix $P_j^\perp$, for $k \in \mathbb{M}_{j+1}$ and $\ell \in \mathbb{M}_j$, is given by

$$(P_j^\perp)_{k,\ell} = \langle \phi_{j+1,k}^\perp, \phi_{j,\ell}^\perp \rangle.$$

Moreover, the entries in each column and row of $P_j^\perp$ decay exponentially. Thereby, the entries in the rows and columns of the matrix product $P_j^\perp (P_j^\perp)^T$ decay exponentially as well. On the other hand, for every $k, \ell \in \mathbb{M}_j$, the (k,ℓ)-entry of the matrix product $(P_j^\perp)^T P_j^\perp$ is given by

$$c_{k,\ell} = \sum_{s \in \mathbb{M}_{j+1}} \langle \phi_{j+1,s}^\perp, \phi_{j,k}^\perp \rangle \ \langle \phi_{j+1,s}^\perp, \phi_{j,\ell}^\perp \rangle = \langle \phi_{j,k}^\perp, \phi_{j,\ell}^\perp \rangle = \delta_{k,\ell},$$

where we have applied the Plancherel identity for the elements $\phi_{j,k}^\perp, \phi_{j,\ell}^\perp \in V_{j+1}$. This leads to the conclusion

$$(P_j^\perp)^T P_j^\perp = I. \tag{4.13}$$

Let Θ_j be the function vector in (4.12), where every $\theta_{j,k}$ decays exponentially, since the columns of $I - P_j^\perp (P_j^\perp)^T$ do so. Moreover, $\theta_{j,k}$ is an element of V_{j+1} by definition, and is also an element of W_j, since the identity (4.13) shows that the mixed Gramian vanishes; that is, we have

$$\langle \Theta_j^T, \Phi_j^\perp \rangle = (I - P_j^\perp (P_j^\perp)^T) \langle (\Phi_{j+1}^\perp)^T, \Phi_{j+1}^\perp \rangle P_j^\perp = (I - P_j^\perp (P_j^\perp)^T) P_j^\perp = 0.$$

Another application of (4.13) leads to

$$\begin{aligned}
\Theta_j(x)\Theta_j^T(y) &= \Phi_{j+1}^{\perp}(x)(I - P_j^{\perp}(P_j^{\perp})^T)^2(\Phi_{j+1}^{\perp}(y))^T \\
&= \Phi_{j+1}^{\perp}(x)\left(I - 2P_j^{\perp}(P_j^{\perp})^T + P_j^{\perp}(P_j^{\perp}))^T P_j^{\perp}(P_j^{\perp}))^T\right)(\Phi_{j+1}^{\perp}(y))^T \\
&= \Phi_{j+1}^{\perp}(x)\left(I - P_j^{\perp}(P_j^{\perp})^T\right)(\Phi_{j+1}^{\perp}(y))^T \\
&= \Phi_{j+1}^{\perp}(x)(\Phi_{j+1}^{\perp}(y))^T - \Phi_j^{\perp}(x)(\Phi_j^{\perp}(y))^T \\
&= K_{j+1}(x,y) - K_j(x,y).
\end{aligned}$$

Hence, the orthoprojection kernel of W_j is given by (4.10). This implies that Θ_j constitutes a normalized tight frame of W_j. Indeed, for every $f \in W_j$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{M}_{j+1}} |\langle f, \theta_{j,k}\rangle|^2 &= \langle f, \Theta_j\rangle\ \langle \Theta_j^T, f\rangle \\
&= \int_I f(x)\int_I f(y)(K_{j+1} - K_j)(x,y)\,dy\,dx \\
&= \int_I |f(x)|^2\,dx.
\end{aligned}$$

Finally let us show that every $\theta_{j,k}$ has m vanishing moments. For $0 \le \nu \le m-1$, it follows from Assumption A2 and (2.10) that the sequences

$$[g_{j,k}]_{k\in\mathbb{M}_j} := \langle x^\nu, \Phi_j^{\perp}\rangle, \qquad [g_{j+1,k}]_{k\in\mathbb{M}_{j+1}} := \langle x^\nu, \Phi_{j+1}^{\perp}\rangle$$

satisfy the relation

$$[g_{j,k}]_{k\in\mathbb{M}_j}(P_j^{\perp})^T = [g_{j+1,k}]_{k\in\mathbb{M}_{j+1}}.$$

Hence, we obtain

$$\langle x^\nu, \Theta_j\rangle = \langle x^\nu, \Phi_{j+1}^{\perp}\rangle(I - P_j^{\perp}(P_j^{\perp})^T) = [g_{j+1,k}] - [g_{j,k}](P_j^{\perp})^T = 0.$$

This completes the proof of the theorem. ■

Remark. Since the spaces W_j, $j \in \mathbf{Z}$, are pairwise orthogonal, the family $\{\Theta_j\}$ constitutes a tight frame of $L_2(I)$.

In order to find more general tight NMRA frames with vanishing moments, particularly those with compact support, we now make use of the concept of approximate duals.

Theorem 5. *Assume that the NMRA $(V_j)_{j\in\mathbf{Z}}$ and the associated Riesz bases $\{\Phi_j\}$ satisfy Assumptions A1–A3, where the integers m and L in Assumptions A2–A3 satisfy $1 \le L \le m$. Furthermore, let S_j, $j \in \mathbf{Z}$, be spsd matrices which define certain approximate duals of Φ_j of order L and satisfy*

$$S_{j+1} - P_j S_j P_j^T \ge 0. \tag{4.14}$$

Then there exists an spsd matrix Z_j which defines a bounded operator on $\ell_2(\tilde{\mathbb{M}}_{j+1})$ such that

$$K_{S_{j+1}} - K_{S_j}(x,y) = \frac{\partial^{2L}}{\partial x^L \partial y^L} \Xi_{j+1}(x)\ Z_j\ \Xi_{j+1}^T(y), \tag{4.15}$$

where $\{\Xi_j\}$ are the Riesz basis as described in Assumption A3. Moreover, if $S_{j+1} - P_j S_j P_j^T$ is banded, then the rows and columns of Z_j have finite support or decay at least exponentially.

Proof: Theorem 1 gives

$$(K_{j+1} - K_{S_{j+1}})(x,y) = \frac{\partial^{2L}}{\partial x^L \partial y^L} \Xi_{j+1}(x)\ A_{j+1} \Xi_{j+1}^T(y),$$

where A_{j+1} defines a bounded operator on $\ell_2(\tilde{\mathbb{M}}_{j+1})$. Moreover, since $\{\Xi_j\}$ defines an NMRA of $L_2(I)$, the refinement relation

$$\Xi_j = \Xi_{j+1} \tilde{P}_j$$

holds, where the rows and columns of the matrix $\tilde{P}_j$ decay exponentially. (Note that the entries of this matrix are $\tilde{p}_{k,\ell} = \langle \xi_{j+1,k}, \tilde{\xi}_{j,\ell} \rangle$, and all the functions $\tilde{\xi}_{j+1,k}$ have exponential decay as a consequence of Assumption A1.) Therefore, Theorem 1 gives

$$(K_j - K_{S_j})(x,y) = \frac{\partial^{2L}}{\partial x^L \partial y^L} \Xi_{j+1}(x)\ \tilde{P}_j A_j \tilde{P}_j^T \Xi_{j+1}^T(y),$$

and $\tilde{P}_j A_j \tilde{P}_j^T$ is a bounded operator on $\ell_2(\tilde{\mathbb{M}}_{j+1})$. Furthermore, by Assumptions A2–A3 and Theorem 4, we obtain

$$(K_{j+1} - K_j)(x,y) = \Theta_{j+1}(x) \Theta_{j+1}^T(y) = \frac{\partial^{2L}}{\partial x^L \partial y^L} \Xi_{j+1}(x)\ B_{j+1} \Xi_{j+1}^T(y),$$

where B_{j+1} defines a bounded operator on $\ell_2(\mathbb{M}_{j+1})$ as well. If we let

$$Z_j := -A_{j+1} + \tilde{P}_j A_j \tilde{P}_j^T + B_{j+1},$$

the first result of the theorem follows. The second result can be established in a similar way as the decay property of $a_{k,\ell}$ in Proposition 2. Recall from Assumption A1 the definition of the parameter h_j. If $S_{j+1} - P_j S_j P_j^T$ has bandwidth r, we obtain

$$(K_{S_{j+1}} - K_{S_j})(x,y) = 0 \quad \text{for} \quad |x-y| > rh_{j+1}.$$

It follows, as in the proof of Proposition 2, that

$$H(x,y) := \Xi_{j+1}(x)\ Z_j\ \Xi_{j+1}^T(y)$$

vanishes for all $|x-y| > rh_{j+1}$ as well. The entries of the matrix Z_j are given by

$$z_{j;k,\ell} = \int_I \int_I H(x,y)\tilde{\xi}_{j+1,k}(x)\tilde{\xi}_{j+1,\ell}(y)\,dy\,dx. \tag{4.16}$$

Since the functions $\tilde{\xi}_{j+1,k}$ decay exponentially and constitute the dual basis of a strictly local Riesz basis, the above integral decays exponentially as $|k-\ell|$ tends to infinity. ■

Remark. In the last statement of the theorem, Z_j can be shown to be banded, if there exists a strictly local biorthogonal basis $\Omega_{j+1} = [\omega_{j+1,k}]_{k\in\tilde{\mathbb{M}}_{j+1}} \subset L_2(I)$ of Ξ_{j+1}. Indeed, the entries $z_{j;k,\ell}$ in (4.16) can then be written as

$$z_{j;k,\ell} = \int_I \int_I H(x,y)\omega_{j+1,k}(x)\omega_{j+1,\ell}(y)\,dy\,dx,$$

and this integral vanishes for sufficiently large values of $k-\ell$.

The results in Theorem 3 and Theorem 5 have the following consequence.

Corollary 1. *Let the assumptions of Theorem 5 be satisfied, and assume, in addition, that S_j are banded matrices which satisfy properties (i) and (ii) of Theorem 3. If the spsd matrix Z_j in (4.15) has a factorization of the form*

$$Z_j = \widehat{Q}_j \widehat{Q}_j^T, \tag{4.17}$$

where $\widehat{Q}_j$ has dimensions $\tilde{\mathbb{M}}_{j+1} \times \mathbb{N}_j$ and all columns of $\widehat{Q}_j$ decay at least exponentially, then the functions $\psi_{j,k}$ in

$$\Psi_j(x) = [\psi_{j,k}(x)]_{k\in\mathbb{N}_j} := \frac{d^L}{dx^L}\Xi_{j+1}(x)\ \widehat{Q}_j \subset V_{j+1}$$

decay exponentially, have L vanishing moments, and $\{\Psi_j\}_{j\in\mathbf{Z}}$ constitutes a tight NMRA frame of $L_2(I)$.

Note that we are mainly interested in the construction of tight NMRA frames, where all the wavelets $\psi_{j,k}$ have *compact support* and L vanishing moments. Solutions can be found by following the procedure below:

1. Find banded spsd matrices S_j such that $\Phi_j S_j$ are approximate duals of Φ_j, and such that the positivity constraint (4.14) is satisfied.
2. Find the matrix Z_j in (4.15) for the difference of the kernels $K_{S_{j+1}} - K_{S_j}$; in many cases Z_j is banded (see the remark preceding Corollary 1).
3. Find a factorization $Z_j = \widehat{Q}_j\widehat{Q}_j^T$ in (4.17) where the matrix $\widehat{Q}_j$ has finitely many nonzero entries in each column.

For univariate splines of arbitrary order m, we will develop a method in Section 5 to realize all three steps of the construction in an explicit manner. In general, however, we do not know if the Assumptions A1–A3 are sufficient to guarantee the existence of strictly local tight NMRA frames.

Next we include a result which, for the one-sided unbounded interval $I = [0,\infty)$, explains that there exists a factorization of a banded spsd matrix $Z = QQ^T$ with a banded lower triangular matrix Q, just like the Cholesky factorization for finite matrices. Our proof, however, cannot be easily extended to the bi-infinite case $I = \mathbb{R}$, as it depends on the Gram-Schmidt orthogonalization of the rows of an infinite matrix.

Proposition 3. *Let $Z = [z_{k,\ell}]_{k,\ell\geq 1}$ be an infinite spsd matrix with bandwidth r. (In particular, Z defines a bounded non-negative operator on $\ell_2(\mathbb{N})$.) Then there is a banded* *lower triangular matrix Q, with the same bandwidth r, such that*

$$Z = QQ^T.$$

Proof: Let $V = Z^{1/2}$ be the spsd matrix with $V^2 = Z$. We denote by $v_1, v_2, \ldots,$ the row vectors of V. Let $W = [w_k]_{k \in \mathbf{N}}$ be the matrix, whose rows are the non-zero vectors which result from the Gram-Schmidt orthogonalization of the vectors $v_1, v_2, \ldots$ Hence, we skip row k whenever

$$v_k \in \operatorname{span}\{v_1, \ldots, v_{k-1}\}. \tag{4.18}$$

Let C be the matrix carrying the coefficients of the orthogonalization; in other words,

$$V = CW \qquad \text{and} \qquad Z = V^2 = VV^T = CWW^TC^T.$$

By the construction, we have $WW^T = I$. Moreover, if we insert zero columns into C for all indices k which satisfy (4.18), the resulting matrix Q is lower triangular and satisfies $Z = CC^T = QQ^T$. Since Z has bandwidth r, the vectors v_k satisfy

$$v_k \cdot v_\ell = 0 \quad \text{for all} \quad 1 \le \ell < k - r.$$

This implies that

$$q_{k,\ell} = 0 \quad \text{for all} \quad 1 \le \ell < k - r.$$

Therefore, Q has bandwidth r as well. ■

A partial converse of Theorem 5 is given below to conclude this section.

Theorem 6. *Assume that $(V_j)_{j \in \mathbf{Z}}$ constitutes an NMRA of $L_2(I)$ and the associated Riesz bases $\{\Phi_j\}_{j \in \mathbf{Z}}$ satisfy Assumptions A1–A3. Let $\{\Psi_j\}_{j \in \mathbf{Z}} = \{\Phi_{j+1} Q_j\}_{j \in \mathbf{Z}}$ be a tight NMRA frame of $L_2(I)$ and S_j, $j \in \mathbf{Z}$, be the spsd matrices satisfying conditions (i)–(iii) of Theorem 3. If all the wavelets $\psi_{j,k}$ have exponential decay and L vanishing moments,* *where $1 \le L \le m$, and if $\Phi_{j_0} S_{j_0}$ is an approximate dual of order L of Φ_{j_0} for at least one index $j_0 \in \mathbf{Z}$, then for all $j \in \mathbf{Z}$, $\Phi_j S_j$ is an approximate dual of order L of Φ_j.*

Proof: We obtain from Theorem 3 and Assumption A3 that

$$(K_{S_{j+1}} - K_{S_j})(x,y) = \Psi_j(x)\Psi_j^T(y) = \frac{\partial^{2L}}{\partial x^L \partial y^L}\, \Xi_{j+1}(x) A_j A_j^T \Xi_{j+1}^T(y),$$

where $A_j A_j^T$ defines a bounded operator on $\ell_2(\tilde{\mathbb{M}}_{j+1})$. Moreover, Theorem 4 assures that

$$(K_{j+1} - K_j)(x,y) = \Theta_j(x)\Theta_j^T(y) = \frac{\partial^{2L}}{\partial x^L \partial y^L}\, \Xi_{j+1}(x) B_j B_j^T \Xi_{j+1}^T(y),$$

where $B_j B_j^T$ defines a bounded operator on the same ℓ_2-space. By the fact that

$$(K_{j+1} - K_{S_{j+1}})(x,y) = (K_j - K_{S_j})(x,y) - (K_{S_{j+1}} - K_{S_j})(x,y) + (K_{j+1} - K_j)(x,y),$$

we may conclude that S_{j+1} defines an approximate dual of order L if and only if S_j does. Therefore, knowing that at least one S_{j_0} defines an approximate dual is enough to conclude that all the S_j define approximate duals of order L. ∎

5. Tight NMRA frames of spline functions

In this section, we follow the theory of tight NMRA frames developed in the previous sections and study in great depth the theory, with constructive proofs, of such frames of spline functions, particularly those with compact support and desirable order of vanishing moments. To facilitate our presentation, this section is divided into 3 subsections, with the first for laying the ground work, the second for formulating the necessary matrix calculus, and the third, which is the longest, for describing the minimally supported approximate dual (Theorem 9) and the resulting construction of tight NMRA frames of splines.

5.1. Nonstationary spline MRA

Let us first give a very brief review of the pertinent properties of spline NMRA, with special emphasis on the conditions in Assumptions A1–A3. A more complete discussion of the properties of B-splines, and spline functions in general, can be found in [1,13,15]. (See also [4; Section 4] for splines on a bounded interval).

For $I = \mathbb{R}$, the NMRA of splines of order $m \in \mathbb{N}$ is based on nested knot vectors

$$\mathbf{t}_j \subset \mathbf{t}_{j+1} \tag{5.1}$$

where each $\mathbf{t}_j$ is a bi-infinite and non-decreasing sequence

$$\mathbf{t}_j = [\ldots \le t_{-1}^{(j)} \le t_0^{(j)} \le t_1^{(j)} \le \ldots] \tag{5.2}$$

which satisfies

$$t_k^{(j)} < t_{k+m}^{(j)} \qquad \text{for all} \quad k \in \mathbf{Z}, \tag{5.3}$$

and

$$\lim_{k\to-\infty} t_k^{(j)} = -\infty, \qquad \lim_{k\to\infty} t_k^{(j)} = \infty. \tag{5.4}$$

On the other hand, for $I = [0,\infty)$, the knot vectors have the form

$$t_j = [t_{-m+1}^{(j)} = \cdots = t_0^{(j)} = 0 < t_1^{(j)} \le t_2^{(j)} \le \ldots] \tag{5.5}$$

and satisfy (5.3) and the second relation of (5.4). To unify notations, we let $\mathbb{M}$ be either $\mathbf{Z}$ or $\{-m+1, -m+2, \ldots\}$.

Note that we allow the knots to have multiplicities greater than 1. The relation $\mathbf{t}_j \subset \mathbf{t}_{j+1}$ is to be understood in the sense of ordered sets: $\mathbf{t}_{j+1}$ is obtained from $\mathbf{t}_j$ by inserting new knots or by raising the multiplicity of existing knots. Moreover, we allow that the number of knots $t_k^{(j+1)}$ that are inserted into the interval $[t_r^{(j)}, t_{r+1}^{(j)})$, $r \in \mathbf{Z}$, varies from 0 to a maximum of n_j per interval.

The L_2-normalized B-splines of order m for the knot vector $\mathbf{t}_j$ are given by

$$N_{j;m,k}^B := \left(\frac{t_{k+m}^{(j)} - t_k^{(j)}}{m}\right)^{-1/2} N_{j;m,k}, \qquad k \in \mathbb{M},$$

where

$$N_{j;m,k}(x) = (t^{(j)}_{k+m} - t^{(j)}_k)[t^{(j)}_k, \ldots, t^{(j)}_{k+m}](\cdot - x)^{m-1}_+$$

denotes the normalized B-splines which constitute a non-negative partition of unity. Here, $[t_k, \ldots, t_{k+m}]f$ represents the m-th divided difference of f with knots $t_k, \ldots, t_{k+m}$. It is well known that (regardless of the geometry of the knots) the family

$$\Phi_{j;m} := \{N^B_{j;m,k};\ k \in \mathbb{M}\}$$

is a Riesz basis of the corresponding space V_j of spline functions and the Riesz bounds A and B in (2.5) can be chosen depending only on m, see [1; p.156,9; p.145]. Moreover, the support of $N^B_{j;m,k}$ is the interval $[t^{(j)}_k, t^{(j)}_{k+m}]$; hence, if

$$0 < \alpha_j := \inf_k (t^{(j)}_{k+m} - t^{(j)}_k) \leq \sup_k (t^{(j)}_{k+m} - t^{(j)}_k) =: h_j < \infty, \tag{5.6}$$

then $\Phi_{j;m}$ is a strictly local Riesz basis of V_j. $\{\Phi_j\}$ generates an NMRA of $L_2(I)$, provided that

$$\lim_{j\to\infty} h_j = 0 \quad \text{and} \quad \lim_{j\to\infty} \alpha_j = \infty. \tag{5.7}$$

Under these assumptions, all the conditions in Assumption A1 are satisfied.

We also mention that B-splines of order m satisfy the Marsden identity

$$\frac{x^\nu}{\nu!} = \sum_{k\in\mathbb{M}} g^{(\nu)}_{j,k} N_{j;m,k}(x), \qquad 0 \leq \nu \leq m-1, \tag{5.8}$$

where the coefficients $g^{(\nu)}_{j,k}$ are homogeneous and symmetric polynomials of degree ν in the variables $t^{(j)}_{k+1}, \ldots, t^{(j)}_{k+m-1}$. Hence, Assumption A2 is satisfied for B-splines, where the parameter m is equal to the order m of the B-spline basis $\Phi_{j;m}$.

In view of Assumption A3, we define the family

$$\Xi_j := \Phi_{j,m+L} = \{N^B_{j;m+L,k};\ k \in \mathbb{M}\} \tag{5.9}$$

of B-splines of order $m + L$ whose knot vector is $\mathbf{t}_j$. As explained before, Ξ_j constitutes a strictly local Riesz basis. Since all the knots in $\mathbf{t}_j$ have multiplicity at most m, all the B-splines $N^B_{j;m+L,k}$ have at least $L-1$ absolutely continuous derivatives. Therefore, we have $\Xi_j \subset H^L(I)$. Moreover, if $I = [0, \infty)$, then the first $L - 1$ derivatives of all the B-splines $N^B_{j;m+L,k}$ vanish at 0.

As for condition (ii) in Assumption A3, we note that the derivative of a normalized B-spline of order $r + 1 > m$ is given by

$$N'_{j;r+1,k} = d^{-1}_{j;r,k} N_{j;r,k} - d^{-1}_{j;r,k+1} N_{j;r,k+1}, \qquad k \in \mathbb{M}, \tag{5.10}$$

where $d_{j;r,k}$ are the divided knot differences $(t^{(j)}_{k+r} - t^{(j)}_k)/r$. The normalization of the B-splines in L_2 leads further to

$$(N^B_{j;r+1,k})' = (d_{j;r+1,k} d_{j;r,k})^{-1/2} N^B_{j;r,k} - (d_{j;r+1,k} d_{j;r,k+1})^{-1/2} N^B_{j;r,k+1}, \qquad k \in \mathbb{M}. \tag{5.11}$$

Written in matrix form, the recursive application of (5.10) gives

$$\frac{d^\nu}{dx^\nu} \Phi_{j;m+\nu}(x) = \Phi_{j;m}(x)\ E^B_{j;m,\nu}, \tag{5.12}$$

where

$$E^B_{j;m,\nu} := D^B_{j;m} D^B_{j;m+1} \cdots D^B_{j;m+\nu-1}, \tag{5.13}$$

$$D^B_{j;r} := \operatorname{diag}\left[d^{-1/2}_{j;r,k}\right]_{k \in \mathbb{M}} \Delta \operatorname{diag}\left[d^{-1/2}_{j;r+1,k}\right]_{k \in \mathbb{M}}, \qquad r \geq m, \tag{5.14}$$

and Δ is the infinite or bi-infinite matrix that represents the first order difference, with entries given by

$$\Delta_{k,\ell} = \begin{cases} 1, & \text{for } k = \ell, \\ -1, & \text{for } k = \ell + 1, \\ 0, & \text{otherwise}. \end{cases} \tag{5.15}$$

Next, observe that an element $f = \Phi_{j;m}\mathbf{u}$, with $\mathbf{u} \in \ell_2(\mathbb{M})$, has L vanishing moments, if and only if

$$f(x) = \frac{d^L}{dx^L}\ \Phi_{j;m+L}(x)\mathbf{v} \quad \text{and} \quad \mathbf{u} = E^B_{j;m,L}\mathbf{v}. \tag{5.16}$$

Moreover, if $u_k = 0$ for all $k < i_1$ and $k > i_2$, then $\mathbf{v}$ can be so chosen such that $v_k = 0$ for all $k < i_1$ and $k > i_2 - L$, or, if $\mathbf{u}$ decays exponentially, then $\mathbf{v}$ does as well. This shows that all the conditions of Assumption A3 are satisfied.

The L_2-normalized B-splines satisfy the refinement equation

$$\Phi_{j;m} = \Phi_{j+1;m} P^B_{j;m} \tag{5.17}$$

where the matrix $P^B_{j;m}$ has finitely many non-negative entries in each row and column. Its upper/lower profile is defined by two strictly increasing index sequences $u(k)$ and $\ell(k)$, such that

$$\{t^{(j)}_k, \ldots, t^{(j)}_{k+m}\} \subset \{t^{(j+1)}_{u(k)}, \ldots, t^{(j+1)}_{\ell(k)+m}\},$$

where the subset notation is again used for ordered sets, which means that all the elements are counted with their multiplicity. In other words, the upper/lower profile of $P^B_{j;m}$ is defined by the fact that only the B-splines in $\Phi_{j+1;m}$, whose support is contained in the support of $N^B_{j;m,k}$, appear in the refinement relation for this B-spline.

The commutation relation for $P^B_{j;m}$ and $D^B_{j;m}$ is given by

$$P^B_{j;m} D^B_{j;m} = D^B_{j+1;m} P^B_{j;m+1}, \tag{5.18}$$

which further yields

$$P^B_{j;m} E^B_{j;m,\nu} = E^B_{j+1;m,\nu} P^B_{j;m+\nu}, \tag{5.19}$$

with $E^B_{j;m,\nu}$ as in (5.13).

5.2. Matrix calculus of approximate duals of B-splines

In this subsection, we develop the necessary matrix calculus for the characterization and construction of approximate duals of the L_2-normalized B-spline basis $\Phi_{\mathbf{t};m}$ with respect to a (one-sided) infinite or bi-infinite knot vector $\mathbf{t}$ which satisfies (5.3)–(5.4). Let $\Gamma(\mathbf{t})$ denote the Gramian of $\Phi_{\mathbf{t};m}$.

The following result is an immediate consequence of Theorem 1 and (5.16).

Theorem 7. *A banded symmetric matrix S (with bounded entries) defines an approximate dual of order L of the B-spline basis $\Phi_{\mathbf{t};m}$, if and only if there exists a symmetric matrix A, of exponential decay, which defines a bounded operator on $\ell_2(\mathbb{M})$, such that*

$$\Gamma^{-1}(\mathbf{t}) - S = E^B_{\mathbf{t};m,L} A (E^B_{\mathbf{t};m,L})^T. \tag{5.20}$$

The following result highlights the matrix product on the right-hand side of (5.20). The result is comparable with a Taylor expansion in real analysis. For its proof, we introduce the row vector of the first moments

$$M_{\mathbf{t};r} := \left[\int_I N^B_{\mathbf{t};r,k} \right]_{k \in \mathbb{M}} = [d^{1/2}_{\mathbf{t};r,k}]_{k\in\mathbb{M}},$$

where $d_{\mathbf{t};r,k} > 0$ is the divided knot difference $(t_{k+r} - t_k)/r$ and $r \geq m$.

Theorem 8. *Let G be a symmetric matrix with exponential decay and $\mathbf{t}$ a knot vector that satisfies (5.3)–(5.4) and (5.6). Then for any positive integer L and $r \geq m$, there exist unique diagonal matrices $G_0, \ldots, G_{L-1}$ and a unique symmetric matrix X_L, of exponential decay, such that*

$$G = G_0 + E^B_{\mathbf{t};r,1}\, G_1\, (E^B_{\mathbf{t};r,1})^T + \cdots + E^B_{\mathbf{t};r,L-1}\, G_{L-1}\, (E^B_{\mathbf{t};r,L-1})^T + E^B_{\mathbf{t};r,L}\, X_L\, (E^B_{\mathbf{t};r,L})^T. \tag{5.21}$$

Furthermore, $G_0, \ldots, G_{L-1}$ and X_L are uniquely determined by G and $\mathbf{t}$.

Proof: We prove this result by induction. For $L = 1$, since each entry of $M_{\mathbf{t};r}$ is non-zero, there exists a unique diagonal matrix G_0, such that $M_{\mathbf{t};r}G = M_{\mathbf{t};r}G_0$. Moreover, the condition (5.6) guarantees that the diagonal elements of G_0 are uniformly bounded. By Corollary 2 below, there exists a unique symmetric matrix X_1, with exponential decay, such that

$$G - G_0 = E^B_{\mathbf{t};r,1}\, X_1\, (E^B_{\mathbf{t};r,1})^T, \tag{5.22}$$

and this establishes (5.21) for $L = 1$.

For the induction step, let us assume that $L > 1$ and $G_0, \ldots, G_{L-2}, X_{L-1}$ are the unique matrices which give the representation

$$G = G_0 + E^B_{\mathbf{t};r,1}\ G_1\ (E^B_{\mathbf{t};r,1})^T + \cdots + E^B_{\mathbf{t};r,L-2}\ G_{L-2}\ (E^B_{\mathbf{t};r,L-2})^T + E^B_{\mathbf{t};r,L-1}\ X_{L-1}\ (E^B_{\mathbf{t};r,L-1})^T. \tag{5.23}$$

By an application of the result for $L = 1$, there exists a diagonal matrix G_{L-1} and a symmetric matrix X_L, of exponential decay, such that

$$X_{L-1} = G_{L-1} + E^B_{\mathbf{t};r+L-1,1}\ X_L\ (E^B_{\mathbf{t};r+L-1,1})^T. \tag{5.24}$$

Then, by inserting (5.24) into (5.23) and making use of the relation $E^B_{\mathbf{t};r,L-1} E^B_{\mathbf{t};r+L-1,1} = E^B_{\mathbf{t};r,L}$, we obtain the representation (5.21). The uniqueness of this representation is obvious. ■

In the following, let $\mathbb{1}$ denote the infinite or bi-infinite row vector with the value 1 in each entry.

Lemma 1. *Let $G = [g_{i,k}]_{i,k\in\mathbb{M}}$ be a symmetric matrix with exponential decay; i.e.,*

$$|g_{i,k}| \le c_1 \lambda^{|i-k|}, \quad i,k \in \mathbb{M}, \tag{5.25}$$

where $c_1 > 0$ and $0 < \lambda < 1$. If G satisfies $\mathbb{1}\ G = 0$, there exists a unique symmetric matrix $A = [a_{i,k}]_{i,k\in\mathbb{M}}$, with $|a_{i,k}| \le c_2 \lambda^{|i-k|}$ for all $i,k \in \mathbb{M}$, such that

$$G = \Delta\ A\ \Delta^T. \tag{5.26}$$

Proof: If $\mathbb{M} = \mathbb{N}$ we can expand the matrix G by zero blocks to a symmetric matrix with index set $\mathbf{Z}$. Hence, we only need to consider the case $\mathbb{M} = \mathbf{Z}$. We first define the matrix $Y = [y_{i,k}]_{i,k\in\mathbf{Z}}$, where

$$y_{i,k} = \sum_{\ell=-\infty}^{i} g_{\ell,k}. \tag{5.27}$$

For all $i \le k$, the relation (5.25) implies that

$$|y_{i,k}| \le c_1 \sum_{\ell=-\infty}^{i} \lambda^{|k-\ell|} = \frac{c_1}{1-\lambda}\lambda^{|k-i|}. \tag{5.28}$$

Moreover, the condition $1\!\!\mathrm{I}\, G = 0$ gives

$$y_{i,k} = -\sum_{\ell=i+1}^{\infty} g_{\ell,k}, \tag{5.29}$$

and this implies that (5.28) holds for all $i > k$ as well. Thus we have a matrix Y which satisfies (5.28) and

$$G = \Delta Y.$$

All other solutions $\tilde{Y}$ to this identity have the property that $\tilde{Y} - Y$ has constant columns. Hence, Y is the only solution with exponentially decaying columns.

Next we wish to show that $1\!\!\mathrm{I}Y^T = 0$ in order to apply the same technique as in the first step for defining the matrix A. For every $i \in \mathbf{Z}$, we obtain from (5.27)–(5.29) that

$$\sum_{k=-\infty}^{\infty} y_{i,k} = -\sum_{k=-\infty}^{i}\left(\sum_{\ell=i+1}^{\infty} g_{\ell,k}\right) + \sum_{k=i+1}^{\infty}\left(\sum_{\ell=-\infty}^{i} g_{\ell,k}\right)$$

and

$$\sum_{k=i+1}^{\infty}\sum_{\ell=-\infty}^{i} |g_{\ell,k}| \le c_1 \sum_{k=i+1}^{\infty}\sum_{\ell=-\infty}^{i} \lambda^{|\ell-k|} = \frac{c_1\lambda}{(1-\lambda)^2}.$$

Therefore, the interchange of the summation and the symmetry of G give

$$\sum_{k=i+1}^{\infty}\left(\sum_{\ell=-\infty}^{i} g_{\ell,k}\right) = \sum_{\ell=-\infty}^{i}\left(\sum_{k=i+1}^{\infty} g_{\ell,k}\right) = \sum_{\ell=-\infty}^{i}\left(\sum_{k=i+1}^{\infty} g_{k,\ell}\right).$$

Consequently, the identity $1\!\!\mathrm{I}Y^T = 0$ holds. The same argument as in the first step shows that the matrix $A = [a_{i,k}]_{i,k\in\mathbf{Z}}$, where

$$a_{i,k} = \sum_{\ell=-\infty}^{k} y_{i,\ell} = \sum_{\ell=-\infty}^{k}\sum_{m=-\infty}^{i} g_{m,\ell}, \quad i,k \in \mathbf{Z}, \tag{5.30}$$

satisfies the decay condition

$$|a_{i,k}| \le \frac{c_1}{(1-\lambda)^2}\lambda^{|k-i|}, \quad i,k \in \mathbf{Z},$$

and $Y^T = \Delta A$. This gives

$$G = G^T = Y^T \Delta^T = \Delta A \Delta^T.$$

Moreover, the uniqueness and the symmetry of A follow easily. ■

Corollary 2. *Let $\mathbf{t}$ be a knot vector that satisfies (5.3)–(5.4) and (5.6), and let $r \ge m$. If G is a symmetric matrix with exponential decay that satisfies $M_{\mathbf{t};r}G = 0$, then there exists a unique symmetric matrix $A = [a_{i,k}]_{i,k\in\mathbb{M}}$, of exponential decay, such that*

$$G = D^B_{\mathbf{t};r}\ A\ (D^B_{\mathbf{t};r})^T. \tag{5.31}$$

Proof: We define the symmetric matrices

$$\tilde{G} := \mathrm{diag}\,[d^{1/2}_{\mathbf{t};r,k}]_k\ G\ \mathrm{diag}\,[d^{1/2}_{\mathbf{t};r,k}]_k, \quad \tilde{A} := \mathrm{diag}\,[d^{-1/2}_{\mathbf{t};r+1,k}]_k\ A\ \mathrm{diag}\,[d^{-1/2}_{\mathbf{t};r+1,k}]_k.$$

Then the identity $M_{\mathbf{t};r}G = 0$ is equivalent to $\mathbb{I}\ \tilde{G} = 0$. Moreover, by (5.14), the identity (5.31) is equivalent to $\tilde{G} = \Delta\ \tilde{A}\ \Delta^T$. Since it follows from (5.6) that $\tilde{G}$ decays exponentially, we obtain, by Lemma 1, that $\tilde{A}$ is uniquely determined by (5.31) and decays exponentially. As a consequence of (5.6), A decays exponentially and satisfies (5.31). ■

In the same way, the above proofs yield that banded matrices have finite expansions of the form (5.21) with diagonal matrices G_k alone. Recall that matrices with bandwidth 1 are diagonal, with bandwidth 2 are tridiagonal, etc. We state the following without proof.

Proposition 4. *Let G be a symmetric banded matrix with bandwidth L, $\mathbf{t}$ a knot vector that satisfies (5.3)–(5.4), and $r \ge m$. Then there exist diagonal matrices $G_0, \ldots, G_{L-1}$, such that*

$$G = G_0 + E^B_{\mathbf{t};r,1}\ G_1\ (E^B_{\mathbf{t};r,1})^T + \cdots + E^B_{\mathbf{t};r,L-1}\ G_{L-1}\ (E^B_{\mathbf{t};r,L-1})^T. \tag{5.32}$$

Furthermore, $G_0, \ldots, G_{L-1}$ are uniquely determined by G and $\mathbf{t}$.

5.3. Approximate duals with minimum support

As in [4], we consider the homogeneous polynomials $F_\nu : \mathbb{R}^r \to \mathbb{R}$, defined by

$$F_\nu(x_1, \ldots, x_r) = \frac{2^{-\nu}}{\nu!} \sum_{\substack{1 \le i_1, \ldots, i_{2\nu} \le r, \\ i_1, \ldots, i_{2\nu} \ \text{distinct}}} \prod_{j=1}^{\nu} (x_{i_{2j-1}} - x_{i_{2j}})^2. \tag{5.33}$$

Without causing any confusion, we make use of the same symbol F_ν for any number of arguments. Hence, for $r < 2\nu$, F_ν becomes the zero function, in accordance with the fact that the sum in (5.33) is empty. For notational consistency, we set $F_0 \equiv 1$, regardless of the number r of arguments.

For $r \ge 2\nu$, it follows from the definition that F_ν is a symmetric and homogeneous polynomial of degree 2ν; i.e.,

$$F_\nu(\alpha x_1, \ldots, \alpha x_r) = \alpha^{2\nu} F_\nu(x_1, \ldots, x_r), \qquad F_\nu(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) = F_\nu(x_1, \ldots, x_r),$$

for every $\alpha \in \mathbb{R}$ and every permutation σ. Moreover, F_ν is invariant under a constant shift of the arguments $(x_1, \ldots, x_r) \mapsto (x_1 - c, \ldots, x_r - c)$, and its coordinate degree in each of its variables is 2. Several recursion relations for F_ν were proved in [4]. It is worthwhile to mention that F_ν is also a polynomial in the centered moments

$$\sigma_\mu(x_1, \ldots, x_r) = \sum_{k=1}^{r} (x_k - \overline{x})^\mu, \qquad 2 \le \mu \le \nu,$$

where $\overline{x}$ is the mean of $x_1, \ldots, x_r$. This allows for a very fast computation of F_ν.

In order to establish representations of the minimally supported approximate duals of $\Phi_{\mathbf{t};m} := \Phi_{j;m}$, with knot vector $\mathbf{t} := \mathbf{t}_j$, we introduce the sequences

$$\beta_{m,k}^{(\nu)}(\mathbf{t}) := \frac{m!(m-\nu-1)!}{(m+\nu)!(m+\nu-1)!} F_\nu(t_{k+1}, \ldots, t_{k+m+\nu-1}), \tag{5.34}$$

where $1 \le \nu \le m-1$ and $k \in \mathbb{M}$, and the diagonal matrices

$$U_\nu^B(\mathbf{t}) := \operatorname{diag}(\beta_{m,k}^{(\nu)}(\mathbf{t});\ k \in \mathbb{M}). \tag{5.35}$$

The spsd matrix $S_L^B(\mathbf{t})$, for $1 \le L \le m$, is defined by

$$S_L^B(\mathbf{t}) = I + \sum_{\nu=1}^{L-1} E_{\mathbf{t};m,\nu}^B U_\nu^B(\mathbf{t})(E_{\mathbf{t};m,\nu}^B)^T, \tag{5.36}$$

where we use the notation $E_{\mathbf{t};m,\nu}^B$ instead of using the index j to denote the dependency on the knot vector. It is easy to see that $S_L^B(\mathbf{t})$ is a symmetric positive definite (infinite or bi-infinite) matrix with bandwidth L. Moreover, the kernel $K_{S_L^B(\mathbf{t})}$ in (2.11) has the form

$$K_{S_L^B(\mathbf{t})}(x,y) = \Phi_{\mathbf{t};m}(x)\Phi_{\mathbf{t};m}^T(y) + \sum_{\nu=1}^{L-1}\sum_{k\in\mathbb{M}} \beta_{m,k}^{(\nu)}(\mathbf{t})\frac{\partial^{2\nu}}{\partial x^\nu \partial y^\nu}\ N_{\mathbf{t};m+\nu,k}^B(x)N_{\mathbf{t};m+\nu,k}^B(y). \tag{5.37}$$

The following theorem, which is the main result in this section, is an extension of the corresponding result on a bounded interval $I = [a,b]$ established in [4; Theorem 5], where the knot vector $\mathbf{t}$ has m-fold stacked knots at both endpoints a and b. (We remark that the formulation of the matrix S_L in [4] is given in terms of the normalized B-splines $N_{j;m,k}$, whereas we choose the L_2-normalized B-splines $N_{j;m,k}^B$ in this present paper for ease of notations.)

Theorem 9. *Let I denote the unbounded interval $[0,\infty)$ or $\mathbb{R}$ and let $1 \le L \le m$. The matrix $S_L^B(\mathbf{t})$ in (5.36) defines an approximate dual of order L relative to the B-spline basis $\Phi_{\mathbf{t};m}$.*

The proof of this theorem is divided into two parts. For the (one-sided) infinite interval $I = [0,\infty)$, the result can be reduced to our earlier results in [4]; for the bi-infinite interval a new proof by a "knot removal" argument will be developed in Theorem 11 below.

Proof of Theorem 9 for $I = [0,\infty)$: We make use of the abbreviation $K_S := K_{S_L^B(\mathbf{t})}$. By Proposition 1, it is sufficient to show that

$$\int_0^\infty x^\mu K_S(x,y)\,dx = y^\mu, \quad 0 \le \mu \le L-1,\ y \in [0,\infty). \tag{5.38}$$

When we fix $y \in [t_r, t_{r+1})$, where $r \geq 0$, the function $K(x) := K_S(x,y)$ is given by

$$K(x) = \sum_{\nu=0}^{L-1} \sum_{k=r-m-\nu+1}^{r} \beta_{m,k}^{(\nu)}(\mathbf{t}) \frac{\partial^{2\nu}}{\partial x^\nu \partial y^\nu} \; N_{j;m+\nu,k}^B(x) N_{j;m+\nu,k}^B(y).$$

Therefore, K cannot be distinguished from the function

$$\tilde{K}(x) := K_{\tilde{S}}(x,y),$$

where $\tilde{S}$ is the matrix of the minimally supported approximate dual with the finite knot vector

$$\tilde{\mathbf{t}} := \{t_{-m+1} = \cdots = t_0 = 0 < t_1 \leq \cdots \leq t_{r+m+L} = \tilde{t}_{r+m+L+1} = \cdots = \tilde{t}_{r+m+L+m-1}\}.$$

The result in [4; Theorem 5] shows that (5.38) is satisfied. This completes the proof of Theorem 9 for $I = [0, \infty)$. ■

Before we can give a proof of Theorem 9 for $I = \mathbb{R}$, and in order to develop relevant further results for the case $I = [0, \infty)$, the following extension of the "knot insertion" method to infinite knot vectors is needed. Instead of single knot insertion into a finite knot vector, as was done in the proof of Theorem 5 of [4], we allow the insertion of infinitely many knots in a single step, if these knots are enough separated from each other. More precisely, we call the knot vector $\tilde{\mathbf{t}}$ a simple refinement of $\mathbf{t}$, if

$$\tilde{\mathbf{t}} = [\tilde{t}_k]_{k \in \mathbb{M}} = [\ldots, t_{\rho_i}, \tau_i, t_{\rho_i+1}, \ldots, t_{\rho_{i+1}}, \tau_{i+1}, t_{\rho_{i+1}+1}, \ldots,], \quad \text{where} \quad \rho_{i+1} - \rho_i \geq 2m. \tag{5.39}$$

Moreover, without loss of generality, we assume that $\tilde{t}_{\rho_0} = t_{\rho_0}$ and $t_{\rho_i+1} - t_{\rho_i} > 0$ for all i. The relation between $\{t_k\}$ and $\{\tilde{t}_k\}$ can then be formulated as

$$\begin{aligned} \tilde{t}_{\rho_i+i} &= t_{\rho_i}, \\ \tilde{t}_{\rho_i+i+1} &= \tau_i, \\ \tilde{t}_{k+i+1} &= t_k, \qquad \text{for } \rho_i + 1 \leq k \leq \rho_{i+1} - 1. \end{aligned} \tag{5.40}$$

For an admissible refinement of the knot vectors $\mathbf{t}_j$, in Subsection 5.1, we impose the condition that the number of knots of $\mathbf{t}_{j+1} \setminus \mathbf{t}_j$ in each interval $[t_r^{(j)}, t_{r+1}^{(j)})$ be bounded by a constant n_j. More precisely, we require that there exists a strictly increasing sequence (σ_k) such that

$$t_k^{(j)} = t_{\sigma_k}^{(j+1)} \quad \text{and} \quad \sigma_{k+1} - \sigma_k \leq n_j + 1. \tag{5.41}$$

It is obvious that this condition implies that, in a finite number of steps, we can pass from $\mathbf{t}_j$ to $\mathbf{t}_{j+1}$ by successive steps of simple refinement; *i.e.*, there exist $J \leq 2mn_j$ nested knot vectors $\mathbf{t}_1^\sharp, \ldots, \mathbf{t}_J^\sharp$, with

$$\mathbf{t}_j =: \mathbf{t}_0^\sharp \subset \mathbf{t}_1^\sharp \subset \cdots \subset \mathbf{t}_J^\sharp \subset \mathbf{t}_{J+1}^\sharp := \mathbf{t}_{j+1},$$

such that, for each $i = 0, 1, \ldots, J$, $\mathbf{t}_{i+1}^\sharp$ is a simple refinement of $\mathbf{t}_i^\sharp$.

Several steps of the proof of Theorem 9, for the case $I = \mathbb{R}$, and parts of the construction of tight NMRA spline frames are presented first for the case of a **simple** refinement $\mathbf{t} \subset \tilde{\mathbf{t}}$. First, we return to the definition of $\beta_{m,k}^{(\nu)}(\mathbf{t})$ and simplify the notation by writing

$$\beta_k^{(\nu)} = \beta_{m,k}^{(\nu)}(\mathbf{t}), \quad \tilde{\beta}_k^{(\nu)} = \beta_{m,k}^{(\nu)}(\tilde{\mathbf{t}}), \qquad k \in \mathbf{Z}.$$

Likewise, we use the short-hand notations

$$U_\nu := U_\nu^B(\mathbf{t}), \quad \tilde{U}_\nu := U_\nu^B(\tilde{\mathbf{t}}), \quad \tilde{D}_r := D_{\tilde{\mathbf{t}};r}^B, \quad \tilde{E}_{r,s} := E_{\tilde{\mathbf{t}};r,s}^B.$$

It is easy to see that

$$\beta_k^{(\nu)} = \tilde{\beta}_{k+i+1}^{(\nu)}, \qquad \text{for } \rho_i \leq k \leq \rho_{i+1} + 1 - m - \nu. \tag{5.42}$$

By the assumption (5.39) on the simple refinement, the following result can be proven in the same way as in our previous paper [4; Lemma 4], as the insertion of the knots τ_i in (5.39) does not interfere with each other. Therefore, we omit the proof of the result.

Lemma 2. *For all $i \in \mathbf{Z}$ and $\rho_i + 2 - m - \nu \le k \le \rho_i$, we have*

$$\tilde{\beta}_{k+i}^{(\nu)} = \frac{(t_{k+m+\nu-1} - \tau_i)\beta_{k-1}^{(\nu)}}{t_{k+m+\nu-1} - t_k} + \frac{(\tau_i - t_k)\beta_k^{(\nu)}}{t_{k+m+\nu-1} - t_k} - \frac{(t_{k+m+\nu-1} - \tau)(\tau - t_k)\beta_k^{(\nu-1)}}{(m+\nu)(m+\nu-1)} \tag{5.43}$$

We next describe the block diagonal structure of the refinement matrix $P_{\mathbf{t},\tilde{\mathbf{t}};r}$ for $m \le r = m + \nu \le 2m - 1$. Let

$$P^i_{\mathbf{t},\tilde{\mathbf{t}};m+\nu} := \begin{bmatrix} 1 & & & & \\ 1 - a^i_{2,m+\nu} & a^i_{2,m+\nu} & & & \\ & \ddots & \ddots & & \\ & & 1 - a^i_{m+\nu,m+\nu} & a^i_{m+\nu,m+\nu} \\ & & & 1 \end{bmatrix}, \tag{5.44}$$

where

$$a^i_{j,m+\nu} := \frac{\tau_i - t_{\rho_i - m - \nu + j}}{t_{\rho_i + j - 1} - t_{\rho_i - m - \nu + j}}, \quad j = 2, \ldots, m + \nu. \tag{5.45}$$

Then the column indices of the columns of $P^i_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ run from $\rho_i - m - \nu + 1$ to ρ_i, and the row indices of the rows of $P^i_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ start from $\rho_i + i - m - \nu + 1$ and end with $\rho_i + i$. Let

$$I^i_{m+\nu} := I_{\rho_{i+1} - \rho_i - m - \nu}. \tag{5.46}$$

If $\rho_{i+1} - \rho_i = 2m$ and $m = \nu$, then $I^i_{m+\nu}$ is empty. Similarly, the column indices of the columns of $I^i_{m+\nu}$ run from $\rho_i + 1$ to $\rho_{i+1} - m - \nu$, and the row indices of the rows of $I^i_{m+\nu}$ start from $\rho_i + i + 1$ and end with $\rho_{i+1} + i - m - \nu$. Then the refinement matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ is block diagonal of the form

$$P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu} = \mathrm{diag}[\ldots, I^{i-1}_{m+\nu}, P^i_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}, I^i_{m+\nu}, P^{i+1}_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}, \ldots]. \tag{5.47}$$

Lemma 3. *Suppose that the knot vectors $\mathbf{t}$ and $\tilde{\mathbf{t}}$ are given as in (5.40) with the constraint (5.39). Let diagonal matrices $V_\nu = V^\nu_{\mathbf{t},\tilde{\mathbf{t}}}$, $0 \le \nu \le m$, be defined by $V_0 = 0$ and, for $1 \le \nu \le m$, by the diagonal entries*

$$v^{(\nu)}_{k+i} := \begin{cases} \dfrac{(t_{k+m+\nu-1} - \tau_i)(\tau_i - t_k)\beta_k^{(\nu-1)}}{(m+\nu-1)(t_{k+m+\nu-1} - t_k)}, & \rho_i + 2 - m - \nu \le k \le \rho_i,\ i \in \mathbf{Z} \\ 0, & \textit{otherwise.} \end{cases} \tag{5.48}$$

Then V_ν is positive semi-definite and satisfies

$$V_\nu + \tilde{U}_\nu - P_{m+\nu}\ U_\nu\ P_{m+\nu}^T = \tilde{D}_{m+\nu}\ V_{\nu+1}\ \tilde{D}_{m+\nu}^T, \quad 0 \le \nu \le m-1. \tag{5.49}$$

Furthermore, the sequence of matrices V_ν, $0 \le \nu \le m$, is uniquely determined by the identity (5.49).

Proof: Since each of the matrices in (5.49) can be written as a block diagonal matrix with block sizes compatible to those of $P_{m+\nu}$ in (5.47), the proof of (5.49) can be reduced to the case with only a single block, and this case has already been taken care of in [4; Lemma 5], without being affected by the endpoints of a bounded interval. ■

The result of Lemma 3 has the following consequence.

Proposition 5. *Under the same assumptions as in Lemma 3 on the knot vectors $\mathbf{t}$ and $\tilde{\mathbf{t}}$, the matrix $\tilde{S}_L - P_{\mathbf{t},\tilde{\mathbf{t}};m} S_L P_{\mathbf{t},\tilde{\mathbf{t}};m}^T$, for each $L = 1, \ldots, m$, is positive semi-definite and satisfies*

$$\tilde{S}_L^B - P_{\mathbf{t},\tilde{\mathbf{t}};m}^B S_L^B (P_{\mathbf{t},\tilde{\mathbf{t}};m}^B)^T = \tilde{E}_{m,L}^B V_L (\tilde{E}_{m,L}^B)^T, \tag{5.50}$$

where the matrices $V_L, L = 1, \ldots, m$, are those in Lemma 3.

The proof of the previous Proposition is based on the definition of the matrices S_L and $\tilde{S}_L$, as in (5.36). Since it is similar to the proof of [4; Lemma 6], it is omitted here.

We now come back to the consideration of a general **admissible** knot refinement $\mathbf{t}_j \subset \mathbf{t}_{j+1}$. The intermediate simple refinements

$$\mathbf{t}_j =: \mathbf{t}_0^\sharp \subset \mathbf{t}_1^\sharp \subset \cdots \subset \mathbf{t}_J^\sharp \subset \mathbf{t}_{J+1}^\sharp := \mathbf{t}_{j+1} \tag{5.51}$$

lead to the following result.

Theorem 10. *Let $\mathbf{t}_{j+1}$ be an admissible refinement of $\mathbf{t}_j$ and $1 \le L \le m$. Then the matrix $S_L^B(\mathbf{t}_{j+1}) - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\ S_L^B(\mathbf{t}_j)\ \ (P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\ )^T$ is positive semi-definite. Moreover, if $\mathbf{t}_k^\sharp$, $1 \le k \le J$, define successive simple refinements as in (5.51), the matrix has the representation*

$$S_L^B(\mathbf{t}_{j+1}) - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\ S_L^B(\mathbf{t}_j)\ \ (P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\ )^T = E^B_{\mathbf{t}_{j+1};m,L}\ Z_L\ (E^B_{\mathbf{t}_{j+1};m,L})^T, \tag{5.52}$$

where

$$Z_L = Z_L(\mathbf{t}_j, \mathbf{t}_{j+1}) := \sum_{k=1}^{J+1} P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m+L}\ V_{k,L}\ (P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m+L})^T, \tag{5.53}$$

and $V_{k,L}$ are diagonal matrices with non-negative entries. Furthermore, Z_L can be written as $\sum_{k=1}^{J+1} \widehat{Q}_k \widehat{Q}_k^T$, where $\widehat{Q}_k$ are banded lower triangular matrices with bandwidth $k+1$.

Proof: For two adjacent knot vectors $\mathbf{t}_{k-1}^\sharp$ and $\mathbf{t}_k^\sharp$, define $V_{k,L}$ as in Lemma 3 for $k = 1, \ldots, J+1$. We write the left-hand side of (5.52) as a telescoping sum and make use of Proposition 5 and the commutation relation (5.19), in order to obtain

$$\begin{aligned}
&S_L^B(\mathbf{t}_{j+1}) - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\ S_L^B(\mathbf{t})\ \ (P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\ )^T \\
&\qquad = \sum_{k=1}^{J+1} [P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m} S_L^B(\mathbf{t}_k^\sharp)\ (P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m})^T - P^B_{\mathbf{t}_{k-1}^\sharp,\mathbf{t}_{j+1};m} S_L^B(\mathbf{t}_{k-1}^\sharp)\ (P^B_{\mathbf{t}_{k-1}^\sharp,\mathbf{t}_{j+1};m})^T] \\
&\qquad = \sum_{k=1}^{J+1} P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m} [S_L^B(\mathbf{t}_k^\sharp) - P^B_{\mathbf{t}_{k-1}^\sharp,\mathbf{t}_k^\sharp;m} S_L^B(\mathbf{t}_{k-1}^\sharp)\ (P^B_{\mathbf{t}_{k-1}^\sharp,\mathbf{t}_k^\sharp;m})^T]\ (P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m})^T \\
&\qquad = \sum_{k=1}^{J+1} P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m}\ E^B_{\mathbf{t}_k^\sharp;m,L}\ V_{k,L}\ (E^B_{\mathbf{t}_k^\sharp;m,L})^T\ (P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m})^T \\
&\qquad = \sum_{k=1}^{J+1} E^B_{\mathbf{t}_{j+1};m,L}\ P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m+L}\ V_{k,L}\ (P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m+L})^T\ (E^B_{\mathbf{t}_{j+1};m,L})^T.
\end{aligned}$$

This proves the identity (5.52) with Z_L in (5.53). In addition, the matrices

$$\widehat{Q}_k = P^B_{\mathbf{t}_k^\sharp,\mathbf{t}_{j+1};m+L}\ V_{k,L}^{1/2}, \qquad \text{for } k = 1, \ldots, J+1, \tag{5.54}$$

are banded lower triangular matrices with bandwidth $k+1$. They provide the desired decomposition of Z_L in the theorem. ■

The next result is the key for the proof of Theorem 9 for the bi-infinite case. It provides us with the necessary argument for "knot removal" and is more powerful than the "knot insertion" argument in our paper [4].

Theorem 11. *Let* $\mathbf{t}_{j+1}$ *be an admissible refinement of* $\mathbf{t}_j$*. Then the matrix* $S_{j+1} := S_L^B(\mathbf{t}_{j+1})$ *defines an approximate dual of order* L *of the B-spline basis* $\Phi_{j+1;m}$*, if and only if the matrix* $S_j := S_L^B(\mathbf{t}_j)$ *defines an approximate dual of order* L *of the B-spline basis* $\Phi_{j;m}$*.*

Proof: We first observe that

$$\begin{aligned}\Gamma^{-1}(\mathbf{t}_{j+1}) - S_{j+1} = & \\ & (\Gamma^{-1}(\mathbf{t}_{j+1}) - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\Gamma^{-1}(\mathbf{t}_j)(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T) \\ & + P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}(\Gamma^{-1}(\mathbf{t}_j) - S_j)(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T - (S_{j+1} - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}S_j(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T).\end{aligned} \tag{5.55}$$

By Theorem 4, (4.10), and (5.16), there exists a symmetric matrix X^1, of exponential decay, such that

$$\Gamma^{-1}(\mathbf{t}_{j+1}) - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}\Gamma^{-1}(\mathbf{t}_j)(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T = E^B_{\mathbf{t}_{j+1};m,L}X^1(E^B_{\mathbf{t}_{j+1};m,L})^T.$$

Moreover, by Theorem 10, there exists a symmetric banded matrix Z_L, such that

$$S_{j+1} - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}S_j(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T = E^B_{\mathbf{t}_{j+1};m,L}Z_L(E^B_{\mathbf{t}_{j+1};m,L})^T.$$

In order to prove the "if"-statement of the theorem, we assume that S_j defines an approximate dual of order L of the B-spline basis $\Phi_{j;m}$. By Theorem 7 there exists a symmetric matrix X_j, of exponential decay, such that

$$\Gamma^{-1}(\mathbf{t}_j) - S_j = E^B_{\mathbf{t}_j;m,L}X_j(E^B_{\mathbf{t}_j;m,L})^T. \tag{5.56}$$

Therefore, it follows from (5.55) and the commutation relation (5.19) that

$$\Gamma^{-1}(\mathbf{t}_{j+1}) - S_{j+1} = E^B_{\mathbf{t}_{j+1};m,L}[X^1 - Z_L + P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m+L}X_j(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m+L})^T](E^B_{\mathbf{t}_{j+1};m,L})^T.$$

Hence, by Theorem 7, S_{j+1} defines an approximate dual of order L of the B-spline basis $\Phi_{j+1;m}$.

Conversely, if S_{j+1} defines an approximate dual of order L of the B-spline basis $\Phi_{j+1;m}$, we obtain from (5.55) and the identity (5.56), where we replace j by $j+1$, that

$$P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m}(\Gamma^{-1}(\mathbf{t}_j) - S_j)(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T = E^B_{\mathbf{t}_{j+1};m,L}X^0(E^B_{\mathbf{t}_{j+1};m,L})^T, \tag{5.57}$$

where $X^0 := X_{j+1} - X^1 + Z_L$. Since $\Gamma^{-1}(\mathbf{t}_j)$ decays exponentially and S_j is banded, by Theorem 8 there exist unique diagonal matrices $W_0, \dots, W_{L-1}$, and a unique symmetric matrix X_L of exponential decay, such that

$$\begin{aligned} &\Gamma^{-1}(\mathbf{t}_j) - S_j = \\ &\qquad W_0 + \cdots + E^B_{\mathbf{t}_j;m,L-1}\, W_{L-1}\, (E^B_{\mathbf{t}_j;m,L-1})^T + E^B_{\mathbf{t}_j;m,L}\, X_L\, (E^B_{\mathbf{t}_j;m,L})^T. \end{aligned} \tag{5.58}$$

We need to show that $W_k = 0$ for $0 \le k \le L-1$. First, we consider the case where $\mathbf{t}_{j+1}$ is a simple refinement of $\mathbf{t}_j$. The case of an admissible refinement then follows by induction.

Let $\mathbf{t}_{j+1}$ be a simple refinement of $\mathbf{t}_j$ as in (5.39) and denote the refinement matrix of order $r \ge m$ by $P_r := P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};r}$. Note that P_m has the structure as described in (5.47) and (5.44). Therefore, $P_m\mathbf{v} = 0$ implies $\mathbf{v} = 0$. Moreover, we write $E_{m,\nu} := E^B_{\mathbf{t}_{j+1};m,\nu}$. Multiplication of both sides of (5.58) by P_m and P_m^T, an application of (5.57), and the commutation relation (5.19) give

$$\begin{aligned} &E_{m,L}X^0E^T_{m,L} = \\ &\qquad P_mW_0P_m^T + E_{m,1}P_{m+1}W_1P^T_{m+1}E^T_{m,1} + \cdots \\ &\qquad + E_{m,L-1}P_{m+L-1}W_{L-1}P^T_{m+L-1}E^T_{m,L-1} + E_{m,L}P_{m+L}X_LP^T_{m+L}E^T_{m,L}. \end{aligned} \tag{5.59}$$

We use induction in order to show that the diagonal matrices $W_0, \dots, W_{L-1}$ vanish. The row vector $M_{\mathbf{t}_{j+1};m}$ satisfies the identities

$$M_{\mathbf{t}_{j+1};m}E_{m,\nu} = \left[\int_I \frac{d^\nu}{dx^\nu} N^B_{j+1;m+\nu,k}\right] = 0, \quad \nu \ge 1,$$

$$M_{\mathbf{t}_{j+1};m}P_m = M_{\mathbf{t}_j;m}.$$

Therefore, multiplication of identity (5.59) by the row vector $M_{\mathbf{t}_{j+1};m}$ from the left leads to

$$M_{\mathbf{t}_{j+1};m} P_m W_0 P_m^T = 0. \tag{5.60}$$

As mentioned above, the special form of P_m implies that $M_{\mathbf{t}_{j+1};m} P_m W_0 = 0$. Since all components of the vector $M_{\mathbf{t}_{j+1};m} P_m = M_{\mathbf{t}_j;m}$ are nonzero, the diagonal matrix W_0 must vanish.

Assume now, that the matrices $W_0, \ldots, W_k$ vanish, where $0 \le k < L-1$. Then (5.59) becomes

$$\begin{aligned} &E_{m,L} X^0 E_{m,L}^T = \\ &\qquad E_{m,k+1} P_{m+k+1} W_{k+1} P_{m+k+1}^T E_{m,k+1}^T + \cdots \\ &\qquad + E_{m,L-1} P_{m+L-1} W_{L-1} P_{m+L-1}^T E_{m,L-1}^T + E_{m,L} P_{m+L} X_L P_{m+L}^T E_{m,L}^T. \end{aligned} \tag{5.61}$$

The repeated application of the assertion of uniqueness in Corollary 2 leads to a cancellation of the matrix factor $E_{m,k+1}$ and its transpose in (5.61). Hence, we obtain

$$\begin{aligned} &E_{m+k+1,L-k-1} X^0 E_{m+k+1,L-k-1}^T = \\ &\qquad P_{m+k+1} W_{k+1} P_{m+k+1}^T + \cdots \\ &\qquad + E_{m+k+1,L-k-2} P_{m+L-1} W_{L-1} P_{m+L-1}^T E_{m+k+1,L-k-2}^T \\ &\qquad + E_{m+k+1,L-k-1} P_{m+L} X_L P_{m+L}^T E_{m+k+1,L-k-1}^T. \end{aligned} \tag{5.62}$$

Then, using the same argument as above, with $M_{\mathbf{t}_{j+1};m+k+1}$ instead of $M_{\mathbf{t}_{j+1};m}$, we obtain $W_{k+1} = 0$. Thus we have shown that S_j defines an approximate dual of $\Phi_{j;m}$, if $\mathbf{t}_{j+1}$ is a simple refinement of $\mathbf{t}_j$.

For an admissible refinement

$$\mathbf{t}_j =: \mathbf{t}_0^\sharp \subset \mathbf{t}_1^\sharp \subset \cdots \subset \mathbf{t}_J^\sharp \subset \mathbf{t}_{J+1}^\sharp := \mathbf{t}_{j+1},$$

where $\mathbf{t}_k^\sharp$ is a simple refinement of $\mathbf{t}_{k-1}^\sharp$, $1 \le k \le J+1$, we first obtain that $S_L^B(\mathbf{t}_J^\sharp)$ defines an approximate dual of order L of the corresponding B-spline basis with respect to the knot vector $\mathbf{t}^{\sharp}_J$, by the argument for simple refinements. By repeating this argument, we finally obtain that $S^B_L(\mathbf{t}_j)$ defines an approximate dual of the B-spline basis $\Phi_{j;m}$. This completes the proof of Theorem 11. ∎

We now return to the proof of Theorem 9 for the case where $I = \mathbb{R}$.

Proof of Theorem 9 for $I = \mathbb{R}$: Let $\mathbf{t}$ be a bi-infinite knot vector as in (5.2)–(5.4) and (5.6). We define the knot vector $\tilde{\mathbf{t}} \supset \mathbf{t}$ by inserting several copies of t_0 into $\mathbf{t}$, such that the multiplicity of t_0 in $\tilde{\mathbf{t}}$ is m. Thus the basis functions $\Phi_{\tilde{\mathbf{t}};m}$ can be treated as two disjoint sets of basis functions, $\Phi_{\tilde{\mathbf{t}}^1;m}$ on $(-\infty, t_0]$ and $\Phi_{\tilde{\mathbf{t}}^2;m}$ on $[t_0, \infty)$, where

$$\tilde{\mathbf{t}}^1 := [\ldots, t_{-1}, \underbrace{t_0, \ldots, t_0}_{m-\text{fold}}], \qquad \tilde{\mathbf{t}}^2 := [\underbrace{t_0, \ldots, t_0}_{m-\text{fold}}, t_1, \ldots].$$

Moreover, the Gramian matrix $\Gamma(\tilde{\mathbf{t}})$ for $\Phi_{\tilde{\mathbf{t}};m}$ is a block diagonal matrix

$$\Gamma(\tilde{\mathbf{t}}) = \begin{bmatrix} \Gamma(\tilde{\mathbf{t}}^1) & \\ & \Gamma(\tilde{\mathbf{t}}^2) \end{bmatrix}.$$

Since the result of Theorem 9 was already shown for both intervals $(-\infty, t_0]$ and $[t_0, \infty)$, the matrices $S^B_L(\tilde{\mathbf{t}}^1)$ and $S^B_L(\tilde{\mathbf{t}}^2)$ define approximate duals of order L for the respective B-spline basis $\Phi_{\tilde{\mathbf{t}}^1;m}$ and $\Phi_{\tilde{\mathbf{t}}^2;m}$. From Theorem 7, it follows that

$$\begin{aligned} \Gamma^{-1}(\tilde{\mathbf{t}}^1) - S^B_L(\tilde{\mathbf{t}}^1) &= E^B_{\tilde{\mathbf{t}}^1;m,L} X(\tilde{\mathbf{t}}^1)(E^B_{\tilde{\mathbf{t}}^1;m,L})^T, \\ \Gamma^{-1}(\tilde{\mathbf{t}}^2) - S^B_L(\tilde{\mathbf{t}}^2) &= E^B_{\tilde{\mathbf{t}}^2;m,L} X(\tilde{\mathbf{t}}^2)(E^B_{\tilde{\mathbf{t}}^2;m,L})^T \end{aligned},$$

where $X(\tilde{\mathbf{t}}^1)$ and $X(\tilde{\mathbf{t}}^2)$ are symmetric matrices of exponential decay. Since the matrices $S^B_L(\tilde{\mathbf{t}})$ and $E^B_{\tilde{\mathbf{t}};m,L}$ have the diagonal block form

$$S^B_L(\tilde{\mathbf{t}}) = \begin{bmatrix} S^B_L(\tilde{\mathbf{t}}^1) & \\ & S^B_L(\tilde{\mathbf{t}}^2) \end{bmatrix} \qquad \text{and} \quad E^B_{\tilde{\mathbf{t}};m,L} = \begin{bmatrix} E^B_{\tilde{\mathbf{t}}^1;m,L} & \\ & E^B_{\tilde{\mathbf{t}}^2;m,L} \end{bmatrix},$$

it follows that

$$\Gamma^{-1}(\tilde{\mathbf{t}}) - S^B_L(\tilde{\mathbf{t}}) = E^B_{\tilde{\mathbf{t}};m,L} X(\tilde{\mathbf{t}})(E^B_{\tilde{\mathbf{t}};m,L})^T,$$

where $X(\tilde{\mathbf{t}})$ is the block diagonal matrix with diagonal blocks $X_{\tilde{\mathbf{t}}^1;m,L}$ and $X_{\tilde{\mathbf{t}}^2;m,L}$. Therefore, $S_L(\tilde{\mathbf{t}})$ yields an approximate dual of order L for $\Phi_{\tilde{\mathbf{t}};m}$. By Theorem 11, $S_L(\mathbf{t})$ also defines an approximate dual of order L for $\Phi_{\mathbf{t};m}$. This concludes the proof of Theorem 9. ■

Further results on the approximate dual $S_L^B(\mathbf{t})$, for both the one-sided infinite and bi-infinite case, can now be derived easily.

Theorem 12. *Let $\mathbf{t}$ be a knot vector which satisfies (5.2)–(5.6). If R is an spsd matrix that defines an approximate dual of order L of $\Phi_{\mathbf{t};m}$, and R has bandwidth L, then $R = S_L^B(\mathbf{t})$; i.e., $S_L^B(\mathbf{t})$ defines a minimally supported approximate dual of order L for $\Phi_{\mathbf{t};m}$.*

Proof: Let R be a matrix as in the theorem. By the assumption and Theorem 7, there exist symmetric matrices X_1 and X_2, with exponential decay, such that

$$\Gamma^{-1}(\mathbf{t}) - S_L^B(\mathbf{t}) = E^B_{\mathbf{t};m,L}\ X_1\ (E^B_{\mathbf{t};m,L})^T, \qquad \Gamma^{-1}(\mathbf{t}) - R = E^B_{\mathbf{t};m,L}\ X_2\ (E^B_{\mathbf{t};m,L})^T.$$

It follows that

$$S_L^B(\mathbf{t}) - R = E^B_{\mathbf{t};m,L}\ (X_1 - X_2)(E^B_{\mathbf{t};m,L})^T, \tag{5.63}$$

and this matrix has bandwidth at most L. By Theorem 8, there exist unique diagonal matrices $G_0, \dots, G_{L-1}$ and a unique symmetric matrix Y of exponential decay, such that

$$S_L^B(\mathbf{t}) - R = G_0 + E^B_{\mathbf{t};m,1}\ G_1\ (E^B_{\mathbf{t};m,1})^T + \cdots + E^B_{\mathbf{t};m,L-1}\ G_{L-1}\ (E^B_{\mathbf{t};m,L-1})^T + E^B_{\mathbf{t};m,L}\ Y\ (E^B_{\mathbf{t};m,L})^T. \tag{5.64}$$

Moreover, by Proposition 4, the matrix Y is the zero matrix. By a comparison of (5.63) and (5.64) we find that $S_L^B(\mathbf{t}) - R = 0$. ■

The following result describes the uniform boundedness of the kernels $K_{S_L^B(\mathbf{t})}$, regardless of the knot vector $\mathbf{t}$ and the interval $I = [0, \infty)$ or $I = \mathbb{R}$. The proof is completely analogous to the case of a bounded interval, see [4; Theorem 8].

Theorem 13. *For any knot vector as in (5.2)–(5.4), the kernel $K_{S_L^B(\mathbf{t})}$ satisfies (4.7) and (4.8), where the upper bound C does not depend on the knot vector.*

As a consequence of the last theorem, we can show that items (i) and (ii) of Theorem 3 are satisfied if we choose the minimally supported approximate dual $S_j = S_L^B(\mathbf{t}_j)$ for the construction of tight NMRA frames of splines.

Theorem 14. *Let $\mathbf{t}_j \subset \mathbf{t}_{j+1}$, $j \in \mathbf{Z}$, be knot vectors which satisfy (5.2)–(5.6). Then the quadratic forms*

$$T_j f := [\langle f, N_{j;m,k}\rangle]_{k\in\mathbb{M}_j}\ S_L^B(\mathbf{t}_j)\ [\langle f, N_{j;m,k}\rangle]_{k\in\mathbb{M}_j}$$

are uniformly bounded on $L_2(I)$ and

$$\lim_{j\to\infty} T_j f = \|f\|^2, \qquad \lim_{j\to-\infty} T_j f = 0,$$

holds for all $f \in L_2(I)$.

Proof: The uniform boundedness of T_j directly follows from Theorem 13. The same reasoning as in (4.7)–(4.9) leads to the density result. Uniform boundedness of the quadratic forms T_j, as j tends to $-\infty$, implies that $T_j f$ tends to 0 in $L_2(I)$ for $j \to -\infty$. ■

We summarize the procedure for the construction of tight NMRA frames of spline functions with L vanishing moments based on the results of Theorem 3, Theorem 5, and Corollary 1.

Suppose every refinement $\mathbf{t}_j \subset \mathbf{t}_{j+1}$ is admissible as in (5.51).

1. Construct $S_L^B(\mathbf{t}_{j+1})$ and $S_L^B(\mathbf{t}_j)$ as defined in (5.36). Then the matrix

$$S_L^B(\mathbf{t}_{j+1}) - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m} S_L^B(\mathbf{t}_j)(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T$$

is spsd by Theorem 10.

2. Compute a symmetric factorization in the form of

$$S_L^B(\mathbf{t}_{j+1}) - P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m} S_L^B(\mathbf{t}_j)(P^B_{\mathbf{t}_j,\mathbf{t}_{j+1};m})^T = E^B_{\mathbf{t}_{j+1};m,L}\left(\sum_{k=1}^{J+1} \widehat{Q}_{j,k}\widehat{Q}_{j,k}^T\right)(E^B_{\mathbf{t}_{j+1};m,L})^T,$$

(again by applying Theorem 10), where the matrices $\widehat{Q}_{j,k}$ result from breaking up the knot refinement $\mathbf{t}_j \subset \mathbf{t}_{j+1}$ into $J+1$ steps of simple knot insertion. In particular, all the matrices $\widehat{Q}_{j,k}$ are banded and lower triangular.

3. Then the collection of families

$$\Psi_j := \{\psi_{j,k}\} = \bigcup_{k=1}^{J+1} \Phi_{j;m} E^B_{\mathbf{t}_{j+1};m,L} \widehat{Q}_{j,k}$$

defines a tight NMRA frame of spline wavelets $\psi_{j,k}$, which have compact support and L vanishing moments.

Following this procedure, we will construct two concrete examples of tight NMRA frames of linear and cubic splines in the next section.

Finally, we remark that the general time-domain approach introduced in this paper allows certain flexibilities over the standard Fourier approach for the stationary setting. It is therefore interesting to know if this new approach can be applied to settle some of the unanswered questions on stationary tight frames on $\mathbb{R}$, such as the problem of minimum support for 3 symmetric/antisymmetric frame generators with maximum order of vanishing moments, a question raised in [10].

6. Examples of Tight Frames of Spline-Wavelets

In this section, we demonstrate our results in Section 5 with examples on linear and cubic splines.

6.1. Piecewise linear tight frames

Let $(\mathbf{t}_j)_{j\in\mathbf{Z}}$ be a nested sequence of knot vectors on $\mathbb{R}$ and meshsizes $h(\mathbf{t}_j)$ tending to zero. Here, we consider piecewise linear spline-wavelets with 2 vanishing moments, that is, $m = L = 2$. The matrices $S^B_2(\mathbf{t}_j)$ in (5.36) are given by

$$S^B_2(\mathbf{t}_j) = I + E^B_{2,1}(\mathbf{t}_j) U^B_1(\mathbf{t}_j) (E^B_{2,1}(\mathbf{t}_j))^T,$$

where

$$U_1^B(\mathbf{t}_j) = \mathrm{diag}\,[\cdots, \frac{(t_{k+2}^{(j)} - t_{k+1}^{(j)})^2}{6}, \cdots],$$

$$E_{2,1}^B(\mathbf{t}_j) = \mathrm{diag}\,[\cdots, \left(\frac{2}{t_{k+2} - t_k}\right)^{1/2}, \cdots]\ \Delta\ \mathrm{diag}\,[\cdots, \left(\frac{3}{t_{k+3} - t_k}\right)^{1/2}, \cdots].$$

It is sufficient to describe the construction of the wavelet family $\Psi_0 = \{\psi_{0,k}\}$, since the families Ψ_j, $j \neq 1$, are constructed analogously. In the following, we develop an explicit formulation of the wavelets $\psi_{0,k}$ for the case that two adjacent knot vectors satisfy the condition $t_{2k}^{(1)} = t_k^{(0)}$ and each knot is a simple knot. For convenience, the superscript (1) of $t_k^{(1)}$ will be dropped from now on. In this case, the factorization

$$S_2^B(\mathbf{t}_1) - P_{\mathbf{t}_0,\mathbf{t}_1;2}^B S_2^B(\mathbf{t}_0)(P_{\mathbf{t}_0,\mathbf{t}_1;2}^B)^T = E_{\mathbf{t}_1;2,2}^B Z_2 (E_{\mathbf{t}_1;2,2}^B)^T$$

is obtained for a symmetric matrix $Z_2 = Z_2(\mathbf{t}_0, \mathbf{t}_1)$ with bandwidth 3. Z_2 can be decomposed into the form of $\widehat{Q}_0 \widehat{Q}_0^T$, where

$$\widehat{Q}_0 = R_1 \begin{bmatrix} \ddots & & & & & \\ & 1 & t_1 - t_0 & & & \\ & & t_5 - t_1 & & & \\ & & t_6 - t_5 & 1 & t_3 - t_2 & \\ & & & & t_7 - t_3 & \\ & & & & t_8 - t_7 & \ddots \end{bmatrix} R_2 \qquad (6.1)$$

and where R_1 and R_2 are diagonal matrices with diagonal entries given by

$$R_{1;k,k} = \frac{4}{t_{k+2} - t_{k-2}}, \qquad k \in \mathbf{Z},$$

$$R_{2;k,k} = \frac{(t_{k+1} - t_{k-1})\sqrt{(t_{k+3} - t_k)(t_k - t_{k-3})}}{12\sqrt{2(t_{k+3} - t_{k-3})}}, \quad \text{if } k \text{ is odd},$$

and

$$R_{2;k,k} = \frac{1}{12\sqrt{2}} \Big((t_{k+2} - t_{k-1})(t_{k+1} - t_{k-2}) \times$$
$$\big((t_k - t_{k-1})(t_k - t_{k-2})(t_{k+2} - t_{k+1}) + (t_{k+1} - t_k)(t_{k+2} - t_k)(t_{k-1} - t_{k-2})\big)\Big)^{1/2}$$

if k is even. The wavelet family Ψ_0 is then defined by the coefficient matrix

$$Q_0 := E_{\mathbf{t}_1;2,2}\widehat{Q}_0 =: [\ldots, \mathbf{q}_{2k}, \mathbf{q}_{2k+1}, \ldots] \cdot R_2,$$

where R_2 is the diagonal matrix in (6.1) and the column vectors $\mathbf{q}_{2k}$ and $\mathbf{q}_{2k+1}$ are given by

$$\mathbf{q}_{2k}^T = \Bigg[\mathbf{0}, \\ \frac{24}{(t_{2k}-t_{2k-2})(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-2})}, \frac{24(t_{2k-2}+t_{2k-1}-t_{2k+1}-t_{2k+2})}{(t_{2k+1}-t_{2k-1})(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-1})(t_{2k+2}-t_{2k-2})}, \\ \frac{24}{(t_{2k+2}-t_{2k})(t_{2k+2}-t_{2k-1})(t_{2k+2}-t_{2k-2})}, \mathbf{0}\Bigg],$$

and

$$\mathbf{q}_{2k+1}^T = \Bigg[\mathbf{0}, \frac{24(t_{2k-1}-t_{2k-2})}{(t_{2k}-t_{2k-2})(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-2})}, \frac{24}{(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-2})}, \\ \frac{24(t_{2k-2}+t_{2k}-t_{2k+2}-t_{2k+4})}{(t_{2k+2}-t_{2k-2})(t_{2k+2}-t_{2k})(t_{2k+4}-t_{2k})}, \frac{24}{(t_{2k+4}-t_{2k+1})(t_{2k+4}-t_{2k})}, \\ \frac{24(t_{2k+4}-t_{2k+3})}{(t_{2k+4}-t_{2k+2})(t_{2k+4}-t_{2k+1})(t_{2k+4}-t_{2k})}, \mathbf{0}\Bigg].$$

In the special case, where the knots in $\mathbf{t}_0$ are equidistant (with stepsize h_0) and the new knots are placed in the middle of each knot interval, our construction leads to

$$Q_0 = \frac{1}{12\sqrt{h_0}} \begin{bmatrix} \ddots & & & & & \\ & 6 & \sqrt{6} & & & \\ & -12 & 2\sqrt{6} & & & \\ & 6 & -6\sqrt{6} & 6 & \sqrt{6} & \\ & & 2\sqrt{6} & -12 & 2\sqrt{6} & \\ & & \sqrt{6} & 6 & -6\sqrt{6} & \\ & & & & 2\sqrt{6} & \ddots \end{bmatrix}.$$

The wavelets are shifts (by integer multiples of h_0) of the two generators $\psi_{0,0}$ and $\psi_{0,1}$, namely

$$\psi_{0,2k}(x) = \psi_{0,0}(x - kh_0), \quad k \in \mathbf{Z},$$

$$\psi_{0,2k+1}(x) = \psi_{0,1}(x - kh_0), \quad k \in \mathbf{Z}.$$

Moreover, all of these interior wavelets are symmetric. If we fix the stepsize $h_0 = 1$, then these generators are identical with the functions ψ^1 and ψ^2 that were constructed in the shift-invariant (i.e. stationary) setting for $L_2(\mathbb{R})$ in [3].

6.2. Piecewise cubic tight frames with double knots

Let V_0 be the space of all splines of order 4 with knot vector

$$\mathbf{t}_0 = \{\ldots, 0, 0, 1, 1, 2, 2, \ldots\},$$

and $\mathbf{t}_1$ is the refinement with double knots at the half integers, that is,

$$\mathbf{t}_1 = \{\ldots, 0, 0, 1/2, 1/2, 1, 1, 3/2, 3/2, 2, 2, \ldots\}.$$

In order to achieve 4 vanishing moments for the tight frame, we need the following diagonal matrices in (5.35), namely

$$\begin{aligned}
U_0(\mathbf{t}_0) &= \mathrm{diag}(\ldots, 4, 4, 2, 2, \ldots),\\
U_1(\mathbf{t}_0) &= \tfrac{1}{9}\mathrm{diag}(\ldots, 3, 1, 3, 1, \ldots),\\
U_2(\mathbf{t}_0) &= \tfrac{11}{900}\mathrm{diag}(\ldots, 1, 1, \ldots),\\
U_3(\mathbf{t}_0) &= \tfrac{1}{2700}\mathrm{diag}(\ldots, \tfrac{43}{12}, 1, \tfrac{43}{12}, 1, \ldots),
\end{aligned}$$

and

$$U_\nu(\mathbf{t}_1) = 2^{1-2\nu}U_\nu(\mathbf{t}_0), \qquad 0 \le \nu \le 3.$$

Then the matrix Z_4 in

$$S_4^B(\mathbf{t}_1) - P^B_{\mathbf{t}_0,\mathbf{t}_1;4}S_4^B(\mathbf{t}_0)(P^B_{\mathbf{t}_0,\mathbf{t}_1;4})^T = E^B_{\mathbf{t}_1;4,4}Z_4(E^B_{\mathbf{t}_1;4,4})^T$$

can be written as

$$Z_4 = (I-K_3)(I-K_2)(I-K_1)\tilde{Z}_4(I-K_1^T)(I-K_2^T)(I-K_3^T),$$

(with tridiagonal nilpotent matrices K_i) lead to a matrix $\tilde{Z}_4$ with bandwidth 4. The factorization of $\tilde{Z}_4$ leads to 5 wavelet frame generators $\psi^i \in V_1$, $1 \le i \le 5$, with

$$\psi^i = \sum_{s=0}^{8} \hat{q}_s^{(i)} \frac{d^4}{dx^4} N_{\mathbf{t}_1,8;s}, \quad 1 \le i \le 5, \tag{6.2}$$

where the coefficients are listed in Table 1 and their graphs are depicted in Figure 1. Note that ψ^2, ψ^4, ψ^5 are symmetric and ψ^1, ψ^3 are antisymmetric. The supports of these generators are

$$\operatorname{supp}\psi^1 = \operatorname{supp}\psi^2 = [0,4], \quad \operatorname{supp}\psi^3 = \operatorname{supp}\psi^4 = \operatorname{supp}\psi^5 = [1,4].$$

The spline wavelets ψ^1, ψ^2 have simple knots at 0 and 4, and double knots at $.5, 1, \ldots, 3.5$, while ψ^3 and ψ^4 have double knots at $1, 1.5, \ldots, 4$. The spline wavelet ψ^5 has simple knots at $1, 4$ and double knots at $1.5, 2, \ldots, 3.5$.

i	$\hat{q}_0^{(i)}$	$\hat{q}_1^{(i)}$	$\hat{q}_2^{(i)}$	$\hat{q}_3^{(i)}$	$\hat{q}_4^{(i)}$	$\hat{q}_5^{(i)}$	$\hat{q}_6^{(i)}$	$\hat{q}_7^{(i)}$	$\hat{q}_8^{(i)}$
1	0.092642	0.370569	1.852847	0.989527	−0.989527	−1.852847	−0.370569	−0.092642	
2	0.126349	0.505395	2.526977	3.156191	3.156191	2.526977	0.505395	0.126349	
3				0.526730	1.601752	0.086252	−0.086252	−1.601752	−0.526730
4				0.580480	2.180883	1.757771	1.757771	2.180883	0.580480
5					0.869741	3.478964	3.478964	0.869741	

Table 1. Coefficients ($*1000$) of wavelets ψ^i in expansion (6.2).

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Figure 1. Generators of wavelets of piecewise cubic tight frame with double knots.

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