Nonstationary Tight Wavelet Frames, I: Bounded Intervals[1)]

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Abstract

The notion of tight (wavelet) frames could be viewed as a generalization of orthonormal wavelets. By allowing redundancy, we gain the necessary flexibility to achieve such properties as "symmetry" for compactly supported wavelets and, more importantly, to be able to extend the classical theory of spline functions with arbitrary knots to a new theory of spline-wavelets that possess such important properties as local support and vanishing moments of order up to the same order of the associated B-splines. This paper is devoted to develop the mathematical foundation of a general theory of such tight frames of non-stationary wavelets on a bounded interval, with spline-wavelets on nested knot sequences of arbitrary non-degenerate knots, having an appropriate number of knots stacked at the end-points, as canonical examples. In a forthcoming paper under preparation, we develop a parallel theory for the study of nonstationary tight frames on an unbounded interval, and particularly the real line, which precisely generalizes the recent work [7,18] from the shift-invariance setting to a general nonstationary theory. In this regard, it is important to point out that, in contrast to orthonormal wavelets, tight frames on a bounded interval, even for the stationary setting in general, cannot be constructed simply by using the tight frame generators for the real line in [7,18] and introducing certain appropriate boundary functions. In other words, the general theories for tight frames on bounded and unbounded intervals are somewhat different, and the results in this paper cannot be derived directly from those of our forthcoming paper. The intent of this paper and the forthcoming one is to build a mathematical foundation for further future research in this direction. There are certainly many interesting unanswered questions, including those concerning minimum support, minimum cardinality of frame elements on each level, "symmetry", and order of approximation of truncated frame series. In addition, generalization of our development to sibling frames already encounters the obstacle of achieving Bessel bounds to assure the frame structure.

1. Introduction

Adaptation of Daubechies' wavelets [16] to yield "locally supported" orthonormal bases of $L_2(I)$ for a bounded interval $I := [a, b]$, simply by introducing the necessary additional wavelets of the same order near the boundary points of I (see [4,5,14]), inherits the affine structure as well as certain limitations of Daubechies' compactly supported orthonormal wavelets for $L_2(\mathbb{R})$. In particular, the lack of symmetry prevents the possibility of linear-phase filtering in applications to signal processing, and the non-existence of an analytical formulation, such as NURBS [27], gives rise to complications in system design in CAD/CAM applications for meeting certain precise specifications of extremely stringent tolerance allowance. As a continuation in the development of MRA frames, initiated by Ron and Shen [28,29], it was shown, in two recent parallel independent developments [7,18], that compactly supported orthonormal wavelet bases of $L_2(\mathbb{R})$ can be replaced by compactly supported tight frames to achieve symmetry and analytical formulations (such as cardinal splines of any order $m \geq 2$), while retaining the same order of vanishing moments (such as m, for the m^{th} order cardinal spline-wavelet tight frames).

In this paper, we observe that it is not possible, in general, to adopt the above-mentioned tight frames [7,18] as interior wavelets for formulating the tight frames of $L_2(I)$, and therefore, go ahead to develop a general theory, along with specific constructive schemes, for the study of tight frames of $L_2(I)$ that consist of "locally supported" functions (to be called wavelets) which possess the arbitrarily desirable order of vanishing moments. Furthermore, this new theory will extend the affine structure to achieve truly nonstationary formulations, such as m^{th} order splines with arbitrary knots in I, for each of the multi-levels (of spline spaces on nested knot sequences), and only rely on the structure of nested finite-dimensional subspaces $V_0 \subset V_1 \subset \cdots$ of $L_2(I)$ that exhaust all of $L_2(I)$ in the sense of L_2-closure. More precisely, for each $j = 0, 1, \ldots$, the space V_j is the algebraic span of some locally supported functions $\phi_{j,k}$, and the wavelets $\psi_{j,\ell}$ that constitute the j^{th} level W_j of the tight frame of $L_2(I)$ are also locally supported, being functions chosen from V_{j+1} that span all of W_j, such that $W_j + V_j = V_{j+1}$. Here, the notion of local support simply means that the lengths of the support intervals of $\phi_{j,k}$ and $\psi_{j,\ell}$ tend to zero, as $j \to \infty$, uniformly in k and ℓ, respectively; although in the actual construction of $\psi_{j,\ell}$ in terms of $\phi_{j+1,k'}$, we will restrict the supports of $\psi_{j,\ell}$ so that they are comparable in size with the supports of the corresponding relevant $\phi_{j,k}$ and $\phi_{j+1,k'}$. For example, when the normalized B-splines of order $m \geq 1$ are used as $\phi_{j,k}$, the only requirements are that the knot sequences (also called knot vectors) $\mathbf{t}_j = \{t_{j,k}\}$ of $\phi_{j,k}$ are nested in the sense of $\mathbf{t}_0 \subset \mathbf{t}_1 \subset \cdots$ and that they are dense in I, meaning that $\max_k(t_{j,k+1} - t_{j,k}) \to 0$ as $j \to \infty$; and the support of each $\psi_{j,\ell}$ for this spline setting is comparable in size with the quantities $(t_{j,k+m} - t_{j,k})$ and $(t_{j+1,k'+m} - t_{j+1,k'})$ for the appropriate indices $k = k(\ell)$ and $k' = k'(\ell)$. The length of this support interval will depend on the desirable order of vanishing moments of $\psi_{j,k}$ (such as any L, where $1 \leq L \leq m$, for this spline discussion).

To achieve the desirable order of vanishing moments, the concept of vanishing moment recovery (VMR) introduced in our earlier paper [7] (or the notion of fundamental function of multiresolution introduced in [28] and adopted in [18] for recovering vanishing moments) is extended to matrix formulation, namely some symmetric positive definite matrices S_j for the j^{th} levels. The wavelets $\psi_{j,\ell}$ are to be formulated in terms of S_j and S_{j+1}, in addition to the $\phi_{j+1,k'}$'s and their relationship with the $\phi_{j,k}$'s. As for the ground level V_0, since we do not wish to re-formulate the $\phi_{0,k}$'s, the notion of tight frames is slightly modified in this paper to mean

$$T_0 f + \sum_{j\geq 0} \sum_k |\langle f, \psi_{j,k}\rangle|^2 = \|f\|^2, \qquad f \in L_2(I), \tag{1.1}$$

where T_0 is defined by the quadratic form

$$T_0 f := [\langle f, \phi_{0,k}\rangle]\, S_0\, [\langle f, \phi_{0,k}\rangle]^T, \tag{1.1}$$

and the wavelets $\{\psi_{j,k}\}$ are so normalized that the tight frame constant (or bound) is 1.

The paper is organized as follows. A general theory of tight frames of nonstationary wavelets for $L_2(I)$ is developed in Sections 2 and 3, with the notion of approximate duals introduced and studied in some details in Section 3. In order to apply this theory to spline functions on arbitrary nested knot sequences and develop useful constructive schemes and specific formulations, the necessary preliminary material on B-splines is discussed in Section 4. The ingredients of particular interest in this paper are introducing the notion of approximate duals and establishing an explicit formulation that possesses certain positivity properties for the approximate duals of B-splines on arbitrary knots. These main results are presented in Section 5 which is divided into 7 subsections to facilitate the presentation of this section. In addition, two technical results on Bernstein polynomials, which are needed in Section 5, are proved in Section 9. The construction of tight frames of spline-wavelets and the analysis of the support of the wavelets are the contents of Section 6. Examples of linear and cubic spline-wavelet frames are presented in Section 7, and a MATLAB program for the computation of approximate duals of B-splines is recorded in Section 8. In Section 10, we show, with two illustrative cardinal cubic spline examples, that in general tight frames on a bounded interval cannot be constructed by adopting the frame basis functions from a tight frame for an unbounded interval as interior basis functions and introducing certain boundary functions. Some of the results of this paper have been announced without proof in the survey article [11].

2. Characterization and Existence of Nonstationary Tight Frames

We begin with the specification of the generic setting of a nonstationary multiresolution analysis. Let $I = [a, b]$ be a bounded interval in $\mathbb{R}$, and

$$V_0 \subset V_1 \subset \cdots \subset L_2(I)$$

a nested sequence of finite-dimensional subspaces, such that

$$\operatorname{clos}_{L_2}\Big(\bigcup_{j\ge 0} V_j\Big) = L_2(I),$$

and that for each $j \geq 0$, the space V_j is spanned by

$$\Phi_j := \Big[\phi_{j,k};\ 1 \leq k \leq M_j\Big], \tag{2.1}$$

where $M_j \geq \dim V_j$. We consider Φ_j in (2.1) as a row vector and let P_j be an $M_{j+1} \times M_j$ real matrix that describes the "refinement" relation

$$\Phi_j = \Phi_{j+1} P_j \tag{2.2}$$

of $V_j \subset V_{j+1}$. In this paper, since we are concerned with the study of tight frames of wavelets with vanishing moments, we assume that V_0 contains the set Π_{L-1} of all polynomials of degree up to $L-1$. For a more homogeneous formulation of results, we use the notation $\mathbb{M}_j = \{1, \ldots, M_j\}$.

Note that linear independence or stability of the families Φ_j is of no concern in this setting, but will be assumed only for convenience in our presentation. Moreover, we do not require any conditions of "uniform" refinement, as usually assumed in the wavelet literature. In particular, we do not assume the spaces V_j to be shift-invariant, nor do we assume dilation invariance. On the other hand, for the wavelets to be useful in applications, we require the following *localization property* of the refinable function vectors.

Definition 1. *The function family $\{\Phi_j\}_{j\geq 0}$ is said to be* locally supported, *if the sequence*

$$h(\Phi_j) := \max_{k \in \mathbb{M}_j} \mathit{length}(\mathit{supp}\, \phi_{j,k}) \tag{2.3}$$

converges to zero.

We will consider matrices Q_j of dimensions $M_{j+1} \times N_j$ (and use the notation $\mathbb{N}_j = \{1, \ldots, N_j\}$), such that the family

$$\{\Psi_j\}_{j\geq 0} := \{\Phi_{j+1}\, Q_j\}_{j\geq 0} \tag{2.4}$$

also satisfies the localization property as defined above. Of special interest, we further consider $Q_j = [q_{i,k}^{(j)}]$ with

$$q_{i,k}^{(j)} = 0 \quad \text{for all} \quad i < i_j(k) \quad \text{and} \quad i > i_j(k) + m_2, \tag{2.5}$$

where $i_j(k)$, $k \in \mathbb{N}_j$, are nondecreasing sequences such that $i_j(k+m_1) > i_j(k)$. In particular, when $i_j(k) = 2k$, the condition (2.5) defines "2-slanted" matrices discussed in [15]. The above notation allows

$$\psi_{j,k} = \sum_{i \in \mathbb{M}_{j+1}} q_{i,k}^{(j)} \phi_{j+1,i} \tag{2.6}$$

to be associated with a reference index $i_j(k)$ that refers to its first nonzero coefficient in (2.6). Furthermore, the condition (2.5) assures that every $\psi_{j,k}$ is a linear combination of at most $m_2 + 1$ consecutive elements of Φ_{j+1}; hence, Ψ_j is locally supported, as defined by (2.3). We consider this as the typical setting for wavelet frames in the nonstationary setting.

Even for this general (nonstationary) setting, we will say that $\{\Phi_j\}$ in (2.1) generates a multiresolution approximation (MRA) of $L_2(I)$ and the tight frame, to be introduced later, an MRA tight frame of $L_2(I)$. A typical example of a nonstationary MRA is $\{V_j\}$, where for each $j \geq 0$, V_j is the space of spline functions of order $m \geq 2$ with respect to some knot vector $\mathbf{t}_j$, with m stacked knots at both endpoints of I, while the interior knots may be nonuniformly spaced and have variable multiplicities from 1 to m, and where the knot vectors are nested, i.e. $\mathbf{t}_0 \subset \mathbf{t}_1 \subset \cdots$, and dense in I. The families Φ_j and Φ_{j+1} can be chosen to be properly normalized B-splines, and the matrix P_j in (2.2) is the refinement matrix that can be computed by applying the Oslo-algorithm. Explicit representation of certain P_j's are given in [10]. If the maximal knot difference tends to zero, then Φ_j defines a locally supported family. A typical family $\Psi_j = \Phi_{j+1}Q_j$ will be defined, where we use a fixed number m_1 (which is often 2 or 3) of frame elements for each "new" knot that is inserted from $\mathbf{t}_j$ to $\mathbf{t}_{j+1}$. The matrix Q_j has m_1 consecutive columns that define $\psi_{j,k} \in V_{j+1}$ whose support contains the same new knot and which are linear combinations of at most $m_2 + 1$ consecutive B-splines. Then Q_j satisfies the conditions of (2.5). Let us relate (2.5) to the case of a stationary MRA on $L_2(\mathbb{R})$, where we find m_1 functions $\psi^1, \ldots, \psi^{m_1}$, whose shifts and dilates generate the tight frame of $L_2(\mathbb{R})$. This setting can be expressed in terms of the families $\Psi_j = \Phi_{j+1} Q_j$, where Q_j is a block Toeplitz matrix that is defined by merging the columns of the two-slanted matrices Q^i, $1 \le i \le m_1$, that appear in the two-scale relation

$$[\psi^i(\cdot - k)]_{k \in \mathbf{Z}} = \Phi_1 Q^i, \qquad 1 \le i \le m_1.$$

More details about spline spaces are given in Section 4, and a comprehensive study of nonstationary spline-wavelet tight frames is given in Sections 5 and 6.

Of particular importance for our investigation is the construction of certain symmetric positive semi-definite (spsd) matrices that give rise to the following operations. These matrices may be considered as extension of the notion of VMR functions in our earlier paper [7].

Definition 2. *Let Φ_j be a finite family with cardinality M_j in $L_2(I)$. For any spsd matrix*

$$S_j = [s_{k,\ell}^{(j)}]_{k,\ell \in \mathbb{M}_j},$$

consider the quadratic form T_j, defined by

$$T_j f := \Big[\langle f, \phi_{j,k}\rangle\Big]_{k \in \mathbb{M}_j} S_j \Big[\langle f, \phi_{j,k}\rangle\Big]_{k \in \mathbb{M}_j}^T, \qquad f \in L_2(I), \tag{2.7}$$

and the corresponding kernel K_{S_j}, defined by

$$K_{S_j}(x,y) := \sum_{k,\ell \in \mathbb{M}_j} s_{k,\ell}^{(j)} \phi_{j,k}(x) \phi_{j,\ell}(y). \tag{2.8}$$

Note that the kernel K_{S_j} is symmetric, i.e., $K_{S_j}(x,y) = K_{S_j}(y,x)$. Moreover T_j and K_{S_j} are related by

$$T_j f = \int_I f(x) \int_I f(y) K_{S_j}(x,y)\, dy\, dx, \quad f \in L_2(I). \tag{2.9}$$

Our aim in this section is to give a definition and characterization of nonstationary MRA tight frames of $L_2(I)$ that correspond to the locally supported function vectors Φ_j. We assume that the ground level component $T_0 f$ of f is given by an spsd matrix S_0 as in Definition 2 and consider the family

$$\Psi_j := [\psi_{j,k}]_{k \in \mathbb{N}_j} = \Phi_{j+1} Q_j, \quad j \ge 0, \tag{2.10}$$

of wavelets in the following notion of tight (wavelet) frames.

Definition 3. *Assume that $\{\Phi_j\}_{j\geq 0}$ is a locally supported family and S_0 is an spsd matrix, that defines the quadratic form T_0 in (2.7). Then the family $\{\Psi_j\}_{j\geq 0} = \{\Phi_{j+1}Q_j\}_{j\geq 0}$ constitutes an* MRA tight frame *of $L_2(I)$ with respect to T_0, if*

$$T_0 f + \sum_{j\geq 0}\sum_{k\in \mathbb{N}_j} |\langle f, \psi_{j,k}\rangle|^2 = \|f\|^2, \quad \text{for all} \quad f \in L_2(I). \tag{2.11}$$

Note that $T_0 f \leq \|f\|^2$, for $f \in L_2(I)$, is a necessary condition for the existence of a tight frame relative to T_0. The number N_j of frame elements (or wavelets) in Ψ_j serves as a free parameter in the construction of tight frames. In particular, this number can be chosen to be larger than (dim V_{j+1} − dim V_j), which is precisely the number of wavelets if redundancy is not considered. For the study of tight frames, it is more practical to consider the numbers N_j to be bounded by a constant c multiple of (dim V_{j+1} − dim V_j) with $c > 1$. Moreover, in the absence of scaling invariance among the spaces V_j, the numbers dim V_j may increase irregularly, e.g. if adaptive refinement of the subspaces V_j of $L_2(I)$ is considered. In the typical example of spline spaces, where the property of nestedness of the spaces is assured by the insertion of additional knots into a given knot vector $\mathbf{t}_j$, it is often desirable to consider the number of wavelets in Ψ_j to be proportional to the number of new knots in the knot vector $\mathbf{t}_{j+1}$.

The importance of including the quadratic form T_0 in this definition will become clear, when we discuss vanishing moments of the families Ψ_j. First we give a general characterization of tight MRA-frames, which provides analogous results as developed in [7; Theorem 1] and [18; Proposition 1.11] (where only one direction of the implication is shown) for the shift invariant (i.e., stationary) setting in $L_2(\mathbb{R})$.

Theorem 1. *Let $\{\Phi_j\}_{j\geq 0}$ be a locally supported family and S_0 an spsd matrix such that $\|T_0 f\| \leq \|f\|^2$ for all $f \in L_2(I)$. Then $\{\Psi_j\}_{j\geq 0} = \{\Phi_{j+1}Q_j\}_{j\geq 0}$ defines an MRA tight frame with respect to T_0, in the sense of Definition 3, if and only if there exist spsd matrices S_j of dimensions $M_j \times M_j$, $j \geq 1$, such that the following conditions hold:*

(i) The quadratic forms T_j in (2.7) satisfy

$$\lim_{j\to\infty} T_j f = \|f\|^2, \qquad f \in L_2(I). \tag{2.12}$$

(ii) For each $j \geq 0$, Q_j, S_j, and S_{j+1} satisfy the identity

$$S_{j+1} - P_j S_j P_j^T = Q_j Q_j^T. \tag{2.13}$$

Proof: We first assume that $\psi_{j,k}$, $j \geq 0$, $k \in \mathbb{N}_j$, define an MRA tight frame with respect to T_0, and each family Ψ_j is defined by a matrix Q_j in (2.10). If we define the matrices S_j recursively by

$$S_{j+1} = P_j S_j P_j^T + Q_j Q_j^T, \quad j \geq 0, \tag{2.14}$$

then (ii) is satisfied automatically. It is easily seen that each S_j is an spsd matrix of the correct size and

$$T_{J+1} f = T_J f + \sum_{k\in\mathbb{N}_J} |\langle f, \psi_{J,k}\rangle|^2 = T_0 f + \sum_{j=0}^{J} \sum_{k\in\mathbb{N}_j} |\langle f, \psi_{j,k}\rangle|^2, \qquad J \geq 0. \tag{2.15}$$

Hence, the quadratic form T_J is bounded from above by the identity, and property (i) holds as well. Thus, we have proved one direction of the theorem. To establish the converse, assume that the spsd matrices S_j, $j \geq 1$, are given and satisfy (i)–(ii). Then the identity (2.15) is a direct consequence of condition (ii), and (i) implies that taking the limit for $J \to \infty$ on both sides of (2.15) leads to the tight frame condition. ■

Remark 1. Operators of the form (2.9) are well studied in the Functional Analysis literature. For example, the following three conditions are sufficient for the validity of property (i) in Theorem 1:

$$\int_I |K_{S_j}(x,y)|\, dy \leq C \qquad \text{a.e. } x \in I, \quad j \geq 0, \tag{2.16}$$

for some constant $C > 0$,

$$\int_I K_{S_j}(x,y)\, dy = 1, \qquad \text{a.e. } x \in I, \quad j \geq 0, \tag{2.17}$$

and

$$\lim_{j\to\infty}\int_{|x-y|>\epsilon} |K_{S_j}(x,y)|\,dy = 0, \qquad j \geq 0, \tag{2.18}$$

for any $\epsilon > 0$. We remark that condition (2.18), by itself, is satisfied, if the matrices S_j have a fixed maximal bandwidth $r > 0$ and $\{\Phi_j\}_{j\geq 0}$ is locally supported, since the integral in (2.18) is zero for sufficiently large j. We return to the construction of kernels K_{S_j} of this type in the next section.

There is a simple way to see that the ground level T_0 is relevant to the order of vanishing moments of the (frame) wavelets $\psi_{j,k}$. Indeed, assuming that all of the wavelets $\psi_{j,k}$ have vanishing moments of order $L \geq 1$ and that $\Pi_{L-1} \subset V_0$, we see that the tight frame condition (2.11) then implies

$$T_0 f = \|f\|^2 \quad \text{for all} \quad f \in \Pi_{L-1}. \tag{2.19}$$

On the other hand, the condition

$$\int_I f(y) K_{S_0}(x,y)\,dy = f(x), \quad f \in \Pi_{L-1},\ x \in I, \tag{2.20}$$

implies, by (2.9), that $T_0 f = \|f\|^2$ for all $f \in \Pi_{L-1}$ as well. Note that (2.20), with $L = 1$, is identical to the property (2.17), which is an integral part of the approximation properties of the sequence of kernels K_{S_j}. Hence, conditions (2.19) and (2.20) offer two points of view for the characterization of tight MRA-frames with L vanishing moments.

Theorem 2. *Let S_0 be an spsd matrix such that $\|T_0 f\| \leq \|f\|^2$ for all $f \in L_2(I)$ and let $\{\Psi_j\}_{j\geq 0} = \{\Phi_{j+1} Q_j\}_{j\geq 0}$. Then the following statements hold.*

(a) The functions $\psi_{j,k}$, $j \geq 0$, $k \in \mathbb{N}_j$, have L vanishing moments and define an MRA tight frame with respect to T_0, if and only if there exist spsd matrices S_j of dimensions $M_j \times M_j$, $j \geq 1$, such that conditions (i)–(ii) of Theorem 1 hold and that

(iii) $$T_j f = \|f\|^2 \quad \textit{for all} \quad f \in \Pi_{L-1},\ j \geq 0.$$

(b) *Under the additional assumption that the kernel K_{S_0} satisfies (2.20), the result in part (a) is valid with property (iii) replaced by*

(iii') $$\int_I f(y)K_{S_j}(x,y)\,dy = f(x), \quad f \in \Pi_{L-1},\ x \in I,\ j \geq 1.$$

Proof: In comparison with Theorem 1, we only have to establish the claim that for all of the wavelets $\psi_{j,k}$ to have L vanishing moments, it is necessary and sufficient that property (iii), or its replacement (iii'), is satisfied. If the vanishing moment condition is satisfied for all $\psi_{j,k}$ of the tight MRA-frame, then $T_0 f = \|f\|^2$ holds for all $f \in \Pi_{L-1}$, by (2.11). The recursive definition of S_j, $j \geq 1$, in the proof of Theorem 1 leads to the identity (2.15), and this gives $T_j f = T_0 f = \|f\|^2$ for all $j \geq 1$ and $f \in \Pi_{L-1}$. Likewise, the stronger condition (2.20) is inherited by K_{S_j}. Conversely, if all of the operators T_j satisfy $T_j f = \|f\|^2$ for $f \in \Pi_{L-1}$ (or if the stronger condition (iii') is satisfied for K_{S_j}, $j \geq 0$), then identity (2.15) implies that

$$|\langle f, \psi_{j,k}\rangle|^2 \leq T_{j+1}f - T_j f = 0, \qquad f \in \Pi_{L-1},$$

for all $j \geq 0$ and $k \in \mathbb{N}_j$. Hence, the wavelets $\psi_{j,k}$ have L vanishing moments. ■

Remark 2. The result in the previous theorem does not extend to the case of unbounded intervals without additional requirements on the functions $\psi_{j,k}$. The problem arises, since Π_{L-1} is not a subspace of $L_2(\mathbb{R})$, and therefore the tight frame condition (2.11) cannot be directly combined with the vanishing moment condition. The study of such tight frames for an unbounded interval is indeed not a direct modification of the study in this paper and is therefore treated in a separate forthcoming paper [8].

Remark 3. In a sequence of papers, Ciesielski and Figiel [13] constructed spline functions on $[a,b]$ which constitute a Riesz bases of a Sobolev subspace of $L_2(a,b)$ with various boundary conditions. These splines, however, are not locally supported with respect to the B-spline basis. Our results in Sections 5–6 devise a method for the construction of splines that are locally supported and constitute a tight MRA frame.

3. Dual Bases and Approximate Duals

In this section we provide some background material concerning the conditions stated in Theorems 1 and 2. In the first part of this section, a formulation in terms of some integral kernels for $L_2(I)$ is chosen. In the second part, an equivalent matrix formulation is developed that is useful for the specific considerations to be discussed in Section 5. For convenience, we only restrict our attention to the assumption that Φ_j is a basis of the space V_j. Under this assumption, the Gramian matrix

$$\Gamma_j = [\langle \phi_{j,k}, \phi_{j,\ell} \rangle]_{k,\ell \in \mathbb{M}_j}$$

is symmetric positive definite, and its dual basis $\widetilde{\Phi}_j$ is given by the function vector

$$\widetilde{\Phi}_j = [\widetilde{\phi}_{j,k}]_{k \in \mathbb{M}_j} = \Phi_j \, \Gamma_j^{-1}. \tag{3.1}$$

3.1. Definition of approximate duals and basic results

It is well known that the reproducing kernel K_j of the space V_j is given by

$$K_j(x,y) = \Phi_j(x) \; \widetilde{\Phi}_j(y)^T = \sum_{k \in \mathbb{M}_j} \phi_{j,k}(x) \widetilde{\phi}_{j,k}(y), \qquad x, y \in I. \tag{3.2}$$

Thus, for any $f \in V_j$, the identity

$$\|f\|^2 = \int_I f(x) \int_I f(y) K_j(x,y) \, dy \, dx = [\langle f, \phi_{j,k} \rangle]_k \; \Gamma_j^{-1} \; [\langle f, \phi_{j,k} \rangle]_k^T$$

holds. Moreover, the corresponding orthogonal projections of $L_2(I)$ onto V_j and its orthogonal complementary subspace relative to V_{j+1} are given by

$$f \mapsto \int_I f(y) K_j(\cdot, y) \, dy; \tag{3.3}$$

$$f \mapsto \int_I f(y) (K_{j+1}(\cdot, y) - K_j(\cdot, y)) \, dy, \tag{3.4}$$

respectively. Here, we recall that the kernel $(K_{j+1} - K_j)$ is often employed for the construction of orthonormal or semi-orthogonal wavelets; in fact, an orthonormal wavelet basis $\{\eta_{j,k}\}$ for the MRA $\{V_j\}_{j\geq 0}$ satisfies

$$K_{j+1}(x,y) - K_j(x,y) = \sum_k \eta_{j,k}(x)\eta_{j,k}(y).$$

Now, since Φ_j is supposed to be refinable with respect to Φ_{j+1} in the sense of (2.2), we can also write

$$K_{j+1}(x,y) - K_j(x,y) = \Phi_{j+1}(x)(\Gamma_{j+1}^{-1} - P_j\Gamma_j^{-1}P_j^T)\Phi_{j+1}(y)^T. \tag{3.5}$$

In particular, the matrix $\Gamma_{j+1}^{-1} - P_j\Gamma_j^{-1}P_j^T$ is always positive semi-definite.

The notion of approximate duals to be introduced in this paper is also used to define linear operators of the form (3.3), with the reproducing kernel K_j replaced by

$$K_S(x,y) = \Phi_j(x)\ S\ \Phi_j(y)^T,$$

for some spsd matrix S.

Definition 4. *Let $\Phi = (\phi_k)_{k\in\mathbb{M}}$ be a basis of a finite-dimensional subspace V of $L_2(I)$ and $L \geq 1$ an integer such that $\Pi_{L-1} \subset V$. For an spsd matrix S, the function vector*

$$\Phi^S = (\phi_k^S)_{k\in\mathbb{M}} = \Phi \cdot S$$

is called an approximate dual *of order L, if*

$$f = \int_I f(y)K_S(\cdot,y)\,dy = \sum_{k\in\mathbb{M}} \langle f, \phi_k^S\rangle\phi_k \quad \text{for all} \quad f \in \Pi_{L-1}, \tag{3.6}$$

where K_S is defined in (2.8), with S_j replaced by S and the superscript j suppressed.

We note that identity (3.6) is equivalent to

$$\langle f, \phi_k^S\rangle = \langle f, \tilde{\phi}_k\rangle, \qquad f \in \Pi_{L-1},$$

where $[\tilde{\phi}_k]_{k\in\mathbb{M}} = \Phi\,\Gamma^{-1}$ denotes the dual basis of Φ in V.

Remark 4. Operators of the form

$$Qf = \sum_{k\in\mathbb{M}} \lambda_k(f)\phi_k$$

where λ_k are continuous linear functionals on $L_p(I)$, have been extensively studied in the literature of spline approximation (see e.g. [1,3,32]). If V contains the polynomial space Π_{L-1} and $Qf = f$ for all $f \in \Pi_{L-1}$, then Q is often called a quasi-interpolation [1,2] or quasi-projection operator [24]. Therefore, condition (iii') in Theorem 2 relates the construction of tight MRA frames to the construction of special quasi-projection operators.

We can now rephrase Theorem 2(b) in terms of the new terminology of approximate duals.

Corollary 1. *Let S_0 in (1.1) be an spsd matrix that defines an approximate dual of Φ_0 such that $T_0 f \le \|f\|^2$ for all $f \in L_2(I)$. Also, let $\{\Psi_j\}_{j\ge0} = \{\Phi_{j+1}Q_j\}_{j\ge0}$ and $\Psi_j = \{\psi_{j,k}\}$. Then the wavelets $\psi_{j,k}$, $j \ge 0$, $k \in \mathbb{N}_j$, have L vanishing moments and define an MRA tight frame in the sense of Definition 3, if and only if there exist spsd matrices S_j of dimension $M_j \times M_j$, $j \ge 1$, such that conditions (i)–(ii) of Theorem 1 hold and S_j defines an approximate dual of Φ_j.*

3.2. Matrix formulation and vanishing moments

In parallel to the previous formulation in terms of integral kernels on $L_2(I)$, we give an equivalent matrix formulation of some of the conditions. For this purpose, we need the following requirement for the bases $\{\Phi_j\}$, namely: there exist matrices $E_{j,L} \in \mathbb{R}^{M_j\times\tilde{M}_j}$, with suitable $\tilde{M}_j \in \mathbb{N}$, that have the following properties:

- *a function $\eta = \Phi_j\mathbf{u} \in V_j$, with $\mathbf{u} \in \mathbb{R}^{M_j}$, has vanishing moments of order L if and only if there exists a vector $\mathbf{v} \in \mathbb{R}^{\tilde{M}_j}$ such that $\mathbf{u} = E_{j,L}\mathbf{v}$;* (3.7)
- *there exist matrices $\tilde{P}_j \in \mathbb{R}^{\tilde{M}_{j+1}\times\tilde{M}_j}$ such that $P_j E_{j,L} = E_{j+1,L}\tilde{P}_j$.*

This assumption is made in anticipation of our study of the structure of spline spaces to be presented in the next section. Typically, $E_{j,L}$ is not invertible, but rather represents a difference operator of order L. (The matrix $E_{j,L}$ is analogous to the Laurent polynomial factor $(1-z)^L$ in the shift-invariant setting.) The second property is known as a "commutation property" in the literature on subdivision schemes, see [17]

We now state three conditions on the spsd matrices S_j and explain their relation to Theorems 1 and 2. The conditions are

$$(S_{j+1} - P_j S_j P_j^T) \;\text{ is spsd;} \tag{3.8}$$

$$(\Gamma_j^{-1} - S_j) \;\text{ is spsd;} \tag{3.9}$$

$$\Gamma_j^{-1} - S_j = E_{j,L} X_j E_{j,L}^T \;\text{ for some symmetric matrix } X_j. \tag{3.10}$$

It is clear that the last two conditions can be combined to one condition by requiring that X_j in (3.10) be an spsd matrix.

The condition (3.8) is necessary and sufficient for the existence of matrices Q_j in condition (ii) of Theorem 1.

We next show that the condition (3.9) is equivalent to the property

$$T_j f \leq \|f\|^2, \qquad f \in L_2(I), \tag{3.11}$$

of T_j, which is necessary for the family $\{\Psi_j\}$ to constitute a tight MRA-frame of $L_2(I)$ with respect to T_0. Indeed, if (3.11) holds, then for any $f \in V_j$, we have

$$\|f\|^2 - T_j f = \Big[\langle f, \phi_{j,k}\rangle\Big]_{k\in\mathbb{M}_j} (\Gamma_j^{-1} - S_j) \Big[\langle f, \phi_{j,k}\rangle\Big]_{k\in\mathbb{M}_j}^T \geq 0.$$

Hence, since the moment sequences exhaust the finite-dimensional sequence space $\ell_2(\mathbb{M}_j)$, the matrix $\Gamma_j^{-1} - S_j$ must be positive semi-definite. Conversely, positive semi-definiteness of $\Gamma_j^{-1} - S_j$ implies that

$$T_j f = \Big[\langle f, \phi_{j,k}\rangle\Big]_{k\in\mathbb{M}_j} S_j \Big[\langle f, \phi_{j,k}\rangle\Big]_{k\in\mathbb{M}_j}^T \leq \Big[\langle f, \phi_{j,k}\rangle\Big]_{k\in\mathbb{M}_j} \Gamma_j^{-1} \Big[\langle f, \phi_{j,k}\rangle\Big]_{k\in\mathbb{M}_j}^T \leq \|f\|^2$$

for all $f \in L_2(I)$, since the third expression is the norm of the orthogonal projection of f onto V_j.

Finally, we claim that the condition (3.10) is equivalent to (3.6).

Proof of claim: Here, we drop the index j for simplicity. Indeed, if (3.10) is satisfied, we obtain

$$K(x,y) - K_S(x,y) = (\Phi(x)E_L)\ X\ (\Phi(y)E_L)^T = \sum_{k,\ell} x_{k,\ell}\theta_k(x)\theta_\ell(y),$$

where the notation

$$[\theta_k]_{1 \le k \le \tilde{M}} := \Phi E_L$$

is used. By the definition of E_L, all functions θ_k have vanishing moments of order L, and, for all $f \in \Pi_{L-1}$, we obtain

$$0 = \int_I f(y)(K(x,y) - K_S(x,y))\,dy = f(x) - \int_I f(y)K_S(x,y)\,dy. \tag{3.12}$$

Therefore, S defines an approximate dual. Conversely, if S is an spsd matrix that defines an approximate dual Φ^S, then

$$A := \Gamma^{-1} - S$$

is a symmetric matrix. Since the relation (3.12) is valid for all $f \in \Pi_{L-1}$, the null space of A contains all vectors of the form $[\langle f, \phi_k\rangle]_{k\in\mathbb{M}}^T$ with $f \in \Pi_{L-1}$. The spectral decomposition

$$A = \sum_{k=1}^{r} \lambda_k \mathbf{u}_k \mathbf{u}_k^T$$

can be used, where λ_k are the nonzero eigenvalues of A with norm-one eigenvectors $\mathbf{u}_k$ and r is the rank of A. Clearly, the vectors $\mathbf{u}_k$ are orthogonal to the null space of A. We then define

$$\theta_k := \Phi\ \mathbf{u}_k \in V, \qquad 1 \le k \le r,$$

and obtain, for all $f \in \Pi_{L-1}$, that

$$\langle f, \theta_k\rangle = \ [\langle f, \phi_\ell\rangle]\ \mathbf{u}_k = 0.$$

This shows that all of the functions θ_k have L vanishing moments. By the definition of E_L, there exist vectors $\mathbf{v}_k \in \mathbb{R}^{\tilde{M}}$, $1 \le k \le r$, such that $\mathbf{u}_k = E_L \mathbf{v}_k$. If we insert this identity into the spectral decomposition of A, we obtain (3.10) by defining X to be the matrix

$$X = \sum_{k=1}^{r} \lambda_k \mathbf{v}_k \mathbf{v}_k^T. \quad \blacksquare$$

An important consequence can be drawn by combining the conditions (3.8) and (3.10). Recall that the bases Φ_j and Φ_{j+1} are related by the refinement relation $\Phi_j = \Phi_{j+1} P_j$ in (2.2). So, if $\Pi_{L-1} \subset V_j$ (as assumed in Definition 4), then a similar argument as before gives

$$\Gamma_{j+1}^{-1} - P_j \Gamma_j^{-1} P_j^T = E_{j+1,L} Y_{j+1} E_{j+1,L}^T, \tag{3.13}$$

where Y_{j+1} is an spsd matrix. If S_{j+1} and S_j are spsd matrices that define certain approximate duals of Φ_{j+1} and Φ_j, respectively, we can combine (3.10) and (3.13) to get

$$\begin{aligned} S_{j+1} - P_j S_j P_j^T &= -(\Gamma_{j+1}^{-1} - S_{j+1}) + \Gamma_{j+1}^{-1} - P_j \Gamma_j^{-1} P_j^T + P_j(\Gamma_j^{-1} - S_j) P_j^T \\ &= E_{j+1,L}\,(Y_{j+1} - X_{j+1})\,E_{j+1,L}^T + P_j E_{j,L} X_j E_{j,L}^T P_j^T \\ &= E_{j+1,L} \left(Y_{j+1} - X_{j+1} + \tilde{P}_j X_j \tilde{P}_j^T \right) E_{j+1,L}^T. \end{aligned}$$

Furthermore, if the condition (3.8) is valid as well, then there exists a factorization of the form

$$S_{j+1} - P_j S_j P_j^T = (E_{j+1,L} \widehat{Q}_j)\ (E_{j+1,L} \widehat{Q}_j)^T$$

that provides a special form for the matrix $Q_j = E_{j+1,L} \widehat{Q}_j$ in condition (ii) of Theorem 1. Therefore, the individual functions of the vector $\Psi_j = \Phi_j E_{j+1,L} \widehat{Q}_j$ have vanishing moments of order L.

We summarize the findings of the matrix formulation in the following result, where we also make use of the statements in Remark 1 in Section 2.

Theorem 3. *Let $\{\Phi_j\}_{j\geq 0}$ be a family of locally supported bases that satisfy the refinement relation (2.2) and S_j be spsd matrices of dimensions $M_j \times M_j$, such that the conditions (3.8)–(3.10) are satisfied for all $j \geq 0$. Then the families Φ^{S_j} are approximate duals of order L. Moreover, a factorization of the form*

$$S_{j+1} - P_j S_j P_j^T = (E_{j+1,L}\widehat{Q}_j)\,(E_{j+1,L}\widehat{Q}_j)^T = Q_j Q_j^T \tag{3.14}$$

with real matrices $Q_j = E_{j+1,L}\widehat{Q}_j$ exists. If, in addition, the kernels K_{S_j} satisfy (2.16) and (2.18), then the function vectors $\Psi_j = \Phi_{j+1}E_{j+1,L}\widehat{Q}_j$, $j \geq 0$, define a tight MRA-frame relative to T_0 and all wavelets $\psi_{j,k}$ have vanishing moments of order L.

We remark that property (2.18) is automatically satisfied, if the S_j's are banded with bandwidth r independent of j.

This completes the description of the general procedure for the construction of tight MRA-frames with vanishing moments of order L. We see that the essential part is the construction of uniformly bounded approximate duals (to satisfy (2.16)), such that the positivity constraints in (3.8)–(3.9) are satisfied. We will define such duals for B-splines of arbitrary order with arbitrary knot vector in Section 5.

4. Background on Univariate B-splines

Based on the general considerations in Sections 2 and 3, we will develop, throughout the rest of this paper, methods for the construction of tight frames of $L_2(I)$ that are linear combinations of B-splines. In the present section we recall several facts about B-splines and introduce the necessary notations. For a more detailed description we refer the reader to [1,27,32].

Let $m, N \in \mathbb{N}$ and

$$\mathbf{t} = \{t_k;\ -m+1 \leq k \leq N+m\} \tag{4.1}$$

be a knot vector such that

$$t_k \leq t_{k+1} \qquad \text{and} \qquad t_k < t_{k+m} \qquad \text{for all} \quad k, \tag{4.2}$$

$$t_{-m+1} = \cdots = t_0 = a \qquad \text{and} \qquad t_{N+1} = \cdots t_{N+m} = b. \tag{4.3}$$

Note that we consider knot vectors as ordered sets whose elements may have multiplicities up to m. The multiplicity μ_k of a knot $t_k \in \mathbf{t}$ is the number of times this knot is repeated in $\mathbf{t}$. The number m will denote the order (i.e. degree plus 1) of the spline functions, and N is the number of interior knots. The conditions in (4.2)–(4.3) assure that $\mu_k \leq m$ for all k and both boundary knots a and b have multiplicity m, which we shortly denote as "stacked boundary knots".

The normalized B-spline $N_{\mathbf{t};m,k}$ of order m (or degree $m-1$) is a function on $\mathbb{R}$ defined by

$$N_{\mathbf{t};m,k}(x) = (t_{k+m} - t_k)[t_k, \ldots, t_{k+m}](\cdot - x)_+^{m-1}, \qquad k \in \mathbb{M}, \tag{4.4}$$

where $[t_k, \ldots, t_{k+m}]$ denotes the divided difference of order m and $\mathbb{M} = \{-m+1, \ldots, N\}$ denotes the proper index set. It is well known that $N_{\mathbf{t};m,k}$ has support $[t_k, t_{k+m}]$, is strictly positive inside this interval, and is a polynomial of degree $m-1$ in each interval (t_i, t_{i+1}), $k \leq i \leq k+m-1$. Moreover, it has $m - \mu_i - 1$ continuous derivatives at t_i. The integral of $N_{\mathbf{t};m,k}$ is given by

$$\int_{\mathbb{R}} N_{\mathbf{t};m,k}(x)dx = \frac{t_{k+m} - t_k}{m} =: d_{\mathbf{t};m,k}. \tag{4.5}$$

An interesting identity is the representation formula for normalized B-splines that was discovered by Schoenberg and Curry in [31; Lemma 6]. It states that for $r \geq m$ and any complex number z, not purely imaginary, then we have

$$\int_a^b (1 - zx)^{-r-1} N_{\mathbf{t};r,k}(x)\, dx = d_{\mathbf{t};r,k} \prod_{i=k}^{k+r} \frac{1}{1 - zt_i}, \qquad r \geq m, \tag{4.6}$$

near the origin. This identity can be employed as a generating function formula for the moments of the B-spline.

The spline space $\mathcal{S}_{\mathbf{t};m}$ is the space of all piecewise polynomials of degree $m-1$ on I with so-called "breakpoints" $t_k \in \mathbf{t}$ and smoothness $m - \mu_k - 1$ at every knot t_k. The row vector of normalized B-splines

$$\Phi_{\mathbf{t};m} := [N_{\mathbf{t};m,k}]_{k\in\mathbb{M}} \tag{4.7}$$

is a basis of $\mathcal{S}_{\mathbf{t};m}$. Moreover, under the normalization

$$\Phi^B_{\mathbf{t};m} = [N^B_{\mathbf{t};m,k}]_{k\in\mathbb{M}} = [d^{-1/2}_{\mathbf{t};m,k} N_{\mathbf{t};m,k}]_{k\in\mathbb{M}},$$

this family defines a Riesz basis of $\mathcal{S}_{\mathbf{t};m}$, and its upper and lower Riesz bounds can be chosen to be independent of the knot vector $\mathbf{t}$, see [1; p.156,19; p.145]; more precisely, there exists a constant $D_m > 0$ which depends on m, but not on the knot vector, such that

$$D_m \|\{c_k\}_{k\in\mathbb{M}}\|^2_{\ell_2} \le \Big\| \sum_{k\in\mathbb{M}} c_k N^B_{\mathbf{t};m,k} \Big\|^2 \le \|\{c_k\}_{k\in\mathbb{M}}\|^2_{\ell_2}, \qquad \{c_k\}_k \in \ell_2(\mathbb{M}). \tag{4.8}$$

The Gramian matrix Γ^B of $\Phi^B_{\mathbf{t};m}$, given by

$$\Gamma^B = \int_I \Phi^B_{\mathbf{t};m}(x)^T \Phi^B_{\mathbf{t};m}(x)\, dx = \Big[(d_{\mathbf{t};m,k} d_{\mathbf{t};m,\ell})^{-1/2} \langle N_{\mathbf{t};m,k}, N_{\mathbf{t};m,\ell} \rangle \Big]_{k,\ell\in\mathbb{M}}$$

is a symmetric positive definite banded matrix, whose upper and lower bounds are the Riesz bounds of $\Phi^B_{\mathbf{t};m}$. (It is also known to be totally positive.) As in (3.1), we can define the dual basis $\widetilde{\Phi} = \Phi^B_{\mathbf{t};m}(\Gamma^B)^{-1}$ and the reproducing kernel

$$K(x,y) = \Phi^B_{\mathbf{t};m}(x) \widetilde{\Phi}(y)^T.$$

Note that K also defines the kernel of the orthogonal projection of $L_2(I)$ onto $\mathcal{S}_{\mathbf{t};m}$. The result of the recent proof of de Boor's conjecture by A. Shadrin [33] can be stated as follows: there exists a constant C_m that does not depend on the knot vector or the interval I, such that

$$\sup_{x\in I} \int_I |K(x,y)|\, dy \le C_m. \tag{4.9}$$

(Indeed, the expression on the left-hand side of (4.9) gives the operator norm of the ortho-projection operator as an operator from $L_\infty(I)$ to $L_\infty(I)$. This operator norm was shown by Shadrin to be bounded by a constant that does not depend on the knot vector or I.) Our construction in the next section will yield approximate duals whose kernel K_S has the same property, see Section 5.7.

The B-splines lead to a partition of unity and, more generally, to Marsden's identity:

$$\frac{(y-x)^s}{s!} = \sum_{k\in\mathbb{M}} g_{\mathbf{t};m,k}^{(m-1-s)}(y) N_{\mathbf{t};m,k}(x), \qquad 0 \le s \le m-1, \quad x,y\in I, \tag{4.10}$$

where

$$g_{\mathbf{t};m,k}(y) = \frac{1}{(m-1)!}(y-t_{k+1})\dots(y-t_{k+m-1})$$

is a polynomial that depends only on m and the interior knots of $N_{\mathbf{t};m,k}$. In particular, we obtain

$$\frac{x^s}{s!} = \sum_{k\in\mathbb{M}} G_s(t_{k+1},\dots,t_{k+m-1}) N_{\mathbf{t};m,k}(x), \qquad 0 \le s \le m-1, \tag{4.11}$$

where the coefficients

$$G_s(t_{k+1},\dots,t_{k+m-1}) = (-1)^s g_{\mathbf{t};m,k}^{(m-1-s)}(0)$$

are homogeneous and symmetric polynomials of degree s with respect to the "variables" $t_{k+1},\dots,t_{k+m-1}$; i.e.

$$G_s(\alpha t_1,\dots,\alpha t_{m-1}) = \alpha^s G_s(t_1,\dots,t_{m-1}), \qquad G_s(t_{\sigma(1)},\dots,t_{\sigma(m-1)}) = G_s(t_1,\dots,t_{m-1}),$$

for every $\alpha\in\mathbb{R}$ and every permutation σ. A similar structure will be found to exist for the approximate dual of B-splines that we consider in Section 5.

Next we develop the matrix formulation (3.7) needed for the description of linear combinations of B-splines which have vanishing moments of a certain order. When we make use of B-splines of higher order $r > m$ with respect to the same knot vector $\mathbf{t}$, we need to observe that the stacked knots at both endpoints of I have multiplicity m (and not r). Therefore, the B-splines have at most m-fold knots at the endpoints a and b, which implies that the functions and their $r-m-1$ first derivatives vanish at a and b. It is well known that the derivative of a normalized B-spline of order $r+1>m$ satisfies the recurrence relation

$$N'_{\mathbf{t};r+1,k} = d^{-1}_{\mathbf{t};r,k} N_{\mathbf{t};r,k} - d^{-1}_{\mathbf{t};r,k+1} N_{\mathbf{t};r,k+1}, \qquad k, k+r+1-m \in \mathbb{M}, \tag{4.12}$$

where $d_{\mathbf{t};r,k}$ are the divided knot differences $(t_{k+r}-t_k)/r$ as in (4.5). Written in matrix form, the recursive application of (4.12) gives

$$\frac{d^\nu}{dx^\nu}\Phi_{\mathbf{t};m+\nu}(x) = \Phi_{t;m}(x)\, \underbrace{D_{\mathbf{t};m}\cdots D_{\mathbf{t};m+\nu-1}}_{=E_{\mathbf{t};m,\nu}}, \tag{4.13}$$

where the matrices $D_{\mathbf{t};r}$ are bi-diagonal and can be defined as

$$D_{\mathbf{t};r} := \operatorname{diag}\left[d^{-1}_{\mathbf{t};r,-m+1}, \ldots, d^{-1}_{\mathbf{t};r,N+m-r}\right] \Delta_{N+m-(r-m)}, \qquad r \geq m, \tag{4.14}$$

with

$$\Delta_n := \begin{pmatrix} 1 & & & \\ -1 & 1 & \mathbf{0} & \\ \mathbf{0} & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}_{n\times(n-1)}. \tag{4.15}$$

Note that the vector $\Phi_{\mathbf{t};m+\nu}$ on the left-hand side of (4.13) has ν fewer entries than $\Phi_{\mathbf{t};m}$. The recursion for the L_2-normalized splines is given by

$$\frac{d^\nu}{dx^\nu}\Phi^B_{\mathbf{t};m+\nu}(x) = \Phi^B_{t;m}(x)\, \underbrace{\operatorname{diag}\left[d^{1/2}_{\mathbf{t};m,k}\right]_k D_{\mathbf{t};m}\cdots D_{\mathbf{t};m+\nu-1}\operatorname{diag}\left[d^{-1/2}_{\mathbf{t};m+\nu,k}\right]_k}_{=E^B_{\mathbf{t};m,\nu}}. \tag{4.16}$$

The identities (4.13) and (4.16) are particularly useful for the study of vanishing moments of order $L \geq 1$, meaning that

$$\int_I x^\nu f(x)dx = 0 \qquad \text{for all} \quad 0 \leq \nu \leq L-1. \tag{4.17}$$

For the study of splines, we make use of the fact that a spline $s \in \mathcal{S}_{\mathbf{t},m}$ has L vanishing moments, if and only if it is the L-th derivative of a spline S of order $m+L$ with respect to the same knot vector $\mathbf{t}$, and S can be chosen such that its derivatives $S^{(\nu)}$, $0 \leq \nu \leq L-1$, vanish at both endpoints of I, and observe that the multiplicity of the knots at a and b remain to be m (and not the order $m+L$ of S). We need the following result.

Lemma 1. *A spline* $s = \Phi^B_{\mathbf{t};m}\mathbf{u}$, $\mathbf{u} = [u_k]_{-m+1\le k\le N}$, *has* L *vanishing moments, if and only if there exists a column vector* $\mathbf{v} = [v_k]_{-m+1\le k\le N-L}$, *such that*

$$\mathbf{u} = E^B_{\mathbf{t};m,L}\mathbf{v}, \tag{4.18}$$

Moreover, if $u_k = 0$ *for all* $k < i_1$ *and/or* $k > i_2$, *then* $\mathbf{v}$ *can be so chosen that* $v_k = 0$ *for all* $k < i_1$ *and/or* $k > i_2 - L$. *The same result is valid when the superscript* B *is dropped.*

Proof: We can choose $\mathbf{v}$ as the coefficient vector of the spline S of order $m + L$ with knot vector $\mathbf{t}$ such that $s = S^{(L)}$, where S satisfies homogeneous boundary conditions mentioned above. Equation (4.18) is a direct consequence of (4.16). The additional conditions on the coefficient sequence $\mathbf{u}$ imply that the support of s is contained in $[t_{i_1}, t_{i_2+m}]$. Hence, the support of S is confined to the same interval, which determines the support of its coefficient sequence as claimed. ■

Let us now assume that two knot vectors $\mathbf{t} \subset \tilde{\mathbf{t}}$ that satisfy condition (4.2) are given, where the subset notation is used for ordered sets: new knots of multiplicity $\le m$ can be inserted into $\mathbf{t}$, or the multiplicity $\mu_k < m$ of an existing knot t_k in $\mathbf{t}$ can be increased. The index sets of the bases $\Phi_{\mathbf{t};m}$ and $\Phi_{\tilde{\mathbf{t}};m}$ are denoted by $\mathbb{M}$ and $\tilde{\mathbb{M}}$, respectively, and we allow for arbitrary (finite) refinements of the knot vector $\mathbf{t}$.

The B-splines satisfy the refinement equation

$$\Phi_{\mathbf{t};m} = \Phi_{\tilde{\mathbf{t}};m} P_{\mathbf{t},\tilde{\mathbf{t}};m}, \tag{4.19}$$

where the matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ has nonnegative entries, with each row summing to 1, and is sparse in the following sense: if $\ell(k)$ and $u(k)$ denote strictly increasing sequences such that

$$\{t_k, \ldots, t_{k+m}\} \subset \{\tilde{t}_{\ell(k)}, \ldots, \tilde{t}_{u(k)+m}\},$$

then the entries $p_{i,k}$ in the k-th column of $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ are zero, if $i < \ell(k)$ or $i > u(k)$. In other words, only the B-splines in $\Phi_{\tilde{\mathbf{t}};m}$, whose support is contained in the support of $N_{\mathbf{t};m,k}$, appear in the refinement relation for this B-spline. (The row indices of $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ refer to the basis functions in $\Phi_{\tilde{\mathbf{t}};m}$, and the column indices refer to the basis functions in $\Phi_{\mathbf{t};m}$, respectively.) The useful relation

$$[d_{\tilde{\mathbf{t}};m,k};\ k \in \tilde{\mathbb{M}}]P_{\mathbf{t},\tilde{\mathbf{t}};m} = [d_{\mathbf{t};m,k};\ k \in \mathbb{M}] \tag{4.20}$$

immediately follows from (4.5).

We consider, in particular details, the special case where $\tilde{\mathbf{t}} \setminus \mathbf{t} = \{\tau\}$ is a singleton and $\tau \in [t_\rho, t_{\rho+1})$. In this case, we have

$$P_{\mathbf{t},\tilde{\mathbf{t}};m} = \begin{bmatrix} \ddots & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & b_2 & a_2 & & & \\ & & & \ddots & \ddots & & \\ & & & & b_m & a_m & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & \ddots \end{bmatrix}, \tag{4.21}$$

where

$$a_i = \frac{\tau - t_{\rho-m+i}}{t_{\rho+i-1} - t_{\rho-m+i}}, \qquad b_i = 1 - a_i \geq 0, \qquad i = 2, \ldots, m, \tag{4.22}$$

and a_i has row and column index $\rho - m + i$. The same identities are valid if m is replaced by an integer $m + \nu > m$ in (4.19)–(4.22). A technical difference may arise if $\rho \leq \nu$, or if $\rho \geq N - \nu$. This means that the inserted knot is close to the left or right endpoints of I. (Recall that the numbering of the knots is given such that the first and last interior knots of $\mathbf{t}$ have indices 1 and N, respectively.) If $\rho \leq \nu$, the matrix in (4.21) must be truncated on the left so that its first column has the subdiagonal entry $b_{\nu-\rho+2}$. Similarly, if $\rho \geq N - \nu$, the matrix in (4.21) must be truncated on the right so that its last column has the diagonal entry $a_{m+N-\rho}$. The row sums of the first and last row of the matrix $P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ may then be less than 1.

Since both knot vectors are finite, we can proceed with knot insertion from $\mathbf{t}$ to $\tilde{\mathbf{t}}$ in a finite number of steps, such that at most one new knot is inserted per interval $[t_k, t_{k+m+1}]$ for each step. This explains that $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ has a factorization into matrices that are block diagonal with blocks of the form (4.21). Another important algorithm for insertion of several knots, which describes a recursion of $P_{\mathbf{t},\tilde{\mathbf{t}};m}$ with respect to m, is known as the Oslo-algorithm. Note that the L_2-normalized basis satisfies the refinement equation

$$\Phi^B_{\mathbf{t};m} = \Phi^B_{\tilde{\mathbf{t}};m} P^B_{\mathbf{t},\tilde{\mathbf{t}};m}, \quad \text{where} \quad P^B_{\mathbf{t},\tilde{\mathbf{t}};m} = \operatorname{diag}\left[d^{1/2}_{\tilde{\mathbf{t}};m,k}\right]_k P_{\mathbf{t},\tilde{\mathbf{t}};m} \operatorname{diag}\left[d^{-1/2}_{\mathbf{t};m,k}\right]_k. \tag{4.23}$$

The following result is a version of the "commutation" relation for refinable functions in the case of B-splines.

Lemma 2. *For all $r \geq m$, the identity*

$$D_{\tilde{\mathbf{t}},r} P_{\mathbf{t},\tilde{\mathbf{t}};r+1} = P_{\mathbf{t},\tilde{\mathbf{t}};r} D_{\mathbf{t},r}, \tag{4.24}$$

holds and

$$E_{\tilde{\mathbf{t}};m,\nu} P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu} = P_{\mathbf{t},\tilde{\mathbf{t}};m} E_{\mathbf{t};m,\nu}, \qquad E^B_{\tilde{\mathbf{t}};m,\nu} P^B_{\mathbf{t},\tilde{\mathbf{t}};m+\nu} = P^B_{\mathbf{t},\tilde{\mathbf{t}};m} E^B_{\mathbf{t};m,\nu}. \tag{4.25}$$

Proof: The recurrence relation for the derivative (4.13) and the scaling relation (4.19) give

$$\begin{aligned}\Phi_{\tilde{\mathbf{t}};r}(x) D_{\tilde{\mathbf{t}},r} P_{\mathbf{t},\tilde{\mathbf{t}};r+1} &= \frac{d}{dx}\Phi_{\tilde{\mathbf{t}};r+1}(x) P_{\mathbf{t},\tilde{\mathbf{t}};r+1} = \frac{d}{dx}\Phi_{\mathbf{t};r+1}(x) \\ &= \Phi_{\mathbf{t};r}(x) D_{\mathbf{t},r} = \Phi_{\tilde{\mathbf{t}};r}(x) P_{\mathbf{t},\tilde{\mathbf{t}};r} D_{\mathbf{t},r}.\end{aligned}$$

Identity (4.24) follows from the fact that $\Phi_{\tilde{\mathbf{t}};r}$ (or its L_2-normalization, if the interval I is unbounded) is a (Riesz) basis. The identities in (4.25) follow by recursive application of (4.24). ■

5. Minimally Supported Approximate Duals of B-splines

This section is devoted to the development of an explicit formulation of the unique approximate duals of B-splines with minimum support, as well as all necessary results for the construction of tight frames of spline-wavelets on a bounded interval. The section is divided into 7 subsections to facilitate our presentation.

5.1. Preliminary results

Analogous to the Marsden coefficients in (4.11), we define homogeneous polynomials $F_\nu : \mathbb{R}^r \to \mathbb{R}$ by

$$F_\nu(x_1, \ldots, x_r) = \frac{2^{-\nu}}{\nu!} \sum_{\substack{1 \le i_1, \ldots, i_{2\nu} \le r, \\ i_1, \ldots, i_{2\nu} \text{ distinct}}} \prod_{j=1}^{\nu} (x_{i_{2j-1}} - x_{i_{2j}})^2. \tag{5.1}$$

Without causing any confusion, we abuse the use of the notation of F_ν, by allowing different numbers of arguments. In addition, the notation $F_\nu(\{x_1, \ldots, x_r\} \setminus \{x_{i_1}, \ldots, x_{i_s}\})$ will be employed in order to denote the function defined for $r-s$ variables by leaving out $x_{i_1}, \ldots, x_{i_s}$. If $r < 2\nu$, F_ν is defined to be the zero function, in accordance with the fact that the sum in (5.1) is empty. We also let $F_0 \equiv 1$ regardless of the number of arguments.

For $r \ge 2\nu$, it follows from the definition that F_ν is a symmetric and homogeneous polynomial of degree 2ν; i.e.,

$$F_\nu(\alpha x_1, \ldots, \alpha x_r) = \alpha^{2\nu} F_\nu(x_1, \ldots, x_r), \qquad F_\nu(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) = F_\nu(x_1, \ldots, x_r),$$

for every $\alpha \in \mathbb{R}$ and every permutation σ. It is also clear that F_ν is invariant under a constant shift of the arguments $(x_1, \ldots, x_r) \mapsto (x_1 - c, \ldots, x_r - c)$, and its coordinate degree in each of its variables is 2. The following result describes several other properties of F_ν.

Lemma 3. *For every $\nu \ge 1$ and $r \ge 2\nu$ the following identities hold:*

(i) Recursion with respect to r and ν:

$$F_\nu(x_1, \ldots, x_r) = F_\nu(x_1, \ldots, x_{r-1}) + \sum_{i=1}^{r-1} (x_r - x_i)^2 F_{\nu-1}(\{x_1, \ldots, x_{r-1}\} \setminus \{x_i\}). \tag{5.2}$$

(ii) Recursion with respect to ν:

$$F_\nu(x_1, \ldots, x_r) = \frac{1}{\nu} \sum_{1 \le i_1 < i_2 \le r} (x_{i_1} - x_{i_2})^2 F_{\nu-1}(\{x_1, \ldots, x_r\} \setminus \{x_{i_1}, x_{i_2}\}). \tag{5.3}$$

(iii) Recursion with respect to r:

$$(r-2\nu)F_\nu(x_1,\dots,x_r)=\sum_{i=1}^{r}F_\nu(\{x_1,\dots,x_r\}\setminus\{x_i\}),\tag{5.4}$$

and, more generally, for any $1\le k\le r$,

$$\binom{r-2\nu}{k}F_\nu(x_1,\dots,x_r)=\sum_{1\le i_1<\dots<i_k\le r}F_\nu(\{x_1,\dots,x_r\}\setminus\{x_{i_1},\dots,x_{i_k}\}).\tag{5.5}$$

(iv) For any $\alpha,\beta\in\mathbb{R}$, $\alpha+\beta=1$,

$$\begin{aligned}F_\nu(x_1,\dots,x_r,\alpha x+\beta y)=&\alpha F_\nu(x_1,\dots,x_r,x)+\beta F_\nu(x_1,\dots,x_r,y)-\\&(r+2-2\nu)\alpha\beta(x-y)^2F_{\nu-1}(x_1,\dots,x_r).\end{aligned}\tag{5.6}$$

For $r\ge 2\nu$ and $x_1\le x_2\le\dots\le x_r$,

$$F_\nu(x_1,\dots,x_r)\le\frac{2^{-\nu}r!}{\nu!(r-2\nu)!}\,(x_r-x_\nu)^2(x_{r-1}-x_{\nu-1})^2\cdots(x_{r-\nu+1}-x_1)^2.\tag{5.7}$$

Proof: Let $r\ge 2\nu$. The recursion in (i) follows directly from (5.1). In order to show (ii) and (iii), we introduce the notation of a typical summand in (5.1) by setting

$$y(x_{i_1},\dots,x_{i_{2\nu}}):=(x_{i_1}-x_{i_2})^2\cdots(x_{i_{2\nu-1}}-x_{i_{2\nu}})^2,\tag{5.8}$$

and observe that the total number of summands in (5.1) is $r!/(r-2\nu)!$. Since the product remains the same, if we rearrange its ν factors or switch the two terms of any of the ν differences, there are $2^\nu\nu!$ summands that express the same homogeneous polynomial. Therefore, F_ν can be rewritten as

$$F_\nu(x_1,\dots,x_r)=\sum_{\substack{1\le i_1,\dots,i_{2\nu}\le r\text{ distinct}\\ i_1>i_3>\dots>i_{2\nu-1}\\ i_{2j-1}>i_{2j}\text{ for }1\le j\le\nu}}y(x_{i_1},\dots,x_{i_{2\nu}}),\tag{5.9}$$

where the conditions on the ordering of the indices $i_1,\dots,i_{2\nu}$ are used to select a unique representer for each summand. Now, the proof of (ii) goes as follows. Both sides in (5.3) are composed of multiples of $y(x_{i_1}, \ldots, x_{i_{2\nu}})$, where $i_1, \ldots, i_{2\nu}$ can be assumed to satisfy the constraints of the indices in (5.9). While such terms appear once on the left-hand side of (5.3), they appear precisely ν times on the right-hand side of (5.3) as a result of permuting the order of the factors. This fact necessitates the factor $1/\nu$ in front of the summation. A similar argument is used in order to prove (iii). Here, we note that both sides vanish, by definition, if $r < 2\nu + k$.

The proof of (iv) is based on the identity

$$(\alpha x + \beta y - x_i)^2 = \alpha(x - x_i)^2 + \beta(y - x_i)^2 - \alpha\beta(y - x)^2,$$

which holds for all real x, y, x_i and $\alpha + \beta = 1$. Note that the recursion (5.2) also holds for $r < 2\nu$. Hence, we obtain, by (5.2), that

$$F_\nu(x_1, \ldots, x_r, \alpha x + \beta y) = F_\nu(x_1, \ldots, x_r) + \sum_{i=1}^{r} (\alpha x + \beta y - x_i)^2 F_{\nu-1}(\{x_1, \ldots, x_r\} \setminus \{x_i\}).$$

Likewise, the assumption that $\alpha + \beta = 1$ and the recursion (5.2) together give

$$\begin{aligned} &\alpha F_\nu(x_1, \ldots, x_r, x) + \beta F_\nu(x_1, \ldots, x_r, y) = \\ &\qquad F_\nu(x_1, \ldots, x_r) + \sum_{i=1}^{r} \left[\alpha(x - x_i)^2 + \beta(y - x_i)^2\right] F_{\nu-1}(\{x_1, \ldots, x_r\} \setminus \{x_i\}). \end{aligned}$$

By applying these two identities, we have

$$\begin{aligned} &F_\nu(x_1, \ldots, x_r, \alpha x + \beta y) - \alpha F_\nu(x_1, \ldots, x_r, x) - \beta F_\nu(x_1, \ldots, x_r, y) = \\ &\qquad - \alpha\beta(y - x)^2 \sum_{i=1}^{r} F_{\nu-1}(\{x_1, \ldots, x_r\} \setminus \{x_i\}), \end{aligned}$$

which is the same as (5.6) by an application of (5.4).

For the inequality (5.7), we make use of the simple fact that, for all real numbers a, b, c, d,

$$a \le b \le c \le d \implies (d - a)(c - b) \le (d - b)(c - a),$$

namely for $x_1 \leq x_2 \leq \cdots \leq x_r$, the products in (5.8) satisfy

$$y(x_{i_1}, \ldots, x_{i_{2\nu}}) \leq (x_r - x_\nu)^2 (x_{r-1} - x_{\nu-1})^2 \cdots (x_{r-\nu+1} - x_1)^2 =: y_{\max}(\nu; x_1, \ldots, x_r), \quad (5.10)$$

which gives the upper bound estimate

$$F_\nu(x_1, \ldots, x_r) \leq \frac{2^{-\nu} r!}{\nu!(r-2\nu)!} y_{\max}(\nu; x_1, \ldots, x_r),$$

or equivalently, the inequality (5.7). ■

The invariance properties of F_ν are sufficient to guarantee that $F_\nu(x_1, \ldots, x_r)$ is a polynomial of the centered moments

$$\sigma_\ell := \frac{1}{r} \sum_{k=1}^{r} (x_k - \overline{x})^\ell, \qquad 2 \leq \ell \leq 2\nu,$$

where $\overline{x} = (x_1 + \cdots + x_r)/r$. We have

$$\begin{aligned}
F_1(x_1, \ldots, x_r) \;&= r^2 \sigma_2, \\
2F_2(x_1, \ldots, x_r) \;&= r^2(r^2 - 3r + 3)\ \sigma_2^2 - r^2(r-1)\ \sigma_4, \\
6F_3(x_1, \ldots, x_r) \;&= r^3(r-2)(r^2 - 7r + 15)\ \sigma_2^3 - 3r^2(r-2)(r^2 - 5r + 10)\ \sigma_4\sigma_2 - \\
&\qquad 2r^2(3r^2 - 15r + 20)\ \sigma_3^2 + 2r^2(r-1)(r-2)\ \sigma_6,
\end{aligned}$$

$$\begin{aligned}
24F_4(x_1, \ldots, x_r) \;&= r^4(r^4 - 18r^3 + 125r^2 - 384r + 441)\ \sigma_2^4 - \\
&\qquad 6r^3(r^4 - 16r^3 + 104r^2 - 305r + 336)\ \sigma_4\sigma_2^2 + \\
&\qquad 3r^2(r^4 - 14r^3 + 95r^2 - 322r + 420)\ \sigma_4^2 + \\
&\qquad 8r^2(r-2)(r-3)(r^2 - 7r + 21)\ \sigma_6\sigma_2 - \\
&\qquad 8r^3(r-3)(3r^2 - 24r + 56)\ \sigma_3^2\sigma_2 + \\
&\qquad 48\ r^2(r-3)(r^2 - 7r + 14)\ \sigma_5\sigma_3 - 6r^2(r-1)(r-2)(r-3)\ \sigma_8,
\end{aligned}$$

$$
\begin{aligned}
120F_5(x_1,\ldots,x_r) \ &= r^5(r-4)(r^4-26r^3+261r^2-1176r+2025)\ \sigma_2^5 - \\
&\quad 10r^4(r-4)(r^4-24r^3+230r^2-999r+1674)\ \sigma_4\sigma_2^3 + \\
&\quad 20r^3(r-4)(r^4-20r^3+168r^2-645r+972)\ \sigma_6\sigma_2^2 + \\
&\quad 15r^3(r-4)(r^4-22r^3+211r^2-942r+1620)\sigma_4^2\sigma_2 - \\
&\quad 20r^4(3r^4-60r^3+470r^2-1665r+2232)\ \sigma_3^2\sigma_2^2 - \\
&\quad 30r^2(r-2)(r-3)(r-4)(r^2-9r+36)\ \sigma_8\sigma_2 - \\
&\quad 20r^2(r-4)(r^4-18r^3+173r^2-828r+1512)\ \sigma_6\sigma_4 + \\
&\quad 240\ r^3(r^4-19r^3+143r^2-493r+648)\ \sigma_5\sigma_3\sigma_2 + \\
&\quad 20r^4(r-4)(3r^2-30r+83)\ \sigma_4\sigma_3^2 - \\
&\quad 24r^2(5r^4-90r^3+655r^2-2250r+3024)\ \sigma_5^2 - \\
&\quad 240\ r^2(r-3)(r-4)(r^2-9r+24)\ \sigma_7\sigma_3 + \\
&\quad 24r^2(r-1)(r-2)(r-3)(r-4)\ \sigma_{10}.
\end{aligned}
$$

5.2. Definition of the minimally supported approximate duals

In order to establish representations of the minimally supported approximate duals, we need to introduce some notations. For a given knot sequence $\mathbf{t}$, let

$$\beta_{m,k}^{(0)}(\mathbf{t}) := 1, \qquad -m+1 \le k \le N, \tag{5.11}$$

$$\beta_{m,k}^{(\nu)}(\mathbf{t}) := \frac{m!(m-\nu-1)!}{(m+\nu)!(m+\nu-1)!} F_\nu(t_{k+1},\ldots,t_{k+m+\nu-1}), \tag{5.12}$$

where $1 \le \nu \le m-1$ and $-m+1 \le k \le N-\nu+1$. Here, F_ν is the homogeneous polynomial defined in (5.1). Moreover, we define

$$u_{m,k}^{(\nu)}(\mathbf{t}) := \frac{m+\nu}{t_{k+m+\nu}-t_k}\ \beta_{m,k}^{(\nu)}(\mathbf{t}), \quad \nu = 0,\ldots,m-1, \tag{5.13}$$

and consider the diagonal matrices

$$U_\nu(\mathbf{t}) := \operatorname{diag}\,(u_{m,k}^{(\nu)}(\mathbf{t});\ -m+1 \le k \le N-\nu). \tag{5.14}$$

The approximate dual of order L, for $1 \leq L \leq m$, is then given by

$$S_L(\mathbf{t}) = U_0(\mathbf{t}) + \sum_{\nu=1}^{L-1} D_{\mathbf{t};m} \cdots D_{\mathbf{t};m+\nu-1} U_\nu(\mathbf{t}) D_{\mathbf{t};m+\nu-1}^T \cdots D_{\mathbf{t};m}^T, \tag{5.15}$$

where $D_{\mathbf{t};r}, r \geq m$ are defined in (4.14). It is easy to see that this $(m+N) \times (m+N)$ matrix is symmetric, nonsingular and banded with bandwidth L. Moreover, the kernel K_{S_L} in (2.8) has the form

$$K_{S_L}(x,y) = \sum_{\nu=0}^{L-1} \sum_{k=-m+1}^{N-\nu} u_{m,k}^{(\nu)}(\mathbf{t}) \frac{d^{2\nu}}{dx^\nu dy^\nu} N_{\mathbf{t};m+\nu,k}(x) N_{\mathbf{t};m+\nu,k}(y). \tag{5.16}$$

In the following subsections, we will show that $\Phi_{\mathbf{t};m} S_L$ is the minimally supported approximate dual of $\Phi_{\mathbf{t};m}$ of order L, the kernel K_{S_L} satisfies (2.16) (where the upper bound C does not depend on the knot vector or the length of I), and that conditions (3.8)–(3.9) are satisfied for approximate duals with respect to nested knot vectors. Hence, the construction of tight MRA frames can be performed with the sequence of the so-defined matrices $S_{j,L}$. The main step of the proof makes use of knot insertion. Therefore, as a starting point for our induction argument, we first prove the result for the polynomial space Π_{m-1} on the interval $[a,b]$.

5.3. Approximate duals of Bernstein polynomials

Here, we restrict our attention to the simplest case where the knot vector $\mathbf{t}$ has no interior knot ($N=0$); that is,

$$t_{-m+1} = \cdots = t_0 = a < b = t_1 = \cdots = t_m.$$

In this case, the B-spline basis $N_{\mathbf{t};m,k}$, $-m+1 \leq k \leq 0$, of order m is identical to the basis of Bernstein polynomials of degree $n := m-1$ on the interval $[a,b]$, given by

$$B_{n,k}(x) := (b-a)^{-n} \binom{n}{k} (x-a)^k (b-x)^{n-k}, \qquad 0 \leq k \leq n = m-1;$$

that is, $N_{\mathbf{t};m,k-m+1} = B_{n,k}$. Of course, if we let $B^0_{n,k}(x)$ denote the Bernstein polynomials on $[0,1]$, that is

$$B^0_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \le k \le n,$$

then $B_{n,k}(x) = B^0_{n,k}(\frac{x-a}{b-a})$ for $x \in [a,b]$. So, as usual, we can study the special case of the Bernstein polynomials on $[0,1]$ without any loss of generality.

In this special case, for $1 \le L \le m = n+1$, the kernel in (5.16) has the form

$$K_{S_L}(x,y) = \sum_{\nu=0}^{L-1} \sum_{k=0}^{n-\nu} u^{(\nu)}_{m,k-n}(\mathbf{t}) \frac{d^{2\nu}}{dx^\nu dy^\nu} B^0_{n+\nu,k+\nu}(x) B^0_{n+\nu,k+\nu}(y). \tag{5.17}$$

The evaluation of the coefficients

$$u^{(\nu)}_{m,k-n}(\mathbf{t}) = (m+\nu)\beta^{(\nu)}_{m,k-n}(\mathbf{t}) = \frac{m!(m-\nu-1)!}{(m+\nu-1)!(m+\nu-1)!} F_\nu(t_{k-n+1}, \dots, t_{k+\nu})$$

makes use of the closed form expression for

$$F_\nu(t_{k-n+1}, \dots, t_{k+\nu}) = F_\nu(\underbrace{0, \dots, 0}_{n-k}, \underbrace{1, \dots, 1}_{k+\nu}) = \nu! \binom{n-k}{\nu} \binom{k+\nu}{\nu},$$

which can be obtained either directly from (5.1) or by an application of Lemma 3. This gives

$$K_{S_L}(x,y) = \sum_{\nu=0}^{L-1} \sum_{k=0}^{n-\nu} \frac{\nu!(n+1)!(n-\nu)!}{[(n+\nu)!]^2} \binom{n-k}{\nu} \binom{k+\nu}{\nu} \frac{d^{2\nu}}{dx^\nu dy^\nu} B^0_{n+\nu,k+\nu}(x) B^0_{n+\nu,k+\nu}(y). \tag{5.18}$$

In order to prove that S_L defines an approximate dual of order L, we will find a representation for the reproducing kernel of Π_n, considered as a subspace of $L_2(0,1)$, which is similar to (5.18). Note that an approximate dual of order $L = m$ must be identical to the dual basis of the Bernstein polynomials.

Representations of the dual basis of Bernstein polynomials on $[0,1]$ are given in [25,12,34], but we need a new representation as given in the following theorem for the purpose of formulating approximate duals in terms of partial sums. There does not seem to be any immediate connection between the representations in [25,12,34] and ours.

Theorem 4. *The Bernstein polynomial basis $\{B^0_{n,k};\ 0 \le k \le n\}$ of degree $n \ge 1$ possesses the Sobolev space orthogonality property*

$$(n+1)\sum_{i=0}^{n}\frac{(n-i)!}{i!n!}\int_0^1 x^i(1-x)^i\frac{d^i}{dx^i}B^0_{n,k}(x)\frac{d^i}{dx^i}B^0_{n,\ell}(x)\,dx = \delta_{k,\ell}, \qquad 0 \le k,\ell \le n. \quad (5.19)$$

Moreover, the polynomials

$$C_{n,k}(x) := (n+1)\sum_{i=0}^{n}(-1)^i\frac{(n-i)!}{i!n!}\frac{d^i}{dx^i}\left(x^i(1-x)^i\frac{d^i}{dx^i}B^0_{n,k}(x)\right), \quad 0 \le k \le n, \qquad (5.20)$$

constitute the dual basis of the Bernstein polynomial basis, and

$$K(x,y) = K(y,x) = \sum_{k=0}^{n} B^0_{n,k}(x)C_{n,k}(y) \qquad (5.21)$$

defines the reproducing kernel of the space of polynomials of degree n with respect to the ordinary inner product on $[0,1]$.

The proof of this result will be given in Section 9.

Remark 5. After we communicated our result to Margareta Heilmann of the University of Wuppertal, she discovered (using Maple) that for low degree n, the Sobolev orthogonality property in (5.19) can be strengthened into the identity

$$\sum_{i=0}^{n}\frac{(n-i)!}{i!n!}x^i(1-x)^i\frac{d^i}{dx^i}B^0_{n,k}(x)\frac{d^i}{dx^i}B^0_{n,\ell}(x) = \delta_{k,\ell}\ B^0_{n,k}(x).$$

This identity is then proved to hold for every degree n and even extended to multivariate Bernstein polynomials on a d-dimensional simplex, with a proper adaptation of the differential operator in [22].

The reproducing kernel in (5.21) can be written in another form, which is more suitable for our subsequent arguments.

Corollary 2. *The reproducing kernel $K(x,y)$ in Theorem 4 has the equivalent form*

$$K(x,y) = \sum_{\nu=0}^{n}\sum_{k=0}^{n-\nu}\frac{\nu!(n+1)!(n-\nu)!}{[(n+\nu)!]^2}\binom{n-k}{\nu}\binom{k+\nu}{\nu}\frac{d^{2\nu}}{dx^\nu dy^\nu}B^0_{n+\nu,k+\nu}(x)B^0_{n+\nu,k+\nu}(y). \qquad (5.22)$$

The proof of this result will also be given in Section 9. The importance of our formulation in (5.22) is that the kernel K_{S_L} in (5.18) is obtained as a partial sum of the reproducing kernel K. Since K reproduces all polynomials in Π_n, i.e.

$$\int_0^1 f(y)K(x,y)\,dy = f(x), \qquad f \in \Pi_n,$$

and the terms for $\nu \geq L$ in (5.22) annihilate all polynomials in Π_{L-1}, the kernel K_{S_L} reproduces all polynomials in Π_{L-1}. In other words, we have shown that S_L in (5.15) defines an approximate dual of order L, in the Bernstein case.

The matrix formulation in Section 3 can be given in terms of the inverse Gramian of the Bernstein basis. The next result is a direct consequence of Corollary 2 and (4.13).

Corollary 3. *Let $\mathcal{G}_0$ be the Gramian of the Bernstein basis $(B^0_{n,k};\ 0 \leq k \leq n)$ on $[0,1]$. Then*

$$\mathcal{G}_0^{-1} = (n+1)I_{n+1} + (n+1)\sum_{\nu=1}^{n} \Delta_{n+1}\cdots\Delta_{n+2-\nu}A_\nu\Delta^T_{n+2-\nu}\cdots\Delta^T_{n+1}, \tag{5.23}$$

where Δ_r is defined in (4.15) and $A_\nu = \operatorname{diag}\left(\alpha^{(\nu)}_{n+1,0},\ldots,\alpha^{(\nu)}_{n+1,n-\nu}\right)$ is a diagonal matrix with entries

$$\alpha^{(\nu)}_{n+1,k} := \frac{\binom{k+\nu}{\nu}\binom{n-k}{\nu}}{\binom{n}{\nu}}, \qquad 0 \leq k \leq n-\nu. \tag{5.24}$$

More generally, the inverse Gramian of the Bernstein polynomials $B_{n,k}$ on the interval $I=[a,b]$ is given by

$$\mathcal{G}^{-1} = \frac{n+1}{b-a}\left[I_{n+1} + \sum_{\nu=1}^{n} \Delta_{n+1}\cdots\Delta_{n+2-\nu}A_\nu\Delta^T_{n+2-\nu}\cdots\Delta^T_{n+1}\right]. \tag{5.25}$$

Identity (5.25) shows, in perhaps the most appropriate way, how the construction of the matrix S_L in (5.15) validates identity (3.10). More precisely, the right-hand side of (5.25) defines a successive approximation of the inverse Gramian by means of banded matrices.

The first term is diagonal, the next term ($\nu = 1$) is tridiagonal, etc. To be specific, let S_L be the partial sum in (5.25), so that

$$S_L = \frac{n+1}{b-a} \begin{cases} I_{n+1}, & \text{if } L = 1, \\ I_{n+1} + \sum_{\nu=1}^{L-1} \Delta_{n+1} \cdots \Delta_{n+2-\nu} A_\nu \Delta_{n+2-\nu}^T \cdots \Delta_{n+1}^T, & \text{if } L = 2, \ldots, n+1. \end{cases}$$

If we write $S_L = [\ s_{ij}^L\ ]_{0 \le i,j \le n}$, then

$$s_{ij}^L = (-1)^{i+j} \frac{n+1}{b-a} \sum_{\nu=0}^{L-1} \sum_{\ell=0}^{n-\nu} \binom{\nu}{i-\ell} \binom{\nu}{j-\ell} \alpha_{n+1,\ell}^{(\nu)}.$$

Since $\mathcal{G}^{-1} = S_{n+1}$, writing $\mathcal{G}^{-1} = [\ g_{ij}\ ]_{0 \le i,j \le n}$, we obtain

$$g_{ij} = (-1)^{i+j} \frac{n+1}{b-a} \sum_{\nu=0}^{n} \sum_{\ell=0}^{n-\nu} \binom{\nu}{i-\ell} \binom{\nu}{j-\ell} \alpha_{n+1,\ell}^{(\nu)}.$$

Therefore, the difference between $\mathcal{G}^{-1}$ and S_L, for $1 \le L \le n$, admits a factorization of the form

$$\mathcal{G}^{-1} - S_L = \Delta_{n+1} \cdots \Delta_{n+2-L} X_L \Delta_{n+2-L}^T \cdots \Delta_{n+1}^T, \tag{5.26}$$

where

$$X_L := \frac{n+1}{b-a} \left[A_L + \sum_{\nu=L+1}^{n} \Delta_{n+1-L} \cdots \Delta_{n+2-\nu} A_\nu \Delta_{n+2-\nu}^T \cdots \Delta_{n+1-L}^T \right]. \tag{5.27}$$

If we write $X_L = [\ x_{ij}^L\ ]_{0 \le i,j \le n-L}$, then

$$x_{ij}^L = (-1)^{i+j} \frac{n+1}{b-a} \sum_{\nu=L}^{n} \sum_{\ell=0}^{n-\nu} \binom{\nu-L}{i-\ell} \binom{\nu-L}{j-\ell} \alpha_{n+1,\ell}^{(\nu)}.$$

The factorization in (5.26) governs the construction of approximate duals of B-splines as shown in (3.10), except for a different normalization of the factors Δ_r.

5.4. Induction proof for B-splines

In this subsection, we show that $S_L := S_L(\mathbf{t})$ in (5.15) defines an approximate dual $\Phi_{\mathbf{t};m} \cdot S_L$ of order L, for an arbitrary knot sequence

$$\mathbf{t} := [\underbrace{a, \ldots, a}_{m}, t_1, \ldots, t_N, \underbrace{b, \ldots, b}_{m}] \tag{5.28}$$

with $t_k < t_{k+m}$ for all k. Let $\Gamma(\mathbf{t})$ denote the Gramian of $\Phi_{\mathbf{t};m}$.

Theorem 5. *For $1 \le L \le m$, let $S_L := S_L(\mathbf{t})$ be defined as in (5.15). Then $\Phi_{\mathbf{t};m} \cdot S_L$ is an approximate dual of order L that corresponds to the B-spline basis $\Phi_{\mathbf{t};m}$ in the sense of Definition 4. That is,*

$$\Gamma^{-1}(\mathbf{t}) - S_L(\mathbf{t}) = D_{\mathbf{t};m} \cdots D_{\mathbf{t};m+L-1} X_L(\mathbf{t}) D^T_{\mathbf{t};m+L-1} \cdots D^T_{\mathbf{t};m}, \tag{5.29}$$

for some symmetric matrix $X_L(\mathbf{t})$.

In order to prove Theorem 5, we use arguments about knot insertion. An intermediate result is concerned with the approximate duals relative to two knot vectors $\mathbf{t} \subset \tilde{\mathbf{t}}$, where $\mathbf{t}$ is as in (5.28) and

$$\tilde{\mathbf{t}} = [\underbrace{a, \ldots, a}_{m}, \tilde{t}_1, \ldots, \tilde{t}_{N+M}, \underbrace{b, \ldots, b}_{m}]. \tag{5.30}$$

We first introduce the notation of the intermediate knot vectors

$$\mathbf{t} =: \mathbf{t}_0 \subset \mathbf{t}_1 \subset \cdots \subset \mathbf{t}_M := \tilde{\mathbf{t}}, \tag{5.31}$$

such that $\mathbf{t}_{k+1} \setminus \mathbf{t}_k$, $k = 0, \ldots, M-1$, is a singleton. In the following, we encounter the refinement matrices $P_{\mathbf{t}_k, \tilde{\mathbf{t}}; m+L}$ between the intermediate knot vector $\mathbf{t}_k$ and the final refinement $\tilde{\mathbf{t}}$, for splines of order $m+L$. As usual, we assume that all knots of $\tilde{\mathbf{t}}$ have multiplicity at most m.

Theorem 6. *For $L = 1, \ldots, m$, the matrix $S_L(\tilde{\mathbf{t}}) - P_{\mathbf{t},\tilde{\mathbf{t}};m}\ S_L(\mathbf{t})\ \ P^T_{\mathbf{t},\tilde{\mathbf{t}};m}$ is positive semi-definite and has the representation*

$$S_L(\tilde{\mathbf{t}}) - P_{\mathbf{t},\tilde{\mathbf{t}};m}\ S_L(\mathbf{t})\ \ P^T_{\mathbf{t},\tilde{\mathbf{t}};m} = E_{\tilde{\mathbf{t}};m,L}\ Z_L\ E^T_{\tilde{\mathbf{t}};m,L}, \tag{5.32}$$

where

$$Z_L = Z_L(\mathbf{t}, \tilde{\mathbf{t}}) := \sum_{k=1}^{M} P_{\mathbf{t}_k,\tilde{\mathbf{t}};m+L}\ V_L(\mathbf{t}_k)\ P^T_{\mathbf{t}_k,\tilde{\mathbf{t}};m+L}, \tag{5.33}$$

and $V_L(\mathbf{t}_k)$ are diagonal matrices with nonnegative entries.

Theorem 6 is of independent interest for the construction of tight frames, as it confirms the positivity condition (3.8) for the difference of two consecutive approximate duals $S_L(\mathbf{t}_j)$

and $S_L(\mathbf{t}_{j+1})$ for nested knot vectors $\cdots \subset \mathbf{t}_j \subset \mathbf{t}_{j+1} \subset \cdots$. Since the proof of Theorem 5 depends on Theorem 6, we start with the proof of Theorem 6.

The proof works by successive insertion of single knots. Let us consider the special case

$$\tilde{\mathbf{t}} = [\underbrace{a, \ldots, a}_{m}, \tilde{t}_1, \ldots, \tilde{t}_{N+1}, \underbrace{b, \ldots, b}_{m}] = [a, \ldots, a, t_1, \ldots, t_\rho, \tau, t_{\rho+1}, \ldots, t_N, b, \ldots, b], \quad (5.34)$$

where only one new knot τ is inserted in the interval $[t_\rho, t_{\rho+1})$. Of course, t_ρ is assumed to be a knot of multiplicity at most m in the refined knot vector $\tilde{\mathbf{t}}$ as well. Note that $\tilde{t}_k = t_k$ for $-m+1 \le k \le \rho$, $\tilde{t}_{\rho+1} = \tau$, and $\tilde{t}_k = t_{k-1}$ for $\rho+1 \le k \le N+m+1$. The refinement relation (4.19), with $m+\nu$ in place of m and the matrix $P_{m+\nu} := P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ as in (4.21), plays an important role in our derivation of Theorem 6.

For simplicity, we denote

$$\beta_k^{(\nu)} = \beta_{m,k}^{(\nu)}(\mathbf{t}), \qquad k = 1-m, \ldots, N-\nu,$$
$$\tilde{\beta}_k^{(\nu)} = \beta_{m,k}^{(\nu)}(\tilde{\mathbf{t}}), \qquad k = 1-m, \ldots, N-\nu+1.$$

Likewise, we use the short-hand notations $U_\nu := U_\nu(\mathbf{t})$, $\tilde{U}_\nu := U_\nu(\tilde{\mathbf{t}})$, $\tilde{D}_r := D_{\tilde{\mathbf{t}};r}$, $\tilde{E}_{r,s} := E_{\tilde{\mathbf{t}};r,s}$. By (5.12) and appealing to the symmetry of the functions F_ν in (5.1), we can write

$$\begin{aligned} \tilde{\beta}_k^{(\nu)} &= \frac{m!(m-\nu-1)!}{(m+\nu)!(m+\nu-1)!}\, F_\nu(\tilde{t}_{k+1}, \ldots, \tilde{t}_{k+m+\nu-1}) \\ &= \frac{m!(m-\nu-1)!}{(m+\nu)!(m+\nu-1)!} \times \\ &\begin{cases} F_\nu(t_{k+1}, \ldots, t_{k+m+\nu-1}), & \text{if } 1-m \le k \le \rho+1-m-\nu, \\ F_\nu(t_{k+1}, t_{k+m+\nu-2}, \tau), & \text{if } \max(\rho+2-m-\nu, 1-m) \le k \le \min(\rho, N-\nu+1), \\ F_\nu(t_k, \ldots, t_{k+m+\nu-2}), & \text{if } \rho+1 \le k \le N-\nu+1. \end{cases} \end{aligned} \quad (5.35)$$

When $\rho < \nu$, the first case of (5.35) does not occur, and when $\rho > N-\nu$, the last case does not occur. By a comparison of (5.12) and (5.35), we obtain

$$\tilde{\beta}_k^{(\nu)} = \begin{cases} \beta_k^{(\nu)}, & 1-m \le k \le \rho+1-m-\nu, \\ \beta_{k-1}^{(\nu)}, & k = \rho+1, \ldots, N-\nu+1. \end{cases} \quad (5.36)$$

The terms with the remaining indices

$$\max(\rho + 2 - m - \nu, 1 - m) \le k \le \min(\rho, N - \nu + 1) \tag{5.37}$$

are treated in the next lemma.

Lemma 4. *For k in (5.37),*

$$\tilde{\beta}_k^{(\nu)} = \frac{(t_{k+m+\nu-1} - \tau)\beta_{k-1}^{(\nu)}}{t_{k+m+\nu-1} - t_k} + \frac{(\tau - t_k)\beta_k^{(\nu)}}{t_{k+m+\nu-1} - t_k} - \frac{(t_{k+m+\nu-1} - \tau)(\tau - t_k)\beta_k^{(\nu-1)}}{(m+\nu)(m+\nu-1)}, \tag{5.38}$$

where $\beta_k^{(\nu)} = 0$ if $k < 1 - m$ or $k > N - \nu$.

The proof of this result is delayed to Section 5.5. The key step of the proof of Theorem 6 is the next lemma.

Lemma 5. *Let diagonal matrices $V_\nu = V_\nu(\tilde{\mathbf{t}})$, $0 \le \nu \le m$, of dimension $(m + N + 1 - \nu) \times (m + N + 1 - \nu)$, be defined by $V_0 = 0$ and, for $1 \le \nu \le m$, by the diagonal entries*

$$v_k^{(\nu)} := \begin{cases} \dfrac{(\tilde{t}_{k+m+\nu} - \tau)(\tau - \tilde{t}_k)\beta_k^{(\nu-1)}(\tilde{\mathbf{t}} \setminus \{\tau\})}{(m+\nu-1)(\tilde{t}_{k+m+\nu} - \tilde{t}_k)}, & k \text{ in (5.37)} \\ 0, & \text{otherwise.} \end{cases} \tag{5.39}$$

Then V_ν is positive semi-definite and satisfies

$$V_\nu + \tilde{U}_\nu - P_{m+\nu}\, U_\nu\, P_{m+\nu}^T = \tilde{D}_{m+\nu}\, V_{\nu+1}\, \tilde{D}_{m+\nu}^T, \quad 0 \le \nu \le m - 1. \tag{5.40}$$

Furthermore, the sequence of matrices V_ν, $0 \le \nu \le m$, is uniquely determined by the identity (5.40).

The proof of Lemma 5 is also delayed to Section 5.5. The commutation property (4.25) comes into play when we form the sums

$$S_L = S_L(\mathbf{t}) = U_0 + \sum_{\nu=1}^{L-1} E_{m,\nu} U_\nu E_{m,\nu}^T$$

and

$$\tilde{S}_L = S_L(\tilde{\mathbf{t}}) = \tilde{U}_0 + \sum_{\nu=1}^{L-1} \tilde{E}_{m,\nu} \tilde{U}_\nu \tilde{E}_{m,\nu}^T,$$

for $1 \le L \le m$, as in (5.15). For the insertion of a single knot, as in (5.34), Theorem 6 is shown by the following result.

Lemma 6. *For $L = 1, \ldots, m$, the matrix $\tilde{S}_L - P_m S_L P_m^T$ is positive semi-definite and satisfies*

$$\tilde{S}_L - P_m S_L P_m^T = \tilde{E}_{m,L} V_L \tilde{E}_{m,L}^T, \tag{5.41}$$

where V_L is the matrix in Lemma 5.

Proof: We use induction on L. The result for $L = 1$ is given in Lemma 5, where we let $\nu = 0$ in (5.40) and make use of $V_0 = 0$. For $1 \leq L \leq m - 1$, the definition (5.15) leads to

$$\tilde{S}_{L+1} - P_m S_{L+1} P_m^T = \tilde{S}_L - P_m S_L P_m^T + \tilde{E}_{m,L} \tilde{U}_L \tilde{E}_{m,L}^T - P_m E_{m,L} U_L E_{m,L}^T P_m^T.$$

The commutation relation (4.25) gives

$$P_m E_{m,L} = \tilde{E}_{m,L} P_{m+L}.$$

Then by the induction hypothesis (5.41), we obtain

$$\begin{aligned} \tilde{S}_{L+1} - P_m S_{L+1} P_m^T &= \tilde{E}_{m,L} (V_L + \tilde{U}_L - P_{m+L} U_L P_{m+L}^T) \tilde{E}_{m,L}^T \\ &= \tilde{E}_{m,L+1} V_{L+1} \tilde{E}_{m,L+1}^T. \end{aligned}$$

The last step is again an application of (5.40). Thus we have proved (5.41). Lemma 5 implies that V_ν, $1 \leq \nu \leq m$, is positive semi-definite, and therefore the matrix $\tilde{S}_L - P_m S_L P_m^T$ is also positive semi-definite. ■

Next we show that the result in Lemma 6 can be extended to general knot refinements on a bounded interval, and thereby prove Theorem 6. Therefore, we go back to general knot vectors $\mathbf{t} \subset \tilde{\mathbf{t}}$ in (5.28), (5.30) and define the intermediate knot vectors $\mathbf{t}_k$, $0 \leq k \leq M$, as in (5.31).

Proof of Theorem 6: We write the left-hand side of (5.32) as a telescoping sum and make use of Lemma 6 and the commutation relation, in order to obtain

$$\begin{aligned} S_L(\tilde{\mathbf{t}}) - P_{\mathbf{t},\tilde{\mathbf{t}};m} S_L(\mathbf{t})\ P^T_{\mathbf{t},\tilde{\mathbf{t}};m} &= \sum_{k=1}^{M} [P_{\mathbf{t}_k,\tilde{\mathbf{t}};m} S_L(\mathbf{t}_k)\ P^T_{\mathbf{t}_k,\tilde{\mathbf{t}};m} - P_{\mathbf{t}_{k-1},\tilde{\mathbf{t}};m} S_L(\mathbf{t}_{k-1})\ P^T_{\mathbf{t}_{k-1},\tilde{\mathbf{t}};m}] \\ &= \sum_{k=1}^{M} P_{\mathbf{t}_k,\tilde{\mathbf{t}};m} [S_L(\mathbf{t}_k) - P_{\mathbf{t}_{k-1},\mathbf{t}_k;m} S_L(\mathbf{t}_{k-1})\ P^T_{\mathbf{t}_{k-1},\mathbf{t}_k;m}]\ P^T_{\mathbf{t}_k,\tilde{\mathbf{t}};m} \\ &= \sum_{k=1}^{M} P_{\mathbf{t}_k,\tilde{\mathbf{t}};m}\ E_{\mathbf{t}_k;m,L}\ V_L(\mathbf{t}_k)\ E^T_{\mathbf{t}_k;m,L}\ P^T_{\mathbf{t}_k,\tilde{\mathbf{t}};m} \\ &= \sum_{k=1}^{M} E_{\tilde{\mathbf{t}};m,L}\ P_{\mathbf{t}_k,\tilde{\mathbf{t}};m+L}\ V_L(\mathbf{t}_k)\ P_{\mathbf{t}_k,\tilde{\mathbf{t}};m+L}\ E^T_{\tilde{\mathbf{t}};m,L}. \end{aligned}$$

This proves the identity (5.32) with Z_L in (5.33). Here, the matrix Z_L is positive semi-definite by the result of Lemma 5. ■

Finally, we can combine the results in Theorem 6 and those from the previous sections, in order to prove Theorem 5. For this purpose, we choose $\mathbf{t}$ to be the knot vector of the Bernstein basis, i.e.

$$\mathbf{t} = [\underbrace{a,\ldots,a}_{m},\underbrace{b,\ldots,b}_{m}] \tag{5.42}$$

and $N = 0$ in (5.28). Then consider the arbitrary knot vector $\tilde{\mathbf{t}}$ in Theorem 5, denoted by

$$\tilde{\mathbf{t}} = [\underbrace{a,\ldots,a}_{m}, t_1,\ldots,t_M, \underbrace{b,\ldots,b}_{m}], \tag{5.43}$$

as a refinement (5.30) of $\mathbf{t}$. As before, let the Gramian matrices of $\Phi_{\mathbf{t};m}$ and $\Phi_{\tilde{\mathbf{t}};m}$ be denoted by $\Gamma(\mathbf{t})$ and $\Gamma(\tilde{\mathbf{t}})$, respectively, and recall from identity (3.13) that there is a positive semi-definite matrix $Y(\mathbf{t},\tilde{\mathbf{t}})$ with

$$\Gamma^{-1}(\tilde{\mathbf{t}}) - P_{\mathbf{t},\tilde{\mathbf{t}};m}\Gamma^{-1}(\mathbf{t})P^T_{\mathbf{t},\tilde{\mathbf{t}};m} = E_{\tilde{\mathbf{t}};m,m} Y(\mathbf{t},\tilde{\mathbf{t}}) E^T_{\tilde{\mathbf{t}};m,m}.$$

Proof of Theorem 5: For the Bernstein case, which is associated with the knot vector $\mathbf{t}$ in (5.42), we have already established in (5.26) that

$$\begin{aligned}\Gamma^{-1}(\mathbf{t}) - S_L(\mathbf{t}) &= \sum_{\nu=L}^{m-1} E_{\mathbf{t};m,\nu}\ U_\nu(\mathbf{t})\ E^T_{\mathbf{t};m,\nu} \\ &= E_{\mathbf{t};m,L} \underbrace{\left[U_L(\mathbf{t}) + \sum_{\nu=1}^{m-L-1} E_{\mathbf{t};m+L,\nu}\ U_{L+\nu}(\mathbf{t})\ E^T_{\mathbf{t};m+L,\nu}\right]}_{=:X_L(\mathbf{t})}\ E^T_{\mathbf{t};m,L}.\end{aligned}$$

For the B-spline basis $\Phi_{\tilde{\mathbf{t}};m}$ on the refined knot vector $\tilde{\mathbf{t}}$ in (5.43), we obtain

$$\begin{aligned}\Gamma^{-1}(\tilde{\mathbf{t}}) - S_L(\tilde{\mathbf{t}}) &= [\Gamma^{-1}(\tilde{\mathbf{t}}) - P_{\mathbf{t},\tilde{\mathbf{t}};m}\ \Gamma^{-1}(\mathbf{t})\ P^T_{\mathbf{t},\tilde{\mathbf{t}};m}] + P_{\mathbf{t},\tilde{\mathbf{t}};m}\ [\Gamma^{-1}(\mathbf{t}) - S_L(\mathbf{t})]\ P^T_{\mathbf{t},\tilde{\mathbf{t}};m} - \\ &\qquad [S_L(\tilde{\mathbf{t}}) - P_{\mathbf{t},\tilde{\mathbf{t}};m}\ S_L(\mathbf{t})\ P^T_{\mathbf{t},\tilde{\mathbf{t}};m}] \\ &= E_{\tilde{\mathbf{t}};m,m}\ Y(\mathbf{t},\tilde{\mathbf{t}})\ E^T_{\tilde{\mathbf{t}};m,m} + P_{\mathbf{t},\tilde{\mathbf{t}};m}\ E_{\mathbf{t};m,L}\ X_L(\mathbf{t})\ E^T_{\mathbf{t};m,L}\ P^T_{\mathbf{t},\tilde{\mathbf{t}};m} - \\ &\qquad E_{\tilde{\mathbf{t}};m,L}\ Z_L(\mathbf{t},\tilde{\mathbf{t}})\ E^T_{\tilde{\mathbf{t}};m,L} \\ &= E_{\tilde{\mathbf{t}};m,L}\ \ [E_{\tilde{\mathbf{t}};m+L,m-L}\ Y(\mathbf{t},\tilde{\mathbf{t}})\ E^T_{\tilde{\mathbf{t}};m+L,m-L} + \\ &\qquad\quad P_{\mathbf{t},\tilde{\mathbf{t}};m+L}\ X_L(\mathbf{t})\ P^T_{\mathbf{t},\tilde{\mathbf{t}};m+L} - Z_L(\mathbf{t},\tilde{\mathbf{t}})]\ \ E^T_{\tilde{\mathbf{t}};m,L}.\end{aligned}$$

The last matrix in brackets is symmetric, and therefore $S_L(\tilde{\mathbf{t}})$ defines an approximate dual of order L. ■

5.5. Proof of lemmas in Section 5.4

Proof of Lemma 4: By the definitions of $\beta_k^{(\nu)}$ and $\tilde{\beta}_k^{(\nu)}$ in (5.12) and (5.35), respectively, we see that (5.38) is equivalent to

$$F_\nu(t_{k+1}, t_{k+m+\nu-2}, \tau) =$$

$$\frac{t_{k+m+\nu-1} - \tau}{t_{k+m+\nu-1} - t_k} F_\nu(t_k, \ldots, t_{k+m+\nu-2}) + \frac{\tau - t_k}{t_{k+m+\nu-1} - t_k} F_\nu(t_{k+1}, \ldots, t_{k+m+\nu-1}) \qquad (5.44)$$

$$- (m-\nu)(t_{k+m+\nu-1} - \tau)(\tau - t_k) F_{\nu-1}(t_{k+1}, \ldots, t_{k+m+\nu-2}).$$

This equivalence is also valid in the extreme cases, when $k = m-1$ or $N+1-\nu$, since by the definition (5.1), we have

$$F_\nu(\underbrace{a,\ldots,a}_{m}, t_1, \ldots, t_{\nu-1}) = 0, \qquad F_\nu(t_{N-\nu+2}, \ldots, t_N, \underbrace{b,\ldots,b}_{m}) = 0.$$

Now, (5.44) immediately follows from (iv) in Lemma 3 with $\alpha = \dfrac{t_{k+m+\nu-1} - \tau}{t_{k+m+\nu-1} - t_k}$, $\beta = \dfrac{\tau - t_k}{t_{k+m+\nu-1} - t_k}$, $x = t_k$, and $y = t_{k+m+\nu-1}$. ■

Proof of Lemma 5: For all k in the range

$$\max(\rho + 2 - m - \nu, 1 - m) \le k \le \min(\rho, N + 1 - \nu), \tag{5.45}$$

the knot $\tau = \tilde{t}_{\rho+1}$ appears in the sequence of knots $(\tilde{t}_{k+1}, \ldots, \tilde{t}_{k+m+\nu-1})$. Therefore,

$$t_k = \tilde{t}_k \le \tau \le \tilde{t}_{k+m+\nu} = t_{k+m+\nu-1}, \tag{5.46}$$

which shows that all diagonal entries $v_k^{(\nu)}$ in (5.39) are nonnegative. Hence, the matrices V_ν are positive semi-definite.

We define the row vectors

$$\begin{aligned}
\mathbf{d}_{m+\nu} &= [d_{m+\nu,k}] := \frac{1}{m+\nu}[(t_{k+m+\nu} - t_k);\ -m+1 \le k \le N-\nu];\\
\tilde{\mathbf{d}}_{m+\nu} &= [\tilde{d}_{m+\nu,k}] := \frac{1}{m+\nu}[(\tilde{t}_{k+m+\nu} - \tilde{t}_k);\ -m+1 \le k \le N+1-\nu]
\end{aligned} \tag{5.47}$$

and recall from (4.14) that

$$\tilde{D}_{m+\nu} := D_{\tilde{\mathbf{t}};m+\nu} = \operatorname{diag}\left(\tilde{\mathbf{d}}_{m+\nu}\right)^{-1} \Delta_{m+N+1-\nu}.$$

The identity (5.40) is equivalent to

$$\begin{aligned}
A &:= \operatorname{diag}\left(\tilde{\mathbf{d}}_{m+\nu}\right)\left(V_\nu + \tilde{U}_\nu - P_{m+\nu}\ U_\nu\ P_{m+\nu}^T\right)\operatorname{diag}\left(\tilde{\mathbf{d}}_{m+\nu}\right)\\
&= \Delta_{m+N+1-\nu}\ V_{\nu+1}\ \Delta_{m+N+1-\nu}^T =: B,
\end{aligned} \tag{5.48}$$

where the matrix A on the left-hand side of (5.48) is a real and symmetric tridiagonal matrix. We first show that its column sums vanish, and therefore A must be of the form

$$A = \begin{bmatrix} c_{1-m} & -c_{1-m} & & & & \\ -c_{1-m} & c_{1-m}+c_{2-m} & -c_{2-m} & & & \\ & -c_{2-m} & c_{2-m}+c_{3-m} & & & \\ & & -c_{3-m} & & & \\ & & & \ddots & & \\ & & & & -c_{N-\nu-1} & \\ & & & & c_{N-\nu-1}+c_{N-\nu} & -c_{N-\nu} \\ & & & & -c_{N-\nu} & c_{N-\nu} \end{bmatrix} \tag{5.49}$$

with real entries c_k, $-m+1 \le k \le N-\nu$. Clearly, A has vanishing column sums if and only if

$$\tilde{\mathbf{d}}_{m+\nu} \left(V_\nu + \tilde{U}_\nu - P_{m+\nu}\ U_\nu\ P_{m+\nu}^T \right) = \mathbf{0}. \tag{5.50}$$

By (4.20), we conclude that $\tilde{\mathbf{d}}_{m+\nu} P_{m+\nu} = \mathbf{d}_{m+\nu}$. Hence, identity (5.50) is equivalent to

$$\tilde{\mathbf{d}}_{m+\nu} \left(V_\nu + \tilde{U}_\nu \right) = \mathbf{d}_{m+\nu}\ U_\nu\ P_{m+\nu}^T. \tag{5.51}$$

The definition in (5.13) gives

$$\begin{aligned} \tilde{\mathbf{d}}_{m+\nu} \tilde{U}_\nu &= [\tilde{\beta}_k^{(\nu)};\ -m+1 \le k \le N+1-\nu], \\ \mathbf{d}_{m+\nu} U_\nu &= [\beta_k^{(\nu)};\ -m+1 \le k \le N-\nu]. \end{aligned} \tag{5.52}$$

For all k in the range (5.45), we obtain, from (5.46),

$$\tilde{d}_{m+\nu,k} = \frac{\tilde{t}_{k+m+\nu} - \tilde{t}_k}{m+\nu} = \frac{t_{k+m+\nu-1} - t_k}{m+\nu}.$$

Hence, (5.39) gives

$$\tilde{d}_{m+\nu,k}\ v_k^{(\nu)} = \begin{cases} \dfrac{(t_{k+m+\nu-1} - \tau)(\tau - t_k)\beta_k^{(\nu-1)}}{(m+\nu-1)(m+\nu)}, & k \text{ as in (5.45)}, \\ 0, & \text{otherwise}. \end{cases}$$

An application of the identity (5.38) yields the entries

$$\tilde{d}_{m+\nu,k}\ (v_k^{(\nu)} + \tilde{u}_k^{(\nu)}) = \begin{cases} \dfrac{t_{k+m+\nu-1} - \tau}{t_{k+m+\nu-1} - t_k}\ \beta_{k-1}^{(\nu)} + \dfrac{\tau - t_k}{t_{k+m+\nu-1} - t_k}\ \beta_k^{(\nu)}, & k \text{ as in (5.45)}, \\ \tilde{\beta}_k^{(\nu)}, & \text{otherwise}, \end{cases} \tag{5.53}$$

of $\tilde{\mathbf{d}}_{m+\nu}(V_\nu + \tilde{U}_\nu)$. We claim that these are precisely the entries of the row vector

$$\mathbf{x} = [x_k] := \mathbf{d}_{m+\nu}\ U_\nu\ P_{m+\nu}^T \tag{5.54}$$

which appears on the right-hand side of (5.51). We use indices $1-m \leq k \leq N+1-\mu$ for its entries x_k.

For all $1-m \leq k < \rho+2-m-\nu$, the row of $P_{m+\nu} = P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ in (4.21) with row index k is a unit vector with entry 1 in its k-th column. From (5.52) and (5.36), we conclude that

$$x_k = \beta_k^{(\nu)} = \tilde{\beta}_k^{(\nu)}.$$

Likewise, for all $\rho+1 \leq k \leq N+1-\nu$, the row of $P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ with row index k is a unit vector with entry 1 in its $(k-1)$-st column. Therefore, we obtain

$$x_k = \beta_{k-1}^{(\nu)} = \tilde{\beta}_k^{(\nu)}.$$

This establishes the equality of the corresponding entries of both sides of identity (5.51) for all indices not in the range shown in (5.45). For the remaining indices k in (5.45), the corresponding row of $P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ has the form

$$[0,\ldots,0,1-a_k,a_k,0,\ldots,0], \tag{5.55}$$

where

$$a_k = \frac{\tau - t_k}{t_{k+m+\nu-1} - t_k} = \frac{\tau - t_k}{\tilde{t}_{k+m+\nu} - \tilde{t}_k} \tag{5.56}$$

appears with column index k. The only exceptions are the first row (with $k = 1-m$) and/or the last row (with $k = N+1-\nu$), which have the form

$$\begin{aligned} &[a_{1-m}, 0, \ldots, 0], \\ &[0, \ldots, 0, 1 - a_{N+1-\nu}], \end{aligned} \tag{5.57}$$

if $\rho < \nu$ and/or $\rho > N-\nu$, respectively. This is due to the truncation of the matrix in (4.21) mentioned in Section 4. Therefore, we obtain

$$x_k = \frac{t_{k+m+\nu-1} - \tau}{t_{k+m+\nu-1} - t_k}\ \beta_{k-1}^{(\nu)} + \frac{\tau - t_k}{t_{k+m+\nu-1} - t_k}\ \beta_k^{(\nu)},$$

in the typical case (5.55), and

$$x_{1-m} = \frac{\tau - t_{1-m}}{t_\nu - t_{1-m}}\ \beta^{(\nu)}_{1-m},$$
$$x_{N+1-\nu} = \frac{t_{N+m} - \tau}{t_{N+m} - t_{N+1-\nu}}\ \beta^{(\nu)}_{N-\nu},$$

if the modifications (5.57) occur. Thus, we have shown that the vector $\mathbf{x}$ in (5.54) agrees with the vector in (5.53), and this completes the proof of the identities (5.50)–(5.51).

In the next step of the proof of Lemma 5, we show that both matrices A and B in (5.48) agree. It is clear that B also has the form (5.49), since it is a real and symmetric tridiagonal matrix whose row and column sums vanish. In order to prove the equality $A = B$, it suffices to show that the subdiagonal elements of both matrices agree. Observe that the subdiagonal entry of the $(k+1)$-st row of B, $1 - m \le k \le N - \nu$, is simply

$$b_k = -v_k^{(\nu+1)}.$$

In particular, $b_k \neq 0$ can occur only if

$$\max(\rho + 1 - m - \nu, 1 - m) \le k \le \min(\rho, N - \nu). \tag{5.58}$$

Also, the subdiagonal entry $-c_k$ of the $(k+1)$-st row of A, as in (5.49), comes from the single matrix

$$-\mathrm{diag}\left(\tilde{\mathbf{d}}_{m+\nu}\right) P_{m+\nu}\ U_\nu\ P^T_{m+\nu}\mathrm{diag}\left(\tilde{\mathbf{d}}_{m+\nu}\right). \tag{5.59}$$

Note that the first, third, and last factor of this product are diagonal matrices. Therefore, a nonzero subdiagonal entry of A can only occur in row $k+1$, if the inner product of rows $k+1$ and k of the matrix $P_{m+\nu} = P_{\mathbf{t},\tilde{\mathbf{t}};m+\nu}$ is nonzero. Due to the special form of this matrix, as shown in (4.21), this can only occur if k satisfies (5.58). Otherwise, the subdiagonal entries of A and B are $-c_k = b_k = 0$. If k satisfies (5.58), then

$$-c_k = -a_k\ (1 - a_{k+1})\ u_k^{(\nu)}\ \tilde{d}_{m+\nu,k+1}\ \tilde{d}_{m+\nu,k}.$$

If we replace a_k and a_{k+1} by the expression on the right-hand side of (5.56), we obtain

$$
\begin{aligned}
-c_k &= -\frac{\tau - t_k}{\tilde{t}_{k+m+\nu} - \tilde{t}_k} \, \frac{t_{k+m+\nu} - \tau}{\tilde{t}_{k+m+\nu+1} - \tilde{t}_{k+1}} \, \beta_k^{(\nu)} \, \frac{m+\nu}{t_{k+m+\nu} - t_k} \, \frac{\tilde{t}_{k+m+\nu+1} - \tilde{t}_{k+1}}{m+\nu} \, \frac{\tilde{t}_{k+m+\nu} - \tilde{t}_k}{m+\nu} \\
&= -\frac{(\tau - t_k)(t_{k+m+\nu} - \tau)\beta_k^{(\nu)}}{(m+\nu)(t_{k+m+\nu} - t_k)},
\end{aligned}
$$

and this is equal to $b_k = -v_k^{(\nu+1)}$, as defined in (5.39). Therefore, both matrices A and B in (5.48) are identical.

Finally, to discuss the uniqueness of the matrices V_ν in (5.40), we consider the uniqueness in the equivalent identity (5.48) instead. The factorization on the right-hand side of (5.48) exists, with a diagonal matrix $V_{\nu+1}$, if and only if the symmetric matrix A has vanishing row and column sums. For each $0 \le \nu \le m-1$, there exists a unique diagonal matrix V_ν such that this property is satisfied. This also determines V_m in a unique way. ■

5.6. Minimally supported approximate duals

We show that the matrix $S_L := S_L(\mathbf{t})$ in (5.15) is the only symmetric matrix with bandwidth at most L such that $\Phi_{\mathbf{t};m} S_L$ is an approximate dual of $\Phi_{\mathbf{t};m}$ of order L. This result is based on the variation-diminishing property of the B-spline basis.

Theorem 7. *Let $\mathbf{t}$ be a knot vector as in (4.1)–(4.3). If R is a symmetric matrix of size $(m+N) \times (m+N)$ and bandwidth at most L such that $\Phi_{\mathbf{t};m} R$ is an approximate dual of $\Phi_{\mathbf{t};m}$ of order L, then R must be the matrix in (5.15).*

Proof: Let R be a matrix that satisfies all the assumptions in the theorem, and let us assume that $Z = [z_{k,\ell}] := S_L - R$ is nonzero. The index range of the rows and columns of all matrices is chosen to be $-m+1 \le k, \ell \le N$. Let $\hat{k}$ be the index of the first nonzero row of Z; hence,

$$
s(y) := \sum_{\ell=-m+1}^{N} z_{\hat{k},\ell} N_{\mathbf{t};m,\ell}(y)
$$

is a nonzero spline. Due to symmetry and bandwidth of Z, we have

$$z_{\hat{k},\ell} = 0 \qquad \text{for all} \qquad \ell < \hat{k} \ \text{ and } \ \ell \geq \hat{k} + L,$$

which gives

$$s(y) = \sum_{\ell=\hat{k}}^{\hat{k}+L-1} z_{\hat{k},\ell} N_{\mathbf{t};m,\ell}(y).$$

Since s is a linear combination of at most L consecutive B-splines, it can have at most $L-1$ vanishing moments due to the variation-diminishing property of the B-spline basis, see [1; p.156]. This observation leads to a contradiction, as we first show for the case where $t_{\hat{k}} < t_{\hat{k}+1}$. Indeed, for any $x \in (t_{\hat{k}}, t_{\hat{k}+1})$, we have $N_{\mathbf{t};m,k}(x) = 0$ for $k > \hat{k}$. In addition, by our choice of $\hat{k}$ we know that $z_{k,\ell} = 0$ for all $k < \hat{k}$ and all ℓ. Therefore, we obtain, for such x, that

$$K_Z(x,y) := \sum_{k,\ell=-m+1}^{N} z_{k,\ell} N_{\mathbf{t};m,k}(x) N_{\mathbf{t};m,\ell}(y) = N_{\mathbf{t};m,\hat{k}}(x) s(y). \tag{5.60}$$

Now, the polynomial reproduction property of both kernels K_{S_L} and K_R implies that

$$0 = \int_a^b y^\nu K_Z(x,y)\, dy = N_{\mathbf{t};m,\hat{k}}(x) \int_a^b y^\nu s(y)\, dy, \qquad 0 \leq \nu \leq L-1, \tag{5.61}$$

so that the spline s must have L vanishing moments, which is a contradiction to the aforementioned variation-diminishing property.

The general case, where $t_{\hat{k}} = \ldots = t_{\hat{k}+\rho-1} < t_{\hat{k}+\rho}$ is a multiple knot, is treated similarly. For the evaluation of $K_Z(x,y)$ in (5.60), we substitute the one-sided partial derivative

$$\frac{\partial^{m-\rho}}{\partial x^{m-\rho}} K_Z(t_{\hat{k}}+, y) = \sum_{k,\ell=-m+1}^{N} z_{k,\ell} N_{\mathbf{t};m,k}^{(m-\rho)}(x) N_{\mathbf{t};m,\ell}(y) = N_{\mathbf{t};m,\hat{k}}^{(m-\rho)}(t_{\hat{k}}+) s(y).$$

Note that the value $N_{\mathbf{t};m,\hat{k}}^{(m-\rho)}(t_{\hat{k}}+)$ is nonzero, while $N_{\mathbf{t};m,k}^{(m-\rho)}(t_{\hat{k}}+) = 0$ for all $k > \hat{k}$. The rest of the argument that involves the polynomial reproduction property remains the same. This completes the proof of the theorem. ■

5.7. Boundedness of the kernel K_{S_L}

Let $\mathbf{t}$ be a knot vector as in the previous section (see (4.1)–(4.3)). Then the kernel K_{S_L} in (5.16), with the positive definite matrix $S_L = S_L(\mathbf{t})$ in (5.15), has the form

$$K_{S_L} = \sum_{\nu=0}^{L-1} K_{\mathbf{t};m}^{(\nu)},$$

where

$$K_{\mathbf{t};m}^{(\nu)}(x,y) := \frac{\partial^{2\nu}}{\partial x^\nu \partial y^\nu} \sum_{k=-m+1}^{N-\nu} \frac{m+\nu}{t_{k+m+\nu}-t_k}\, \beta_{m,k}^{(\nu)}(\mathbf{t}) N_{\mathbf{t};m+\nu,k}(x) N_{\mathbf{t};m+\nu,k}(y). \tag{5.62}$$

The next result shows that K_{S_L} satisfies the estimate (2.16) with an absolute constant C_m that does not depend on the knot vector or the interval I.

Theorem 8. *The kernels $K_{\mathbf{t};m}^{(\nu)}$ satisfy $K_{\mathbf{t};m}^{(0)} \geq 0$,*

$$\int_I K_{\mathbf{t};m}^{(\nu)}(x,y)\, dy = \delta_\nu, \qquad x \in I, \tag{5.63}$$

and

$$\int_I |K_{\mathbf{t};m}^{(\nu)}(x,y)|\, dy \leq \frac{2^\nu (m+\nu-1)!}{\nu!(m-1)!}, \qquad x \in I,\ 0 \leq \nu \leq L-1. \tag{5.64}$$

Proof: Recall that $\beta_{m,k}^{(0)}(\mathbf{t}) = 1$ (see (5.11)). The first relation $K_{\mathbf{t};m}^{(0)} \geq 0$ is obvious, and the identity (5.63) for $\nu = 0$ readily follows from

$$\int_I N_{\mathbf{t};m,k}(y)\, dy = \frac{t_{k+m}-t_k}{m}$$

and the partition of unity, see (4.10). Identity (5.63) for $\nu \geq 1$ follows from

$$\int_I \frac{d^\nu}{dy^\nu} N_{\mathbf{t};m+\nu,k}(y)\, dy = 0,$$

since every B-spline $N_{\mathbf{t};m+\nu,k}$ has compact support in I and vanishes at both endpoints of I.

Next we introduce the auxiliary kernels

$$\kappa_\nu(x,y) = \sum_{k=-m+1}^{N-\nu} \frac{m+\nu}{t_{k+m+\nu}-t_k}\ \beta_{m,k}^{(\nu)}(\mathbf{t}) \left|\frac{d^\nu}{dx^\nu} N_{\mathbf{t};m+\nu,k}(x)\right| \ \left|\frac{d^\nu}{dy^\nu} N_{\mathbf{t};m+\nu,k}(y)\right|. \tag{5.65}$$

Clearly, we have that $\kappa_0 = K_{\mathbf{t};m}^{(0)}$ and $\kappa_\nu \geq |K_{\mathbf{t};m}^{(\nu)}|$ for $\nu \geq 1$. The upper estimate in (5.7), with $r = m+\nu-1$, leads to

$$\begin{aligned}\kappa_\nu(x,y) \leq \frac{2^{-\nu} m!}{\nu!(m+\nu-1)!} \sum_{k=-m+1}^{N-\nu} \frac{y_{\max}(\nu; t_{k+1},\ldots,t_{k+m+\nu-1})}{t_{k+m+\nu}-t_k} \times \\ \left|\frac{d^\nu}{dx^\nu} N_{\mathbf{t};m+\nu,k}(x)\right| \ \left|\frac{d^\nu}{dy^\nu} N_{\mathbf{t};m+\nu,k}(y)\right|.\end{aligned} \tag{5.66}$$

where

$$y_{\max}(\nu; t_{k+1},\ldots,t_{k+m+\nu-1}) = (t_{k+m+\nu-1}-t_{k+\nu})^2 (t_{k+m+\nu-2}-t_{k+\nu-1})^2 \cdots (t_{k+m}-t_{k+1})^2$$

is defined in (5.10). The differentiation formula (4.12) can be applied recursively, in order to generate the central inequalities

$$\sqrt{y_{\max}(\nu; x_{k+1},\ldots,x_{k+m+\nu-1})} \left|\frac{d^\nu}{dx^\nu} N_{\mathbf{t};m+\nu,k}(x)\right| \leq \frac{(m+\nu-1)!}{(m-1)!} \sum_{i=0}^{\nu} \binom{\nu}{i} N_{\mathbf{t};m,k+i}(x),$$

$$\frac{\sqrt{y_{\max}(\nu; x_{k+1},\ldots,x_{k+m+\nu-1})}}{t_{k+m+\nu}-t_k} \left|\frac{d^\nu}{dy^\nu} N_{\mathbf{t};m+\nu,k}(y)\right| \leq \frac{(m+\nu-1)!}{(m-1)!} \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{N_{\mathbf{t};m,k+i}(y)}{t_{k+i+m}-t_{k+i}}.$$

The last sum, by (4.5), has the integral

$$\int_I \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{N_{\mathbf{t};m,k+i}(y)}{t_{k+i+m}-t_{k+i}}\ dy = \frac{2^\nu}{m}.$$

Applying this result and making use of the partition of unity relation, we obtain

$$\int_I \kappa_\nu(x,y)\,dy \leq \frac{(m+\nu-1)!}{\nu!(m-1)!} \sum_{k=-m+1}^{N-\nu} \sum_{i=0}^{\nu} \binom{\nu}{i} N_{\mathbf{t};m,k+i}(x) \leq \frac{2^\nu (m+\nu-1)!}{\nu!(m-1)!}.$$

This establishes the uniform bound on the kernel $K_{\mathbf{t},m}^{(\nu)}$ in (5.64). We have thus completed the proof of the theorem. ■

In summary, we see that the integral kernel K_{S_L} satisfies the estimate (2.16), where the constant

$$C := \sum_{\nu=0}^{L-1} 2^{\nu} \binom{m+\nu-1}{\nu}$$

does not depend on $\mathbf{t}$ or I.

Remark 6. The existence and uniqueness of minimally supported approximate duals of B-splines (see Section 5.6) was proven for all odd $1 \leq L \leq m$ by Sablonnière and Sbibih [30; Theorem 1]. The explicit representation of the approximate dual was only found for the cases $L = 1$ and $(L = 3,\ m \leq 4)$ in [30], where a much more complicated formulation is given. Our results in this section, among others, provide the explicit formulation of $S_L(\mathbf{t})$ for all m and L. Moreover, the conjecture that an upper bound in (2.16) exists, which does not depend on the knot vector and the length of the interval (proven only for $L = 3$ and $m = 3, 4$ in [30]) is a direct consequence of Theorem 8.

6. Construction of Tight Frames of Spline-Wavelets and Study of Their Supports

The results in Theorems 5–8 can be integrated into the general results on tight frames described in Section 2 as follows. Let $\mathbf{t}_j$, $j \geq 0$, be a nested sequence of knot vectors, such that (4.2)–(4.3) are satisfied and that the maximal knot spacings

$$h(\mathbf{t}_j) := \max_k \{t_{k+1}^{(j)} - t_k^{(j)}\} \tag{6.1}$$

converge to zero. Also, as before, let the B-splines $N_{j;m,k}$ with knots given by $\mathbf{t}_j$ provide the bases of the MRA spline spaces V_j of $L_2(I)$.

As a consequence of Theorem 8, the uniform boundedness of the kernel $K_{S_L(\mathbf{t}_j)}$ leads to the following result.

Theorem 9. *Let $1 \leq L \leq m$, $\mathbf{t}_0 \subset \mathbf{t}_1 \subset \cdots$ be knot vectors with $h(\mathbf{t}_j)$ tending to zero, and $S_L(\mathbf{t}_j)$ be the matrix in (5.15). Then the quadratic forms*

$$T_j f := [\langle f, N_{j;m,k}\rangle]_{k\in\mathbb{M}_j}\ S_L(\mathbf{t}_j)\ [\langle f, N_{j;m,k}\rangle]_{k\in\mathbb{M}_j}$$

are uniformly bounded on $L_2(I)$, and

$$\lim_{j\to\infty} T_j f = \|f\|^2, \qquad f \in L_2(a,b).$$

For the reasoning that leads to this result, see (2.16)–(2.18). Theorem 9 explains that the condition (i) in Theorem 1 is always satisfied, if we choose the matrix $S_L(\mathbf{t}_j)$ in (5.15) to formulate the minimally supported approximate duals of order L of the B-spline bases.

Next, we observe that Theorem 6 already provides for the construction of the matrices Q_j that defines the tight frame

$$\Psi_j := \Phi_{\mathbf{t}_{j+1};m} Q_j$$

of $L_2(I)$ relative to T_0, consisting of wavelets $\psi_{j,k}$ with L vanishing moments.

Theorem 10. *Under the same assumptions as in Theorem 9, there is a factorization*

$$S_L(\mathbf{t}_{j+1}) - P_{\mathbf{t}_j,\mathbf{t}_{j+1};m} S_L(\mathbf{t}_j) P^T_{\mathbf{t}_j,\mathbf{t}_{j+1};m} = (E_{\mathbf{t}_{j+1};m,L}\widehat{Q}_j)(E_{\mathbf{t}_{j+1};m,L}\widehat{Q}_j)^T = Q_j Q_j^T \qquad (6.2)$$

where $Q_j = E_{\mathbf{t}_{j+1};m,L}\widehat{Q}_j$ is of dimension $(N_{j+1}+m)\times(N_{j+1}+m-L)$. The families $\Psi_j := \Phi_{\mathbf{t}_{j+1};m}Q_j$, $j \ge 0$, of cardinality $(N_{j+1}+m-L)$, constitute a tight frame of $L_2(I)$ relative to T_0, such that all the wavelets $\psi_{j,k} \in \Psi_j$, $j \ge 0$, have L vanishing moments.

Proof: The Cholesky factorization of the matrix $Z_L = Z_L(\mathbf{t}_j, \mathbf{t}_{j+1}) =: \hat{Q}_j \hat{Q}_j^T$ in (5.32), with lower triangular matrix $\hat{Q}_j$, provides the factorization in (6.2). ■

The sparsity of the matrices $Q_j = [q_{i,k}^{(j)}]$ in (6.2) determines the support of the tight frame spline-wavelets. The length of the support of $\psi_{j,k}$ is easily determined by counting the number of consecutive B-splines in its representation

$$\psi_{j,k} = \sum_{i=u_k(Q_j)}^{\ell_k(Q_j)} q_{i,k}^{(j)} N_{j+1;m,i},$$

where the sequences $\ell_k(C)$ and $u_k(C)$, for a sparse rectangular matrix C, denote the lower and upper profiles of nonzero entries, namely: $u_k(C)$ is the row index of the first nonzero entry in the k-th column of C, and $\ell_k(C)$ is the index of the last nonzero entry of that column. (In our applications, we also assume that both sequences are (weakly) increasing and ignore columns of all zeros.) It is well known that the Cholesky decomposition $C = LL^T$ of an spsd matrix C defines a lower triangular matrix L whose lower profile $\ell_k(L)$ is bounded from above by the least monotonic majorant of the lower profile $\ell_k(C)$, due to the "fill-in" of Gaussian elimination, and whose upper profile is $u_k(L) = k$. For later use, we also define the right profile $r_i(C)$ of C, which gives the column index of the last nonzero entry of the i-th row. Note that

$$u_{r_i(C)}(C) \leq i \leq \ell_{r_i(C)}(C) \tag{6.3}$$

holds for all row indices i of C.

In order to determine the lower and upper profiles of the matrix Q_j in (6.2), we make use of the sparsity pattern of the matrix $P_j := P_{\mathbf{t}_j,\mathbf{t}_{j+1};m}$ in the refinement equation (4.21) relative to the knot vectors $\mathbf{t}_j \subset \mathbf{t}_{j+1}$. It turns out that

$$\begin{aligned} \mu_k &:= u_k(P_j), \\ \lambda_k &:= \ell_k(P_j), \end{aligned} \qquad -m+1 \leq k \leq N_j,$$

satisfy the relation

$$\{t_k^{(j)}, \ldots, t_{k+m+L-1}^{(j)}\} \subset \{t_{\mu_k}^{(j+1)}, \ldots, t_{\lambda_{k+L-1}+m}^{(j+1)}\}, \tag{6.4}$$

where the subset notation is to be understood in the sense of ordered sets and $\lambda_{k+L-1} + m$ and μ_k are minimum and maximum numbers, respectively, for the inclusion relation (6.4). In other words, the B-splines $N_{j+1;m,i} \in V_{j+1}$, with $\mu_k \leq i \leq \lambda_{k+L-1}$, are the only ones needed for the representation of the subfamily $N_{j;m,k}, \ldots, N_{j;m,k+L-1}$ in the refinement relation (2.2). Therefore, by counting the number of relevant B-splines in V_{j+1}, we obtain

$$\lambda_{k+L-1} - \mu_k + 1 = L + \#((\mathbf{t}_{j+1} \setminus \mathbf{t}_j) \cap (t_k^{(j)}, t_{k+m+L-1}^{(j)})), \tag{6.5}$$

which is L plus the number of new knots in the open interval $(t_k^{(j)}, t_{k+m+L-1}^{(j)})$. Moreover, with

$$\rho_k := r_k(P_j), \qquad -m+1 \leq k \leq N_{j+1}, \tag{6.6}$$

it follows from (6.3) and (6.5) that

$$\lambda_{\rho_k+L-1} - k + 1 \le \lambda_{\rho_k+L-1} - \mu_{\rho_k} + 1 \le L + \#((\mathbf{t}_{j+1} \setminus \mathbf{t}_j) \cap (t_{\rho_k}^{(j)}, t_{\rho_k+m+L-1}^{(j)})). \tag{6.7}$$

This provides the necessary notation and background for the following result.

Theorem 11. *Let $\mathbf{t}_j \subset \mathbf{t}_{j+1}$ be two nested knot vectors and $Q_j = E_{\mathbf{t};m,L}\widehat{Q}_j$ be as in Theorem 10, where $\hat{Q}_j$ is the lower triangular Cholesky factor of $Z_L(\mathbf{t}_j, \mathbf{t}_{j+1})$. Then the upper and lower profiles of Q_j are given by*

$$u_k(Q_j) \ge k, \quad \ell_k(Q_j) \le \lambda_{\rho_k+L-1}, \qquad -m+1 \le k \le N_{j+1} - L, \tag{6.8}$$

where $\lambda_k = \ell_k(P_{\mathbf{t}_j,\mathbf{t}_{j+1};m})$ and $\rho_k = r_k(P_{\mathbf{t}_j,\mathbf{t}_{j+1};m})$. Furthermore, the number of nonzero coefficients in the k-th column is bounded by the expression on the right-hand side of (6.7), and the wavelet $\psi_{j,k}$ is a spline in V_{j+1} with support contained in $[t_k^{(j+1)}, t_{\rho_k+m+L-1}^{(j)}]$.

Proof: First we recall from Section 5.3 that $S_j := S_L(\mathbf{t}_j)$ is a banded matrix with upper and lower bandwidth L; *i.e.*

$$u_k(S_j) = \max\{1-m, k-L+1\}, \quad \ell_k(S_j) = \min\{N_j, k+L-1\}, \qquad 1-m \le k \le N_j.$$

Since S_j is positive definite, the Cholesky factorization $S_j = CC^T$ exists, where C is a lower triangular matrix with lower bandwidth L. Therefore, the upper and lower profiles of the product $D := P_{\mathbf{t}_j,\mathbf{t}_{j+1};m}C$ are bounded by

$$u_k(D) \ge \mu_k, \quad \ell_k(D) \le \lambda_{k+L-1}, \qquad -m+1 \le k \le N_j,$$

where the numbers μ_k and λ_k are defined in (6.3). We denote the rows of D by $\mathbf{d}_i$, $-m+1 \le i \le N_{j+1}$, and observe that $r_i(D) = r_i(P_{\mathbf{t}_j,\mathbf{t}_{j+1};m}) = \rho_i$, as in (6.6).

The matrix on the left-hand side of (6.2) is $F := S_{j+1} - DD^T$. The pattern of nonzero entries of S_{j+1} is a subpattern of the corresponding pattern of DD^T. Therefore, it suffices to find bounds for the upper and lower profiles of DD^T. Nonzero entries of this matrix occur only when rows $\mathbf{d}_i$ and $\mathbf{d}_{\hat{i}}$ of D have an overlapping pattern of nonzero elements. By the symmetry of DD^T, we can restrict our attention to its upper triangular part. For the row indices $-m+1 \le i \le N_{j+1}$, we have $\mathbf{d}_i \cdot \mathbf{d}_{\hat{i}} = 0$ if $\hat{i} > \lambda_{\rho_i+L-1}$, since $\mathbf{d}_i$ has zeros in all columns $\rho_i < k \le N_j$ and $\mathbf{d}_{\hat{i}}$ has zeros in all columns $-m+1 \le k \le \rho_i$. Therefore, the right and lower profiles of $F = S_{j+1} - DD^T$, by the symmetry of F, are bounded by

$$r_i(F),\ \ell_i(F) \le \lambda_{\rho_i+L-1}, \qquad -m+1 \le i \le N_{j+1}.$$

Consequently, the matrix $Z_j = Z_L(\mathbf{t}_j, \mathbf{t}_{j+1})$ in (5.32), after elimination of the L-th order differences $E_{\mathbf{t}_{j+1};m,L}$ and $E^T_{\mathbf{t}_{j+1};m,L}$, has reduced right and lower profiles

$$r_i(Z_j),\ \ell_i(Z_j) \le \lambda_{\rho_i+L-1} - L, \qquad -m+1 \le i \le N_{j+1} - L.$$

The Cholesky factor $\hat{Q}_j$ of Z_j is a lower triangular matrix with the same bound for its lower profile. Multiplication of $\hat{Q}_j$ by the matrix $E_{\mathbf{t}_{j+1};m,L}$ gives the matrix Q_j with upper and lower profiles

$$u_i(Q_j) \ge i, \quad \ell_i(Q_j) \le \lambda_{\rho_i+L-1}, \qquad -m+1 \le i \le N_{j+1} - L.$$

This completes the proof of Theorem 11. ■

7. Examples of Tight Frames of Spline-Wavelets

In this section, we demonstrate our results in Sections 5 and 6 by including examples on linear and cubic splines.

7.1. Piecewise linear tight frames

Let $(\mathbf{t}_j)_{j\geq 0}$ be a nested sequence of knot vectors with double knots at a and b and meshsizes $h(\mathbf{t}_j)$ tending to zero. Here, we consider piecewise linear spline-wavelets with 2 vanishing moments, so that $m = L = 2$. The matrices $S_2(\mathbf{t}_j)$ in (5.15) are tridiagonal matrices of dimension $N_j + 2$, and the diagonal matrices $U_0(\mathbf{t}_j)$ and $U_1(\mathbf{t}_j)$ in (5.14) have diagonal entries

$$
\begin{aligned}
u_{2,k}^{(0)}(\mathbf{t}_j) &= \frac{2}{t_{k+2}^{(j)} - t_k^{(j)}}, \qquad -1 \leq k \leq N_j,\\
u_{2,k}^{(1)}(\mathbf{t}_j) &= \frac{(t_{k+2}^{(j)} - t_{k+1}^{(j)})^2}{2(t_{k+3}^{(j)} - t_k^{(j)})}, \qquad -1 \leq k \leq N_j - 1.
\end{aligned}
$$

It is sufficient to describe the construction of the wavelet family $\Psi_0 = \{\psi_{0,k}\}$, since the families Ψ_j, $j \geq 1$, are constructed analogously. In the following, we develop an explicit formulation of the wavelets $\psi_{0,k}$ for the special case, where all interior knots are simple and one "new" knot is placed between two adjacent knots of $\mathbf{t}_0$; in other words, we assume that

$$
a = \underbrace{t_{-1}^{(1)}}_{=t_{-1}^{(0)}} = \underbrace{t_0^{(1)}}_{=t_0^{(0)}} < t_1^{(1)} < \underbrace{t_2^{(1)}}_{=t_1^{(0)}} < \cdots < \underbrace{t_{2N_0}^{(1)}}_{=t_{N_0}^{(0)}} < t_{2N_0+1}^{(1)} < \underbrace{t_{2N_0+2}^{(1)}}_{=t_{N_0+1}^{(0)}} = \underbrace{t_{2N_0+3}^{(1)}}_{=t_{N_0+2}^{(0)}} = b.
$$

For convenience, the superscript (1) of $t_k^{(1)}$ will be dropped from now on. In this case, the factorization

$$
S_2(\mathbf{t}_1) - P_{\mathbf{t}_0,\mathbf{t}_1;2} S_2(\mathbf{t}_0) P_{\mathbf{t}_0,\mathbf{t}_1;2}^T = E_{\mathbf{t}_1;2,2} Z_2 E_{\mathbf{t}_1;2,2}^T
$$

is obtained where $Z_2 = Z_2(\mathbf{t}_0, \mathbf{t}_1)$ is the symmetric matrix of dimension $N_1 = 2N_0 + 1$ in (5.32) which, in this special case, has bandwidth 3. Instead of a Cholesky factorization of

Z_2, here we choose a more economical factorization $Z_2 = \hat{Q}_0 \hat{Q}_0^T$, where

$$\hat{Q}_0 = R_1 \begin{bmatrix} t_3 - a & & & & & & & \\ t_4 - t_3 & 1 & t_1 - t_0 & & & & & \\ & & t_5 - t_1 & & & & & \\ & & t_6 - t_5 & 1 & t_3 - t_2 & & & \\ & & & & t_7 - t_3 & & & \\ & & & & t_8 - t_7 & \ddots & & \\ & & & & & & 1 & t_{2N_0-1} - t_{2N_0-2} \\ & & & & & & & b - t_{2N_0-1} \end{bmatrix} R_2 \tag{7.1}$$

and where R_1 and R_2 are diagonal matrices with diagonal entries (indexed from 1 to $2N_0+1$) given by

$$R_{1;k,k} = \frac{4}{t_{k+2} - t_{k-2}}, \qquad 1 \le k \le 2N_0 + 1,$$

$$R_{2;k,k} = \frac{(t_{k+1} - t_{k-1})\sqrt{(t_{k+3} - t_k)(t_k - t_{k-3})}}{12\sqrt{2(t_{k+3} - t_{k-3})}}, \quad k = 1, 3, \ldots, 2N_0 + 1,$$

and

$$R_{2;k,k} = \frac{1}{12\sqrt{2}}\Big((t_{k+2} - t_{k-1})(t_{k+1} - t_{k-2})\times$$
$$\big((t_k - t_{k-1})(t_k - t_{k-2})(t_{k+2} - t_{k+1}) + (t_{k+1} - t_k)(t_{k+2} - t_k)(t_{k-1} - t_{k-2})\big)\Big)^{1/2}$$

for all $k = 2, 4, \ldots, 2N_0$. Here, we let $t_{-2} := a$ and $t_{2N_0+4} := b$. The wavelet family Ψ_0 is then defined by the coefficient matrix

$$Q_0 := E_{\mathbf{t}_1;2,2} \hat{Q}_0 =: [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{2N_0+1}] \cdot R_2,$$

where R_2 is the diagonal matrix in (7.1) and the column vectors $\mathbf{q}_k$ of dimension $(2N_0 + 3)$ are given by

$$\mathbf{q}_1^T = \left[\tfrac{24}{(t_1-t_{-1})(t_2-t_{-1})},\ \tfrac{24(t_{-1}+t_0-t_2-t_4)}{(t_4-t_0)(t_2-t_0)(t_2-t_{-1})},\ \tfrac{24}{(t_4-t_1)(t_4-t_0)},\ \tfrac{24(t_4-t_3)}{(t_4-t_2)(t_4-t_1)(t_4-t_0)},\ \mathbf{0}_{2N_0-1}\right],$$

$$\mathbf{q}_{2N_0+1}^T = \Big[\mathbf{0}_{2N_0-1},$$
$$\tfrac{24(t_{2N_0-1}-t_{2N_0-2})}{(t_{2N_0}-t_{2N_0-2})(t_{2N_0+1}-t_{2N_0-2})(t_{2N_0+2}-t_{2N_0-2})},\ \tfrac{24}{(t_{2N_0+1}-t_{2N_0-2})(t_{2N_0+2}-t_{2N_0-2})},$$
$$\tfrac{24(t_{2N_0-2}+t_{2N_0}-t_{2N_0+2}-t_{2N_0+3})}{(t_{2N_0+3}-t_{2N_0})(t_{2N_0+2}-t_{2N_0})(t_{2N_0+2}-t_{2N_0-2})},\ \tfrac{24}{(t_{2N_0+3}-t_{2N_0+1})(t_{2N_0+3}-t_{2N_0})}\Big],$$

where the symbol $\mathbf{0}_\ell$ denotes the zero-vector of dimension ℓ, and, for $1 \le k \le N_0$,

$$\mathbf{q}_{2k}^T = \Big[\mathbf{0}_{2k-1}, \frac{24}{(t_{2k}-t_{2k-2})(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-2})}, \frac{24(t_{2k-2}+t_{2k-1}-t_{2k+1}-t_{2k+2})}{(t_{2k+1}-t_{2k-1})(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-1})(t_{2k+2}-t_{2k-2})}, \frac{24}{(t_{2k+2}-t_{2k})(t_{2k+2}-t_{2k-1})(t_{2k+2}-t_{2k-2})}, \mathbf{0}_{2N_0-2k+1}\Big],$$

while for $1 \le k \le N_0 - 1$,

$$\mathbf{q}_{2k+1}^T = \Big[\mathbf{0}_{2k-1}, \frac{24(t_{2k-1}-t_{2k-2})}{(t_{2k}-t_{2k-2})(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-2})}, \frac{24}{(t_{2k+1}-t_{2k-2})(t_{2k+2}-t_{2k-2})}, \frac{24(t_{2k-2}+t_{2k}-t_{2k+2}-t_{2k+4})}{(t_{2k+2}-t_{2k-2})(t_{2k+2}-t_{2k})(t_{2k+4}-t_{2k})}, \frac{24}{(t_{2k+4}-t_{2k+1})(t_{2k+4}-t_{2k})}, \frac{24(t_{2k+4}-t_{2k+3})}{(t_{2k+4}-t_{2k+2})(t_{2k+4}-t_{2k+1})(t_{2k+4}-t_{2k})}, \mathbf{0}_{2N_0-2k-1}\Big].$$

Up to multiplication by the diagonal entries of the matrix R_2, the vectors $\mathbf{q}_k$ represent the coefficient sequences of the wavelets $\psi_{0,k}$ for $1 \le k \le 2N_0 + 1$. Hence, we conclude that

- the wavelets $\psi_{0,2k}$, $1 \le k \le N_0$, have a 3-tap coefficient sequence, support $[t_{k-1}^{(0)}, t_{k+1}^{(0)}]$ and 5 simple knots $t_{k-1}^{(0)} = t_{2k-2}^{(1)} < t_{2k-1}^{(1)} < \cdots < t_{2k+2}^{(1)} = t_{k+1}^{(0)}$; up to their normalization, they are uniquely determined by their property of having two vanishing moments;

- the wavelets $\psi_{0,2k+1}$, $1 \le k \le N_0 - 1$, have a 5-tap coefficient sequence, support in $[t_{k-1}^{(0)}, t_{k+2}^{(0)}]$, and 7 simple knots $t_{k-1}^{(0)} = t_{2k-2}^{(1)} < \cdots < t_{2k+4}^{(1)} = t_{k+2}^{(0)}$; by inspecting the coefficient sequence $\mathbf{q}_{2k+1}$, we observe that the second and next-to-last knots of $\psi_{0,2k+1}$ are inactive, i.e. the wavelet is a linear polynomial in $[t_{2k-2}^{(1)}, t_{2k}^{(1)}]$ and $[t_{2k+2}^{(1)}, t_{2k+4}^{(1)}]$; under this constraint, and up to the normalization constant, the wavelets are uniquely determined by the property of having two vanishing moments;

- the boundary wavelet $\psi_{0,1}$ has a double knot at a and 4 simple knots $t_1^{(1)}, \ldots, t_4^{(1)}$; we also observe that $\psi_{0,1}$ is a linear polynomial in $[t_2^{(1)}, t_4^{(1)}]$ and thereby determined, up to the normalization, by the property of having two vanishing moments;

- the boundary wavelet $\psi_{0,2N_0+1}$ has a double knot at b, simple knots at $t^{(1)}_{2N_0-2}, \ldots, t^{(1)}_{2N_0+1}$ and is a linear polynomial in $[t^{(1)}_{2N_0-2}, t^{(1)}_{2N_0}]$; up to the normalization, it is uniquely determined by the property of having two vanishing moments.

In the special case, where the interior knots in $\mathbf{t}_0$ are equidistant (with stepsize h_0) and the new knots are placed in the middle of each knot interval, our construction leads to

$$Q_0 = \frac{1}{12\sqrt{h_0}} \left[\begin{array}{cccccccccc}
12\sqrt{3} & & & & & & & & & \\
-9\sqrt{3} & 6 & \sqrt{6} & & & & & & & \\
2\sqrt{3} & -12 & 2\sqrt{6} & & & & & & & \\
\sqrt{3} & 6 & -6\sqrt{6} & 6 & \sqrt{6} & & & & & \\
 & & 2\sqrt{6} & -12 & 2\sqrt{6} & & & & & \\
 & & \sqrt{6} & 6 & -6\sqrt{6} & & & & & \\
 & & & & 2\sqrt{6} & \ddots & & & & \\
 & & & & \sqrt{6} & & & & & \\
 & & & & & & 6 & \sqrt{6} & & \\
 & & & & & & -12 & 2\sqrt{6} & & \\
 & & & & & & 6 & -6\sqrt{6} & 6 & \sqrt{3} \\
 & & & & & & & 2\sqrt{6} & -12 & 2\sqrt{3} \\
 & & & & & & & \sqrt{6} & 6 & -9\sqrt{3} \\
 & & & & & & & & & 12\sqrt{3}
\end{array} \right].$$

The interior wavelets (with coefficient sequences in columns 2 to $2N_0$) are shifts (by integer multiples of h_0) of the two generators $\psi_{0,2}$ and $\psi_{0,3}$, namely

$$\psi_{0,2k+2}(x) = \psi_{0,2}(x - kh_0), \quad 1 \le k \le N_0 - 1,$$

$$\psi_{0,2k+3}(x) = \psi_{0,3}(x - kh_0), \quad 1 \le k \le N_0 - 2.$$

Moreover, all of these interior wavelets are symmetric. If we fix the stepsize $h_0 = 1$, then these generators are identical with the functions ψ^1 and ψ^2 that were constructed in the shift-invariant (i.e. stationary) setting for $L_2(\mathbb{R})$ in [7]. The current construction reveals that the adaptation to the bounded interval $[a, b]$ by assigning one boundary wavelet at each endpoint of the interval works successfully in this particular example. However, this does not apply to the general setting as will be discussed in Section 10.

7.2. Piecewise cubic tight frames with simple interior knots

For simplicity of the presentation, we restrict to the case of simple equidistant interior knots of stepsize $h_0 = 1$ in the interval $I = [0, N+1]$; *i.e.*,

$$\begin{aligned}
\mathbf{t}_0 &= \{0,0,0,0,1,2,\ldots,N,N+1,N+1,N+1,N+1\}, \\
\mathbf{t}_1 &= \{0,0,0,0,\frac{1}{2},1,\frac{3}{2},\ldots,N,N+\frac{1}{2},N+1,N+1,N+1,N+1\}.
\end{aligned} \tag{7.2}$$

The method described in Section 6, for $L = 4$ vanishing moments, employs the Cholesky factorization of the matrix $Z_4 = Z_4(\mathbf{t}_0, \mathbf{t}_1)$ in (5.32), which has dimension $N_1 := 2N+1$. This leads to the definition of N_1 non-symmetric wavelets with 4 vanishing moments. As in the previous subsection, we choose an alternative factorization method that we will describe in some detail. In particular, we will choose a larger number of wavelets, namely $3N - 14$ interior wavelets and 6 boundary wavelets for each endpoint, in order to obtain symmetry and shift-invariance at the same time for the interior wavelets. Moreover, the construction is scale-invariant, in that the same coefficient sequences (for interior and boundary wavelets in Ψ_j and with proper scaling by $2^{j/2}$) can be employed for all scales $j \geq 0$, if uniform refinement of the knot vector is used by inserting the midpoint between two adjacent knots in $\mathbf{t}_j$ for the definition of $\mathbf{t}_{j+1}$.

For the particular knot vector in (7.2), the diagonal matrices $U_\nu(\mathbf{t}_0)$, $0 \leq \nu \leq 3$, in (5.14), of dimension $(N+4-\nu) \times (N+4-\nu)$, are given by

$$\begin{aligned}
U_0 &= \mathrm{diag}(4, 2, \tfrac{4}{3}, 1, 1, \ldots, 1, \tfrac{4}{3}, 2, 4), \\
U_1 &= \tfrac{1}{3}\mathrm{diag}(\tfrac{3}{8}, \tfrac{11}{12}, \tfrac{5}{4}, 1, 1, \ldots, 1, \tfrac{5}{4}, \tfrac{11}{12}, \tfrac{3}{8}), \\
U_2 &= \tfrac{31}{360}\mathrm{diag}(\tfrac{24}{155}, \tfrac{45}{62}, \tfrac{6}{5}, 1, 1, \ldots, 1, \tfrac{6}{5}, \tfrac{45}{62}, \tfrac{24}{155}), \\
U_3 &= \tfrac{311}{15120}\mathrm{diag}(\tfrac{189}{1555}, \tfrac{1092}{1555}, \tfrac{7}{6}, 1, 1, \ldots, 1, \tfrac{7}{6}, \tfrac{1092}{1555}, \tfrac{189}{1555}).
\end{aligned}$$

Note that with the exception of 3 values in each of the upper and lower corners, the diagonal entries in $U_0, \ldots, U_3$ are constants. The matrices $U_\nu(\mathbf{t}_1)$ are of larger size, but have the same diagonal entries, up to multiplication by $2^{1-2\nu}$, $0 \leq \nu \leq 3$. The matrix Z_4, of dimension

$N_1 \times N_1$ in (5.32), is positive definite and has bandwidth 7. In order to find an economical factorization of this matrix, two symmetric reductions

$$\tilde{Z}_4 = (I - K_2)(I - K_1)Z_4(I - K_1^T)(I - K_2^T), \qquad \text{with } I = I_{N_1},$$

are performed in order to obtain the matrix $\tilde{Z}_4$ with bandwidth 3. Here, K_1 and K_2 are tridiagonal matrices with zero main diagonal; K_1 has nonzero entries $1/8$ in the upper and lower diagonal of rows $4, 6, \ldots, N_1 - 3$, and K_2 has nonzero entries $2/5$ in the upper and lower diagonals of rows $5, 7, \ldots, N_1 - 4$ and $1/6$ in the upper (resp. lower) diagonal of row 3 (resp. $N_1 - 2$).

Our initial attempt for a factorization of $\tilde{Z}_4$ by using the same interior wavelets as for a tight frame of $L^2(\mathbb{R})$ with 2 generators, as given in [7,18], failed. More details are given in Section 10.2. Instead, we find a new factorization $\tilde{Z}_4 = BB^T$, with $B = [B_\ell, B_i, B_r]$, where B_i is an $N_1 \times (3N - 14)$ block given by

$$B_i = \begin{bmatrix}
0 & & & & & & & & & \\
0 & & & & & & & & & \\
0 & & & & & & & & & \\
0 & & & & & & & & & \\
a & & & & & & & & & \\
b & & d & & & & & & & \\
a & c & e & a & & & & & & \\
 & & d & b & & d & & & & \\
 & & & a & c & e & & & & \\
 & & & & & d & & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & & & d & \\
 & & & & & & & c & e & a \\
 & & & & & & & & d & b \\
 & & & & & & & & & a \\
 & & & & & & & & & 0 \\
 & & & & & & & & & 0 \\
 & & & & & & & & & 0 \\
 & & & & & & & & & 0
\end{bmatrix}$$

with

$$a = \sqrt{45248/125}/r, \quad b = \sqrt{187152/5}/r, \quad c = \sqrt{263168/5 - e^2}/r$$
$$d = \sqrt{24880}/r, \quad e = (773536/25 - ab)/(dr), \quad r = 1536\sqrt{21}.$$

This block gives rise to the interior wavelets. Each of B_ℓ and B_r consists of 6 columns and gives rise to the boundary wavelets. The above formulation depicts the symmetry and shift-invariance of the interior wavelets. The columns of the matrix

$$Q_0 = E_{\mathbf{t}_1;4,4}(I+K_1)(I+K_2)B$$

constitute the coefficients of all the wavelets

$$\psi_{0,k} = \sum_{s=-m+1}^{N_1} q_{k,s} N_{\mathbf{t}_1,m;s}, \quad 1 \le k \le 3N-2,$$

in Ψ_0 in their B-spline expansions in terms of the B-spline basis $\Phi_{\mathbf{t}_1,m}$. Another representation can be formulated by using the column vectors of $\widehat{Q}_0 = (I+K_1)(I+K_2)B$ as the coefficients of $\psi_{0,k}$ in the expansion

$$\psi_{0,k} = \sum_{s=-3}^{N_1-4} \hat{q}_{k,s} \frac{d^4}{dx^4} N_{\mathbf{t}_1,8;s} \tag{7.3}$$

with respect to 4-th order derivatives of the corresponding B-splines of order 8. The coefficients in this latter expansion are given in Tables 1 and 2, where Table 1 lists the coefficients $\hat{q}_k^{(i)}$ of the 3 generators ψ^i for the interior wavelets. From this information, it is clear that the supports of ψ^1, ψ^2, ψ^3 are

$$\operatorname{supp}\psi^1 = [0,6], \quad \operatorname{supp}\psi^2 = [1,6], \quad \operatorname{supp}\psi^3 = [0,7].$$

All of the $3N-14$ interior wavelets are given by

$$\psi^i(\cdot - k), \qquad i = 1,2,3, \quad 0 \le k \le N-6, \qquad \psi^1(\cdot - N + 5).$$

The graphs of ψ^i, $i = 1,2,3$, are shown in Figure 1. Table 2 lists the coefficients in (7.3) of the 6 boundary wavelets for the left endpoint of the interval. The first three of these functions have a knot of multiplicity 4 at zero and their supports are $[0,2.5]$, $[0,3]$, $[0,4]$, respectively. The fourth boundary wavelet has a triple knot at 0 and its support is $[0,5]$.

i	$\hat{q}_{k,0}$	$\hat{q}_{k,1}$	$\hat{q}_{k,2}$	$\hat{q}_{k,3}$	$\hat{q}_{k,4}$	$\hat{q}_{k,5}$	$\hat{q}_{k,6}$
1	0.171217	1.369738	3.091033	1.369738	0.171217		
2			0.267942	2.143537	0.267942		
3	0.112045	0.896364	2.883961	4.248047	2.883961	0.896364	0.112045

Table 1. Coefficients (∗100) of interior wavelets $\psi^i = \psi_{0,6+i}$ in expansion (7.3).

k	$\hat{q}_{k,-3}$	$\hat{q}_{k,-2}$	$\hat{q}_{k,-2}$	$\hat{q}_{k,0}$	$\hat{q}_{k,0}$	$\hat{q}_{k,2}$	$\hat{q}_{k,2}$	$\hat{q}_{k,4}$
1	0.468951							
2	0.208884	1.193513						
3	0.046733	0.588400	2.238939	0.279867				
4		0.217826	1.577574	3.111291	1.110131	0.138766		
5			0.051599	0.511502	2.242497	0.280312		
6			0.245479	1.950235	3.573393	2.594618	0.818264	0.102283

Table 2. Coefficients (∗100) of boundary wavelets ψ_k in expansion (7.3).

The last two boundary wavelets have a double knot at 0 and their supports are $[0,5]$, $[0,6]$, respectively. The reflection of these functions yields the 6 boundary wavelets at the other endpoint $N+1$. The graphs of the boundary wavelets for the left endpoint are shown in Figure 2.

Remark 7. The three generators ψ^i, $i = 1,2,3$, in the previous example also generate a tight frame $\{\psi_{j,k} := 2^{j/2}\psi^i(2^j \cdot -k);\ j,k \in \mathbf{Z}\}$ of $L^2(\mathbb{R})$. This construction yields three symmetric generators with 4 vanishing moments and coefficient sequences (in terms of the B-spline basis $\Phi_{\mathbf{t}_1;4}$) of 7, 9, and 11 nonzero coefficients, respectively. This underlines the fact that our general method is also useful for constructing tight frames in the shift-invariant setting discussed in [7] as well as symmetric ones as in [21], but with smaller support and the same order of vanishing moments. It is also worthwhile to observe that the constant diagonal entries of U_ν appear in the shift-invariant setting as the coefficients of VMR Laurent polynomials in [7].

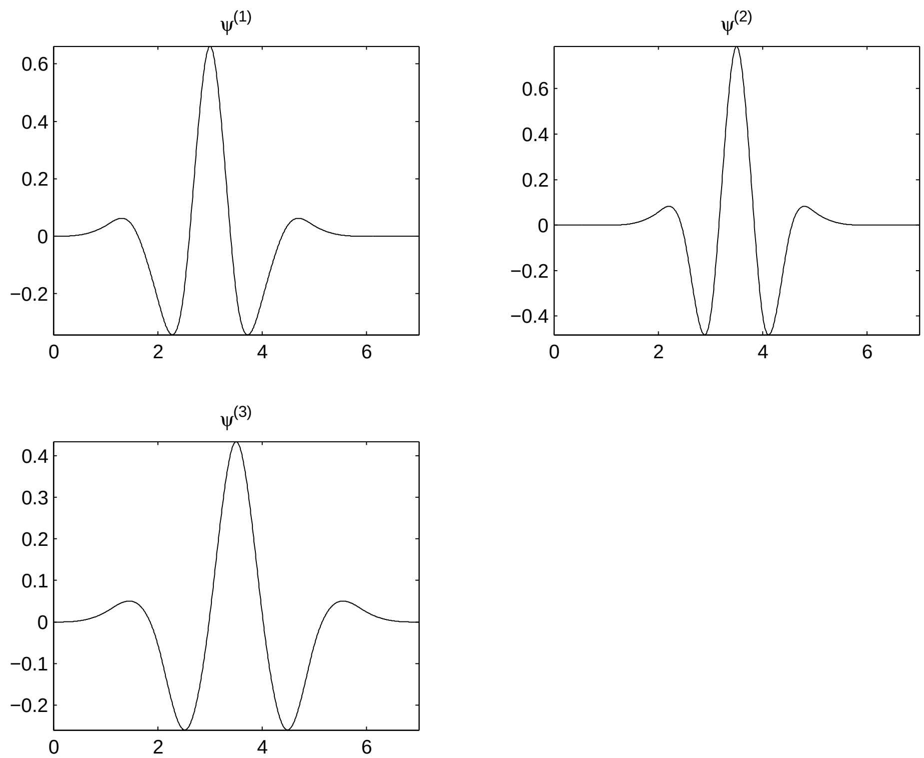

Figure 1. Generators of interior wavelets of piecewise cubic tight frame with simple interior knots.

7.3. Piecewise cubic tight frames with double knots

We assume as in Section 7.2 that $[a, b] = [0, N+1]$, where N is an integer, so that V_0 is the space of all splines of order 4 and with knot vector

$$\mathbf{t}_0 = \{0, 0, 0, 0, 1, 1, 2, 2, \ldots, N, N, N+1, N+1, N+1, N+1\},$$

and $\mathbf{t}_1$ is the refinement with double knots at the half integers. Note that the dimension of V_0 is $2N+4$ and the dimension of V_1 is $4N+6$. Instead of the generic Cholesky factorization of the matrix $Z_4 = Z_4(\mathbf{t}_0, \mathbf{t}_1)$ in (5.32), we describe next an alternate factorization that defines symmetric/anti-symmetric interior wavelets that are shifts of 5 functions $\psi^i \in V_1$, $1 \le i \le 5$. At each interval endpoint, we define 7 boundary wavelets.

The construction is described by the following procedure. First, we compute the diag-

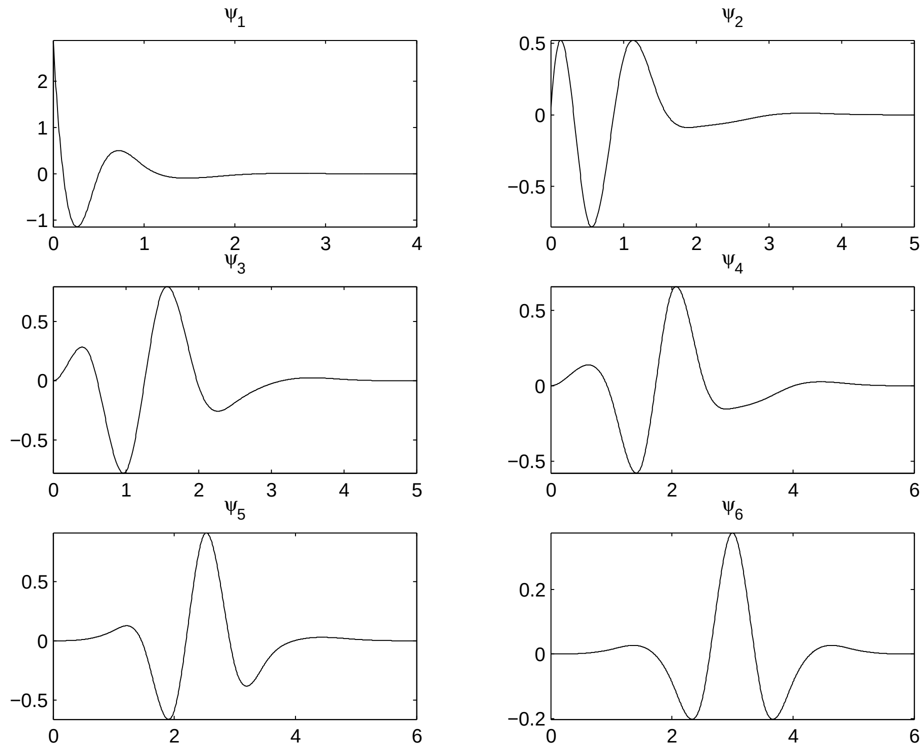

Figure 2. Boundary wavelets of piecewise cubic tight frame with simple interior knots.

onal matrices in (5.14), namely

$$U_0(\mathbf{t}_0) = \mathrm{diag}(4, 4, 2, 2, \ldots, 2, 4, 4),$$

$$U_1(\mathbf{t}_0) = \tfrac{1}{9}\mathrm{diag}(\tfrac{9}{4}, \tfrac{3}{2}, 3, 1, 3, 1, \ldots, 3, 1, 3, \tfrac{3}{2}, \tfrac{9}{4}),$$

$$U_2(\mathbf{t}_0) = \tfrac{11}{900}\mathrm{diag}(\tfrac{9}{22}, \tfrac{3}{2}, 1, 1, \ldots, 1, \tfrac{3}{2}, \tfrac{9}{22}),$$

$$U_3(\mathbf{t}_0) = \tfrac{1}{2700}\mathrm{diag}(\tfrac{3}{2}, \tfrac{4}{3}, \tfrac{43}{12}, 1, \tfrac{43}{12}, 1, \ldots, \tfrac{43}{12}, 1, \tfrac{43}{12}, \tfrac{4}{3}, \tfrac{3}{2}).$$

The matrices $U_\nu(\mathbf{t}_1)$ are of larger size and have the same diagonal entries up to multiplication by $2^{1-2\nu}$, $0 \le \nu \le 3$. The matrix Z_4 in (5.32) has dimension $N_1 \times N_1$, is positive definite, and has bandwidth 8. Similar to the case of simple knots as discussed above, three symmetric reductions

$$\tilde{Z}_4 = (I - K_3)(I - K_2)(I - K_1)Z_0(I - K_1^T)(I - K_2^T)(I - K_3^T), \qquad I = I_{N_1},$$

(with tridiagonal nilpotent matrices K_i) lead to a matrix $\tilde{Z}_4$ with bandwidth 4. The factorization of $\tilde{Z}_4$ leads to the definition of 7 boundary wavelets at each endpoint of the interval and 5 interior wavelet generators $\psi^i \in V_1$, $1 \le i \le 5$, with

$$\psi_{0,7+5k+i} = \psi^i(\cdot - k), \quad 1 \le i \le 5,\ 0 \le k \le N-4, \qquad \psi_{0,5N-8+i} = \psi^i(x-N+3), \quad i = 1, 2.$$

We give the coefficients of the representation

$$\psi^i = \sum_{s=0}^{8} \hat{q}_s^{(i)} \frac{d}{dx^4} N_{\mathbf{t}_1,8;s}, \quad 1 \le i \le 5, \tag{7.4}$$

in Table 3 and depict their graphs in Figure 3. Note that ψ^2, ψ^4, ψ^5 are symmetric and ψ^1, ψ^3 are antisymmetric. The supports of these generators are

$$\operatorname{supp} \psi^1 = \operatorname{supp} \psi^2 = [0,4], \quad \operatorname{supp} \psi^3 = \operatorname{supp} \psi^4 = \operatorname{supp} \psi^5 = [1,4].$$

The spline wavelets ψ^1, ψ^2 have simple knots at 0 and 4, and double knots at $.5, 1, \ldots, 3.5$, while ψ^3 and ψ^4 have double knots at $1, 1.5, \ldots, 4$. The spline wavelet ψ^5 has simple knots at $1, 4$ and double knots at $1.5, 2, \ldots, 3.5$. The total number of interior wavelets is $5N - 13$.

The coefficients $\hat{q}_{k,s}$ of the 7 boundary wavelets $\psi_{0,k}$, $1 \le k \le 7$, at the left endpoint are given in Table 4, and their graphs are shown in Figure 4. The boundary wavelets at the right endpoint are the mirror images of the wavelets on the left endpoint. We thus obtain a total of $5N + 1$ wavelets in V_1, all of which have four vanishing moments.

i	$\hat{q}_0^{(i)}$	$\hat{q}_1^{(i)}$	$\hat{q}_2^{(i)}$	$\hat{q}_3^{(i)}$	$\hat{q}_4^{(i)}$	$\hat{q}_5^{(i)}$	$\hat{q}_6^{(i)}$	$\hat{q}_7^{(i)}$	$\hat{q}_8^{(i)}$
1	0.092642	0.370569	1.852847	0.989527	−0.989527	−1.852847	−0.370569	−0.092642	
2	0.126349	0.505395	2.526977	3.156191	3.156191	2.526977	0.505395	0.126349	
3				0.526730	1.601752	0.086252	−0.086252	−1.601752	−0.526730
4				0.580480	2.180883	1.757771	1.757771	2.180883	0.580480
5					0.869741	3.478964	3.478964	0.869741	

Table 3. Coefficients (∗1000) of interior wavelets $\psi^i = \psi_{0,7+i}$ in expansion (7.4).

Remark 8. The consideration of splines with double knots at all integers leads to an MRA generated by two functions. Therefore, our consideration in this subsection can be viewed as a construction of tight frames of "multiwavelets" for the bounded interval. We remark that

k	$\hat{q}_{k,-3}$	$\hat{q}_{k,-2}$	$\hat{q}_{k,-1}$	$\hat{q}_{k,0}$	$\hat{q}_{k,1}$	$\hat{q}_{k,2}$	$\hat{q}_{k,3}$	$\hat{q}_{k,4}$
1	1.030983	1.417601	0.644364	0.096655				
2		1.964342	1.523281	0.719836	0.300617	0.060123	0.015031	
3			2.170762	1.104518	0.574380	0.134319	0.038137	0.001519
4			0.909528	3.566099	2.804337	1.352000	0.523422	0.061807
5				0.987016	3.948064	3.102908	1.320567	0.181613
6				0.100948	0.403790	2.018952	1.126572	0.207278
7							2.193554	0.731185

Table 4. Coefficients ($*1000$) of boundary wavelets ψ_k in expansion (7.4).

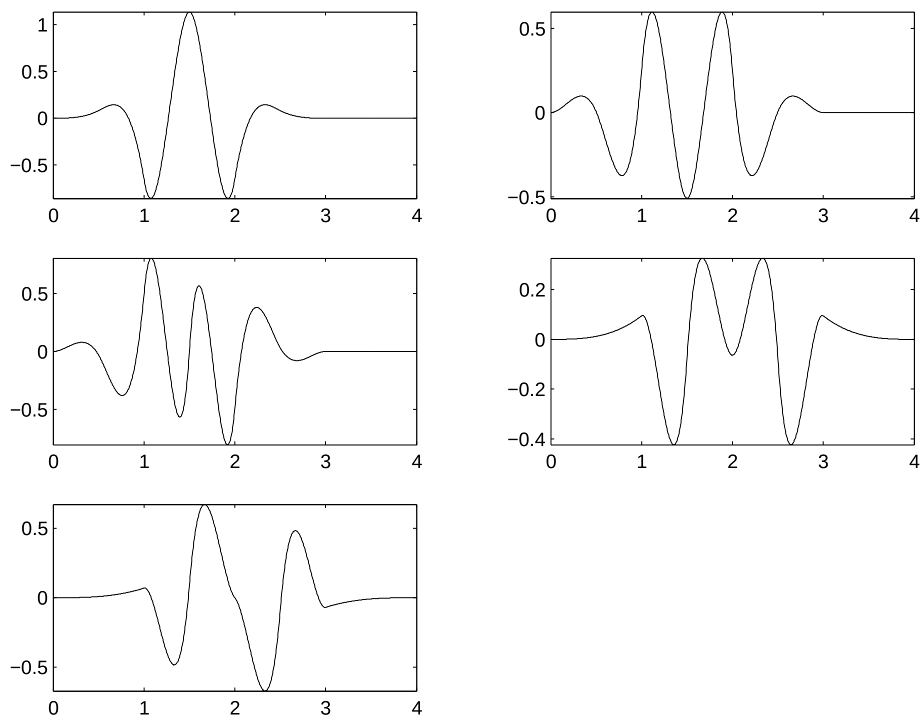

Figure 3. Generators of interior wavelets of piecewise cubic tight frame with double knots.

the Fourier transform approach for the study of tight multiwavelet frames in the shift- and dilation-invariant setting on $L_2(\mathbb{R})$ was recently given in [26], where the discussion is devoted to the study of existence and characterization. In particular, there are no examples of tight frames with higher vanishing moments by using the Fourier approach directly. On the other hand, by working in the time domain, the example of this subsection leads to a normalized tight frame of $L^2(\mathbb{R})$ with 5 generators each having 4 vanishing moments. Its reformulation

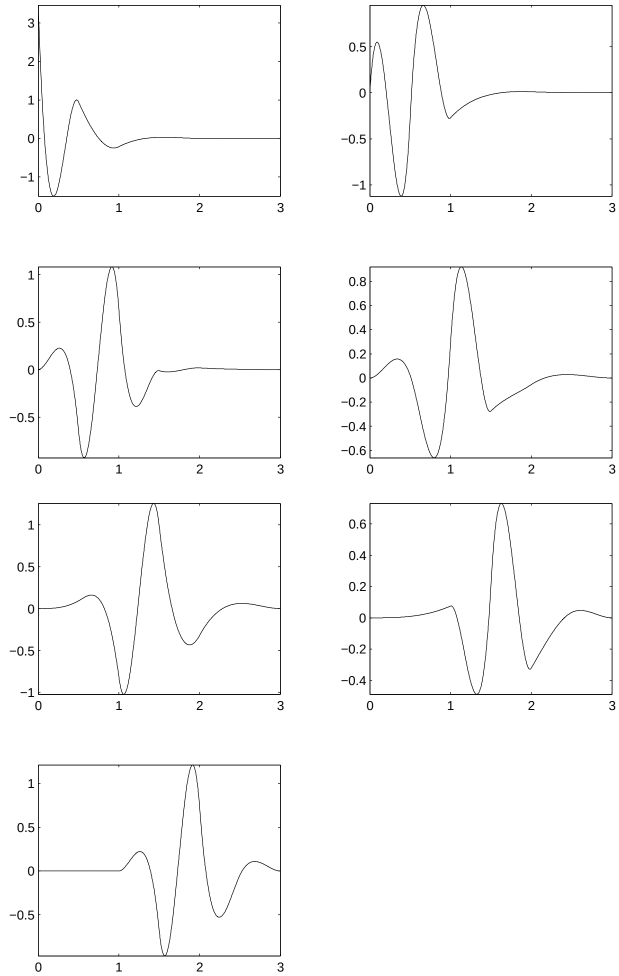

Figure 4. Boundary wavelets of piecewise cubic tight frame with double knots.

in the Fourier domain can be found in [11; Ex. 7.3.1]. Furthermore, the consideration in [26] of VMR Laurent polynomial matrices for the construction of "multiwavelet frames" with L vanishing moments, which are the Fourier analogue of our matrices S_L, is not suited to achieve minimally supported approximate duals, and there is no discussion in [26] of the positivity conditions (3.10)–(3.11) either.

In summary, as an application of our approach, we provide a unified framework for the construction of tight frames of spline-wavelets regardless of the multiplicity of the knots and the rule of knot insertion. In the specialized stationary setting, an approximate dual $\Phi_{\mathbf{Z};m}S_L$ is defined by a biinfinite block Toeplitz matrix S_L, whose analogue in the Fourier approach is a VMR Laurent polynomial matrix of size $r \times r$, where r denotes the (uniform) multiplicity of the equidistant knots. In general, the arbitrariness of the refinement allows for the use of scaling parameters $M > 2$ as in [9] at the same time as multiple knots can be considered. The advantage of our time-domain approach lies in the fact that techniques from matrix linear algebra replace some "ad-hoc" factorization techniques for Laurent polynomial matrices. Preliminary results and examples were given in [11], and the forthcoming paper [8] is devoted to frames of $L_2(\mathbb{R})$ and $L_2(0,\infty)$.

8. Matlab Program for Computing Approximate Duals

The computation of the positive definite matrix S_L that defines the minimally supported approximate dual of order L of the B-spline basis $\Phi_{\mathbf{t};m}$ is given in MATLAB syntax. The vector `knots` is the knot vector (with multiplicity m for the boundary knots while all other knots have multiplicities $\leq m$), where `m` is the order of the B-spline basis, and `mu` is the order L of the approximate dual.

```
function S = make_S(knots,m,mu)
   % compute approximate dual of order mu
   % for B-spline basis of order m
   % use Horner-like scheme for S
   S = make_U(knots,m,mu-1);
           % produce the diagonal matrix U_mu-1
   for nu=mu-2:-1:0
      D = make_D(knots,m+nu);
           % produce the  difference matrix D_knots;m+nu
      S  = D*S*D' + make_U(knots,m,nu);
   end

function U = make_U(knots,m,mu)
   % compute F_2 mu by means of centered moments
   % and normalize to give U_mu
   % currently only for mu=0,1,2,3
   N = length(knots)-m;
           % dimension of spline space
```

```
   temp_knots = knots(2:end - 1);
   udiag = (m+mu)./(knots(m+mu+1:end)-knots(1:end-m-mu));
   switch mu
    case 0,
        beta = ones(1,N);
    case 1,
        a=make_moment(temp_knots,2,m+mu-1);
        beta = (m*a)/((m+1)*(m-1));
    case 2,
        a=make_moment(temp_knots,2,m+mu-1);
        b=make_moment(temp_knots,4,m+mu-1);
        beta = ((m^2-m+1)*a.^2-m*b)/(2*(m+2)*m*(m-1)*(m-2));
    case 3,
        a=make_moment(temp_knots,2,m+mu-1);
        b=make_moment(temp_knots,3,m+mu-1);
        c=make_moment(temp_knots,4,m+mu-1);
        d=make_moment(temp_knots,6,m+mu-1);
        c1 = (m^2-3*m+5)*(m+2)/(6*(m+3)*(m+1)^2*(m-1)*(m-2)*(m-3));
        c2 = -(m^2-m+4)/(2*(m+3)*(m+1)^2*(m-1)*(m-2)*(m-3));
        c3 = -(3*m^2-3*m+2)/(3*(m+3)*(m+1)^2*m*(m-1)*(m-2)*(m-3));
        c4 = 1/(3*(m+3)*(m+1)*(m-1)*(m-2)*(m-3));
        beta = c1*a.^3 + c2*a.*c + c3*b.^2 + c4*d;
     end
    udiag=beta.*udiag;
    U=spdiags(udiag(:),[0],length(udiag),length(udiag));

function a = make_moment(knots,nu,k)
   % compute centered moments of degree nu
   % for all sets of k consecutive knots
   t=knots(:);
   lt=length(t);
   tmp=repmat(t,1,k);
   tmp=tmp(:);
   trep=zeros(lt+1,k);
   trep(:)=[tmp;zeros(k,1)];
   trep=trep';
       % now contains, in each column, k consecutive knots
   tstar=sum(trep)/k;
```

```
        % the mean value
    a=sum((trep-repmat(tstar,k,1)).^nu)/k;
        % the centered moment of degree nu
    a=a(1:lt-k+1);

function D = make_D(knots,order)
    % compute matrix D for derivatives
    % find diagonal entries first
    A = order./(knots(order+1:end)-knots(1:end-order));
    A = A(:);
    D = spdiags([A, -[A(2:end);0] ],[0,-1],length(A),length(A)-1);
```

9. Proofs of Theorem 4 and Corollary 2

Proof of Theorem 4: First, we show that all the three statements in Theorem 4 are equivalent. Let $C_{n,k}$ be defined as in (5.20). Clearly, $C_{n,k}$ is a polynomial of degree at most n. The equivalence of the last two statements of the theorem is a well-known fact about reproducing kernel Hilbert spaces. Furthermore, integration by parts gives

$$\int_0^1 B^0_{n,k}(x) C_{n,\ell}(x)\, dx = (n+1) \sum_{i=0}^{n} \frac{(n-i)!}{i!n!} \int_0^1 x^i (1-x)^i \frac{d^i}{dx^i} B^0_{n,k}(x) \frac{d^i}{dx^i} B^0_{n,\ell}(x)\, dx.$$

This shows that the first and second assertions of the theorem are also equivalent. Therefore, it is sufficient to prove that the kernel $K(x,y)$ in (5.21) satisfies

$$\int_0^1 x^\nu K(x,y)\, dx = y^\nu, \quad 0 \le \nu \le n,\ y \in [0,1]. \tag{9.1}$$

For this purpose, we let $0 \le \nu \le n$ and consider the integral

$$\int_0^1 x^\nu K(x,y)\, dx = \sum_{k=0}^{n} B^0_{n,k}(y) \int_0^1 x^\nu C_{n,k}(x)\, dx. \tag{9.2}$$

Integration by parts leads to

$$\int_0^1 x^\nu C_{n,k}(x)\, dx = (n+1) \sum_{i=0}^{\nu} \frac{(n-i)!}{n!} \binom{\nu}{i} \int_0^1 x^\nu (1-x)^i \frac{d^i}{dx^i} B^0_{n,k}(x)\, dx. \tag{9.3}$$

The well-known relation for derivatives of the Bernstein polynomials gives

$$\frac{d^i}{dx^i} B^0_{n,k} = \frac{n!}{(n-i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} B^0_{n-i,k-j},$$

where, as usual, we set $B^0_{r,s} := 0$ for integers r, s with $s < 0$ or $s > r$. Similarly, we use the standard notation for binomial coefficients $\binom{r}{s} = 0$ for $s < 0$ or $s > r$. These notations help us in rearranging the sums in order to obtain

$$\begin{aligned} &\sum_{i=0}^{\nu} \frac{(n-i)!}{n!} \binom{\nu}{i} x^\nu (1-x)^i \frac{d^i}{dx^i} B^0_{n,k}(x) = \\ &\qquad \sum_{j=0}^{\nu} \sum_{i=0}^{\nu} (-1)^{i-j} \binom{\nu}{i} \binom{i}{j} \binom{n-i}{k-j} x^{\nu+k-j} (1-x)^{n-k+j}. \end{aligned} \tag{9.4}$$

For the inner sum on the right-hand side of (9.4), we use the identity $\binom{\nu}{i}\binom{i}{j} = \binom{\nu}{j}\binom{\nu-j}{i-j}$, which is valid for all $0 \le i, j \le \nu$. Then we obtain

$$\sum_{i=0}^{\nu} (-1)^{i-j} \binom{\nu}{i} \binom{i}{j} \binom{n-i}{k-j} = \binom{\nu}{j} \sum_{i=0}^{\nu} (-1)^{i-j} \binom{\nu-j}{i-j} \binom{n-i}{k-j} = \binom{\nu}{j} \binom{n-\nu}{n-k},$$

see [20; equ. (Z.8)] for the value of the last sum. This can be inserted into (9.4) and (9.3) to yield

$$\begin{aligned} \int_0^1 x^\nu C_{n,k}(x)\, dx &= (n+1) \binom{n-\nu}{n-k} \sum_{j=0}^{\nu} \binom{\nu}{j} \int_0^1 x^{\nu+k-j} (1-x)^{n-k+j}\, dx \\ &= (n+1) \binom{n-\nu}{n-k} \sum_{j=0}^{\nu} \binom{\nu}{j} \frac{1}{(n+\nu+1)\binom{n+\nu}{n-k+j}}. \end{aligned} \tag{9.5}$$

The last expression in (9.5) can be simplified by using the identities

$$\binom{\nu}{j} (n-k+j)!(\nu+k-j)! = \nu! k! (n-k)! \binom{n-k+j}{j} \binom{\nu+k-j}{\nu-j}$$

and

$$\sum_{j=0}^{\nu} \binom{r+j}{j} \binom{s-j}{\nu-j} = \binom{r+s+1}{\nu},$$

see [20]. This gives

$$\int_0^1 x^\nu C_{n,k}(x)\,dx = \frac{(n+1)\nu!k!(n-k)!}{(n+\nu+1)!}\binom{n-\nu}{n-k}\binom{n+\nu+1}{\nu} = \frac{\binom{n-\nu}{n-k}}{\binom{n}{k}}. \tag{9.6}$$

Note that we obtain zero on the right-hand side of (9.6), if $\nu > k$. Combining (9.6) and (9.2), we finally obtain

$$\int_0^1 x^\nu K(x,y)\,dx = \sum_{k=\nu}^{n} \binom{n-\nu}{n-k} y^k (1-y)^{n-k} = y^\nu.$$

This shows that $K(x,y)$ in (5.21) is the reproducing kernel of the space of polynomials of degree n on the interval $[0,1]$, completing the proof of Theorem 4. ■

Proof of Corollary 2: We denote the kernel in (5.22) by $K_2(x,y)$. The well-known relation for the derivatives of Bernstein polynomials gives

$$\frac{d^\nu}{dx^\nu}\left[B^0_{n,k}(x);\ 0 \le k \le n\right] = (-1)^\nu \nu! \binom{n}{\nu} \left[B^0_{n-\nu,k}(x);\ 0 \le k \le n-\nu\right] \Delta^T_{n-\nu+2} \cdots \Delta^T_{n+1}$$

where Δ_r is defined in (4.15). The restriction to a subset of the Bernstein basis of degree $n+\nu$ gives

$$\frac{d^\nu}{dx^\nu}\left[B^0_{n+\nu,k+\nu}(x);\ 0 \le k \le n-\nu\right] = \nu! \binom{n+\nu}{\nu} \left[B^0_{n,k}(x);\ 0 \le k \le n\right] \Delta_{n+1} \cdots \Delta_{n-\nu+2}.$$

By combining these two identities, we obtain

$$\begin{aligned}
&\frac{d^\nu}{dy^\nu}\left[B^0_{n-\nu,k}(x);\ 0 \le k \le n-\nu\right] \cdot \left[B^0_{n+\nu,k+\nu}(y);\ 0 \le k \le n-\nu\right]^T = \\
&\quad \nu! \binom{n+\nu}{\nu} \left[B^0_{n-\nu,k}(x);\ 0 \le k \le n-\nu\right] (\Delta_{n+1} \cdots \Delta_{n-\nu+2})^T \left[B^0_{n,k}(y);\ 0 \le k \le n\right]^T = \\
&\quad (-1)^\nu \frac{\binom{n+\nu}{\nu}}{\binom{n}{\nu}} \frac{d^\nu}{dx^\nu} \left[B^0_{n,k}(x);\ 0 \le k \le n\right] \cdot \left[B^0_{n,k}(y);\ 0 \le k \le n\right]^T.
\end{aligned} \tag{9.7}$$

Now, simple calculations show that

$$\frac{\binom{k+\nu}{\nu}\binom{n-k}{\nu}}{\binom{n+\nu}{2\nu}\binom{n+\nu}{\nu}} B^0_{n+\nu,k+\nu}(x) = \frac{(2\nu)!n!}{(n+\nu)!\nu!} x^\nu (1-x)^\nu B^0_{n-\nu,k}(x), \qquad 0 \le k \le n-\nu. \tag{9.8}$$

Finally, identities (9.7), (9.8), and integration by parts yield

$$\begin{aligned}
&\int_0^1 B^0_{n,\ell}(x) K_2(x,y)\,dx \\
&\quad = (n+1)\sum_{\nu=0}^{n} \frac{(-1)^\nu n!}{(n+\nu)!\nu!}\int_0^1 x^\nu(1-x)^\nu \frac{d^\nu}{dx^\nu}B^0_{n,\ell}(x)\sum_{k=0}^{n-\nu} B^0_{n-\nu,k}(x)\frac{d^\nu}{dy^\nu}B^0_{n+\nu,k+\nu}(y)\,dx \\
&\quad = (n+1)\sum_{\nu=0}^{n} \frac{(n-\nu)!}{\nu! n!}\int_0^1 x^\nu(1-x)^\nu \frac{d^\nu}{dx^\nu}B^0_{n,\ell}(x)\sum_{k=0}^{n} \frac{d^\nu}{dx^\nu}B^0_{n,k}(x)B^0_{n,k}(y)\,dx = B^0_{n,\ell}(y).
\end{aligned}$$

In the last step, we have made use of equation (5.19). Thus we have shown that K_2 is the reproducing kernel of the space of all polynomials of degree n on the interval $[0,1]$. ■

10. Lack of Agreement Between Tight Frames on Bounded and Unbounded Intervals

The objective of this section is to point out a somewhat unexpected obstacle for the construction of tight frames on a bounded interval $[0, N+1]$ in that the standard procedure for constructing orthogonal wavelet bases on $[0, N+1]$ cannot be extended to the construction of tight frames in general. Two cardinal cubic spline examples are presented in this section to demonstrate this interesting observation, with the first one on the construction of tight frames using unitary matrix extension (also called "unitary extension principle", UEP, in [28]), and the second one using a nontrivial VMR function to achieve four vanishing moments.

Consider the MRA on $L^2(\mathbb{R})$, defined by

$$V_0 = \operatorname{clos\,span}\{N_4(\cdot - k);\ k \in \mathbf{Z}\}, \qquad V_j = \{f(2^j\cdot);\ f \in V_0\},$$

where N_4 denotes the cubic cardinal B-spline with simple knots at the integers $0, \ldots, 4$. Then we choose a normalized tight wavelet frame of $L^2(\mathbb{R})$, which is defined as the family of functions

$$X := \{\psi_{i;j,k} := 2^{j/2}\psi_i(2^j \cdot - k);\ j,k \in \mathbf{Z},\ i = 1,2\},$$

where ψ_i (called frame generators or framelets) are compactly supported spline functions in V_1, and normalization is to divide the framelets by the square root of the frame bound constant so as to achieve the value 1 for the upper and lower frame bounds.

10.1. UEP cannot be extended directly to construct tight frames on bounded intervals

In this subsection, we consider the case of one vanishing moment, associated with the unitary matrix extension in [28]. A special construction with 2 non-symmetric generators is contained in [6], where the functions

$$\psi_i(x) = \sum_{k=0}^{4} q_{i,k} N_4(2x-k), \qquad i = 1, 2,$$

have coefficient sequences

$$q_{1,0} = 2a(1-4b^2),\ \ q_{1,1} = \frac{3\sqrt{2}}{16r} - \frac{\sqrt{2}b^2}{r},\ \ q_{1,2} = \frac{\sqrt{2}(5+16b^2)}{32r},\ \ q_{1,3} = -\sqrt{2}b,\ \ q_{1,4} = -\frac{\sqrt{2}b}{4},$$

$$q_{2,0} = \frac{\sqrt{2}r}{4},\ \ q_{2,1} = 2a+\sqrt{2}b,\ \ q_{2,2} = \frac{\sqrt{14}a - 3\sqrt{2}b}{4},\ \ q_{2,3} = \sqrt{2}r - \frac{\sqrt{2}}{4r},\ \ q_{2,4} = \frac{\sqrt{2}r}{4} - \frac{\sqrt{2}}{16r}$$

with parameters a, b, c, r defined as

$$a = \frac{\sqrt{8-2\sqrt{14}}}{8}, \quad b = \frac{\sqrt{8+2\sqrt{14}}}{8},$$

$$c = \frac{\sqrt{2}}{4}, \quad r = \sqrt{a^2+c^2}.$$

For unitary matrix extension, the VMR Laurent polynomial $S(z) = 1$ is employed in [28]; it defines the symbol of the identity matrix on $\ell_2(\mathbf{Z})$ which serves as the analogue of the matrices S_0 and S_1 in our construction.

The analogous setting for the interval $[0, N+1]$, with $N \geq 4$, requires the use of the knot vectors $\mathbf{t}_0$ and $\mathbf{t}_1$ in (7.2), together with B-spline bases Φ_0 and Φ_1. Let P_0 denote the refinement matrix that contains the coefficients of the B-splines $N_{\mathbf{t}_0;4,k}$ with respect to the basis Φ_1. Moreover, we define the matrix

$$Q = \begin{bmatrix} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ q_{1,0} & q_{2,0} & & & & & \\ q_{1,1} & q_{2,1} & & & & & \\ q_{1,2} & q_{2,2} & q_{1,0} & q_{2,0} & & & \\ q_{1,3} & q_{2,3} & q_{1,1} & q_{2,1} & & & \\ q_{1,4} & q_{2,4} & q_{1,2} & q_{2,2} & & & \\ & & q_{1,3} & q_{2,3} & & & \\ & & q_{1,4} & q_{2,4} & & & \\ & & & & \ddots & & \\ & & & & & q_{1,0} & q_{2,0} \\ & & & & & \vdots & \vdots \\ & & & & & q_{1,4} & q_{2,4} \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \end{bmatrix},$$

which contains all coefficient sequences of $\psi_1(\cdot - k)$, $\psi_2(\cdot - k)$, $k = 0, \ldots, N-3$, with respect to the basis Φ_1, whose support is contained in the interval $[0, N+1]$. In order to apply the matrix extension method, we must perform an adaptation at the boundary of the identity matrix; namely we define the matrices

$$S_0 = U_0(\mathbf{t}_0) = \operatorname{diag}(4, 2, \tfrac{4}{3}, 1, 1, \ldots, 1, \tfrac{4}{3}, 2, 4) \in \mathbb{R}^{(N+4)\times(N+4)}$$

and

$$S_1 = U_0(\mathbf{t}_1) = 2\operatorname{diag}(4, 2, \tfrac{4}{3}, 1, 1, \ldots, 1, \tfrac{4}{3}, 2, 4) \in \mathbb{R}^{(2N+5)\times(2N+5)}$$

in order to obtain the unique approximate duals of order 1 with minimum support, which, for an arbitrary knot vector $\mathbf{t}$ of cubic B-splines, are defined by the B-splines $N_k^S := \frac{4}{t_{k+4}-t_k} N_{\mathbf{t};4,k}$. If a tight frame could be found by the matrix extension and if all frame elements $\psi_{i;j,k}$, $j \geq 0$, with support in $[0, N+1]$ belonged to the tight frame, then the matrix

$$A := S_1 - P_0 S_0 P_0^T - QQ^T \tag{10.1}$$

must be positive semidefinite. However, our numerical computation with relative precision 10^{-16} shows that $\lambda \approx -0.0037$ is an eigenvalue of this matrix. This has the following consequence. For any function $f \in L^2([0, N+1])$, whose moment sequence $\langle \Phi_1, f\rangle$ with respect to the B-spline basis Φ_1 is an eigenvector of A for the negative eigenvalue λ, we have

$$\|f\|^2 < \sum_{k=-3}^{N} s_{0,k} |\langle f, N_{\mathbf{t}_0;4,k}\rangle|^2 + \sum_{i=1}^{2} \sum_{k=0}^{N-3} |\langle f, \psi_i(\cdot - k)\rangle|^2.$$

Therefore, the frame elements of the normalized tight frame of $L^2(\mathbb{R})$, whose support is contained in $[0, N+1]$, cannot be extended to a normalized tight frame on the interval by choosing appropriate frame elements in V_1 for the boundary.

10.2. Oblique unitary matrix extension cannot be extended directly to construct tight frames on bounded intervals

The obstacle demonstrated by the above example persists in the construction of tight frames with vanishing moments of higher orders. Two nonsymmetric framelets with 4 vanishing moments for a normalized tight frame of $L^2(\mathbb{R})$ were found in [7,18], namely

$$\psi_1(x) = \sum_{k=0}^{8} q_{1,k} N_4(2x - k),$$

$$\psi_2(x) = \sum_{k=0}^{10} q_{2,k} N_4(2x - k),$$

where $q_{i,k}$ are given in Table 1.

k	$q_{1,k}$	$q_{2,k}$
0	0.01630813193478	0
1	0.06523252773913	0
2	0.01887806291643	0.00929635674687
3	−0.25065038702991	0.03718542698750
4	−0.54734482313354	0.22145387636058
5	1.27958545779041	−1.28861285535716
6	−0.29581552065325	1.89020463879447
7	−0.38661217215439	−0.92585345120529
8	0.00188283977587	−0.04053902663799
9	0.07882870625158	0.07749202744882
10	0.01970717656289	0.01937300686220

Table 1: Coefficients of generators of frame with 4 vanishing moments.

For $N \geq 10$, we choose the matrix Q of coefficients for the interior wavelets as

$$Q = \left[\begin{array}{cccccccc}
0 & \cdots & & & & & & \\
0 & \cdots & & & & & & \\
0 & \cdots & & & & & & \\
q_{1,0} & q_{2,0} & 0 & \cdots & & & & \\
q_{1,1} & q_{2,1} & 0 & \cdots & & & & \\
\vdots & & q_{1,0} & q_{2,0} & 0 & \cdots & & \\
q_{1,8} & \vdots & \vdots & & \ddots & & & \\
 & q_{2,9} & & & & & & \\
 & q_{2,10} & q_{1,8} & & & & & \\
 & & & & & & q_{2,0} & \\
 & & & & & & q_{2,1} & \\
 & & & & & & & q_{1,0} \\
 & & & & & & \vdots & \vdots \\
 & & & & & & q_{2,10} & q_{1,8} \\
 & & & & & & \cdots & 0 \\
 & & & & & & \cdots & 0 \\
 & & & & & & \cdots & 0
\end{array}\right].$$

The positive definite matrices S_0, S_1 of the approximate duals of order 4 of Φ_0 and Φ_1, respectively, which have minimal support, are boundary adapted versions of the coefficient matrix that is related to the minimum degree VMR Laurent polynomial in [7,18]. Then it turns out, that the matrix A in (10.1) has eigenvalue $\lambda \approx -0.0019$. This means, as pointed out in Section 10.1, that it is impossible to extend the interior wavelets by some boundary wavelets to form a normalized tight frame.

10.3. Conclusion

We showed in this section that the construction of tight frames on bounded intervals cannot be viewed as an extension of the known methods for the real line. We even claim that, to the contrary, our formulation of nonstationary frames yields many new results for the stationary case. A more detailed account of this claim is given in [11]. For example, in [11] we prove that our examples of symmetric and antisymmetric generators of tight frames in Sections 7.2 and 7.3 also define tight frames of $L^2(\mathbb{R})$. Symmetric tight frame generators with such small supports were not known before. Moreover, for the MRA of splines of order m with knots of multiplicity r, $2 \leq r \leq m-1$, many new results are consequences of our current development. Firstly, the minimum degree VMR Laurent polynomial S, which is the Fourier transform analogue of our matrix S_0 in Theorem 7, was not even known for cubic splines with double knots, while we give a formulation for splines of any order and any multiplicity of knots. (Note that the construction in [26] does not yield the minimum degree Laurent polynomial.) Secondly, the issue of positive definiteness for the purpose of symmetric matrix factorization to yield tight frame generators was completely unsettled. Theorem 6 affirms this property for the minimum degree VMR Laurent polynomial. Finally, as for splines with simple knots, symmetric and antisymmetric generators of tight frames of $L^2(\mathbb{R})$ were not known before for any MRA that is generated by splines with multiple knots, and constructions with the Fourier technique are likely to produce frame generators with larger support than those in Section 7.3. Hence, we expect that our formulation of the nonstationary MRA frames will have significant impact on the construction of tight frames of $L^2(\mathbb{R})$, in particular those concerning multiwavelet frames (which are frames based on multiply generated MRA's). In another development, the analysis of the duals of Bernstein polynomials in Section 5.3 has already lead to new results in multivariate approximation theory [22,23]. While this article was still being completed, the orthogonality result of Theorem 4 could be strengthened into a new pointwise identity for multivariate Bernstein polynomials, see [22].

In summary, we emphasize the significant dissimilarity between the theories of tight frames on bounded and unbounded intervals. While the mathematical foundation for the theory of tight frames on a bounded interval is the subject of discussion in this present paper, the analogous consideration for the setting of an unbounded interval will be presented in our forthcoming paper [8].

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