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# On Codimension Two Subvarieties in Hypersurfaces

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*Dedicated to Spencer Bloch*

**Abstract.** We show that for a smooth hypersurface $X \subset \mathbb{P}^n$ of degree at least 2, there exist arithmetically Cohen-Macaulay (ACM) codimension two subvarieties $Y \subset X$ which are not an intersection $X \cap S$ for a codimension two subvariety $S \subset \mathbb{P}^n$. We also show there exist $Y \subset X$ as above for which the normal bundle sequence for the inclusion $Y \subset X \subset \mathbb{P}^n$ does not split.

## 1 Introduction

In this note, we revisit some questions of Griffiths and Harris from 1985 [GH]:

**Questions** (Griffiths and Harris) Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree $d \geq 6$ and $C \subset X$ be a curve.

1. Is the degree of $C$ a multiple of $d$?
2. Is $C = X \cap S$ for some surface $S \subset \mathbb{P}^4$?

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The motivation for these questions comes from trying to extend the Noether-Lefschetz theorem for surfaces to threefolds. Recall that the Noether-Lefschetz theorem states that if $X$ is a very general surface of degree $d \geq 4$ in $\mathbb{P}^3$, then $\mathrm{Pic}(X) = \mathbb{Z}$, and hence every curve $C$ on $X$ is the complete intersection of $X$ and another surface $S$.

C. Voisin very soon [Vo] proved that the second question had a negative answer by constructing counter-examples on any smooth hypersurface of degree at least 2. She also considered a third question:

**Question** With the same terminology and when $C$ is smooth:

3. Does the exact sequence of normal bundles associated to the inclusions $C \subset X \subset \mathbb{P}^4$:
$$0 \to N_{C/X} \to N_{C/\mathbb{P}^4} \to \mathcal{O}_C(d) \to 0$$
split?

Her counter-examples provided a negative answer to this question as well. The first question, the Degree Conjecture of Griffiths-Harris, is still open. Strong evidence for this conjecture was provided by some elementary but ingenious examples of Kollár ([BCC],Trento examples). In particular he shows that if $\gcd(d,6)=1$ and $d \geq 4$ and $X$ is a very general hypersurface of degree $d^2$ in $\mathbb{P}^4$, then every curve on $X$ has degree a multiple of $d$. In the same vein, van Geemen shows that if $d > 1$ is an odd number and $X$ is a very general hypersurface of degree $54d$, then every curve on $X$ has degree a multiple of $3d$.

The main result of this note is the existence of a large class of counterexamples which subsumes Voisin's counterexamples and places them in the context of arithmetically Cohen-Macaulay (ACM) vector bundles on $X$. It is well known that ACM bundles which are not sums of line bundles can be found on any hypersurface of degree at least 2 [BGS], and for such a bundle, say of rank $r$, on $X$, ACM subvarieties of codimension two can be created on $X$ by considering the dependency locus of $r-1$ general sections. These subvarieties fail to satisfy Questions 2 and 3. We will be working on hypersurfaces in $\mathbb{P}^n$ for any $n \geq 4$ and our constructions of ACM subvarieties may not give smooth ones. Hence in Question 3, we will consider the splitting of the conormal sheaf sequence instead.

## 2 Main results

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 2$ and let $Y \subset X$ be a codimension 2 subscheme. Recall that $Y$ is said to be an arithmetically Cohen-Macaulay (ACM) subscheme of $X$ if $\mathrm{H}^i(X, I_{Y/X}(\nu)) = 0$ for $0 < i \leq \dim Y$ and for any $\nu \in \mathbb{Z}$. Similarly, a vector bundle $E$ on $X$ is said to be ACM if $\mathrm{H}^i(X, E(\nu)) = 0$ for $i \neq 0, \dim X$ and for any $\nu \in \mathbb{Z}$.

Let $\mathcal{F}$ be a coherent sheaf on $X$ and let $s_i \in \mathrm{H}^0(\mathcal{F}(m_i))$ for $1 \leq i \leq k$, be generators for the $\oplus_{\nu\in\mathbb{Z}} \mathrm{H}^0(\mathcal{O}_X(\nu))-$graded module $\oplus_{\nu\in\mathbb{Z}} \mathrm{H}^0(\mathcal{F}(\nu))$. These sections give a surjection $\oplus_{i=1}^k \mathcal{O}_X(-m_i) \twoheadrightarrow \mathcal{F}$ which induces a surjection of global section $\oplus_{i=1}^k \mathrm{H}^0(\mathcal{O}_X(\nu - m_i)) \twoheadrightarrow \mathrm{H}^0(\mathcal{F}(\nu))$ for any $\nu \in \mathbb{Z}$.

Applying this to the ideal sheaf $I_{Y/X}$ of an ACM subscheme of codimension 2 in $X$, we obtain the short exact sequence
$$0 \to G \to \oplus_{i=1}^k \mathcal{O}_X(-m_i) \to I_{Y/X} \to 0,$$

where $G$ is some ACM sheaf on $X$ of rank $k-1$. Since $Y$ is ACM as a subscheme of $X$, it is also ACM as a subscheme of $\mathbb{P}^n$. In particular, $Y$ is locally Cohen-Macaulay. Hence $G$ is a vector bundle by the Auslander-Buchsbaum Theorem (see [Mat] page 155). We will loosely say that $G$ is associated to $Y$.

Conversely, the following Bertini type theorem which goes back to arguments of Kleiman in [Kl] (see also [Ban]) shows that given an ACM bundle $G$ on $X$, we can use $G$ to construct ACM subvarieties $Y$ of codimension 2 in $X$:

**Proposition 2.1** *(Kleiman). Given a bundle $G$ of rank $k-1$ on $X$, a general map $G \to \oplus_{i=1}^{k}\mathcal{O}_X(m_i)$ for sufficiently large $m_i$ will determine the ideal sheaf (up to twist) of a subvariety $Y$ of codimension 2 in $X$ with a resolution of sheaves:*

$$0 \to G \to \oplus_{i=1}^{k}\mathcal{O}_X(m_i) \to I_{Y/X}(m) \to 0.$$

Since the conclusion of Question 2 implies that of Question 3, we will look at just Question 3, in the conormal sheaf version.

Let $X$ be a hypersurface of degree $d$ in $\mathbb{P}^n$ defined by the equation $f = 0$. Let $X_2$ be the thickening of $X$ defined by $f^2 = 0$ in $\mathbb{P}^n$. Given a subvariety $Y$ of codimension 2 in $X$, let $I_{Y/\mathbb{P}}$ (resp. $I_{Y/X}$) denote the ideal sheaf of $Y \subset \mathbb{P}^n$ (resp. $Y \subset X$). The conormal sheaf sequence is

$$0 \to \mathcal{O}_Y(-d) \to I_{Y/\mathbb{P}}/I^2_{Y/\mathbb{P}} \to I_{Y/X}/I^2_{Y/X} \to 0. \tag{2.1}$$

**Lemma 2.2** *For the inclusion $Y \subset X \subset \mathbb{P}^n$, if the sequence of conormal sheaves (2.1) splits, then there exists a subscheme $Y_2 \subset X_2$ containing $Y$ such that*

$$I_{Y_2/X_2}(-d) \xrightarrow{f} I_{Y_2/X_2} \to I_{Y/X} \to 0$$

*is exact. Furthermore, $fI_{Y_2/X_2}(-d) = I_{Y/X}(-d)$.*

**Proof** Suppose sequence (2.1) splits: then we have a surjection

$$I_{Y/\mathbb{P}} \twoheadrightarrow I_{Y/\mathbb{P}}/I^2_{Y/\mathbb{P}} \twoheadrightarrow \mathcal{O}_Y(-d)$$

where the first map is the natural quotient map and the second is the splitting map for the sequence. The kernel of this composition defines a scheme $Y_2$ in $\mathbb{P}^n$. Since this kernel $I_{Y_2/\mathbb{P}}$ contains $I^2_{Y/\mathbb{P}}$ and hence $f^2$, it is clear that $Y \subset Y_2 \subset X_2$.

The splitting of (2.1) also means that $f \in I_{Y/\mathbb{P}}(d)$ maps to $1 \in \mathcal{O}_Y$. We get the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \to & I_{Y_2/\mathbb{P}} & \to & I_{Y/\mathbb{P}} & \to & \mathcal{O}_Y(-d) & \to & 0 \\
 & & \uparrow f^2 & & \uparrow f & & \uparrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}}(-2d) & \xrightarrow{f} & \mathcal{O}_{\mathbb{P}}(-d) & \to & \mathcal{O}_X(-d) & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

This induces

$$0 \to I_{Y/X}(-d) \to I_{Y_2/X_2} \to I_{Y/X} \to 0.$$

In particular, note that $I_{Y/X}(-d)$ is the image of the multiplication map $f : I_{Y_2/X_2}(-d) \to I_{Y_2/X_2}$. □

Now assume that $Y$ is an ACM subvariety on $X$ of codimension 2. The ideal sheaf of $Y$ in $X$ has a resolution

$$0 \to G \to \oplus_{i=1}^{k} \mathcal{O}_X(-m_i) \to I_{Y/X} \to 0,$$

for some ACM bundle $G$ on $X$ associated to $Y$.

**Lemma 2.3** *Suppose the conditions of the previous lemma hold, and in addition $Y$ is an ACM subvariety. Then there is an extension of the ACM bundle $G$ (associated to $Y$) on $X$ to a bundle $\mathcal{G}$ on $X_2$. ie. there is a vector bundle $\mathcal{G}$ on $X_2$ such that the multiplication map $f : \mathcal{G}(-d) \to \mathcal{G}$ induces the exact sequence $0 \to G(-d) \to \mathcal{G} \to G \to 0$.*

**Proof** Since $Y$ is ACM, $H^1(I_{Y/X}(-d+\nu)) = 0, \forall \nu$, hence in the sequence stated in the previous lemma, the right hand map is surjective on the level of sections. Therefore, the map $\oplus_{i=1}^{k} \mathcal{O}_X(-m_i) \to I_{Y/X}$ can be lifted to a map $\oplus_{i=1}^{k} \mathcal{O}_{X_2}(-m_i) \to I_{Y_2/X_2}$. Since a global section of $I_{Y_2/X_2}(\nu)$ maps to zero in $I_{Y/X}$ only if it is a multiple of $f$, by Nakayama's lemma, this lift is surjective at the level of global sections in different twists, and hence on the level of sheaves. Hence there is a commuting diagram of exact sequences:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
I_{Y_2/X_2}(-d) & \to & I_{Y_2/X_2} & \to & I_{Y/X} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\oplus_{i=1}^{k} \mathcal{O}_{X_2}(-m_i-d) & \to & \oplus_{i=1}^{k} \mathcal{O}_{X_2}(-m_i) & \to & \oplus_{i=1}^{k} \mathcal{O}_X(-m_i) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{G}(-d) & \to & \mathcal{G} & \to & G & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

where the sheaf $\mathcal{G}$ is defined as the kernel of the lift, and the map from the left column to the middle column is multiplication by $f$. It is easy to verify that the lowest row induces an exact sequence

$$0 \to G(-d) \to \mathcal{G} \to G \to 0.$$

By Nakayama's lemma, $\mathcal{G}$ is a vector bundle on $X_2$. $\square$

**Proposition 2.4** *Let $E$ be an ACM bundle on $X$. If $E$ extends to a bundle $\mathcal{E}$ on $X_2$, then $E$ is a sum of line bundles.*

**Proof** There is an exact sequence $0 \to E(-d) \to \mathcal{E} \to E \to 0$, where the left hand map is induced by multiplication by $f$ on $\mathcal{E}$. Let $F_0 = \oplus \mathcal{O}_{\mathbb{P}^n}(a_i) \twoheadrightarrow E$ be a surjection induced by the minimal generators of $E$. Since $E$ is ACM, this lifts to a map $F_0 \twoheadrightarrow \mathcal{E}$. This lift is surjective on global sections by Nakayama's lemma (since the sections of $\mathcal{E}$ which are sent to 0 in $E$ are multiples of $f$). Thus we have a diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & E(-d) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & F_1 & \to & F_0 & \to & \mathcal{E} & \to & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \to & G_1 & \to & F_0 & \to & E & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & E(-d) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

$G_1$ and $F_1$ are sums of line bundles on $\mathbb{P}^n$ by Horrocks' Theorem. Furthermore, $G_1 \cong F_0(-d)$. Thus $0 \to F_0(-d) \xrightarrow{\Phi} F_0 \to E \to 0$ is a minimal resolution for $E$ on $\mathbb{P}^n$. As a consquence of this, one checks that $\det \Phi = f^{\operatorname{rank} E}$. On the other hand, the degree of $\det \Phi = d \operatorname{rank} F_0$ and so we have $\operatorname{rank} F_0 = \operatorname{rank} E$. Restricting, this resolution to $X$, we get a surjection $F_0 \otimes \mathcal{O}_X \twoheadrightarrow E$. The ranks of both vector bundles being the same, this implies that this is an isomorphism. $\square$

**Corollary 2.5** *Let $Y \subset X$ be a codimension 2 ACM subvariety. If the conormal sheaf sequence (2.1) splits, then*

- *the ACM bundle $G$ associated to $Y$ is a sum of line bundles,*
- *there is a codimension 2 subvariety $S$ in $\mathbb{P}^n$ such that $Y = X \cap S$.*

**Proof** The first statement follows from Lemma 2.3 and Proposition 2.4. For the second statement, since the bundle $G$ associated to $Y$ is a sum of line bundles $\oplus_{i=1}^{k-1} \mathcal{O}_X(-l_i)$ on $X$, the map $G \to \oplus_{i=1}^{k} \mathcal{O}_X(-m_i)$ can be lifted to a map $\oplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}}(-l_i) \to \oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}}(-m_i)$. The determinantal variety $S$ of codimension 2 in $\mathbb{P}^n$ determined by this map has the property that $Y = X \cap S$. $\square$

In conclusion, we obtain the following collection of counterexamples:

**Corollary 2.6** *If $G$ is an ACM bundle on $X$ which is not a sum of line bundles, and if $Y$ is a subvariety of codimension 2 in $X$ constructed from $G$ as in Proposition 2.1, then $Y$ does not satisfy the conclusion of either Question 2 or Question 3.*

Buchweitz-Greuel-Schreyer have shown [BGS] that any hypersurface of degree at least 2 supports (usually many) non-split ACM bundles. We will give another construction in the next section.

## 3 Remarks

**3.1** The infinitesimal Question 3 was treated by studying the extension of the bundle to the thickened hypersurface $X_2$. This method goes back to Ellingsrud, Gruson, Peskine and Strømme [EGPS]. If we are not interested in the infinitesimal Question 3, but just in the more geometric Question 2, a geometric argument gives an even easier proof of the existence of codimension 2 ACM subvarieties $Y \subset X$ which are not of the form $Y = X \cap Z$ for some codimension 2 subvariety $Z \subset \mathbb{P}^n$.

**Proposition 3.1** *Let $E$ be an ACM bundle on a hypersurface $X$ in $\mathbb{P}^n$ which extends to a sheaf $\mathcal{E}$ on $\mathbb{P}^n$; i.e. there is an exact sequence*

$$0 \to \mathcal{E}(-d) \xrightarrow{f} \mathcal{E} \to E \to 0. \tag{3.1}$$

*Then $E$ is a sum of line bundles.*

**Proof** At each point $p$ on $X$, over the local ring $\mathcal{O}_{\mathbb{P},p}$ the sheaf $\mathcal{E}$ is free, of the same rank as $E$. Hence $\mathcal{E}$ is locally free except at finitely many points. Let $\mathbb{H}$ be a general hyperplane not passing through these points. Let $X' = X \cap \mathbb{H}$, and $\mathcal{E}', E'$ be the restrictions of $\mathcal{E}, E$ to $\mathbb{H}, X'$.

It is enough to show that $E'$ is a sum of line bundles on $X'$. This is because any isomorphism $\oplus \mathcal{O}_{X'}(a_i) \to E'$ can be lifted to an isomorphism $\oplus \mathcal{O}_X(a_i) \to E$, as $H^1(E(\nu)) = 0, \forall\ \nu \in \mathbb{Z}$. The bundle $E'$ on $X'$ is ACM and from the sequence

$$0 \to \mathcal{E}'(-d) \to \mathcal{E}' \to E' \to 0,$$

it is easy to check that $H^i(\mathcal{E}'(\nu)) = 0, \forall\ \nu \in \mathbb{Z}$, for $2 \leq i \leq n-2$. Since $\mathcal{E}'$ is a vector bundle on $\mathbb{H}$, we can dualize the sequence to get

$$0 \to \mathcal{E}'^{\vee}(-d) \to \mathcal{E}'^{\vee} \to E'^{\vee} \to 0.$$

$E'^{\vee}$ is still an ACM bundle, hence $H^i(\mathcal{E}'^{\vee}(\nu)) = 0, \forall\ \nu \in \mathbb{Z}$, and $2 \leq i \leq n-2$.

By Serre duality, we conclude that $\mathcal{E}'$ is an ACM bundle on $\mathbb{H}$, and by Horrocks' theorem, $\mathcal{E}'$ is a sum of line bundles. Hence, its restriction $E'$ is also a sum of line bundles on $X'$. $\square$

**Proposition 3.2** *Let $Y$ be an ACM subvariety of codimension 2 in the hypersurface $X$ such that the associated ACM bundle $G$ is not a sum of line bundles. Then there is no pure subvariety $Z$ of codimension 2 in $\mathbb{P}^n$ such that $Z \cap X = Y$.*

**Proof** Suppose there is such a $Z$. Then there is an exact sequence $0 \to I_{Z/\mathbb{P}}(-d) \to I_{Z/\mathbb{P}} \to I_{Y/X} \to 0$, where the inclusion is multiplication by $f$, the polynomial defining $X$. Since $Z$ has no embedded points, $H^1(I_{Z/\mathbb{P}}(\nu)) = 0$ for $\nu << 0$. Combining this with $H^1(I_{Y/X}(\nu)) = 0, \forall\ \nu \in \mathbb{Z}$, and using the long exact sequence of cohomology, we get $H^1(I_{Z/\mathbb{P}}(\nu)) = 0, \forall\ \nu \in \mathbb{Z}$.

Now suppose $Y$ has the resolution $0 \to G \to \oplus \mathcal{O}_X(-m_i) \to I_{Y/X} \to 0$. From the vanishing just proved, the right hand map can be lifted to a map $\oplus \mathcal{O}_{\mathbb{P}}(-m_i) \to I_{Z/\mathbb{P}}$, which is easily checked to be surjective (at the level of global sections). It follows that if $\mathcal{G}$ is the kernel of this lift, $\mathcal{G}$ is an extension of $G$ to $\mathbb{P}^n$. By the previous proposition, $G$ is a sum of line bundles. This is a contradiction. $\square$

**3.2** Voisin's original example was as follows. Let $P_1$ and $P_2$ be two planes meeting at a point $p$ in $\mathbb{P}^4$. The union $\Sigma$ is a surface which is not locally Cohen-Macaulay at $p$. Let $X$ be a smooth hypersurface of degree $d > 1$ which passes through $p$. $X \cap \Sigma$ is a curve $Z$ in $X$ with an embedded point at $p$. The reduced subscheme $Y$ has the form $Y = C_1 \cup C_2$, where $C_1$ and $C_2$ are plane curves. Voisin argues that $Y$ itself does not have the form $X \cap S$ for any surface $S$ in $\mathbb{P}^4$.

We can treat this example from the point of view of ACM bundles. $I_{Z/X}$ has a resolution on $X$ which is just the restriction of the resolution of the ideal of the union $P_1 \cup P_2$ in $\mathbb{P}^4$, *viz.*

$$0 \to \mathcal{O}_X(-4) \to 4\mathcal{O}_X(-3) \to 4\mathcal{O}_X(-2) \to I_{Z/X} \to 0.$$

From the sequence $0 \to I_{Z/X} \to I_{Y/X} \to k_p \to 0$, it is easy to see that $Y$ is ACM, with a resolution

$$0 \to G \to 4\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-d) \to I_{Y/X} \to 0.$$

$G$ is an ACM bundle. If it were a sum of line bundles, comparing the two resolutions, we find that $h^0(G(2)) = 0$ and $h^0(G(3)) = 4$, hence $G = 4\mathcal{O}_X(-3)$. But then

$G \to 4\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-d)$ cannot be an inclusion. Thus $G$ is an ACM bundle which is not a sum of line bundles.

Voisin's subsequent smooth examples were obtained by placing $Y$ on a smooth surface $T$ contained in $X$ and choosing divisors $Y'$ in the linear series $|Y + mH|$ on $T$. When $m$ is large, $Y'$ can be chosen smooth. In fact, such curves $Y'$ are doubly linked to the original curve $Y$ in $X$, hence they have a similar resolution $G' \to L \to I_{D'/X} \to 0$, where $L$ is a sum of line bundles and where $G'$ equals $G$ up to a twist and a sum of line bundles.

The fact that $G$ above is not a sum of line bundles is related (via the mapping cone of the map of resolutions) to the fact that $k_p$ itself cannot have a finite resolution by sums of line bundles on $X$. This follows from the following proposition which provides another argument for the existence of ACM bundles on arbitrary smooth hypersurfaces of degree $\geq 2$.

**Proposition 3.3** *Let $X$ be a smooth hypersurface in $\mathbb{P}^n$ of degree $\geq 2$ with homogeneous coordinated ring $S_X$. Let $L$ be a linear space (possibly a point or even empty) inside $X$ of codimension $r$, with homogeneous ideal $I(L)$ in $S_X$. A free presentation of $I(L)$ of length $r-2$ will have a kernel whose sheafification is an ACM bundle on $X$ which is not a sum of line bundles.*

**Proof** It should first be understood that the homogeneous ideal $I(L)$ of the empty linear space will be taken as the irrelevant ideal $(X_0, X_1, \ldots, X_n)$. Let the free presentation of $I(L)$ together with the kernel be

$$0 \to M \to F_{r-2} \to \cdots \to F_0 \to I(L) \to 0,$$

where $F_i$ are free graded $S_X$ modules. Its sheafification looks like

$$0 \to \tilde{M} \to \tilde{F}_{r-2} \to \cdots \to \tilde{F}_0 \to I_{L/X} \to 0.$$

Since $L$ is locally Cohen-Macaulay, $\tilde{M}$ is a vector bundle on $X$, and since $L$ is ACM, so is $\tilde{M}$. $M$ equals $\oplus_{\nu \in \mathbb{Z}} H^0(\tilde{M}(\nu))$. Hence, $\tilde{M}$ is a sum of line bundles only if $M$ is a free $S_X$ module.

If $\mathbb{H}$ is a general hyperplane in $\mathbb{P}^n$ which meets $X$ and $L$ transversally along $X_{\mathbb{H}}$ and $L_{\mathbb{H}}$ respectively, the above sequences of modules and sheaves can be restricted to give similar sequences in $\mathbb{H}$. The restriction $\tilde{M}_{\mathbb{H}}$ is an ACM bundle on $X_{\mathbb{H}}$.

Repeat this successively to find a maximal and general linear space $\mathbb{P}$ in $\mathbb{P}^n$ which does not meet $L$. If $X' = X \cap \mathbb{P}$, the restriction of the sequence of $S_X$ modules to $X'$ gives a resolution

$$0 \to M' \to F'_{r-2} \to \cdots \to F'_0 \to S_{X'} \to k \to 0.$$

Localize this sequence of graded $S_{X'}$ modules at the irrelevant ideal $I(L) \cdot S_{X'}$, to look at its behaviour at the vertex of the affine cone over $X'$. $k$ is the residue field of this local ring. Since $X$ and hence $X'$ has degree $\geq 2$, the cone is not smooth at the vertex. By Serre's theorem ([Se], IV-C-3-Cor 2), $k$ cannot have finite projective dimension over this local ring. Hence $M'$ is not a free module. Therefore neither is $M$. $\square$

**3.3** We make a few concluding remarks about Question 1, the Degree Conjecture of Griffiths and Harris. A vector bundle $G$ on a smooth hypersurface $X$ in $\mathbb{P}^4$ has a second Chern class $c_2(G) \in A^2(X)$, the Chow group of codimension 2 cycles. If $h \in A^1(X)$ is the class of the hyperplane section of $X$, the degree of any element $c \in A^2(X)$ will be defined to be the degree of the zero cycle $c \cdot h \in A^3(X)$. (Note that by the Lefschetz theorem, all classes in $A^1(X)$ are multiples of $h$.)

With this notation, if $E$ is any bundle on $X$ and $Y$ is a curve obtained from $E$ with the sequence (*vide* Proposition 2.1)

$$0 \to E \to \oplus_{i=1}^{k} \mathcal{O}_X(m_i) \to I_{Y/X}(m) \to 0,$$

a calculation tells us that the degree $d$ of $X$ divides the degree of $Y$ if and only if $d$ divides the degree of $c_2(E)$.

More generally: let $Y$ be any curve in $X$ and resolve $I_{Y/X}$ to get

$$0 \to E \to \oplus_{i=1}^{l} \mathcal{O}_X(b_i) \to \oplus_{i=1}^{k} \mathcal{O}_X(a_i) \to I_{Y/X} \to 0,$$

where $E$ is an ACM bundle on $X$. Then a similar calculation tells us that the degree $d$ of $X$ divides the degree of $Y$ if and only if $d$ divides the degree of $c_2(E)$.

Hence we may ask the following question which is equivalent to the Degree Conjecture:

**ACM Degree Conjecture** If $X$ is a general hypersurface in $\mathbb{P}^4$ of degree $d \geq 6$, then for any indecomposable ACM vector bundle $E$ on $X$, $d$ divides the degree of $c_2(E)$.

The examples created above in Proposition 3.3 satisfy this, when $L$ has codimension $> 2$ in $X$. In [MRR], this conjecture is settled for ACM bundles of rank 2 on $X$.

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