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ORIGINAL PAPER

On the base case of a conjecture on ACM bundles over hypersurfaces

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Abstract

We obtain an upper bound on the first Chern class and the Castelnuovo-Mumford regularity of an initialized rank 3 ACM bundle on a general hypersurface in $\mathbb{P}^4$. As a corollary, we prove that a general hypersurface in $\mathbb{P}^4$ of degree $d \geqslant 4$ does not support a rank 3 Ulrich bundle. We also make progress on the base case of a generic version of a conjecture by Buchweitz, Greuel and Schreyer.

Keywords Vector bundles · Exterior powers · Hypersurfaces · Arithmetically Cohen-Macaulay

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1 Introduction

We work over an algebraically closed field of characteristic 0. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Let E be a vector bundle on X. We say that E is *arithmetically Cohen Macaulay* (ACM for short) if

$$H^i(X, E(k)) = 0 \text{ for all } k \in \mathbb{Z}, \text{ and } 0 < i < n.$$

By a well known result of Horrocks [7], any ACM bundle on $\mathbb{P}^n$, the case when $d = 1$, is a direct sum of line bundles. For higher degrees, the situation is much less understood. In this context, we have the following well known conjecture,

Conjecture 1 (Buchweitz, Greuel and Schreyer [3]) *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Any ACM bundle E on X of rank $u < 2^e$ for $e := \left\lfloor \frac{n-1}{2} \right\rfloor$, is a sum of line bundles.*

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We refer to [16] and references cited therein for progress on this conjecture. Since early 2000s, beginning with [9], a generic type BGS conjecture has been studied. A precise version was formulated in [15].

Conjecture 2 (Generic BGS) *Let* $X \subset \mathbb{P}^{n+1}$ *be a general hypersurface of sufficiently high degree and* E *be an ACM bundle of rank* u *on* X*. If* $u < 2^s$*, where* $s := \left\lfloor \frac{n+1}{2} \right\rfloor$*, then* E *is a sum of line bundles.*

We refer to [10, 11] and [12] for the rank two case. The case where $\operatorname{rank} E = 3$ and $\dim X \geqslant 4$ was settled in [15]. In this paper, we investigate the generic BGS conjecture for the remaining case i.e., $\operatorname{rank} E = 3$ and $\dim X = 3$.

Recall that a rank u ACM bundle on a hypersurface of degree d is *Ulrich* if the minimal number of generators of the graded module $H^0_*(X, E) := \oplus_{k \in \mathbb{Z}} H^0(X, E(k))$, is $u \cdot d$. There has been considerable interest in Ulrich bundles since the work of Eisenbud and Schreyer [5] in which they conjecture that every smooth projective variety supports an Ulrich bundle. The existence of an Ulrich bundle on a projective variety will imply that the variety has the same cone of cohomology table as the projective space of dimension equal to that variety. We refer the readers to [1] for more details and references.

By the Grothendieck-Lefschetz theorem it follows that a smooth hypersurface of degree $\geqslant 2$ and dimension $\geqslant 3$ does not support an Ulrich line bundle. In [2], it was shown that a general threefold in $\mathbb{P}^4$ of degree $\geqslant 6$ does not support a rank 2 Ulrich bundle. In this paper, we show that

Theorem 1 *Let* $X \subset \mathbb{P}^4$ *be a general hypersurface of degree* $d \geqslant 4$*. Then* X *does not support a rank* 3 *Ulrich bundle.*

By a result in [8], a general cubic threefold supports a family of rank 3 Ulrich bundles; so the degree bound in the above Theorem is sharp. This result is obtained as a corollary to a more general result which gives an upper bound on the first Chern class.

Theorem 2 *Let* $X \subset \mathbb{P}^4$ *be a general hypersurface of degree* $d \geqslant 3$*. Let* E *be an initialized, indecomposable rank 3 ACM bundle on* X*. Then* $c_1(E) \leqslant d$*.*

Next, we prove several instances of the generic BGS conjecture.

Theorem 3 *Let* $X \subset \mathbb{P}^4$ *be a general hypersurface of degree* $\geqslant 3$ *and let* E *be an initialized ACM bundle of rank* 3 *on* X*. Assume that* $\dim H^0(X, E) \neq 1, 2$ *and* $c_1(E) \leqslant 0$*. Then* E *is split.*

Recall that a vector bundle is said to be *simple*, if its only endomorphisms are homotheties i.e. $H^0(X, \mathcal{E}nd(E)) = 1$. For rank 3 ACM bundles with positive first Chern class, we obtain the following dichotomy:

Theorem 4 *Let* $X \subset \mathbb{P}^4$ *be a general hypersurface of degree* $\geqslant 3$*. Let* E *be an initialized, indecomposable rank 3 ACM bundle on* X*. Assume that* $c_1(E)$ *is positive. Then either* E *is a simple bundle or its Castelnuovo-Mumford regularity is* $d - 1$*.*

2 Preliminaries

We recall some standard facts here about ACM bundles over smooth hypersurfaces. More details can be found in Sect. 2 of [15].

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Let E be an arithmetically Cohen-Macaulay bundle on X of rank u. By [4], it admits a minimal resolution over $\mathbb{P}^{n+1}$ of the form:

$$0 \to \widetilde{F}_1 \xrightarrow{\Phi} \widetilde{F}_0 \to E \to 0, \tag{1}$$

where $\widetilde{F}_0$ and $\widetilde{F}_1$ are sums of line bundles on $\mathbb{P}^{n+1}$. Restricting this resolution to X, we get a 4-term exact sequence

$$0 \to E(-d) \to \overline{F}_1 \xrightarrow{\overline{\Phi}} \overline{F}_0 \to E \to 0.$$

Breaking this up into short exact sequences, we get

$$0 \to G \to \overline{F}_0 \to E \to 0 \quad \text{and}, \tag{2}$$
$$0 \to E(-d) \to \overline{F}_1 \to G \to 0. \tag{3}$$

where $G := \text{Image}(\overline{\Phi})$. It can be easily verified that G is also an ACM bundle and has the following minimal resolution over $\mathbb{P}^{n+1}$:

$$0 \to \widetilde{F}_0(-d) \xrightarrow{\Psi} \widetilde{F}_1 \to G \to 0. \tag{4}$$

The exact sequence (2) defines an element $\zeta \in \text{Ext}^1_X(E, G) \cong H^1(X, E^\vee \otimes G)$. By the Krull-Schmidt theorem, $\zeta = 0$ is equivalent to the splitting of E and G. Tensoring this sequence with $E^\vee$, and taking cohomology, we obtain the following long exact sequence of cohomology:

$$0 \to H^0(X, G \otimes E^\vee) \to H^0(X, \overline{F}_0 \otimes E^\vee) \to H^0(X, E \otimes E^\vee) \to H^1(X, G \otimes E^\vee) \to \cdots .$$

It is standard that, under the coboundary map,

$$H^0(X, E \otimes E^\vee) \to H^1(X, G \otimes E^\vee),$$

the identity 1 is mapped to the element ζ.

Similarly, tensoring the sequence (3) with $E^\vee$, and taking cohomology, we get a boundary map $H^1(X, E^\vee \otimes G) \to H^2(X, \mathcal{E}nd E(-d))$ under which the element ζ maps to to η, the *obstruction class* of E (see Remark 1 below).

Remark 1 For any vector bundle E on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, the vanishing of the class $\eta \in H^2(X, \mathcal{E}nd E(-d))$ is necessary and sufficient for E to extend to a bundle E_2 on $X_2 = V(f^2)$ where f is the polynomial defining X (see, for instance, [13] for details). In the case when E is an ACM bundle (of arbitrary rank), it can be easily shown, by elementary arguments, that E splits if and only if $\eta = 0$.

Since $\dim X \geqslant 3$, we have $\text{Pic}(X) \cong \mathbb{Z}$ by the Grothendieck-Lefschetz theorem. Using this isomorphism, we let $e := c_1(E) \in \mathbb{Z}$, so that $\wedge^u E \cong \mathcal{O}_X(e)$.

Remark 2 Throughout this paper, E will denote an indecomposable, rank 3 ACM bundle. In particular, $\wedge^2 E \cong E^\vee(e)$ is also an indecomposable, rank 3 ACM bundle.

2.1 A filtration

We recall a convenient notation from [15],

Definition 1 On $\wedge^r \overline{F}_0$, we have the following filtration (see Ex. II.5.16 of [6]) via sequence (2):

$$\wedge^r G = E_{r,0} \subset E_{r,1} \subset \ldots \subset E_{r,r-1} \subset E_{r,r} = \wedge^r \overline{F}_0.$$

For every pair $i > j$, we define

$$\mathcal{Q}_{r,i,j} := \operatorname{coker}(E_{r,j} \to E_{r,i}).$$

In particular, we have

$$\mathcal{Q}_{r,i,i-1} = \operatorname{coker}(E_{r,i-1} \to E_{r,i}) = \wedge^i E \otimes \wedge^{r-i} G. \tag{5}$$

Thus we have diagrams (for $3 \geqslant i > j > k \geqslant 0$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & E_{3,k} & = & E_{3,k} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & E_{3,j} & \to & E_{3,i} & \to & \mathcal{Q}_{3,i,j} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{Q}_{3,j,k} & \to & \mathcal{Q}_{3,i,k} & \to & \mathcal{Q}_{3,i,j} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array} \tag{6}$$

We are now in a position to state results from [15] and [10] which we will need,

Theorem 5 (Theorem 5.7 of [15]) *Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. Let E be a rank 3 ACM bundle. Then*

(a) $\wedge^3 G$ *is ACM.*
(b) *The graded module $H^1_*(X, \wedge^2 E \otimes G)$ is generated by ζ in degree $-e$.* [1] *Similarly, $H^1_*(X, \mathcal{Q}_{3,2,0})$, and $H^2_*(X, \wedge^2 E \otimes E(-d))$ are also 1-generated.*
(c) *There is a natural map $H^1_*(X, \mathcal{Q}_{3,2,0}) \to H^1_*(X, \wedge^2 E \otimes G)$ which is an isomorphism.*

Proof Parts (a) and (b) are Theorem 5.7 in [15]. By the bottow row of diagram (6), there is a natural map $\mathcal{Q}_{3,2,0} \to \mathcal{Q}_{3,2,1}$ which induces the map in part (c) as $\mathcal{Q}_{3,2,1} = \wedge^2 E \otimes G$ by (5). The proof of Theorem 5.7 in [15] shows explicitly that the induced map of cohomologies

$$H^1_*(X, \mathcal{Q}_{3,2,0}) \to H^1_*(X, \wedge^2 E \otimes G). \tag{7}$$

is surjective (page 30, [15]). By Theorem 5, both these graded modules are 1-generated, therefore this map is an isomorphism. □

Using the fact that $H^2_*(X, \mathcal{E}nd E)$ is generated by η (part (b) above) and Corollary 3.8 in [10], we have

Theorem 6 *Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree $d \geqslant 3$. Let E be an indecomposable ACM bundle on X. Then $H^2(X, \mathcal{E}nd(E)(k)) = 0$ for $k \geqslant 0$.*

[1] See discussion in Sect. 2.

3 Regularity and chern class bounds

Definition 2 We say that a vector bundle E on X is *initialized* if $H^0(X, E(-t)) = 0$ for every $t > 0$, and $H^0(X, E) \neq 0$. It follows that, for an initialized vector bundle E, its first syzygy bundle G satisfies $H^0(X, G(a)) = 0$ for $a \leqslant 0$.

The following result derives what can be termed a poor man's bound for the Castelnuovo-Mumford regularity of an arbitrary rank ACM bundle E. We will later improve this bound for the rank 3 case. We denote the Castelnuovo-Mumford regularity of E by $m(E)$.

Proposition 7 *Let E be an initialized rank u ACM bundle on a smooth degree d hypersurface $X \subset \mathbb{P}^{n+1}$ with first Chern class e. Then $m(E) \leqslant (u-1)d - e - 1$.*

Proof Let $m = (u-1)d-e-1$. It suffices to show that $H^n(X, E(m-n)) = 0$. Equivalently, we need to show that

$$H^0(X, \wedge^{u-1}E(-e-m+d-2)) = H^0(X, \wedge^{u-1}E(-(u-2)d-1)) = 0.$$

As before, let $X_i = V(f^i)$ where f is the polynomial defining X. Taking exterior powers of sequence (1), we get

$$0 \to \wedge^{u-1}\widetilde{F_1} \to \wedge^{u-1}\widetilde{F_0} \to \mathcal{F}_{u-1} \to 0,$$

where $\wedge^{u-1}\widetilde{F_0} = \oplus \mathcal{O}_{\mathbb{P}}(a_{i_1} + a_{i_2} + \ldots + a_{i_{u-1}})$, and $\mathcal{F}_{u-1}$ is a $\mathcal{O}_{X_{u-1}}$-module. Since E is initialized, we have $H^0(\mathbb{P}^{n+1}, \mathcal{F}_{u-1}(-1)) = 0$.

The following sequence, derived in [16,Proposition 3.5] , completes the proof

$$0 \to \wedge^{u-1}E(-(u-2)d) \to \mathcal{F}_{u-1} \to \mathcal{F}_{u-1}|_{X_{u-2}} \to 0.$$

□

Lemma 8 *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and let E be an initialized, rank 3 vector bundle on X with first Chern class e.*

(i) *If $e > 0$ then $h^0(X, E^\vee) \in \{0, 1\}$. Further, in this case $H^0(X, E^\vee(-1)) = 0$.*
(ii) *If $e \leqslant 0$ then $H^0(X, \wedge^2E(-1)) = 0$ and $h^0(X, \wedge^2E) \leqslant h^0(X, E)$.*

Proof (i). Suppose $H^0(X, E^\vee) \neq 0$ and let $s \in H^0(X, E^\vee)$ be a non-zero section with zero scheme $Z := Z(s)$. Consider the associated Koszul complex

$$0 \to \wedge^3E \to \wedge^2E \to E \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Breaking it into short exact sequences

$$0 \to \wedge^3E \to \wedge^2E \to \mathcal{F}' \to 0, \tag{8}$$
$$0 \to \mathcal{F} \to E \to \mathcal{I}_{Z,X} \to 0, \quad \text{and}$$
$$0 \to \mathcal{I}_{Z,X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0. \tag{9}$$

In the above, we have $\mathcal{F}' \subseteq \mathcal{F}$ with equality if and only if s is a *regular* section. Rewriting sequence (8) as $0 \to \mathcal{O}_X(e) \to E^\vee(e) \to \mathcal{F}' \to 0$ and tensoring these sequences by $\mathcal{O}_X(-e)$, we get

$$h^0(X, E^\vee) = h^0(\mathcal{O}_X) + h^0(\mathcal{F}'(-e)) \leqslant h^0(\mathcal{O}_X) + h^0(\mathcal{F}(-e)) \leqslant 1 + h^0(X, E(-e)) = 1.$$

Twisting every term in the above inequalities by $\mathcal{O}_X(-1)$, it is clear that $H^0(X, E^\vee(-1)) = 0$.

(ii). If $H^0(E^\vee) = 0$, then there is nothing to prove as $\wedge^2 E = E^\vee(e)$ and $e \leqslant 0$. Otherwise, from the exact sequences above, we get

$$h^0(\wedge^2 E(-1)) = h^0(\mathcal{O}_X(e-1)) + h^0(\mathcal{F}'(-1)) \leqslant h^0(\mathcal{F}(-1)) \leqslant h^0(E(-1)) = 0.$$

A similar argument proves the last assertion. □

Corollary 9 *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and let E be an initialized, rank 3 ACM bundle on X with first Chern class e. Let m be the Castelnuovo-Mumford regularity of E.*

(i) *If $e > 0$, then $m \leqslant d-1$.*
(ii) *If $e \leqslant 0$, then $m \leqslant d-e-1$.*

Proof (i). Since E is ACM, m is the smallest integer such that $H^n(X, E(m-n)) = 0$. If $e > 0$, then $H^0(X, E^\vee(-1)) = 0$ by Lemma 8. Equivalently, by Serre duality, we have $H^0(X, E^\vee(n-m+d-n-2)) = H^0(X, E^\vee(d-m-2)) = 0$. Thus we must have $m \leqslant d-1$.

(ii). If $e \leqslant 0$, then $H^n(X, E(d-e-1-n)) = 0 \iff H^0(X, \wedge^2 E(-1)) = 0$. The latter vanishing happens by part (ii) of Lemma 8. □

4 Proof of main results

4.1 Rank 3 ulrich bundles

We will now prove that a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geqslant 4$ does not support an Ulrich bundle of rank 3. This result will follow as a consequence of Theorem 2 which we prove now.

Proof of Theorem 2 We first claim that there is an isomorphism of graded modules.

$$H^1_*(X, E_{3,2}) \cong H^2_*(X, \mathcal{E}nd(E)(e-d)). \tag{10}$$

By Definition 1, we have a short exact sequence

$$0 \to E_{3,0} \to E_{3,2} \to \mathcal{Q}_{3,2,0} \to 0.$$

where $E_{3,0} = \wedge^3 G$. By Theorem 5, $\wedge^3 G$ is ACM, therefore we have an isomorphism.

$$H^1_*(X, E_{3,2}) \cong H^1_*(X, \mathcal{Q}_{3,2,0}). \tag{11}$$

Tensoring sequence (3) with $\wedge^2 E$, we get an isomorphism

$$H^1_*(X, \wedge^2 E \otimes G) \cong H^2_*(X, \wedge^2 E \otimes E(-d)). \tag{12}$$

The RHS in (11) is isomorphic to the LHS in (12) (see Theorem 5 (c)). Since $\mathrm{rank}(E) = 3$, we have $\wedge^2 E \cong E^\vee(e)$; putting these together, we have the isomorphism claimed in (10).

Now suppose $e \geqslant d+1$. By Theorem 6, $H^2(X, \mathcal{E}nd E(k)) = 0$ for $k \geqslant 0$. From the above series of isomorphisms, we get

$$H^1(X, E_{3,2}(-1)) \cong H^2(X, \mathcal{E}nd(E)(e-d-1)) = 0.$$

By definition 1, $E_{3,2}$ sits in a sequence

$$0 \to E_{3,2} \to \wedge^3 \overline{F}_0 \to \wedge^3 E \to 0,$$

and so we see that the map

$$H^0(X, \wedge^3 \overline{F}_0(-1)) \twoheadrightarrow H^0(X, \wedge^3 E(-1))$$

is a surjection. This is a contradiction as the term on the left vanishes by the hypothesis that E is initialized whereas the term on the right is non-zero. □

As an immediate application, we have

Corollary 10 (Theorem 1) *A general hypersurface* $X \subset \mathbb{P}^4$ *of degree* $d \geqslant 4$, *does not support a rank* 3 *Ulrich bundle.*

Proof By the proof of Theorem 2 in [14], we know that if E is a rank 3 Ulrich on a degree d hypersurface then $c_1(E) = \dfrac{3(d-1)}{2}$. Combining with Theorem 2, we must have

$$\frac{3(d-1)}{2} \leqslant d, \quad \text{i.e.} \quad d \leqslant 3.$$

□

For possible future use, we note the following

Corollary 11 *Let* E *be an initialized, indecomposable rank* 3 *ACM bundle on a general hypersurface* $X \subset \mathbb{P}^4$ *with first Chern class* e. *Let* m *denote the Castelnuovo-Mumford regularity of* E. *We have the following bounds*

$$\begin{cases} \dfrac{2d-e-3}{3} \leqslant m \leqslant d-1, & e > 0 \\ \dfrac{2d-e-3}{3} \leqslant m \leqslant d-e-1, & e \leqslant 0. \end{cases}$$

Proof Upper bounds are already proved in Corollary 9.

To see the lower bound, we observe that $E^\vee(d-m-1)$ is an initialized rank 3 ACM bundle with first Chern class $-e+3(d-m-1)$. Applying Theorem 2 to $E^\vee(d-m-1)$ gives the lower bound. □

4.2 Rank 3 ACM bundles with $c_1 \leqslant 0$

Remark 3 We will often use $H^0(X, \wedge^3 G) = H^0(X, \wedge^2 G) = 0$. This can be easily seen using the following series of inclusions

$$H^0(X, \wedge^2 G) \hookrightarrow H^0(X, G \otimes G) \hookrightarrow H^0(X, G \otimes \overline{F}_0) = 0.$$

Similarly, one can show $H^0(X, \wedge^3 G) = 0$.

Lemma 12 *Let* $X \subset \mathbb{P}^4$ *be a general hypersurface with* $d \geqslant 3$ *and* E *be an initialized, rank* 3 *ACM bundle on* X. *Assume that* $c_1(E) \leqslant 0$. *Then we have the following*

- $H^0(X, \wedge^2 E \otimes G) = 0$,
- $H^0(X, \wedge^2 G \otimes E) = 0$, *and*
- $H^0(X, E_{3,1}) = 0$.

Proof Consider the cohomology exact sequence associated to (3):

$$H^0(X, \wedge^2 E \otimes \overline{F}_1) \to H^0(X, \wedge^2 E \otimes G) \to H^1(X, \wedge^2 E \otimes E(-d)).$$

By Lemma 8, the first term vanishes. Further, we have

$$H^1(X, \wedge^2 E \otimes E(-d)) \cong H^1(X, \mathcal{E}nd E(e-d)) \cong H^2(X, \mathcal{E}nd E(2d-e-5))^\vee = 0,$$

where the last vanishing is by Theorem 6. Therefore, $H^0(X, \wedge^2 E \otimes G) = 0$.

Let $s \in H^0(X, E)$ be any non-zero section and consider the associated Koszul complex. As in Lemma 8, we may break it up into short exact sequences,

$$0 \to \wedge^3 E^\vee \to \wedge^2 E^\vee \to \mathcal{G}' \to 0, \quad \text{and}$$
$$0 \to \mathcal{G} \to E^\vee \to \mathcal{I}_{C,X} \to 0.$$

Tensoring these sequences with $G(e)$, we get

$$\begin{aligned} h^0(E \otimes G) =& h^0(G) + h^0(\mathcal{G}' \otimes G(e)) \\ &\leqslant h^0(\mathcal{G} \otimes G(e)) \leqslant h^0(E^\vee \otimes G(e)) = h^0(\wedge^2 E \otimes G) = 0. \end{aligned} \tag{13}$$

Consider the following series of vector space inclusions

$$H^0(X, \wedge^2 G \otimes E) \subset H^0(X, G \otimes G \otimes E) \subset H^0(X, G \otimes \overline{F}_0 \otimes E) = 0. \tag{14}$$

The last vanishing is by equation (13).

From the left vertical row of diagram (6), we have a sequence

$$0 \to \wedge^3 G \to E_{3,1} \to \wedge^2 G \otimes E \to 0.$$

Therefore, by Remark (3) and the series of inclusions in (14) above, we have $H^0(X, E_{3,1}) = 0$. □

We are now in a position to prove Theorem 3.

Proof of Theorem 3 By our hypothesis $h^0(E) \geqslant 3$. If $c_1(E) < 0$, then we have

$$H^0(X, E_{3,2}) \cong H^0(X, \wedge^3 \overline{F}_0) \neq 0.$$

If $c_1(E) = 0$, then the natural map

$$H^0(X, \wedge^3 E) \to H^1(X, E^\vee \otimes G)$$

is injective with $1 \to \zeta$. In particular, $H^0(X, E_{3,2}) \neq 0$, whenever $c_1(E) \leqslant 0$.

We will now derive a contradiction, by showing that, under the additional hypothesis of the hypersurface being general, $H^0(X, E_{3,2}) = 0$. To see this, consider the sequence

$$0 \to E_{3,1} \to E_{3,2} \to \wedge^2 E \otimes G \to 0.,$$

and apply Lemma 12. □

4.3 Rank 3 ACM bundles with positive first chern class

Lemma 13 *Let* X *be a general hypersurface* $X \subset \mathbb{P}^4$ *and let* E *be an initialized, indecomposable rank* 3 *ACM bundle with* $c_1(E) > 0$. *Then*

$$h^0(X, \mathcal{E}nd(E)) = h^0(X, E) \cdot h^0(X, E^\vee) + 1.$$

In particular, E *is simple, if and only if,* $h^0(E^\vee) = 0$.

Proof We first claim that $H^0(X, E^\vee \otimes G) = 0$. By Theorem 6,

$$H^1(X, \mathcal{E}nd E(-d)) \cong H^2(X, \mathcal{E}nd E(2d-5)) = 0 \ \text{ for } d \geqslant 3.$$

Since $c_1(E) > 0$, by Lemma 8, we have $H^0(X, E^\vee \otimes \overline{F}_1) = 0$. The claimed vanishing, $H^0(X, E^\vee \otimes G) = 0$ now follows from the exact sequence below (the cohomology sequence associated to (3) tensored with $E^\vee$)

$$H^0(X, E^\vee \otimes \overline{F}_1) \to H^0(X, E^\vee \otimes G) \to H^1(X, \mathcal{E}nd E(-d)).$$

Therefore, we have (from the cohomology sequence associated to (2))

$$0 \to H^0(X, E^\vee \otimes \overline{F}_0) \to H^0(X, \mathcal{E}nd(E)) \to H^1(X, E^\vee \otimes G) \to 0.$$

The last term is a 1-dimensional vector space by Theorem 5 (b). This completes the proof. □

This gives the following

Corollary 14 (Theorem 4) *Let* E *be an initialized, indecomposable rank* 3 *ACM bundle on a general hypersurface* $X \subset \mathbb{P}^4$ *of degree* $d \geqslant 3$. *Assume that* $c_1(E) > 0$. *Then either* E *is simple or* $\text{reg}(E) = d - 1$.

Proof By Corollary 9, $m \leqslant d - 1$. If $H^0(E^\vee) = 0$, then E is simple by Lemma 13. So let $H^0(E^\vee) \neq 0$ and assume that $m < d-1$. Then $E' := E^\vee(d-m-1)$ is a rank 3, initialized, indecomposable ACM bundle on X and

$$H^0(E'^\vee) = H^0(E(1 + m - d)) = 0.$$

Thus, E' is simple which implies that E is simple. □

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Declarations

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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