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GROTHENDIECK-LEFSCHETZ AND NOETHER-LEFSCHETZ FOR BUNDLES

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ABSTRACT. We prove a mild strengthening of a theorem of Česnavičius which gives a criterion for a vector bundle on a smooth complete intersection of dimension at least 3 to split into a sum of line bundles. We also prove an analogous statement for bundles on a general complete intersection surface.

1. INTRODUCTION

1.1. Recently, K. Česnavičius (see [2]) has proved a conjecture of Dao ([4], 7.2.2) generalizing the Grothendieck-Lefschetz theorem to arbitrary rank vector bundles. As an application, the following result is proved.

Theorem 1 ([2], Theorem 1.2). *Let X be a global complete intersection of dimension at least 3. A vector bundle E on X splits into a sum of line bundles of the form $\oplus_i \mathcal{O}_X(a_i)$ if and only if it satisfies the conditions*

$$\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0 = \mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E(\nu)) \qquad \forall\ \nu \in \mathbb{Z}.$$

For odd-dimensional hypersurfaces, Dao proves a stronger result:

Theorem 2 ([4], 8.3.4). *A vector bundle E on an odd dimensional hypersurface of dimension at least 3 splits into a sum of line bundles of the form $\oplus_i \mathcal{O}_X(a_i)$ if and only if*

$$\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0\ \ \forall\ \ \nu \in \mathbb{Z}.$$

The approach of both Dao and Česnavičius is purely algebraic and both these results follow as a consequence of a statement in commutative algebra concerning the depth of modules on the punctured spectrum of a local ring.

The purpose of this short note is twofold: firstly, to situate the above theorems in a purely geometric context (both in terms of the results and techniques) and secondly, to recast the proof in [2] in the language of geometry (as opposed to commutative algebra) to obtain a strengthening of Theorem 1 – see Theorems 4 and 5. This helps us avoid invoking a theorem of Huneke-Wiegand [7] in commutative algebra as done in [2]; instead we invoke a result of Kempf [8] (and its strengthening due to Mohan Kumar [9]). Doing so enables us to stay firmly in the realm of Grothendieck's Lefschetz theory. We also note that the strengthening that we

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obtain is at the cost of a stronger hypothesis – while the result in [2] is valid for arbitrary complete intersections, the results here are stated only for *smooth* complete intersections.

An analogue of Theorem 1 with a similarly weakened hypothesis is shown to be true when X is a *general* surface in $\mathbb{P}^3$ of degree $d \geq 4$ – see Theorem 7. The analogous Noether-Lefschetz theorem for a general complete intersection surface with $K_X \geq 0$ is more along the theorem in [2] (see Theorem 8 for a precise statement).

We have strived to keep the article self-contained by providing as many details as possible (even for some standard proofs) at the risk of annoying the expert in the hope that it may provide an easy entry-point for the beginner.

1.2. On the geometric side, this story begins with the following result of Kempf.

Theorem 3 ([8]). *A vector bundle E on $\mathbb{P}^n$ for $n \geq 2$ splits into a sum of line bundles if and only if*

(1) $\mathrm{H}^1(\mathbb{P}^n, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0$ *for all* $\nu < 0$, *and*
(2) E *extends to a vector bundle on* $\mathbb{P}^{n+1}$.

In [9], it is shown (among other things) that (1) $\Rightarrow$ (2) in the theorem above. A further generalisation of this result to principal bundles appears in [1]. The proofs in all these theorems use Grothendieck's Lefschetz theory (see [3]) in an essential way.

A natural question then is to ask if there is a version of this theorem for hypersurfaces (and more generally complete intersections) in projective space. Framed in this context, Theorem 2 of Dao then suggests a possible version of this theorem. Furthermore, an approach along the lines of Grothendieck's proof of the Lefschetz theorem for Picard groups suggests that the proof of a splitting theorem for bundles on hypersurfaces should consist of the following two steps:

- firstly, to show that under suitable hypotheses, the bundle E on X extends as a bundle to an open set $U \subset \mathbb{P}^n$ and hence a reflexive sheaf F on all of $\mathbb{P}^n$, and
- secondly, show that the hypotheses imply that the sheaf F splits into a sum of line bundles on $\mathbb{P}^n$.

Carrying out these two steps yields us the following result

Theorem 4. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of dimension at least* 3. *A vector bundle E splits into a sum of line bundles if and only if*

$$\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0 = \mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E(\nu)) \ \forall\, \nu < 0.$$

Remark 1. Recall that when X is a *smooth* complete intersection of dimension at least 3, $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(1)$ by the Grothendieck-Lefschetz theorem. Hence the conclusion of the theorem above is the same as Theorem 1.

Remark 2. The (idea of the) proof here and Dao's result ([4], 7.2.2) suggest that the vanishing for H^1 in positive twists in *op. cit.* can perhaps replace the H^2 vanishing above; i.e., that the conditon that $\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0$ for $\nu \geq 0$ implies that the bundle E on X extends to an open set $U \subset \mathbb{P}^n$. Note that when $\dim X = 3$, this is precisely the case as H^1 is dual to H^2. When $X = \mathbb{P}^{n-1}$, such an extension is automatic since one can take the pull-back of E along the projection map $\pi : \mathbb{P}^n \backslash p \to \mathbb{P}^{n-1}$ for any point $p \in \mathbb{P}^n \backslash \mathbb{P}^{n-1}$. In fact, in [9], Mohan Kumar proves that E extends as a bundle to $\mathbb{P}^n$ by considering pull-backs along two distinct points

$p, q \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$, and shows that the vanishing of $\mathrm{H}^1(\mathbb{P}^{n-1}, \mathcal{E}\mathrm{nd}\, E(\nu))$ for all $\nu < 0$ implies, once again by Grothendieck's Lefschetz theory, that both these extensions are isomorphic in some open set $V \subset \mathbb{P}^n \setminus \{p, q\}$ containing $\mathbb{P}^{n-1}$, and hence can be patched to give an extension bundle on $\mathbb{P}^n$.

Using Theorem 4 as the base case, we prove the following mild strengthening of Theorem 1:

Theorem 5. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of dimension at least 3 and multi-degree $(d_1, \cdots, d_r)$ with $1 < d_1 \leq \cdots \leq d_r$. A vector bundle E on X splits into a sum of line bundles if and only if*

(a) $\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0 \;\forall\, \nu < 0$.
(b) $\mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0 \;\forall\, \nu \in \mathbb{Z}$.

It has been noted elsewhere by the authors that Lefschetz theorems seemingly always occur in pairs – a *Grothendieck-Lefschetz theorem* for smooth hypersurfaces in high dimensions, and a *Noether-Lefschetz theorem* for generic hypersurfaces (usually of sufficiently high degree) in lower dimensions. With this in view, we now proceed to the next result, namely a criterion for a bundle on a surface in $\mathbb{P}^3$ to split into a sum of line bundles.

Let E be a bundle on a *generic* hypersurface of degree d in a smooth projective 3-fold Y with ample polarisation $\mathcal{O}_Y(1)$. Let $\eta \in \mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E(-d))$ be the obstruction class whose vanishing is necessary and sufficient for E to extend to a bundle on X_1 – the first order thickening of X in Y (see [10] for details). Theorem 4 obviously fails when $Y = \mathbb{P}^3$ because the hypothesis on H^2, which is the top cohomology of the self-dual bundle $\mathcal{E}\mathrm{nd}\, E$ and hence dual to its zeroth cohomology, can never hold by Serre's theorem. However, the fact that E lives on a generic hypersurface implies that under the Kodaira-Spencer map

$$\mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E(-d)) \to \mathrm{Hom}\left(\mathrm{H}^0(X, \mathcal{O}_X(d)), \mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E)\right), \quad \eta \mapsto (g \mapsto g.\eta)$$

the class η maps to 0, i.e., for any $g \in \mathrm{H}^0(X, \mathcal{O}_X(d))$, $g.\eta = 0$ in $\mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E)$. Using these ideas, in *op. cit.*, we proved the following extension theorem.

Theorem 6 ([10], Theorem 2)**.** *Let Y be a smooth 3-fold and $X \subset Y$ be a general, ample hyperplane section of Y. Let E be a bundle on X such that the "multiplication" map*

$$\mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E \otimes K_X(a)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(b)) \to \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E \otimes K_X(a+b))$$

is surjective $\forall\, a,\ b \geq 0$. Then there exists a Zariski open set $U \subset Y$ containing X and a bundle $\widetilde{E}$ on U such that $\widetilde{E} \otimes \mathcal{O}_U \cong E$.

As a consequence of this, we have the following version of Theorem 4 for surfaces in $\mathbb{P}^3$:

Theorem 7. *Let $X \subset \mathbb{P}^3$ be a general surface of degree $d \geq 4$. A vector bundle E on X splits into a sum of line bundles of the form $\oplus_i \mathcal{O}_X(a_i)$ if it satisfies the following*

(a) $\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0 \;\forall\, \nu < 0$.
(b) *the multiplication map*

$$\mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(a)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(b)) \to \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(a+b))$$

is surjective $\forall\, a,\ b \geq 0$.

Similar techniques also yield the following (slightly weaker) Noether-Lefschetz theorem for bundles on a *general* complete intersection surface.

Theorem 8. *Let $X \subset \mathbb{P}^{r+2}$ be a general complete intersection surface of multi-degree $(d_1, \cdots, d_r)$ such that $\sum_i d_i \geq r+3$. A vector bundle E on X splits into a sum of line bundles of the form $\oplus \mathcal{O}_X(a_i)$ if and only if*

(i) $\mathrm{H}^1(X, \mathcal{E}nd\, E(\nu)) = 0 \ \forall\, \nu \in \mathbb{Z}$.

(ii) *the multiplication map*

$$\mathrm{H}^0(X, \mathcal{E}nd\, E(a)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(b)) \to \mathrm{H}^0(X, \mathcal{E}nd\, E(a+b))$$

is surjective $\forall\, a, b \in \mathbb{Z}$ such that $\mathrm{H}^0(X, \mathcal{E}nd\, E(a)) \neq 0$ and $b \geq 0$.

Remark 3. Note that $\mathcal{O}_X$ is a direct summand of $\mathcal{E}nd\, E$ for any vector bundle E. For condition (b) in Theorem 7 (respectively (ii) in Theorem 8) to hold, we will require it to hold for the structure sheaf $\mathcal{O}_X$ as well. This is how the condition on the (multi-) degree comes into play.

2. Preliminary results

In this section we prove some results which will be used in the proofs of the theorems stated above.

Lemma 1. *Let Y be any ringed space. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of modules on Y such that $H^p(Y, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) = 0$ for all $p \geq 1, q \geq 1$. Then for $i \geq 2$ there exists a long exact sequence*

$$\begin{aligned} \cdots \to H^0(Y, \mathcal{E}xt^{i-1}(\mathcal{F}, \mathcal{G})) \to H^i(Y, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \\ \to Ext^i(\mathcal{F}, \mathcal{G}) \to H^0(Y, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})) \to \cdots \end{aligned} \tag{1}$$

Proof. There exists a local-to-global Ext spectral sequence:

$$E_2^{p,q} = H^p(Y, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow Ext^{p+q}(\mathcal{F}, \mathcal{G}).$$

This means that there is a filtration on $Ext^i(\mathcal{F}, \mathcal{G})$:

$$0 \subseteq F^i \subseteq F^{i-1} \subseteq \cdots \subseteq F^1 \subset F^0 = Ext^i(\mathcal{F}, \mathcal{G})$$

with $E_\infty^{p,i-p} = F^p/F^{p+1}$.

Since this is a first quadrant sequence and by assumption, $E_2^{p,q} = 0$ for $p > 0, q > 0$, we get

$$F^0/F^1 = E_\infty^{0,i} \cong Ker(E_{i+1}^{0,i} \to E_{i+1}^{i+1,0}) \cong Ker(E_2^{0,i} \to E_2^{i+1,0}),$$

$$F^1 = F^2 = \cdots = F^i, \text{ and } F^i \cong E_\infty^{i,0} \cong \frac{E_2^{i,0}}{Im(E_2^{0,i-1} \to E_2^{i,0})}.$$

Putting all this together will yield the claimed sequence. □

Recall that the *singular locus* of any coherent sheaf $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is the locus of points where $\mathcal{F}$ is not locally free.

Lemma 2. *Let Y be a smooth projective variety of dimension $n \geq 3$ and ample polarisation $\mathcal{O}_Y(1)$. Let F be a reflexive sheaf on Y such that the singular locus of F is a finite set of points. Then we have for $i = 0, 1$,*

$$\mathrm{H}^i(Y, F(\nu)) = 0 \ \ \forall\, \nu \ll 0.$$

Proof. For $i = 0$, the statement follows since F is a torsion-free sheaf. Recall that Serre duality gives an isomorphism

$$\mathrm{H}^i(Y, F(\nu))^\vee \cong \mathrm{Ext}^{n-i}(F(\nu), \omega_Y).$$

Since $\dim \mathrm{Sing(F)} = 0$, we see that $\mathrm{H}^p(Y, \mathcal{E}xt^q_Y(F(\nu), \omega_Y)) = 0$ for all $p \geq 1, q \geq 1$. Therefore by Lemma 1 we have the following sequence for $i \geq 2$:

(2)
$$\cdots \to H^i(Y, \mathcal{H}om(F(\nu), \omega_Y)) \to Ext^i(F(\nu), \omega_Y) \to H^0(Y, \mathcal{E}xt^i(F(\nu), \omega_Y)) \to \cdots.$$

Since F is reflexive, $\mathrm{depth}(F_x) \geq 2$ (Proposition 1.3 of [5]), and so by the Auslander-Buchsbaum formula, we have $\mathcal{E}xt^i_Y(F, \omega_Y)) = 0$ for $i > n-2$. Thus, letting $i = n-1$ in equation (2) yields a surjection

$$\mathrm{H}^{n-1}(Y, \mathcal{H}om(F(\nu), \omega_Y)) \twoheadrightarrow \mathrm{Ext}^{n-1}(F(\nu), \omega_Y).$$

By Serre vanishing theorem, $\mathrm{H}^{n-1}(Y, \mathcal{H}om(F(\nu), \omega_Y)) = 0$ for $\nu \ll 0$. Therefore

$$\mathrm{H}^1(Y, F(\nu))^\vee \cong \mathrm{Ext}^{n-1}(F(\nu), \omega_Y) = 0 \quad \text{for } \nu \ll 0. \qquad \square$$

Corollary 1. *Let Y be a smooth, projective variety of dimension $n \geq 3$ with ample polarisation $\mathcal{O}_Y(1)$ and let $X \subset Y$ be a smooth degree d hypersurface. Let E be a bundle on X such that $\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0$ for all $\nu < 0$. Further assume that E extends to a reflexive sheaf F on Y which is a bundle away from at most a finite set (in the complement of X). Then $\mathrm{H}^1(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0$ for $\nu < 0$.*

Proof. We have a short exact sequence

$$0 \to \mathcal{E}\mathrm{nd}\, F(\nu - d) \to \mathcal{E}\mathrm{nd}\, F(\nu) \to \mathcal{E}\mathrm{nd}\, E(\nu) \to 0.$$

Taking cohomology and using the hypothesis, we get a surjection

$$\mathrm{H}^1(Y, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \twoheadrightarrow \mathrm{H}^1(Y, \mathcal{E}\mathrm{nd}\, F(\nu))\ \forall\, \nu < 0.$$

The statement now follows from Lemma 2. $\square$

Lemma 3. *Let $X \subset Y$ be as above. For $i \neq 0$, assume that we have a reflexive sheaf $\mathcal{F}$ such that*

(a) *the map $\mathrm{Ext}^i(\mathcal{F}(\nu), \omega_Y) \to \mathrm{Ext}^i(\mathcal{F}(\nu - d), \omega_Y)$ is an isomorphism $\forall\ \nu < 0$.*
(b) *$\mathcal{E}xt^i(\mathcal{F}, \omega_Y)$ is supported on a finite set in $Y \setminus X$.*

Then the map $\mathrm{Ext}^i(\mathcal{F}(\nu), \omega_Y) \to \mathrm{H}^0(Y, \mathcal{E}xt^i(\mathcal{F}(\nu), \omega_Y))$ is an isomorphism $\forall\ \nu < 0$. In addition, if the map in (a) *is an injection $\forall\ \nu \geq 0$, then so is the map above.*

Proof. The local-to-global spectral sequence in Lemma 1 yields the following sequence:

$$\mathrm{H}^i(Y, \mathcal{F}^\vee \otimes \omega_Y(-\nu)) \to \mathrm{Ext}^i(\mathcal{F}(\nu), \omega_Y)$$
$$\to \mathrm{H}^0(Y, \mathcal{E}xt^i(\mathcal{F}(\nu), \omega_Y)) \to \mathrm{H}^{i+1}(Y, \mathcal{F}^\vee \otimes \omega_Y(-\nu)).$$

For $-\nu \gg 0$, the extreme terms vanish by Serre vanishing, and so we have an isomorphism

$$\mathrm{Ext}^i(\mathcal{F}(\nu), \omega_Y) \cong \mathrm{H}^0(Y, \mathcal{E}xt^i(\mathcal{F}(\nu), \omega_Y)) \ \ \forall\ -\nu \gg 0.$$

On the other hand, the natural inclusion $\mathcal{O}_Y(-d) \hookrightarrow \mathcal{O}_Y$ yields a commutative square:

$$\begin{array}{ccc} \operatorname{Ext}^i(\mathcal{F}(\nu), \omega_Y) & \to & \mathrm{H}^0(Y, \mathcal{E}xt^i(\mathcal{F}(\nu), \omega_Y)) \\ \downarrow & & \downarrow\cong \\ \operatorname{Ext}^i(\mathcal{F}(\nu - d), \omega_Y) & \to & \mathrm{H}^0(Y, \mathcal{E}xt^i(\mathcal{F}(\nu - d), \omega_Y)). \end{array} \tag{3}$$

The right vertical arrow is an isomorphism for all $\nu \in \mathbb{Z}$ since the sheaf $\mathcal{E}xt^i(\mathcal{F}(\nu), \omega_Y)$ is supported on a finite set in the complement of X. The result now follows since the horizontal arrows are isomorphisms for $\nu \ll 0$. □

Corollary 2. *In the situation above, let $Z \subset Y$ be a smooth hypersurface defined by $g \in \mathrm{H}^0(Y, \mathcal{O}_Y(a))$ for some $a \geq 0$ such that $Z \cap \operatorname{Sing}(\mathcal{F}) = \emptyset$. Then the natural map*

$$\operatorname{Ext}^i(\mathcal{F}(\nu), \omega_Y) \to \operatorname{Ext}^i(\mathcal{F}(\nu - a), \omega_Y)$$

is an isomorphism $\forall\ \nu < 0$. In addition, if the map in Lemma 3(a) *is an injection $\forall\ \nu \geq 0$, then so is the map above.*

Proof. Since Z does not meet $\operatorname{Sing}(\mathcal{F})$, we see that multiplication by g induces an isomorphism

$$g\times : \mathcal{E}xt^i(\mathcal{F}(\nu), \omega_Y) \to \mathcal{E}xt^i(\mathcal{F}(\nu - a), \omega_Y).$$

The statement is now immediate from the commutative square (3) with 'd' replaced by 'a' and the vertical maps induced by g. □

3. Proofs of Theorems 4 and 5

In this section we prove the Grothendieck-Lefschetz theorem for vector bundles on hypersurfaces and complete intersections stated in the introduction.

Proof of Theorem 4. Let X_k be the k-th order thickening of X given by the vanishing of f^{k+1} where f is the defining polynomial of X. The obstruction for a bundle F on X_{k-1} to lift to a bundle F' on X_k is given by an element $\eta_k \in \mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E(-kd))$ (i.e., given F, there exists F' on X_k such that $F'|_{X_{k-1}} \cong F$ if and only if $\eta_k = 0$).

The hypothesis that $\mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, E(-\nu)) = 0$ for all $\nu > 0$ implies that the bundle E extends to every thickening X_k of X, and consequently by Grothendieck's Lefschetz conditions (see [3]), there exists a formal bundle $\widehat{E}$ on the formal completion $\widehat{X}$. By Grothendieck's algebraization theorem (see *op. cit.*), there exists an open set $U \subset \mathbb{P}^n$ containing X and a bundle F_U on U such that $F_U \otimes \mathcal{O}_X \cong E$. Let F be the reflexive sheaf obtained by extending F_U as a coherent sheaf on X followed by replacing it with its double dual. Since F is a bundle along X, and X is ample, this means that there is a finite set $S \subset \mathbb{P}^n \setminus X$ such that F is a bundle when restricted to the complement $\mathbb{P}^n \setminus S$. By Corollary 1,

$$\mathrm{H}^1(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0, \ \nu < 0. \tag{4}$$

The hypothesis on E applied to the cohomology sequence associated to the exact sequence

$$0 \to \mathcal{E}\mathrm{nd}\, F(-d) \to \mathcal{E}\mathrm{nd}\, F \to \mathcal{E}\mathrm{nd}\, E \to 0, \tag{5}$$

yields an isomorphism

$$\mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \cong \mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) \ \forall\ \nu < 0. \tag{6}$$

Let $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a general hyperplane section so that $\mathbb{P}^{n-1} \cap S = \emptyset$. Let $\ell = 0$ be its defining equation for some linear polynomial ℓ. The restriction $F' := F_{|\mathbb{P}^{n-1}}$ is a bundle on $\mathbb{P}^{n-1}$. By Corollary 2, we have isomorphisms

$$\ell : \mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, F(\nu - 1)) \xrightarrow{\cong} \mathrm{H}^2(X, \mathcal{E}\mathrm{nd}\, F(\nu)) \ \forall\ \nu < 0.$$

Once again, taking cohomology of the exact sequence

$$0 \to \mathcal{E}\mathrm{nd}\, F(-1) \to \mathcal{E}\mathrm{nd}\, F \to \mathcal{E}\mathrm{nd}\, F' \to 0, \tag{7}$$

it follows from (4) and (6) that

$$\mathrm{H}^1(\mathbb{P}^{n-1}, \mathcal{E}\mathrm{nd}\, F'(\nu)) = 0 \ \ \forall\ \nu < 0.$$

By (Mohan Kumar's refinement of) Kempf's criterion (see [9]), F' splits into a sum of line bundles. We claim that this implies that F splits into a sum of line bundles (in particular, F is a bundle). Consequently, E splits into a sum of line bundles.

The fact that the splitting of F' implies the splitting of F is a standard argument (see [6]). We give a quick proof for the sake of completeness.

Let $\varphi : F' \to \oplus_i \mathcal{O}_{\mathbb{P}^{n-1}}(a_i)$ denote the isomorphism. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & F(-1) & \to & F & \to & F' & \to & 0 \\ & & & & & & \downarrow \varphi & & \\ 0 & \to & \oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i - 1) & \to & \oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i) & \to & \oplus_i \mathcal{O}_{\mathbb{P}^{n-1}}(a_i) & \to & 0. \end{array}$$

We want to claim that the isomorphism φ lifts to an isomorphism

$$\Phi : F \to \oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i)$$

so that the diagram above commutes. Notice that the isomorphism φ is an element of

$$\mathrm{Hom}(F', \oplus_i \mathcal{O}_{\mathbb{P}^{n-1}}(a_i)) \cong \mathrm{H}^0(\mathbb{P}^{n-1}, \oplus_i F'^{\vee}(a_i)).$$

In order to guarantee that φ lifts to a morphism Φ, it is enough to show that, in the cohomology sequence

$$\mathrm{H}^0(\mathbb{P}^n, \oplus_i F^{\vee}(a_i)) \to \mathrm{H}^0(\mathbb{P}^{n-1}, \oplus_i F'^{\vee}(a_i)) \to \mathrm{H}^1(\mathbb{P}^n, \oplus_i F^{\vee}(a_i - 1))$$

the last term vanishes. In fact, much more is true. Since F' splits, this implies that F' has no intermediate cohomology (i.e., $\mathrm{H}^i(\mathbb{P}^{n-1}, F'(\nu)) = 0$ for all $\nu \in \mathbb{Z}$, and $0 < i < n-1$), and so a similar argument as before using the exact sequence

$$0 \to F^{\vee}(-1) \to F^{\vee} \to F'^{\vee} \to 0$$

yields that $F^{\vee}$ and similarly F have no intermediate cohomology. Thus we see that the isomorphism φ lifts to a morphism Φ. All that remains is to show that Φ is an isomorphism, or equivalently that $\det \Phi$ has no zeroes.

Since $\det(\Phi)_{|\mathbb{P}^{n-1}} = det\varphi \neq 0$, this is saying that $\det \Phi \neq 0$ along the (ample) hyperplane $\mathbb{P}^{n-1}$, and so must be non-zero in an open set $V \subset \mathbb{P}^n$ containing this $\mathbb{P}^{n-1}$. Hence Φ is nowhere vanishing. $\square$

Proof of Theorem 5. The proof is by induction on the codimension r of X. We will use as our base case the case when $X \subset \mathbb{P}^n$ is a smooth hypersurface and so we are done by Theorem 4. Let Y be a smooth complete intersection of multi-degree $(d_1, \cdots, d_{r-1})$ containing X as a hypersurface of degree $d := d_r$. The vanishing of

H^2 for $\nu < 0$ implies that E extends as a reflexive sheaf F on Y which, by the H^1 vanishing in the hypothesis and Corollary 1, satisfies

$$\mathrm{H}^1(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0, \quad \text{for } \nu < 0. \tag{8}$$

The hypotheses applied to the cohomology sequence of

$$0 \to \mathcal{E}\mathrm{nd}\, F(\nu - d) \to \mathcal{E}\mathrm{nd}\, F(\nu) \to \mathcal{E}\mathrm{nd}\, E(\nu) \to 0$$

also yields

$$\mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \cong \mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) \ \forall\ \nu < 0, \tag{9}$$

$$\mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \twoheadrightarrow \mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) \ \forall\ \nu \geq 0, \tag{10}$$

$$\mathrm{H}^3(Y, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \hookrightarrow \mathrm{H}^3(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) \ \forall\ \nu \in \mathbb{Z}. \tag{11}$$

The injections in the last statement together with Serre vanishing imply that

$$\mathrm{H}^3(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0 \ \forall\ \nu \in \mathbb{Z}. \tag{12}$$

Now let $\mathbb{P}^{n-1}$ be a general hyperplane with defining polynomial ℓ, and $X' := X \cap \mathbb{P}^{n-1}$. Then $F' := F_{|X'}$ is a bundle on X'. The idea now is to argue as before using the exact sequence

$$0 \to \mathcal{E}\mathrm{nd}\, F(\nu - 1) \xrightarrow{\ell\times} \mathcal{E}\mathrm{nd}\, F(\nu) \to \mathcal{E}\mathrm{nd}\, F'(\nu) \to 0,$$

and to prove that the hypothesis of the Theorem holds for the bundle F' on X'.

By Corollary 2 and Serre duality, we have that the map

$$\ell\times : \mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu - 1)) \to \mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) \tag{13}$$

is an isomorphism for $\nu < 0$ and a surjection for $\nu \geq 0$.

The desired vanishings for the bundle F' follow from (8), (13) and (12). By induction, F' splits, and as before, this implies that F and hence E also split into a sum of line bundles. $\square$

4. Proofs of Theorems 7 and 8

In this section we prove the Noether-Lefschetz type theorems for vector bundles on surfaces in $\mathbb{P}^3$ and surfaces which are complete intersections.

Proof of Theorem 7. Condition (b) implies, by Theorem 6, that E extends as a reflexive sheaf to $\mathbb{P}^3$. As in the previous proof, hypothesis (a) yields surjections

$$\mathrm{H}^1(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \twoheadrightarrow \mathrm{H}^1(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu)) \ \text{ for } \ \nu < 0,$$

which by Corollary 1 yields

$$\mathrm{H}^1(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0 \ \text{ for } \ \nu < 0. \tag{14}$$

Furthermore, from hypothesis (a) and Serre duality we have vanishings $\mathrm{H}^1(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0$ for $\nu < 0$ or $\nu > d - 4$. Hence we have injections

- $\mathrm{H}^2(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \hookrightarrow \mathrm{H}^2(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu))$, $\nu < 0$ and
- $\mathrm{H}^2(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu)) \hookrightarrow \mathrm{H}^2(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu + d))$, $\nu > -4$.

Applying Serre vanishing to the second statement yields $\mathrm{H}^2(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0$ for $\nu > -4$. Using this vanishing in the first statement, we conclude that

$$\mathrm{H}^2(\mathbb{P}^3, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0 \ \text{ for } \ \nu \in \mathbb{Z}. \tag{15}$$

Using the vanishings in (14) and (15) and arguing as in the proof above yields that the restriction of F to a general hyperplane $\mathbb{P}^2$ satisfies Kempf's criterion and hence splits into a sum of line bundles. Thus we are done. $\square$

Proof of Theorem 8. To prove the statement, we proceed as in the proof of the previous theorem. The first goal is to show that E extends to a reflexive sheaf F on Y, a smooth complete intersection 3-fold containing X. Condition (ii) above ensures that such an extension exists. Next, condition (i) ensures that $\mathrm{H}^1(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0 = \mathrm{H}^2(Y, \mathcal{E}\mathrm{nd}\, F(\nu))$ for all $\nu \in \mathbb{Z}$. Hence the restriction of F to a general hypersurface $X' \subset Y$ defined by a linear polynomial ℓ – denoted by F' – satisfies the condition that $\mathrm{H}^1(X', \mathcal{E}\mathrm{nd}\, F'(\nu)) = 0$ for all $\nu \in \mathbb{Z}$. To apply induction, it is now enough to show that condition (ii) holds for F'. To see this, we first prove that the multiplication map

$$\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a)) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(b)) \to \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a+b)) \tag{16}$$

is surjective for a, b such that $\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a)) \neq 0$ and $b \geq 0$. To do this, we note that since $\mathrm{H}^1(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0$ for all $\nu \in \mathbb{Z}$, we have an exact sequence

$$0 \to \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu - d)) \to \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) \to \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(\nu)) \to 0 \ \forall\, \nu \in \mathbb{Z}.$$

Furthermore, if ν_0 is such that $\mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(\nu)) = 0$ for $\nu < \nu_0$, then $\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0$ for $\nu < \nu_0$ as well. If not, we will have an isomorphism $\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu - a)) \cong \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu))$ for $\nu < \nu_0$ and for all $a \geq 0$. Since $\mathcal{E}\mathrm{nd}\, F$ is torsion free, this group vanishes for $\nu \ll 0$, and so we arrive at a contradiction.

Suppose that $s \in \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a+b))$; then we can write $s = f^k s'$ for some $k \geq 0$ and $s' \in \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a+b-kd))$ such that $f \nmid s'$, so that $s' \mapsto \bar{s}' \neq 0$ in $\mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(a+b-kd))$. In the commutative diagram

(17)
$$\begin{array}{ccc}
\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu_0) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(a+b-kd-\nu_0)) & \to & \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a+b-kd)) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(\nu_0)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(a+b-kd-\nu_0)) & \to & \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(a+b-kd))
\end{array}$$

the bottom horizontal map is a surjection (by our hypothesis) and so the element $\bar{s}'$ above lifts to an element $\sum_j \bar{s}_j \otimes g_j \in \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, E(\nu_0)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(a+b-kd-\nu_0))$. This element in turn is the image of an element $\sum_j s_j \otimes g_j \in \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu_0)) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(a+b-kd-\nu_0))$ as the map on global sections is surjective. We claim that $\sum_j g_j.s_j = s' \in \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a+b-kd))$. If not, on one hand, the difference $\sum_j g_j.s_j - s'$ is divisible by f but, on the other hand, $f \nmid s_j$ for any j and hence it does not divide $\sum_j g_j.s_j$, and this is a contradiction. Hence

$$\sum_j s_j \otimes f^k g_j \in \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu_0)) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(a+b-\nu_0)).$$

Finally, associativity of tensor products gives us a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu_0)) \otimes \big(\mathrm{H}^0(Y, \mathcal{O}_Y(a-\nu_0)) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(b))\big) & \to & \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a)) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(b)) \\
\downarrow 1 \otimes \mu & & \downarrow \\
\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(\nu_0)) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(a+b-\nu_0)) & \to & \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a+b)).
\end{array}$$

Since $\mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, E(a)) \neq 0$, this means that $a \geq \nu_0$, which means that $a - \nu_0 \geq 0$. Hence $\mathrm{H}^0(Y, \mathcal{O}_Y(a-\nu_0)) \neq 0$, and so all the cohomology groups are clearly non-zero in the above diagram.

The image of the element $\sum_j s_j \otimes f^k g_j$ under the bottom horizontal arrow coincides with the element obtained by first lifting it via the left vertical surjection (the multiplication map μ is a surjection) and then taking its image under the top horizontal arrow followed by the right vertical arrow. This proves the desired surjectivity in (16).

It can now be checked that if F' is the restriction of F to a general hyperplane $X' := Y \cap \mathbb{P}^{n+r-1}$, then F' is a bundle on X' which satisfies the hypothesis of the theorem – condition (i) implies that $\mathrm{H}^i(Y, \mathcal{E}\mathrm{nd}\, F(\nu)) = 0$ for $i = 1, 2$, and hence $\mathrm{H}^1(X', \mathcal{E}\mathrm{nd}\, F'(\nu)) = 0$ for $\nu \in \mathbb{Z}$, whereas condition (ii) for F' can be verified using the commutative square:

$$\begin{array}{ccc} \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a)) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(b)) & \twoheadrightarrow & \mathrm{H}^0(Y, \mathcal{E}\mathrm{nd}\, F(a+b)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, F'(a)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(b)) & \to & \mathrm{H}^0(X, \mathcal{E}\mathrm{nd}\, F'(a+b)). \end{array} \tag{18}$$

The surjectivity of the top horizontal and right vertical imply the surjectivity of the bottom horizontal arrow. By induction on the multi-degree (or equivalently on the codimension of the complete intersection) , we have that F' splits into a sum of line bundles. As a consequence F and so E split. Thus we are done. $\square$

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