ON THE PICARD BUNDLE

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Abstract. Fix a holomorphic line bundle \( \xi \) over a compact connected Riemann surface \( X \) of genus \( g \), with \( g \geq 2 \), and also fix an integer \( r \) such that \( \text{deg}(\xi) > r(2g-1) \). Let \( \mathcal{M}_\xi(r) \) denote the moduli space of stable vector bundles over \( X \) of rank \( r \) and determinant \( \xi \). The Fourier–Mukai transform, with respect to a Poincaré line bundle on \( X \times J(X) \), of any \( F \in \mathcal{M}_\xi(r) \) is a stable vector bundle on \( J(X) \). This gives an embedding of \( \mathcal{M}_\xi(r) \) in a moduli space associated to \( J(X) \). If \( g = 2 \), then \( \mathcal{M}_\xi(r) \) becomes a Lagrangian subvariety.

Résumé

Sur le fibré de Picard. Soient \( \xi \) un fibré en droites holomorphe sur une surface de Riemann compacte connexe \( X \) de genre \( g \geq 2 \), et \( r \) un entier tel que \( \text{deg}(\xi) > r(2g-1) \). Notons \( \mathcal{M}_\xi(r) \) l’espace de modules des fibrés vectoriels stables sur \( X \), de rang \( r \) et de déterminant \( \xi \). Ayant choisi un fibré de Poincaré sur \( X \times J(X) \), la transformée de Fourier–Mukai associée fait correspondre à un fibré \( F \in \mathcal{M}_\xi(r) \) un fibré vectoriel stable sur \( J(X) \). Ceci fournit un plongement de \( \mathcal{M}_\xi(r) \) dans un espace de modules associé à \( J(X) \). Lorsque \( g = 2 \), \( \mathcal{M}_\xi(r) \) s’identifie ainsi à une sous-variété lagrangienne de cet espace de modules.

1. Introduction

Let \( (X,x_0) \) be a one–pointed compact connected Riemann surface of genus \( g \), with \( g \geq 2 \). Let \( \mathcal{L} \) be the Poincaré line bundle on \( X \times J(X) \) constructed using \( x_0 \), where \( J(X) \) is the Jacobian of \( X \). Fix an integer \( r \geq 2 \) and a holomorphic line bundle \( \xi \) over \( X \) with \( \text{deg}(\xi) > r(2g-1) \). Let \( \mathcal{M}_\xi(r) \) denote the moduli space of stable vector bundles over \( X \) of rank \( r \) and determinant \( \xi \).

In Lemma 2.1 we show that for any \( F \in \mathcal{M}_\xi(r) \),

\[ \mathcal{V}_F := \phi_{J*}(\mathcal{L} \otimes \phi_X^* F) \]

is a stable vector bundle with respect to the canonical polarization on \( J(X) \), where \( \phi_J \) (respectively, \( \phi_X \)) is the projection of \( X \times J(X) \) to \( J(X) \) (respectively, \( X \)).

Rational characteristic classes of \( \mathcal{V}_F \), as well as the line bundle \( \bigwedge^{\text{top}} \mathcal{V}_F \), are independent of \( F \). Let \( \mathcal{M}(J(X)) \) be the moduli space of stable vector bundles \( W \) over \( J(X) \) with \( \text{rank}(W) = \text{rank}(\mathcal{V}_F), c_i(W) = c_i(\mathcal{V}_F) \) and \( \bigwedge^{\text{top}} W = \bigwedge^{\text{top}} \mathcal{V}_F \). The map \( \mathcal{M}_\xi(r) \longrightarrow \mathcal{M}(J(X)) \) defined by \( F \longmapsto \mathcal{V}_F \) is an embedding (see Corollary 2.3).

We next assume that \( g = 2 \), and if degree(\( \xi \)) is even, then also assume that \( r \geq 3 \). Let \( \mathcal{M}^0(J(X)) \subset \mathcal{M}(J(X)) \) be the locus of all \( W \) for which the image of \( C_1(W)^2 - 2 \cdot C_2(W) \in \text{CH}^2(J(X)) \) in the Deligne–Beilinson cohomology vanishes.

Notation: The \( i \)–th Chern class with values in the Chow group will be denoted by \( C_i \).

We show that the image of \( \mathcal{M}_\xi(r) \) lies in \( \mathcal{M}^0(J(X)) \), and \( \mathcal{M}_\xi(r) \) is a Lagrangian subvariety of the symplectic variety \( \mathcal{M}^0(J(X)) \).
2. Fourier–Mukai transform of a stable vector bundle

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. Fix once and for all a point $x_0 \in X$.

Let $J(X) := \text{Pic}^0(X)$ be the Jacobian of $X$. There is a canonical principal polarization on $J(X)$ given by the cup product of $H^1(X, \mathbb{Z})$. All stable vector bundles over $J(X)$ considered here will be with respect to this polarization.

Let $L$ be a holomorphic line bundle over $X \times J(X)$ such that

- for each point $\xi \in J(X)$, the restriction of $L$ to $X \times \{\xi\}$ is in the isomorphism class of holomorphic line bundles represented by $\xi$, and
- the restriction of $L$ to $\{x_0\} \times J(X)$ is a holomorphically trivial line bundle over $J(X)$.

Such a line bundle $L$ exists [1, p. 166–167]. Moreover, from the see–saw theorem (see [7, p. 54, Corollary 6]) it follows that $L$ is unique up to a holomorphic isomorphism. We will call $L$ the Poincaré line bundle for the pointed curve $(X, x_0)$.

Fix an integer $r \geq 2$. Fix a holomorphic line bundle $\xi$ over $X$ with

\begin{equation}
\text{degree}(\xi) > r(2g - 1).
\end{equation}

Let $M_\xi(r)$ denote the moduli space of stable vector bundles $E$ over $X$ with rank $(E) = r$ and $\bigwedge^r E = \xi$.

Let $\phi_J$ (respectively, $\phi_X$) denote the projection of $X \times J(X)$ to $J(X)$ (respectively, $X$).

**Lemma 2.1.** For each vector bundle $F \in M_\xi(r)$,

\[ R^1\phi_{J*}(L \otimes \phi_X^* F) = 0, \]

where $L$ is the Poincaré line bundle. The direct image

\[ V_F := \phi_X^*(L \otimes \phi_X^* F) \]

is a stable vector bundle over $J(X)$ of rank $\delta := \text{degree}(\xi) - r(g - 1)$.

**Proof.** For a stable vector bundle $W$ over $X$ of rank $r$ and degree $d > 2r(g - 1)$, we have $H^0(X, W^* \otimes K_X) = 0$ because a stable vector bundle of negative degree does not admit any nonzero sections. Hence by Serre duality we have $H^1(X, W) = 0$. Therefore, using (1) it follows that $R^1\phi_{J*}(L \otimes \phi_X^* F) = 0$.

Since $R^1\phi_{J*}(L \otimes \phi_X^* F) = 0$, we know that the direct image $V_F$ in the statement of the lemma is a vector bundle of rank $\text{degree}(\xi) - r(g - 1)$.

The stability of $V_F$ is derived from [2, p. 5, Theorem 1.2] as follows. Consider the embedding

\[ f : X \longrightarrow J(X) \]

defined by $x \longmapsto O_X(x_0 - x)$. Therefore,

\begin{equation}
(\text{Id}_X \times f)^* L = O_{X \times X}((\{x_0\} \times X) - \Delta),
\end{equation}

where $\Delta \subset X \times X$ is the diagonal divisor.

Set $E$ in [2, Theorem 1.2] to be $F \otimes O_X(x_0)$. Using (2) it follows that the vector bundle $M_E$ in [2, Theorem 1.2] is identified with $f^* V_F$. From [2, Theorem 1.2] we know
that $f^*\mathcal{V}_F$ is stable. Now using the openness of the stability condition (see [4, p. 635, Theorem 2.8(B)]) it follows that there is a Zarisky open dense subset

$$U \subset J(X)$$

such that for each $z \in U$, the pullback $f^*\tau_z^*\mathcal{V}_F$ is a stable vector bundle, where $\tau_z \in \text{Aut}(J(X))$ is the translation defined by $y \mapsto y + z$.

If $\mathcal{W} \subset \mathcal{V}_F$ violates the stability condition of $\mathcal{V}_F$ for the canonical polarization, then take a point $z_0 \in U$ (see (3)) such that $\tau_{z_0} \circ f$ intersects the Zarisky open dense subset of $J(X)$ over which $\mathcal{W}$ is locally free. Now it is straightforward to check that $f^*\tau_{z_0}^*\mathcal{W} \subset f^*\tau_{z_0}^*\mathcal{V}_F$ contradicts the stability condition of $f^*\tau_{z_0}^*\mathcal{V}_F$. Therefore, we conclude that $\mathcal{V}_F$ is stable. This completes the proof of the lemma. □

Fix a holomorphic line bundle $L$ over $J(X)$ such that $c_1(L)$ coincides with the canonical polarization on $J(X)$. As in [7, p. 123], set

$$M := m^*L \otimes p_1^*L^* \otimes p_2^*L^*$$
onumber

on $J(X) \times J(X)$, where

$$p_i : J(X) \times J(X) \rightarrow J(X)$$

is the projection to the $i$–th factor, and $m$ is the addition map on $J(X)$; the dual abelian variety $J(X)^\vee$ is identified with $J(X)$ using the Poincaré line bundle $\mathcal{L}$. Let

$$\varphi : X \rightarrow J(X)$$

be the morphism defined by $x \mapsto \mathcal{O}_X(x - x_0)$. Then

$$(\varphi \times \text{Id}_{J(X)})^*M = \mathcal{L}.$$  

**Proposition 2.2.** Consider the vector bundle $\mathcal{V}_F$ is Lemma 2.1. For all $i \neq g$,

$$R^ip_{1*}(M^* \otimes p_2^*\mathcal{V}_F) = 0,$$

and

$$R^qp_{1*}(M^* \otimes p_2^*\mathcal{V}_F) = \varphi_*F,$$

where $M$ and $\varphi$ are is defined in (4) and (6) respectively, and $p_1$ and $p_2$ are the projections in (5).

**Proof.** The proof of the proposition is identical to the proof of Theorem 2.2 in [5, p. 156]. We note that the key input is the result in [7, p. 127] which says that $R^ip_{1*}M = 0$ for $i \neq g$, and $R^qp_{1*}M = \mathbb{C}$ is supported at the point $e_0 = \mathcal{O}_X$ with stalk $H^q(J(X) \times J(X), M) \cong \mathbb{C}$. (see also [7, p. 129, Corollary 1]). □

Let $\xi := c_1(\mathcal{V}_F) \in H^2(J(X), \mathbb{Z})$. Note that since $\mathcal{M}_\xi(r)$ is connected, for all $i \geq 0$, the Chern class $c_i(\mathcal{V}_F) \in H^{2i}(J(X), \mathbb{Z})$ is independent of the choice of $F \in \mathcal{M}_\xi(r)$. We have a morphism

$$\alpha : \mathcal{M}_\xi(r) \rightarrow \text{Pic}^\xi(J(X))$$

defined by $E \mapsto \wedge^\delta \mathcal{V}_E$ (see Lemma 2.1). Since $\mathcal{M}_\xi(r)$ is a Zariski open subset of a unirational variety (the moduli space of semistable vector bundles over $X$ of rank $r$ and determinant $\xi$ is unirational), the morphism $\alpha$ constructed above must be a constant one.
Let $\mathcal{M}(J(X))$ denote the moduli space of stable vector bundles $\mathcal{W}$ over $J(X)$ with rank($\mathcal{W}$) = $\delta := \text{degree}(\xi) - r(g - 1)$, $\bigwedge^\top \mathcal{W} = \text{image}(\alpha)$, and $c_i(\mathcal{W}) = c_i(\mathcal{V}_F)$ for all $i \geq 2$.

**Corollary 2.3.** We have a morphism

\[ \beta : \mathcal{M}_\xi(r) \longrightarrow \mathcal{M}(J(X)) \]

defined by $F \longmapsto \mathcal{V}_F$. This morphism $\beta$ is an embedding.

**Proof.** The map $\beta$ is well defined by Lemma 2.1. That $\beta$ is an embedding follows immediately from Proposition 2.2, because we have a morphism

\[ \gamma : \beta(\mathcal{M}_\xi(r)) \longrightarrow \mathcal{M}_\xi(r) \]

defined by $W \longmapsto \varphi^* R^g p_{1*}(M^* \otimes p^*_2 W)$ such that $\gamma \circ \beta$ is the identity map of $\mathcal{M}_\xi(r)$. \( \square \)

3. The case of $g = 2$

Henceforth, we will assume that $g = 2$. If degree($\xi$) is even, then we will also assume that $r > 2$.

**Lemma 3.1.** Take any $F \in \mathcal{M}_\xi(r)$. Then the image of $C_1(\mathcal{V}_F)^2 - 2 \cdot C_2(\mathcal{V}_F) \in \text{CH}^2(J(X))$ in the Deligne–Beilinson cohomology $H^4_B(J(X), \mathbb{Z}(2))$ (see [3, p. 85, Corollary 7.7]) is independent of $F$. More precisely, it vanishes.

**Proof.** Since $H^4_B(J(X), \mathbb{Z}(2))$ is an extension of a discrete group by a complex torus [3, p. 86, (7.9)], and $\mathcal{M}_\xi(r)$ is connected and unirational, there is no nonconstant morphism from $\mathcal{M}_\xi(r)$ to $H^4_B(J(X), \mathbb{Z}(2))$. In particular, the image of $C_1(\mathcal{V}_F)^2 - 2 \cdot C_2(\mathcal{V}_F)$ in $H^4_B(J(X), \mathbb{Z}(2))$ is independent of the choice of $F \in \mathcal{M}_\xi(r)$.

From [8, § 4] (reproduced in [5, p. 164, Theorem 4.3(2)]) we know that $C_1(\mathcal{V}_F) = r \cdot \lambda_x^* \Theta$, where $\Theta \in \text{Pic}^1(X)$ is the theta divisor, and $\lambda_x : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$ is defined by $\zeta \longmapsto \zeta \otimes \mathcal{O}_X(x_0)$. Similarly, $C_2(\mathcal{V}_F) = r^2 \cdot e_0$, where $e_0 = \mathcal{O}_X$ is the identity element. On the other hand, the image of $\Theta^2 - 2e_0$ in $H^4_B(J(X), \mathbb{Z}(2))$ vanishes (see the proof of Theorem 1.3 in [1, p. 212]). \( \square \)

Consider the moduli space $\mathcal{M}(J(X))$ in (7). Let

\[ \mathcal{M}^0(J(X)) \subset \mathcal{M}(J(X)) \]

be the subvariety defined by the locus of all $E$ such that image of

\[ C_1(E)^2 - 2 \cdot C_2(E) \in \text{CH}^2(J(X)) \]

in $H^4_B(J(X), \mathbb{Z}(2))$ vanishes. From Lemma 3.1 we know that the image of the map $\beta$ in Corollary 2.3 lies in $\mathcal{M}^0(J(X))$.

Since $J(X)$ is an abelian surface, the moduli space $\mathcal{M}^0(J(X))$ in (8) is smooth, and it has a canonical symplectic structure [6, p. 102, Corollary 0.2].

**Theorem 3.2.** The image of the embedding $\beta$ in Corollary 2.3 is a Lagrangian subvariety of the symplectic variety $\mathcal{M}^0(J(X))$. 

Proof. We note that $\mathcal{M}_\xi(r)$ is the smooth locus of the moduli space of semistable vector bundles over $X$ of rank $r$ and determinant $\xi$. In particular, $\mathcal{M}_\xi(r)$ is the smooth locus of a normal unirational variety. Therefore, $\mathcal{M}_\xi(r)$ does not admit any nonzero algebraic two–forms. Consequently, the pull back to $\mathcal{M}_\xi(r)$ of the symplectic form on $\mathcal{M}^0(J(X))$ vanishes identically. Therefore, to prove the theorem it suffices to show that

$$\dim \mathcal{M}^0(J(X)) = 2 \cdot \dim \mathcal{M}_\xi(r) = 2(r^2 - 1).$$

Let $\theta \in H^2(J(X), \mathbb{Z})$ denote the canonical polarization. In the proof of Lemma 3.1 we noted that $c_1(V_F) = r \cdot \theta$, and $ch_2(V_F) = c_1(V_F)^2/2 - c_2(V_F) = 0$. Hence $ch_2(\text{End}(V_F))(J(X)) = -r^2$. Therefore, using Hirzebruch–Riemann–Roch,

$$\dim H^1(J(X), \text{End}(V_F)) = r^2 + 2.$$ 

Since $\dim \mathcal{M}^0(J(X)) = \dim \mathcal{M}(J(X)) - 2 = \dim H^1(J(X), \text{End}(V_F)) - 4$, we now conclude that (9) holds. This completes the proof of the theorem. $\square$

References


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