TORSION POINTS AND MATRICES DEFINING ELLIPTIC CURVES

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Abstract. Let $X \subset \mathbb{P}^2$ be an elliptic curve defined over a field $k$ with defining polynomial $f$. We show that any non-trivial torsion point of order $r$, $\text{char } k \neq r$, determines up to equivalence, a unique minimal matrix $\Phi_r$ of size $3r \times 3r$ with linear polynomial entries such that $\det \Phi_r = f^r$. We also show that the identity, thought of as the trivial torsion point of order $r$, determines up to equivalence, a unique minimal matrix $\Psi_r$ of size $(3r - 2) \times (3r - 2)$ with linear and quadratic polynomial entries such that $\det \Psi_r = f^r$.

1. Introduction

Let $X$ be an elliptic curve over $k$ ($\text{char } k \neq 2, 3$), and let $P_0 \in X(k)$ be a fixed point. The linear system $|3P_0|$ defines an embedding $X \subset \mathbb{P}^2$ so that $X$ is defined as the zero locus of a cubic homogeneous polynomial in 3 variables.

Our interest in this note is to study, for a fixed embedding $X \subset \mathbb{P}^2$, matrix representations of the defining polynomial of $X$ which we make precise in a moment.

We start by stating a more general question, which dates back to at least the work of Dickson [6], about defining polynomials of general hypersurfaces in $\mathbb{P}^n$ and then deal with the question in the context of elliptic curves. Given a general homogeneous polynomial of degree $d$ in the variables $x_0, \ldots, x_n$, is it possible to obtain it as the determinant of a matrix whose entries are linear homogeneous polynomials in the same variables. To avoid any kind of degeneracy, we will assume that the matrix is minimal: i.e. no non-zero scalar entries are allowed in the matrix. Recall that the zero locus of any homogeneous polynomial defines a hypersurface in projective $n$-space and thus one is really asking whether a general hypersurface of degree $d$ in $\mathbb{P}^n$ is “determinantal”?

More recently, questions which are slightly different and more general have been a subject of investigation. These include removing the restriction that the entries are linear, so that they now could be arbitrary degree homogeneous polynomials, or demanding that the matrix satisfy additional conditions such as being symmetric. From the point of view of algebraic geometry, a combination of these two questions leads to a study of the existence, or the lack thereof, of certain line bundles on a general hypersurface. Proceeding in this vein, if one were to study a generalisation of these questions to rank 2 bundles, then one is led to the question of whether (or not) a general hypersurface is “Pfaffian” i.e. whether a general polynomial can be obtained as the Pfaffian (= square root of the determinant) of a skew-symmetric matrix with homogeneous polynomial entries. Of course, every polynomial $f$ is the determinant of a $1 \times 1$ matrix with entry $f$, and has trivial Pfaffian representation via the matrix $\begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$. So in this article, we will be interested in non-trivial matrix representations of homogeneous polynomials.

It turns out that when $n \geq 4$, this concrete property of the defining polynomial of a hypersurface has important consequences for the geometry of the hypersurface. We refer the reader

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1By a general polynomial of degree $d$, we mean a point in a suitable open set of the vector space parametrising all degree $d$ polynomials.
to [11, 4] for further details. There are several aspects of this question that have been studied. The two basic questions that have been addressed so far are

(1) Given $n$, determine all those values of $d$ for which a general hypersurface of degree $d$ does not admit any non-trivial determinantal or Pfaffian representations.

(2) In cases where they do admit such representations, is it possible to describe the (moduli) spaces of all such representations?

Notice that given an $n \times n$ matrix $\Phi$ with linear entries and $M, N \in \text{GL}(n, k)$, the determinant of the product $M\Phi N$ is a scalar multiple of the determinant of $\Phi$; hence both $\Phi$ and $M\Phi N$ represent the same hypersurface. Thus we shall only be interested in “suitable” equivalence classes of representations rather than all representations.

The answer to the first question for the determinantal case is given by the classical Noether-Lefschetz theorem, while the Pfaffian case has been recently answered in [8, 9, 11]. The answer to the second question has also been well-studied in the case when there are infinitely many such representations. In this case, the space of all such (inequivalent) representations has the structure of an algebraic variety and explicit descriptions of these spaces are now completely known in many cases (see [3] and the references therein). Thus the only remaining case which has not been studied is when there are only finitely many such (inequivalent) representations. In this case, one would like to know

a) the exact number of such (inequivalent) representations, and

b) explicit descriptions of these representations.

One such case which is well known is the representation of any smooth plane curve as a linear symmetric determinant. A complete description of these representations is given by the theta characteristics on the curve (see Proposition 3).

This essentially leaves the question of Pfaffian representations unanswered. When $(d, n) = (3, 2)$ i.e. the case of elliptic curves in the plane, it turns out that these representations, including the non-linear ones, are completely determined by the points of the elliptic curve.

In this article, we give explicit descriptions of these representations using methods from algebraic geometry. In fact, we go even further and investigate the question of whether the defining polynomial of any elliptic curve in the plane can be represented as the $r$-th ($r \neq \text{char} k$) root of the determinant of a matrix with polynomial entries. This is done by first rephrasing the question in terms of vector bundles on an elliptic curve. We briefly sketch how this is done.

Given a vector bundle $E$ on a smooth hypersurface $X \subset \mathbb{P}^n$, we consider $H^0_i(X, E) := \bigoplus_{\nu \in \mathbb{Z}} H^0(X, E(\nu))$ as a graded $S$-module where $S := H^0_0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is the polynomial ring in $(n + 1)$ variables. By Serre’s theorems, this is a finitely generated $S$-module. Let $s_i \in H^0_i(X, E(a_i))$ for $1 \leq i \leq l$ be a set of minimal generators. Then we have a surjection of $\mathcal{O}_{\mathbb{P}^n}$-sheaves, $L_0 := \bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^n}(-a_i) \rightarrow E$, which induces a surjection $H^0_0(\mathbb{P}^n, L_0) \rightarrow H^0_0(X, E)$ and yields an exact sequence

$$0 \rightarrow L_1 \xrightarrow{\Phi} L_0 \rightarrow E \rightarrow 0,$$

where $L_1$ is the sheaf corresponding to the first syzygy module. Since $X$ is smooth, and $E$ is a vector bundle on $X$, it is Cohen-Macaulay. By the Auslander-Buchsbaum formula (see [7], Chapter 19), we see that $L_1$ is a vector bundle on $\mathbb{P}^n$ satisfying $H^i_1(\mathbb{P}^n, L_1) = 0$. In case $n = 2$, it follows by Horrocks’ theorem that $L_1$ is a sum of line bundles on $\mathbb{P}^n$. Consequently, $\Phi$ is a minimal square matrix whose entries are homogeneous polynomials in 3 variables and the zero locus of whose determinant (which is a homogeneous polynomial) is supported on $X$. Furthermore, if for example, $E$ is a line bundle, then the size of the matrix $\Phi$ is of size at most $3 \times 3$. In case $E$ is rank 2 with $\det E = \mathcal{O}_X(\alpha)$ for some $\alpha \in \mathbb{Z}$, then using the alternate bilinear
pairing \( E \times E \to \det E \), one sees that \( \Phi \) can be taken to be skew-symmetric (see [2]) of size at most \( 6 \times 6 \) and that the Pfaffian of \( \Phi \) is the defining equation of \( X \). Conversely, any such matrix defines a rank 2 bundle \( E \).

As a natural extension of these ideas, we are also able to easily prove that higher order torsion points also yield matrices whose determinant is the \( r \)-th power of the defining polynomial of the elliptic curve, where \( r \) is the order of the torsion point.

When \( n > 2 \), exact sequences as above where \( \Phi \) is a matrix of forms have been studied extensively. In these cases, the vector bundles \( E \) are precisely those which do not have any intermediate cohomology. In fact, the results in this paper are motivated by the study of such bundles on smooth hypersurfaces.

Here is the statement of the main theorem of this note.

**Theorem 1.** Let \( X \) be an elliptic curve in the plane given by a homogeneous cubic polynomial \( f \in k[x_0,x_1,x_2] \). Then up to a constant factor,

1. There are 3 (inequivalent) representations of \( f \) by symmetric linear determinants and these are in one-to-one correspondence with the non-trivial 2-torsion points of \( X \).
2. Every point of the elliptic curve determines a linear Pfaffian representation of \( f \) and they correspond to decomposable rank 2 bundles; in addition to this, there are 3 other linear Pfaffian representations of \( f \); each of which correspond to indecomposable rank 2 bundles. Furthermore, these are completely determined by the non-trivial 2-torsion points of \( X \).
3. There are 2 (inequivalent) representations of \( f \) as Pfaffians of skew-symmetric minimal matrices of size \( 4 \times 4 \).
4. For any \( r \) such that \((r,\text{char} k) = 1\), there is a bijective correspondence between torsion points of order \( r \) and (equivalence classes of) \( 3r \times 3r \) matrices with linear entries whose determinant is \( f^r \). However, for \( r \geq 3 \), we cannot explicitly describe this equivalence.
5. For any \( r \geq 2 \) such that \((r,\text{char} k) = 1\), the identity, thought of as a torsion point of order \( r \), yields an unique (up to equivalence) \((3r - 2) \times (3r - 2)\) matrix with linear and quadratic entries whose determinant is \( f^r \). As in the previous case, for \( r \geq 3 \), we cannot explicitly describe this equivalence.

As mentioned before, the first part of the above theorem is well-known and a proof can be found for instance in [2].

### 2. Preliminaries

Let \( V \) be the 3-dimensional vector space of linear forms on \( \mathbb{P}^2 \). Let \( \mathbb{M}_{3 \times 3}(V) \) denote the space of all \( 3 \times 3 \) matrices with entries in \( V \). Let \( \mathcal{M}_0 \subset \mathbb{M}_{3 \times 3}(V) \) be the open set consisting of matrices whose determinants have smooth zero loci. Let \( G_0 := \text{GL}(3,k) \times \text{GL}(3,k) \). Then \( G_0 \) acts on \( \mathcal{M}_0 \) via \( \Phi \mapsto M\Phi N^t \). This action in turn factors via the group \( G'_0 := G_0 / \mathbb{G}_m \) where \( \mathbb{G}_m \) embeds in \( G_0 \) diagonally. The determinant map,

\[
\det: \mathcal{M}_0 \to \mathbb{P}(\text{Sym}^3 V)
\]

which associates to a matrix \( \Phi \), its determinant \( \det \Phi \), factors via

\[
\tilde{\det}: \mathcal{M}_0 / G'_0 \to \mathbb{P}(\text{Sym}^3 V).
\]

One checks that \( \dim \mathcal{M}_0 / G'_0 = 10 \) and \( \dim \mathbb{P}(\text{Sym}^3 V) = 9 \).

Let \( \mathbb{M}^\text{sym}_{3 \times 3}(V) \) denote the space of all \( 3 \times 3 \) symmetric matrices with entries in \( V \). Let \( \mathcal{M} \subset \mathbb{M}^\text{sym}_{3 \times 3}(V) \) be the open set consisting of matrices whose determinants have smooth zero loci. Let
$G_1 := \text{GL}(3, k)$. It is clear that the determinant map factors as
\[ \widetilde{\text{det}} : \mathcal{M}/G_1 \to \mathbb{P}(\text{Sym}^3 V) \]
where $\mathcal{M}/G_1$ is the quotient for the $G_1$ action on $\mathcal{M}$ given by $\Phi \mapsto M\Phi M^t$. One checks that $\dim \mathcal{M}/G_1 = \dim \mathbb{P}(\text{Sym}^3 V) = 9$.

Finally, let $M_{6\times6}^{ss}(V)$ denote the space of all $6 \times 6$ skew-symmetric matrices with entries in $V$. There is an embedding
\[ M_{3\times3}(V) \hookrightarrow M_{6\times6}^{ss}(V) \]
defined by
\[ \phi \mapsto \left( \begin{array}{cc} 0 & \phi \\ -\phi^t & 0 \end{array} \right). \]
We shall, by abuse of notation, refer to the image of the above embedding by $M_{3\times3}(V)$. Let $G_2 := \text{GL}(6, k)$. It is clear that the Pfaffian map
\[ \text{Pf} : N \to \mathbb{P}(\text{Sym}^3 V) \]
which associates to a matrix $\Phi$, its Pfaffian $\text{Pf}(\Phi) := \sqrt{\text{det} \Phi}$, factors via
\[ \widetilde{\text{Pf}} : N/G_2 \to \mathbb{P}(\text{Sym}^3 V) \]
where $N/G_2$ is the quotient for the $G_2$ action on $N$ is given by $\Phi \mapsto M\Phi M^t$.

For the case of non-degenerate Pfaffian representations, we let
\[ X := \{ [\Phi] \in N/G_2 \mid \Phi \not\sim_{G_2} \left( \begin{array}{cc} 0 & \phi \\ -\phi^t & 0 \end{array} \right), \phi \in M_{6\times6}^{ss}(V) \}, \]
and this gives a map
\[ \widetilde{\text{Pf}}_{\text{indec}} : X \to \mathbb{P}(\text{Sym}^3 V). \]

The purpose of this note is to describe the fibres of these maps explicitly. As mentioned in the introduction, we shall now recast the questions of determinantal or Pfaffian representations of an elliptic curve in terms of the existence of certain vector bundles. We refer the reader to [2], Theorems A and B for a more general statement and to Propositions 3.1 and 5.1 to the case of our interest.

**Proposition 1.** Let $X$ be a smooth curve of degree 3 in $\mathbb{P}^2$, given by an equation $f = 0$. Let $\phi$ be a $3 \times 3$ linear matrix with $f = \text{det} \phi$. Then
\[ A := \text{Coker}[\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}] \]
is a line bundle on $X$ of degree 0 and $H^0(X, A) = 0$.

Conversely, any line bundle $A$ on $X$ of degree 0 and $H^0(X, A) = 0$ admits a minimal resolution
\[ 0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to A \to 0, \]
where $\phi$ is a $3 \times 3$ linear matrix such that $\text{det} \phi = f$.

In the case of skew-symmetric matrices, one has

**Proposition 2.** Let $X$ be a smooth curve of degree 3 in $\mathbb{P}^2$, given by an equation $f = 0$. Let $\Phi$ be a $6 \times 6$ linear matrix with $\text{Pf} \Phi = f$. Then
\[ E := \text{Coker}[\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 6} \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 6}] \]
is a rank 2 vector bundle on $X$ satisfying $\text{det} E = \mathcal{O}_X$ and $H^0(X, E) = 0$. 

Conversely, any rank 2 vector bundle $E$ on $X$ with $\det E = \mathcal{O}_X$ and $\mathcal{H}^0(X, E) = 0$ admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 6} \overset{\Phi}{\to} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 6} \to E \to 0,$$

where $\Phi$ is a $6 \times 6$ linear matrix such that $\text{Pf} \, \Phi = f$.

3. Linear Matrix representations

3.1. Symmetric Determinantal representations. In Proposition 1, if one were to impose the extra condition that the matrix $\phi$ be symmetric, then we have the following classical result which can be found for instance in [2].

**Proposition 3.** Let $C$ be any smooth curve of degree $d$ in the plane, defined by an equation $f = 0$ and $\kappa$ be a non-trivial theta-characteristic on $C$ i.e. $\kappa$ is a line bundle with $\kappa^{\otimes 2} = K_C$ and $\kappa \not\sim \mathcal{O}_X$. Then $\kappa$ admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^d \overset{/\phi}{\to} \mathcal{O}_{\mathbb{P}^2}(-1)^d \to \kappa \to 0,$$

where the matrix $\phi$ is symmetric, has linear polynomial entries and $\det \phi = f$. Conversely, the cokernel of a symmetric matrix $\phi$ as in the above sequence is a non-trivial theta-characteristic $\kappa$ on $C$.

In particular, we have

**Corollary 1.** Every elliptic curve in the plane has three distinct non-trivial theta characteristics which correspond to $\kappa$ in Proposition 3. Hence its defining polynomial has three inequivalent representations as the determinant of a $3 \times 3$ linear symmetric matrix.

**Proof.** Clearly $\deg \kappa = 0$ and hence $\kappa \in \text{Pic}^0(X) \cong X$. The statement follows from the fact that there are exactly four 2-torsion points in an elliptic curve one of which is the identity. \qed

3.2. Linear Pfaffian representations. We begin with an elementary result.

**Lemma 1.** Let $X$ be a smooth projective curve. Then any degree zero line bundle $A \to X$ which has a non-zero section is isomorphic to $\mathcal{O}_X$.

**Proof.** Let $\phi (\neq 0) \in \mathcal{H}^0(X, A)$. This gives rise to a short exact sequence $0 \to \mathcal{O}_X \overset{\phi}{\to} A \to \tau \to 0$. Here $\tau$, which is the cokernel of $\phi$, is a torsion sheaf and hence is supported on a finite set of points. So we have $\deg A = \deg \mathcal{O}_X + \text{length}(\tau)$ which implies that $\tau$ is the zero sheaf or that $\phi$ is an isomorphism. \qed

The following important theorem due to Atiyah (see [1], Theorem 5 and Corollary 1) will play an important role for us.

**Theorem 2.** Let $X$ be an elliptic curve.

1. Then for any $r > 0$, there exists an indecomposable vector bundle $F_r$, unique up to isomorphism, with $h^0(F_r) = 1$. Moreover, $F_0 = \mathcal{O}_X$ and $F_r$ is defined inductively by the exact sequence,

$$0 \to \mathcal{O}_X \to F_r \to F_{r-1} \to 0.$$

2. Let $E$ be any indecomposable rank $r$ bundle of degree 0. Then there exists a line bundle $A$ such that $E \cong F_r \otimes A$ and such that $A^{\otimes r} = \det E$.

3. $F_r \cong F_r^\vee$ (i.e. $F_r$ is self-dual).

**Theorem 3.** With notation as above,
(1) The general fibre of the morphism $\tilde{\text{Pf}}$ has dimension 1.
(2) The morphism $\text{Pf}_{\text{indec}}$ is generically finite of degree 3.

Proof. We will first need to show that the map $\tilde{\text{Pf}}$ is dominant. By Proposition 2, this is equivalent to the fact that any smooth elliptic curve $X$ supports a rank two vector bundle $E$ with $\det E = \mathcal{O}_X$ and $h^0(E) = 0$ and has a minimal resolution of the form

$$0 \rightarrow \mathcal{O}_{p_2}(-2)^{\oplus 6} \rightarrow \mathcal{O}_{p_2}(-1)^{\oplus 6} \rightarrow E \rightarrow 0.$$ 

Take any $L \in \text{Pic}^0(X)$ and let $\phi$ be the matrix in the minimal resolution of $L$ as given in Proposition 1. Then the bundle $L \oplus L^{-1}$ (which has determinant $\mathcal{O}_X$ and $h^0 = 0$) has a minimal resolution with matrix $\Phi = \begin{pmatrix} 0 & \phi \\ -\phi^t & 0 \end{pmatrix}$. Thus the map $\tilde{\text{Pf}}$ is dominant. Since $\text{Pic}^0(X) \cong X$, we see that the fibre is indeed at least 1-dimensional.

To understand the fibres, we will consider two cases: namely when $E \cong A \oplus A'$ is a sum of line bundles and the other case when $E$ is indecomposable. So let $E \cong A \oplus A'$. Since $\det E = \mathcal{O}_X$, $A' \cong A^{-1}$ and so $E \cong A \oplus A^{-1}$. Since $h^0(E) = 0$, this implies that $h^0(A) = h^0(A^{-1})$ and so $\deg A = 0$, $A \not\cong \mathcal{O}_X$. Thus the decomposable Pfaffian representations are precisely the ones in the previous paragraph.

Now suppose that $E$ is indecomposable. Then by Theorem 2, there exists a line bundle $A$ of degree 0 such that $E \cong F_2 \otimes A$. Thus $E$ can be written as a non-trivial extension

$$0 \rightarrow A \rightarrow E \rightarrow A \rightarrow 0.$$ 

Taking determinants, we get $A^{\otimes 2} \cong \mathcal{O}_X$. The indecomposable bundle $E$ which is described by the non-trivial extension

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$$

has $h^0(E) = 1$ and hence is not the desired element. Thus $E$ has to be isomorphic to one of the three non-trivial extensions

$$0 \rightarrow \kappa_i \rightarrow E \rightarrow \kappa_i \rightarrow 0, \quad i = 1, 2, 3,$$

where $\kappa_i$'s are non-trivial 2-torsion elements in $\text{Pic}^0(X)$.

To prove the theorem, we note that any rank 2 bundle $E$ obtained as any one of the above three non-trivial extensions has the following (easy to check) properties:

(i) $E$ is 1-regular (in the sense of Castelnuovo-Mumford),
(ii) $h^0(E(-\nu)) = 0$ for $\nu \geq 0$, and
(iii) $\det E = \mathcal{O}_X$.

1-regularity implies that the minimal generators of $E$ are in degrees 1 and less, while (ii) implies that all the generators are in degree exactly 1. By Riemann-Roch theorem, $h^0(E(1)) = 6$ and so the minimal resolution of any such $E$ is of the form

$$0 \rightarrow L_1 \Phi \mathcal{O}_{p_2}(-1)^{\oplus 6} \rightarrow E \rightarrow 0.$$ 

Taking duals, we have

$$0 \rightarrow \mathcal{O}_{p_2}(1)^{\oplus 6} \rightarrow L_1^\vee \mathcal{O}_{p_2}(-1)^{\oplus 6} \rightarrow E^\vee(3) \rightarrow 0.$$ 

Noting that $E^\vee \cong E$ and tensoring with $\mathcal{O}_{p_2}(-3)$, we see that $L_1^\vee(-3) \cong \mathcal{O}_{p_2}(-1)^{\oplus 6}$ and so the minimal resolution is of the form (2) above.

Thus the total number of equivalence classes of indecomposable Pfaffian representations of a smooth cubic is 3. □
4. Non-linear matrix representations

4.1. Non-linear symmetric determinants.

**Proposition 4.** An elliptic curve $X \subset \mathbb{P}^2$ cannot be defined as the zero set of the determinant of a symmetric $2 \times 2$ matrix with homogeneous polynomial entries.

**Proof.** Any such matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ has determinant of the form $ac - b^2$ where $a, b, c$ are homogeneous polynomials. If $b = 0$, then the determinant is not irreducible, hence the resulting zero locus is not smooth. If $b \neq 0$, the determinant has even degree. Since the elliptic curve is given by a cubic polynomial, the result is obvious. □

4.2. Non-linear Pfaffians.

**Lemma 2 ([8]).** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d$ and $E$ be a rank 2 bundle on $X$ with $\det E = O_X(\alpha)$. If the minimal resolution of $E$ is of the form

$$0 \to L_1 \xrightarrow{\Phi} L_0 \to E \to 0,$$

where $L_1 = \bigoplus_j O_{\mathbb{P}^n}(-n_j)$ and $L_0 = \bigoplus_j O_{\mathbb{P}^n}(-m_i)$, $m_i \geq 0 \forall i$, then $L_1 \cong L_0^\vee \otimes O_{\mathbb{P}^2}(\alpha - d)$. Furthermore, $\Phi$ may be assumed to be skew-symmetric i.e. $\Phi^t = -\Phi$. Conversely, if a bundle $E$ fits into an exact sequence (2) as above, and $\text{Pf} \Phi$ is the defining polynomial of $X$, then $E$ is a rank 2 bundle with $\det E = O_X(\alpha)$.

Any skew-symmetric matrix $\Phi$ whose Pfaffian, denoted by $\text{Pf}(\Phi)$, is the defining equation of the elliptic curve $X$ has to be of even size since the Pfaffian of any odd sized skew-symmetric matrix is zero. This leaves us with two choices viz. $6 \times 6$ which yields a linear Pfaffian representation and $4 \times 4$ which yields a non-linear Pfaffian representation of $X$. Any $4 \times 4$ skew-symmetric matrix has the form

$$\Phi = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & g \\ -c & -e & -g & 0 \end{pmatrix} \text{ with } \text{Pf}(\Phi) = ag - be + cd.$$

In particular, we have

$$\deg a + \deg g = \deg b + \deg e = \deg c + \deg d = 3,$$

and so each non-zero entry is either a linear or quadratic polynomial. We also have

$$\deg(a) = 3 - \alpha - m_1 - m_2, \quad \deg(b) = 3 - \alpha - m_1 - m_3, \quad \deg(c) = 3 - \alpha - m_1 - m_4,$$

$$\deg(d) = 3 - \alpha - m_2 - m_3, \quad \deg(e) = 3 - \alpha - m_2 - m_4, \quad \deg(g) = 3 - \alpha - m_3 - m_4,$$

Finally $\alpha \leq 3$; otherwise, $a, b, c$ would have negative degrees which in particular would mean that they are identically zero.

To determine the possible $\Phi'$s, we look at the possible minimum resolutions

$$0 \to L_1 \xrightarrow{\Phi} L_0 \to E \to 0.$$

By twisting with a suitable $O_X(m)$, we may assume that

$$L_0 = O_{\mathbb{P}^2}^\oplus \bigoplus_{i=1}^{4-\ell} O_{\mathbb{P}^2}(-m_i), \quad l \geq 0, \quad 0 < m_1 \leq \cdots \leq m_{4-\ell}.$$
Lemma 3. The only possible minimum resolutions are of the form

\[ 0 \to L_1 \xrightarrow{\Phi} L_0 \to E \to 0, \]

where

(a) \( L_0 = \mathcal{O}^{\oplus 3}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \), or

(b) \( L_0 = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \).

Proof. We first need to rule out the possibilities \( L_0 = \mathcal{O}^{\oplus 4}_{\mathbb{P}^2} \) and \( L_0 = \mathcal{O}^{\oplus 2}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-m_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-m_2) \) with \( 0 < m_1 \leq m_2 \).

If \( L_0 = \mathcal{O}^{\oplus 4}_{\mathbb{P}^2} \), then this implies that \( L_1 = \mathcal{O}_{\mathbb{P}^2}(-m) \) for some \( m \). Hence the degrees of all the non-zero terms in the matrix \( \Phi \) are equal. This is not possible since \( \text{Pf}(\Phi) \) has odd degree.

Now suppose that \( L_0 = \mathcal{O}^{\oplus 2}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-m_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-m_2) \). Since \( L_1 = L_0^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(\alpha - 3) \), we see that

\[
\begin{align*}
\deg a &= 3 - \alpha, & \deg b &= 3 - \alpha - m_1, & \deg c &= 3 - \alpha - m_2, \\
\deg d &= 3 - \alpha - m_1, & \deg e &= 3 - \alpha - m_2 & \deg g &= 3 - \alpha - m_1 - m_2.
\end{align*}
\]

Since \( \deg a + \deg g = \deg b + \deg e = \deg c + \deg d = 3 \), we see on solving for \( \alpha, m_1 \) and \( m_2 \), that that this is impossible.

Next let \( L_0 = \mathcal{O}^{\oplus 3}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-m) \) with \( m > 0 \). Then \( L_1 = \mathcal{O}_{\mathbb{P}^2}(\alpha - 3) \oplus \mathcal{O}_{\mathbb{P}^2}(m + \alpha - 3) \).

We see that, using the fact that \( \deg a + \deg g = 3 \), this implies that \( \deg a = 3 - \alpha = 2 \) and \( \deg g = m + 3 - \alpha = 2 \). Solving we get, \( m = 1 \) and \( \alpha = 1 \).

Finally suppose that \( L_0 = \mathcal{O}_{\mathbb{P}^2} \oplus \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^2}(-m_i) \) with \( m_3 \geq m_2 \geq m_1 > 0 \). Plugging in for \( \deg a + \deg g = \deg b + \deg e = \deg c + \deg d = 3 \), we get

\[
\begin{align*}
3 - \alpha - m_1 &= 2 & 3 - \alpha - m_2 - m_3 &= 1 \\
3 - \alpha - m_2 &= 2 & 3 - \alpha - m_1 - m_3 &= 1 \\
3 - \alpha - m_3 &= 2 & 3 - \alpha - m_1 - m_2 &= 1
\end{align*}
\]

Solving these, we get \( m_1 = m_2 = m_3 = 1 \) and \( \alpha = 0 \). \( \square \)

Let

\[
(3) \quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\Phi} \mathcal{O}^{\oplus 3}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to E \to 0,
\]

and

\[
(4) \quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\Psi} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2} \to G \to 0
\]

be the two possibilities from the above lemma. We shall refer to \( E \) above (resp. a resolution of \( E \)) as being of the first kind. Similarly, we shall refer to \( G \) above (resp. a resolution of \( G \)) as being of the second kind.

Lemma 4. Let \( E \) be a rank 2 vector bundle of the first kind on \( X \). Then there is a rank 2 bundle \( G \) which fits into an exact sequence

\[
0 \to G \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X^{\oplus 3} \oplus \mathcal{O}_X(-1) \to E \to 0
\]

and such that \( G \) has a minimal resolution of the second kind. Similarly if \( G \) is a rank 2 bundle of the second kind on \( X \), then there is a rank 2 bundle \( E \) which fits into an exact sequence

\[
0 \to E \otimes \mathcal{O}_X(-2) \to \mathcal{O}_X(-1)^{\oplus 3} \oplus \mathcal{O}_X \to G \to 0
\]

and such that \( E \) has a minimal resolution of the first kind.

Equivalently, the matrices \( \Phi \) and \( \Psi \) always occur in pairs and satisfy \( \Phi \Psi = f I_4 = \Psi \Phi \) where \( I_4 \) is the \( 4 \times 4 \) identity matrix and \( f = \text{Pf}(\Phi) = \text{Pf}(\Psi) \).
Proof. Let 

\[ 0 \to L_1 \xrightarrow{\Phi} L_0 \to E \to 0 \]

be a minimal resolution as above. Consider the map \( L_0(-3) \xrightarrow{\psi} L_0 \to E \to 0 \) given by multiplication by the diagonal matrix \( f.I_4 \), where \( f \) is the polynomial defining \( X \). Since the composite map \( L_0(-3) \xrightarrow{\psi} L_0 \to E \) is zero, we get a map \( L_0(-3) \xrightarrow{\Psi} L_1 \) satisfying \( \Phi \circ \Psi = fI_4 \). We will now show that \( \Psi \) is skew-symmetric. This will in particular prove that \( Pf(\Psi) = f \). To do this, we repeat the same process for the map \( \Psi \), to get a map \( \Phi_0 \) such that \( \Psi \circ \Phi_0 = fI_4 \). Multiplying on the right of both sides by the matrix \( \Phi \), we get \( f\Phi_0 = f\Phi \). This implies that \( \Phi_0 = \Phi \). Thus we have \( \Psi \circ \Phi = fI_4 = \Phi \circ \Psi \).

Since \( \Phi^t = -\Phi \), we get \( \Psi^t = -\Psi \). \( \square \)

Remark 1. There is a constructive proof to show that if \( \Phi \) is a skew-symmetric matrix of size \( 2d \times 2d \), then there exists a companion skew-symmetric matrix \( \Psi \) of size \( 2d \times 2d \) such that \( \Psi \circ \Phi = fI_{2d} \) where \( f := Pf(\Phi) = Pf(\Psi) \) and \( I_{2d} \) is the identity matrix of size \( 2d \times 2d \). The matrix \( \Psi \) is obtained from \( \Phi \) in the following way: the \( (i,j) \)-th entry of \( \Psi \) is the Pfaffian of the skew-symmetric matrix obtained by deleting the \( i \)-th and \( j \)-th rows and columns of \( \Phi \). We refer the reader to [10] for details.

Corollary 2. In Lemma 4, \( E \) is indecomposable if and only if \( G \) is so.

Proof. Assume that \( E \) is decomposable. Then \( E \cong A \oplus A^{-1}(\alpha) \) for some degree 0 line bundle \( A \). Here as usual \( \det E = \mathcal{O}_X(1) \). This implies that the matrix \( \Phi \) is degenerate i.e. of the form

\[
\Phi = \begin{pmatrix}
0 & 0 & q_2 & l_3 \\
0 & 0 & q_3 & -l_2 \\
-q_2 & -q_3 & 0 & 0 \\
-l_3 & l_2 & 0 & 0
\end{pmatrix}
\]

Then by Remark 1, we have

\[
\Psi = \begin{pmatrix}
0 & 0 & -l_2 & q_3 \\
0 & 0 & l_3 & q_2 \\
l_2 & -l_3 & 0 & 0 \\
-q_3 & -q_2 & 0 & 0
\end{pmatrix}
\]

and so \( G \) is also decomposable. The converse follows by interchanging \( E \) and \( G \). \( \square \)

The following theorem gives a concrete example of a bundle of the first kind.

Lemma 5. Let \( X \subset \mathbb{P}^2 \) be an elliptic curve and consider the short exact sequence obtained by restricting the Euler sequence on \( \mathbb{P}^2 \) to \( X \):

\[
0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X^{\oplus 3} \to T_{\mathbb{P}^2}(-1)|_X \to 0.
\]

The bundle \( E := T_{\mathbb{P}^2}(-1)|_X \) satisfies the following properties.

(i) \( \det E = \mathcal{O}_X(1) \),

(ii) \( E \) is globally generated with \( h^0(E) = 3 \) and \( h^0(E(-1)) = h^0(E^\vee) = 0 \),

(iii) the Castelnuovo-Mumford regularity of \( E \) is 1,

(iv) \( E \) is indecomposable, and can be written as a non-trivial extension

\[
0 \to \mathcal{O}_X \to E \to \mathcal{O}_X(1) \to 0,
\]

and

(v) \( E \) has a minimal resolution of the first kind.

Proof. (i) follows by taking determinants.
(ii) In the cohomology long exact sequence of (5), we see that the map  
\[ H^1(\mathcal{O}_X(-1)) \rightarrow H^1(\mathcal{O}_X^{\oplus 3}) \]

is the dual of the evaluation map  
\[ H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_X \rightarrow H^0(\mathcal{O}_X(1)). \]

This implies that \( h^0(E) = 3 \) and \( h^1(E) = 0 \). The former implies in particular that \( E \) is globally generated. By Serre duality, \( h^1(E) = h^0(\mathcal{E}^\vee) = h^0(E(-1)) = 0 \).

(iii) \( h^1(E) = 0 \) implies, by definition, that \( E \) is 1-regular.

(iv) Let \( s \in H^0(E) \) be a general section. Since \( E \) is globally generated, the zero locus is pure of codimension 2 and hence nowhere vanishing. Thus we have an exact sequence of bundles  
\[ 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow L \rightarrow 0. \]

Determinant considerations imply that \( L \cong \mathcal{O}_X(1) \). Tensoring this sequence by \( \mathcal{O}_X(-1) \) and taking cohomology, we get a cohomology sequence  
\[ \rightarrow H^0(X, E(-1)) \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X(-1)) \rightarrow \]

Since the first term is zero by (ii) above, this means that the boundary map is non-zero. In particular, the above extension is non-split.

To prove indecomposability, let us suppose that \( E = A \oplus B \). Since \( E \) is globally generated, this means that both \( A \) and \( B \) are globally generated too. Hence \( \text{deg}(A) \geq 0 \) and \( \text{deg}(B) \geq 0 \). Suppose \( \text{deg}(A) = 1 \). Then by Riemann-Roch, \( h^0(A) = 1 \). Global generation implies that there is a surjection \( H^0(A) \otimes \mathcal{O}_X = \mathcal{O}_X \rightarrow A \). This implies that \( A \cong \mathcal{O}_X \), hence a contradiction. Similarly \( \text{deg}(A) \neq 2 \) (for then \( \text{deg}(B) = 1 \) and then we arrive at a contradiction). Thus \( \text{deg}(A) = 0 \) or \( \text{deg}(B) = 0 \). Global generation implies that \( A \cong \mathcal{O}_X \) or \( B \cong \mathcal{O}_X \). Thus we have \( E = \mathcal{O}_X \oplus \mathcal{O}_X(1) \), again a contradiction.

(v) Since the Castelnuovo-Mumford regularity of \( E \) is 1 and \( h^0(E(-\nu)) = 0 \) for \( \nu > 0 \) from (ii) above, the minimal generators of \( E \) lie in degrees 0 and 1. From the cohomology long exact sequence associated to sequence (5), one first checks that not all its generators are in degree 0 i.e.,

\[ \bigoplus_{\nu \in \mathbb{Z}} H^0(\mathcal{O}_X(\nu)^{\oplus 3}) \rightarrow \bigoplus_{\nu \in \mathbb{Z}} H^0(E(\nu)) \]

is not a surjection. This fails for \( \nu = 1 \) and can be seen as follows. We have the long exact sequence of cohomology for the sequence (5) after tensoring with \( \mathcal{O}_X(1) \):

\[ \cdots \rightarrow H^0(\mathcal{O}_X(1)^{\oplus 3}) \rightarrow H^0(E(1)) \rightarrow H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_X(1)) \rightarrow H^1(E(1)) \rightarrow 0. \]

Since \( H^1(\mathcal{O}_X(1)) = 0 \), this implies that \( E \) has one generator in degree 1 which is not generated by the sections in degree 0. The same argument as above shows that these are all the generators and so we have an induced surjection \( \mathcal{O}_X^{\oplus 3} \oplus \mathcal{O}_X(-1) \rightarrow E \). This surjection in turn can be lifted to a minimal resolution of \( E \) on \( \mathbb{P}^2 \):

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow E \rightarrow 0, \]

such that Pf(\( \Phi \)) is the defining polynomial of \( X \).

\[ \square \]

The next result shows the uniqueness of bundles of the first and second kind.

**Theorem 4.** Let \( X \subset \mathbb{P}^2 \) be an elliptic curve.
(a) There exists a rank 2 bundle \( G \), unique up to isomorphism, with \( \text{det} \, G = \mathcal{O}_X \) and \( H^0(X, G(-1)) = 0 \) whose minimal resolution is of the second kind namely

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\Psi} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2} \to G \to 0.
\]

where \( \Psi \) is a \( 4 \times 4 \) skew-symmetric matrix

\[
\Psi = \begin{pmatrix}
0 & l_1 & -l_2 & q_3 \\
-l_1 & 0 & l_3 & q_2 \\
l_2 & -l_3 & 0 & q_1 \\
-q_3 & -q_2 & -q_1 & 0
\end{pmatrix}.
\]

Here the \( l_i \)’s are linear polynomials and \( q_j \)’s are quadratic polynomials and \( X \) is defined by the vanishing of \( \text{Pf}(\Psi) = \sum_{i=1}^{3} l_i q_i \).

(b) There exists a rank 2 bundle \( E \), unique up to isomorphism, with \( \text{det} \, E = \mathcal{O}_X(1) \) and \( H^0(X, E(-1)) = 0 \), whose minimal resolution is of the first kind

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2} \to E \to 0.
\]

where \( \Phi \) is a \( 4 \times 4 \) skew-symmetric matrix

\[
\Phi = \begin{pmatrix}
0 & q_1 & q_2 & l_3 \\
-q_1 & 0 & q_3 & -l_2 \\
-q_2 & -q_3 & 0 & l_1 \\
-l_3 & l_2 & -l_1 & 0
\end{pmatrix}
\]

and such that \( X \) is defined by the vanishing of \( \text{Pf}(\Phi) = \sum_{i=1}^{3} l_i q_i \).

Proof. (a) The existence of a bundle \( E \) of the first kind from Lemma 4, implies by Lemma 5, that there exists an indecomposable rank 2 bundle \( G \) of the second kind. From the minimal resolution (6), it follows that \( \text{det} \, G = \mathcal{O}_X \) and \( h^0(G) = 1 \). By Theorem 2, \( G \) must be isomorphic to the bundle \( F_2 \) defined as the unique non-trivial extension

\[
0 \to \mathcal{O}_X \to G \to \mathcal{O}_X \to 0.
\]

This proves that \( G \) is unique.

(b) Again by Lemma 4, uniqueness of \( G \) implies uniqueness of \( E \).

\( \square \)

5. Higher Order Torsion Points on an Elliptic Curve

**Theorem 5.** Let \( X \subset \mathbb{P}^2 \) be a smooth elliptic curve with defining polynomial \( f \). Let \( \Phi_r \) be a minimal \( 3r \times 3r \) linear matrix with \((r, \text{char} \, k) = 1\). Suppose that

\[
\text{Coker}[\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3r} \xrightarrow{\Phi_r} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3r}]
\]

is an indecomposable rank \( r \) bundle \( E \) with \( \text{det} \, E = \mathcal{O}_X \). Then \( \text{det} \, \Phi_r = f^r \). Furthermore, such \( E \) and \( \Phi_r \) exist and there is a bijective correspondence between the set of such bundles and the non-trivial \( r \)-torsion points of \( X \).

**Proof.** Consider the exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3r} \xrightarrow{\Phi_r} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3r} \to E \to 0,
\]

thought about as a minimal resolution of the bundle \( E \). Then locally since \( E \cong \mathcal{O}_X^{\oplus r} \), \( \Phi_r \) is the diagonal matrix

\[
\Phi_r = (f, f, \ldots, f, 1, \ldots, 1)
\]
with $f$ occurring $r$ times, and so $\det E = f^r$. Since $E$ is indecomposable of degree 0, by Theorem 2, $E \cong F_r \otimes A$ for some line bundle $A$ with $A^\otimes r = \mathcal{O}_X$. Finally $h^0(E) = 0$ implies that $A \not\cong \mathcal{O}_X$.

To prove the converse, we proceed as in the proof of Theorem 3. It is easy to check (the details of which we omit) that any rank $r$ bundle $E$ obtained as a repeated extension of an $r$-torsion line bundle with itself has the following properties:

(i) $E$ is 1-regular (in the sense of Castelnuovo-Mumford),
(ii) $h^0(E(-\nu)) = 0$ for $\nu \geq 0$, and
(iii) $\det(E) = \mathcal{O}_X$.

Conditions (i) and (ii) imply that $E$ has all its minimal generators in degree 1 and Riemann-Roch implies that there are $3r$ of them. So $E$ has a minimal resolution of the form

$$0 \to L_1 \to L_0 \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3r} \to E \to 0,$$

where $L_1$ is a sum of line bundles on $\mathbb{P}^2$. Dualising as before, we get

$$0 \to L_0^\vee \to L_1^\vee \to E^\vee(3) \to 0. \quad (8)$$

Now

$$E^\vee \cong (F_r \otimes A)^\vee \cong F_r^\vee \otimes A^\vee \cong F_r \otimes A'$$

where $A' = A^\vee \cong A^{r-1}$, and the surjection

$$L_1^\vee(-3) \to E^\vee$$

from (8) induces a surjection

$$H^0_1(\mathbb{P}^2, L_1^\vee) \to H^0_0(X, E^\vee).$$

This together with the fact that $\text{rank}(L_1) = \text{rank}(L_0)$, implies that $L_1^\vee \to E^\vee$ is induced by a set of minimal generators. Since $E^\vee$ also satisfies properties (i)-(iii) above and $h^0(E^\vee(1)) = 3r$ by Riemann-Roch, this implies $L_1^\vee(-3) \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3r}$. Thus $L_1 \cong \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3r}$. \qed

**Theorem 6.** Let $X \subset \mathbb{P}^2$ be an elliptic curve and $F_r$ denote the unique indecomposable rank $r$ bundle with $h^0(F_r) = 1$ and $\det F_r = \mathcal{O}_X$. Then $F_r$ has a minimal resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3(r-1)} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)^{\Psi_r} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3(r-1)} \oplus \mathcal{O}_{\mathbb{P}^2} \to F_r \to 0. \quad (9)$$

In particular, $\Psi_r$ is a minimal $(3r-2) \times (3r-2)$ matrix with linear and quadratic polynomial entries such that $\det \Psi_r = f^r$.

**Proof.** The proof is by induction. For the base case of $r = 2$, $F_2$ is the bundle $G$ in Theorem 4 (i). Now suppose that the theorem holds for $F_{r-1}$. Consider the exact sequence

$$0 \to \mathcal{O}_X \to F_r \to F_{r-1} \to 0.$$

Since $F_r^\vee \cong F_r$ for all $r$, we can consider the dual exact sequence

$$0 \to F_{r-1} \to F_r \to \mathcal{O}_X \to 0.$$

Since this is a non-trivial extension, we see that the coboundary map in the long exact sequence of cohomology, $H^0(X, \mathcal{O}_X) \to H^1(X, F_{r-1})$ is non-zero. However, $\forall s > 0$, $H^1(X, F_s(a)) = 0$ for $a > 0$, and so we have a short exact sequence

$$0 \to H^0(X, F_{r-1}(a)) \to H^0(X, F_r(a)) \to H^0(X, \mathcal{O}_X(a)) \to 0 \quad \forall a > 0.$$

The graded module $N := \oplus_{a>0} H^0(X, \mathcal{O}_X(a))$ is generated by its degree 1 elements (and there are three of them). This implies that the minimal generators of $F_{r-1}$ and $N$ together generate $F_r$ and that this set is minimal. Thus we have a surjection

$$\Theta: \left( \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3(r-2)} \oplus \mathcal{O}_{\mathbb{P}^2} \right) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to F_r.$$
This yields an exact sequence

\[ 0 \to L_1 \to L_0 := \mathcal{O}_{\mathbb{P}^2}(-2)^{3(r-1)} \oplus \mathcal{O}_{\mathbb{P}^2} \to F_r \to 0, \]

where \( L_1 \) is the kernel of the map \( \Theta \). Dualising the exact sequence, we get

\[ 0 \to L_0^\vee \to L_1^\vee \to F_r^\vee(3) \to 0. \]

Now \( F_r^\vee \cong F_r \) and since \( L_0 \) is a sum of line bundles, the map of graded modules \( H^0_*(L_1^\vee(-3)) \to H^0_*(F_r) \) is a surjection. This implies, by the same reasoning as in the proof of the above theorem, that \( L_1^\vee(-3) \cong L_0 \) and thus we are done. \( \square \)

References