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The Noether–Lefschetz theorem for the divisor class group

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Abstract

Let X be a normal projective threefold over a field of characteristic zero and $|L|$ be a base-point free, ample linear system on X. Under suitable hypotheses on $(X, |L|)$, we prove that for a very general member $Y \in |L|$, the restriction map on divisor class groups $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is an isomorphism. In particular, we are able to recover the classical Noether–Lefschetz theorem, that a very general hypersurface $X \subset \mathbb{P}^3_{\mathbb{C}}$ of degree $\geqslant 4$ has $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Keywords: Divisor class group; Noether–Lefschetz theorem

We work over an algebraically closed field of characteristic zero, which shall be denoted by k.

Let X be an irreducible normal projective 3-fold defined over k and $\mathcal{O}_X(1)$ be an ample line bundle on X. Let $V \subset \mathrm{H}^0(X, \mathcal{O}_X(1))$ define a base-point free linear system $|V|$, so that the induced morphism $f : X \to \mathbb{P}^N := \mathbb{P}(V)$ is finite. If Y is a general member of this linear system on X, then Y is a normal projective surface, by Bertini's theorem. Since Y is an effective Cartier divisor in X, we have a natural homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ on divisor class groups,

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defined by $[D] \mapsto [D \cap Y]$, where D is any Weil divisor on X which does not have Y as a component, and $[D \cap Y]$ is the intersection cycle. This homomorphism on class groups is the refined Gysin homomorphism $\mathrm{CH}_2(X) \to \mathrm{CH}_1(Y)$ of [4]. The *Noether–Lefschetz problem* in this context is to find conditions on $(X, \mathcal{O}_X(1))$ which imply that the above map $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is an isomorphism.

If X is non-singular, and $\mathcal{O}_X(1)$ is "sufficiently ample," then the *Noether–Lefschetz theorem* asserts that $\mathrm{Cl}(X) = \mathrm{Pic}(X) \to \mathrm{Pic}(Y) = \mathrm{Cl}(Y)$ is an isomorphism for a "very general" choice of Y. Let S be the k-parameter variety for divisors in the linear system. By "very general" we may mean one of two things: either (i) that k is uncountable, and that $Y \in S$ lies in the complement of a countable union of proper subvarieties of S, or (ii) that Y is the geometric generic member of the linear system, defined after making a base change to the algebraic closure of the function field $k(S)$, and the Picard groups are computed after this base change. We will comment further below (see Section 3) about the relation between these conditions. In the case when $k = \mathbb{C}$, $X = \mathbb{P}^3_{\mathbb{C}}$, and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^3}(d)$ with $d \geqslant 4$, we obtain the "classical" Noether–Lefschetz theorem.

Proofs of versions of the Noether–Lefschetz theorem may be found in several places; for example see [3] for a "modern" treatment, including also statements valid in characteristics $p > 0$. All proofs in the literature which are known to us use either the monodromy of Lefschetz pencils, or Hodge theory in some form (see for example [1]). They do not seem to cover the case of divisor class groups of singular varieties, at least in their present forms.

If X is normal, and $\pi : \widetilde{X} \to X$ is a proper birational morphism from a non-singular proper 3-fold $\widetilde{X}$ (e.g., a resolution of singularities of the normal projective 3-fold X), then the sheaf $\pi_*\omega_{\widetilde{X}}$ is a torsion-free coherent sheaf on X, which restricts to $\omega_{X_{\mathrm{reg}}}$ on the regular locus $X_{\mathrm{reg}} = X \setminus X_{\mathrm{sing}}$. Denote this sheaf by K_X. It is well known to be independent of the choice of the birational morphism π (the Grauert–Riemenschneider theorem, see [5] for example), and for non-singular X (or more generally, for X with only rational singularities), coincides with its canonical sheaf.

The main result of this paper is the following.

Theorem 1. *Let X be a normal projective 3-fold over k, with an ample invertible sheaf $\mathcal{O}_X(1)$, and a subspace $V \subset H^0(X, \mathcal{O}_X(1))$, defining a base-point free linear system, and thus giving a morphism $f : X \to \mathbb{P}^N_k$. Assume further that the coherent sheaf $(f_*K_X)(1)$ is generated by its global sections.*

Let Y denote a very general member of the linear system $|V|$. Then the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is an isomorphism.

In particular, the theorem is true for non-singular X such that $\mathcal{O}_X(1)$ is very ample, $|V|$ is the corresponding complete linear system, and $\omega_X(1)$ is generated by global sections. This is certainly true when $\mathcal{O}_X(1)$ is sufficiently ample, and includes the "classical" case $X = \mathbb{P}^3$, $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^3}(d)$ with $d \geqslant 4$.

The proof of the theorem as stated above differs from existing proofs in two ways. Firstly, it is purely algebraic in nature and in the spirit of Grothendieck's proof of the Grothendieck–Lefschetz theorem (see [2,8]); no use is made of monodromy or Hodge theory. Secondly, we explicitly say how "positive" the linear system needs to be. Our algebraic approach has an advantage of yielding a result for the divisor class group, which is perhaps not easily available from the monodromy/Hodge theory approach.

We may compare our result, for smooth X, with the assumption that f is an embedding, such that $\mathrm{H}^2(\mathcal{O}_X) \to \mathrm{H}^2(\mathcal{O}_Y)$ is not surjective, for a general hyperplane section Y; this hypothesis is sufficient to yield the Noether–Lefschetz theorem using the monodromy argument, for example. We do not quite recover this statement, though we do get it for $X = \mathbb{P}^3$. However, we do have a statement for smooth X, for a finite map f which is not an embedding, and it is not clear to us that this can also be obtained by monodromy arguments. In any case, it is not clear to us what the "most general" assertion (in the direction of the theorem) should be, which would include our results, as well as the classical statement obtained by monodromy/Hodge theory.

The approach here is a generalization of a method introduced by the second author and N. Mohan Kumar (see [10]). It consists of first proving a Formal Noether–Lefschetz theorem, and then using this to obtain the "global" Noether–Lefschetz theorem. For other applications of this method, see [9,13,14].

The plan of the paper is as follows. In Section 1, we give a reformulation in terms of Picard groups of desingularizations, which amounts to considering a Noether–Lefschetz problem for a very general member of a big and base-point free linear system on a smooth proper 3-fold. In Section 2, we introduce and prove the Formal Noether–Lefschetz theorem. In Section 3, we show how the Formal Noether–Lefschetz theorem implies the global Noether–Lefschetz theorem.

1. Reformulation

First let X be a normal projective 3-fold over k, and let $\mathcal{O}_X(1)$ be an ample line bundle over X, together with a linear subspace $V \subset \mathrm{H}^0(X, \mathcal{O}_X(1))$ which gives a base point free linear system $|V|$ on X. Let $Y \in |V|$ be a general element of this linear system; by Bertini's theorem, we have $Y_{\mathrm{sing}} = Y \cap X_{\mathrm{sing}}$, and Y is a normal projective surface.

If $\widetilde{X} \xrightarrow{\pi} X$ is a desingularization of X (by which we mean here that π is a proper birational morphism, and $\widetilde{X}$ is non-singular) we have the following (Cartesian) diagram:

$$\begin{array}{ccc} \widetilde{Y} & \hookrightarrow & \widetilde{X} \\ {\scriptstyle \pi|_{\widetilde{Y}}}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \hookrightarrow & X. \end{array}$$

Note that $\widetilde{Y}$ is a general member of the pull-back linear system π^*V on the smooth proper variety $\widetilde{X}$, and therefore is smooth, by Bertini's theorem; hence $\widetilde{Y} \to Y$ is a desingularization of Y. If X is singular, then $\widetilde{Y}$ is a general member of the linear system determined by $\pi^*V \subset \mathrm{H}^0(\widetilde{X}, \pi^*\mathcal{O}_X(1))$ where $\pi^*\mathcal{O}_X(1)$ is not ample, but is *big and base-point free*. Let $g := f \circ \pi$ be the composite morphism $\widetilde{X} \to X \to \mathbb{P}^N$.

Conversely, if $\widetilde{X}$ is a non-singular and proper 3-fold, and $g : \widetilde{X} \to \mathbb{P}^N$ is a morphism which is generically finite (to its image), then we can consider the Stein factorization of g,

$$\widetilde{X} \xrightarrow{\pi} X \xrightarrow{f} \mathbb{P}^N,$$

where X is a normal projective 3-fold, f is finite, and π is a proper birational map. By the exceptional locus of the generically finite proper morphism g, we will mean the union of the positive dimensional components of fibers of the morphism; this coincides with the exceptional locus of the proper birational morphism π obtained by Stein factorization. Notice that there is a closed subset $S \subset X$ of dimension $\leqslant 1$ such that $\widetilde{X} \setminus \pi^{-1}(S) \to X \setminus S$ is an isomorphism. Thus, for any irreducible component E of the g-exceptional locus, we have $\dim g(E) \leqslant 1$.

In this context, we have the following analogue of the Noether–Lefschetz theorem, for Picard groups.

Theorem 2. *Let $\widetilde{X}$ be a non-singular proper 3-fold, and $g : \widetilde{X} \to \mathbb{P}^N_k$ a morphism, generically finite to its image. Assume that the coherent sheaf $g_* K_{\widetilde{X}} \otimes \mathcal{O}_{\mathbb{P}^N}(1)$ is globally generated. Let $\widetilde{Y}$ be the pullback of a very general hyperplane in $\mathbb{P}^N$. Then there is an exact sequence*

$$0 \to A \to \operatorname{Pic}(\widetilde{X}) \to \operatorname{Pic}(\widetilde{Y}) \to B \to 0$$

where A is freely generated by the irreducible divisors in $\widetilde{X}$ which map to points under g, and B is the group generated by the irreducible divisors in $\widetilde{Y}$ which map to points under $g|_{\widetilde{Y}}$. Further, the class of a $g|_{\widetilde{Y}}$-exceptional divisor lies in the image of $\operatorname{Pic}(\widetilde{X}) \to \operatorname{Pic}(\widetilde{Y})$ precisely when it is the restriction of a g-exceptional divisor class in $\operatorname{Pic}(\widetilde{X})$.

Notice that if E is any irreducible component of the g-exceptional locus of $\widetilde{X}$, then since $\widetilde{Y}$ is a general member of a base-point free linear system on $\widetilde{X}$, we have: (i) $E \cap \widetilde{Y} = \emptyset$ if $\dim g(E) = 0$ (this is necessarily the case if $\dim E = 1$), (ii) the irreducible components of the exceptional locus of $\widetilde{Y} \to \mathbb{P}^N_k$ are divisors (curves), and these are irreducible components of $E \cap \widetilde{Y}$, where $E \subset \widetilde{X}$ is an irreducible divisor with $\dim g(E) = 1$. Also, we note that any non-zero exceptional divisor for g determines a non-zero class in $\operatorname{Pic} \widetilde{X}$, and a similar assertion holds for $g|_{\widetilde{Y}}$ and $\operatorname{Pic} \widetilde{Y}$.

With these remarks, one can easily see that Theorem 2 is equivalent to Theorem 1, using the fact (see [12], Section 1 for a more detailed explanation) that for any proper birational morphism $h : \widetilde{V} \to V$ from a non-singular proper variety $\widetilde{V}$ to a normal projective variety V, we have a natural isomorphism

$$\operatorname{Cl}(V) \cong \frac{\operatorname{Pic}(\widetilde{V})}{(\text{subgroup generated by } h\text{-exceptional divisors})}.$$

The surjection $\operatorname{Pic}(\widetilde{V}) \to \operatorname{Cl}(V)$ may be viewed as a particular case of the proper push-forward map on Chow groups (see [4]), and this is compatible with intersection with a Cartier divisor (this is a particular case of functoriality under proper push-forwards of the refined Gysin maps, as constructed in [4], for example).

The equivalence of Theorem 1 and Theorem 2 allows us to say that Theorem 2 for $(\widetilde{X}, g)$ depends only on the corresponding ample linear system $|V|$ on the normal variety X obtained by Stein factorization. In particular, by Hironaka's theorem, we may assume without loss of generality that $\pi : \widetilde{X} \to X$ is a resolution of singularities, whose exceptional locus is a divisor with simple normal crossings, obtained by blowing up an ideal sheaf corresponding to a subscheme supported on X_{sing}; in particular, that there exists an effective divisor E on $\widetilde{X}$ which is π-exceptional, such that $\mathcal{O}_{\widetilde{X}}(-E)$ is π-ample.

Thus, we will prove Theorem 2 below, with the additional hypothesis that the g-exceptional locus is a divisor with simple normal crossings, and that there is a g-exceptional effective divisor E such that $-E$ is g-ample.

We now set up some further notation. Let $V \subset \mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1))$ be as in Theorem 2, and let $S := \mathbb{P}(V^*)$ be the corresponding parameter space for members of the linear system. Let $\mathcal{X} := \widetilde{X} \times S$ and $p : \mathcal{X} \to \widetilde{X}$, $q : \mathcal{X} \to S$ denote the two projection maps. Further, let $\mathcal{Y} \subset \mathcal{X}$ denote the total space of the given family of divisors in $\widetilde{X}$ parametrized by S. If $\mathcal{V}$ be the vector bundle defined by the exact sequence

$$0 \to \mathcal{V} \to V \otimes \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}(1) \to 0,$$

then

$$\mathcal{Y} = \mathbb{P}_{\widetilde{X}}(\mathcal{V}^*) \subset \widetilde{X} \times S.$$

Let $s \in S$ be a (closed) point parametrizing a general smooth divisor $\widetilde{Y}$ in $\widetilde{X}$ in the given linear system (here "general" means "for s lying in some non-empty Zariski open subset of S"). The following result is proved in our earlier paper [12].

Theorem 3. *For a general $\widetilde{Y}$ as above, we have the following.*

(a) *There is an exact sequence*

$$0 \to A \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{Y}) \to C \to 0,$$

and an inclusion $B \hookrightarrow C$, where

  (i) *A is freely generated by irreducible g-exceptional divisors in $\widetilde{X}$ which have* 0-*dimensional image.*
  (ii) *B is freely generated by the irreducible $g|_{\widetilde{Y}}$-exceptional divisors in $\widetilde{Y}$, which have* 0-*dimensional image under g, modulo the group generated by the classes of exceptional divisors of the form $E \cdot \widetilde{Y}$, where E is an irreducible g-exceptional divisor on $\widetilde{X}$ with* $\dim g(E) = 1$.
  (iii) *C is a free abelian group of finite rank.*

(b) *The pair $(\widetilde{X}, \widetilde{Y})$ satisfy Grothendieck's Condition* $\mathrm{Lef}(\widetilde{X}, \widetilde{Y})$, *as well as the condition* $\mathrm{ALeff}(\widetilde{X}, \widetilde{Y})$ (a weak form of Grothendieck's Effective Lefschetz Condition, see [12]).
(c) *If $\widehat{Y}$ denotes the formal scheme obtained by completing $\widetilde{X}$ along $\widetilde{Y}$, then* $\mathrm{Pic}(\widehat{Y}) \to \mathrm{Pic}(\widetilde{Y})$ *is injective, and there is an exact sequence*

$$0 \to A \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widehat{Y}) \to B \to 0$$

with A, B as above (*i.e., as in Theorem* 2).

Thus, the content of the theorem is that, if we are willing to replace $\mathrm{Pic}(\widetilde{Y})$ by $\mathrm{Pic}(\widehat{Y})$, the Picard group of isomorphism classes of formal line bundles on the formal completion of $\widetilde{X}$ along $\widetilde{Y}$, then we do have the conclusion of Theorem 2, for all $\widetilde{Y}$ corresponding to a non-empty Zariski open set in our linear system S. (For this conclusion, we do not need the hypothesis on global generation of the sheaf $f_*K_X(1)$.) The fact that the group C is free abelian, though not explicitly stated in [12], follows from results proved there: the fact that it is finitely generated (see [12, Lemma 3.3]), and that $\mathrm{coker}\,\mathrm{Pic}(\widehat{Y}) \to \mathrm{Pic}(\widetilde{Y})$ is torsion-free. This torsion-freeness (proved in Section 5(ii) of [12]) follows because it is true with $\widehat{Y}$ replaced by each of the schemes $\widetilde{Y}_n$ in the inverse system of schemes defining the formal scheme $\widehat{Y}$; at this finite level, the cokernel of the map on Picard groups is a subgroup of a certain cohomology H^2 (see the exact cohomology sequence appearing in the proof of Lemma 3.3 in [12]) which is a vector space over the ground field k (of characteristic 0).

Another result in [12] (see Section 5(i)) is that the open set of divisors $\widetilde{Y}$ in the linear system $|V|$ for which Theorem 3 holds may be assumed to be invariant under base change to a larger algebraically closed field. As noted there, this follows from the description of that open set given in [12]—after possibly first making a birational modification of $\widetilde{X}$, it is any non-empty open set in S parametrizing divisors $\widetilde{Y}$ which are non-singular, disjoint from irreducible components of the exceptional locus of $\widetilde{X}$ with 0-dimensional image, and which transversally intersect all other exceptional divisors of $\widetilde{X}$.

The upshot of the above is that, to get the Noether–Lefschetz theorem (Theorem 2) itself, we would have to pass from this formal completion $\widehat{Y}$ to $\widetilde{Y}$ itself.

If dimension $\widetilde{X} \geqslant 4$, then for a base-point free and big linear system on $\widetilde{X}$, we have (with similar notation) $\operatorname{Pic}(\widehat{Y}) \cong \operatorname{Pic}(\widetilde{Y})$ from a version of Grauert–Riemenschneider vanishing, as observed in [12]. This resulted in our version of the Grothendieck–Lefschetz theorem in [12].

However, for a 3-fold $\widetilde{X}$, we need a suitable additional hypothesis on the linear system (that the line bundle is "sufficiently positive"), and then we need to restrict to a "very general" $\widetilde{Y}$. This is as in the classical Noether–Lefschetz theorem.

2. The formal Noether–Lefschetz theorem

The goal of this section is to formulate and prove a *Formal Noether–Lefschetz theorem*. This is about comparing the Picard groups of two different kinds of completions. The first one is the completion $\widehat{Y}$ of $\widetilde{X}$ along the subvariety $\widetilde{Y}$, assumed to be a general member of S, which we encountered already in Theorem 3. The other is the completion of $\mathcal{Y}$ along its fiber over the point $s \in S$ corresponding to $\widetilde{Y}$.

Let $\mathfrak{m}$ denote the ideal sheaf defining the point $s \in S$. Further, let $\mathcal{I} \cong \mathcal{O}_{\widetilde{X}}(-1)$ be the ideal sheaf defining $\widetilde{Y}$ in $\widetilde{X}$. One has an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{Y}} \to 0$$

or equivalently on tensoring with $\mathcal{O}_{\widetilde{X}}(1)$,

$$0 \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}(1) \to \mathcal{O}_{\widetilde{Y}}(1) \to 0.$$

Taking cohomology, we get an exact sequence

$$0 \to k \to V \to W \to 0$$

where $W := \operatorname{Im}(V \to \mathrm{H}^0(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(1)))$.

Let Y_n be the nth infinitesimal neighborhood of $\widetilde{Y}$ in $\widetilde{X}$, so that the sequence Y_n gives rise to the formal completion $\widehat{Y}$ of $\widetilde{X}$ along $\widetilde{Y}$. Similarly let $\mathcal{Y}_n$ be the nth infinitesimal neighborhood of $\widetilde{Y} = \mathcal{Y}_s$ in $\mathcal{Y}$, and let $\widehat{\mathcal{Y}}$ be the associated formal completion.

Note that since $\mathcal{Y}_1 \to Y_1 = \widetilde{Y}$ is an isomorphism, we have an inclusion of ideals $\mathcal{I}\mathcal{O}_{\mathcal{Y}} \subset \mathfrak{m}\mathcal{O}_{\mathcal{Y}}$. This implies that there is a morphism of formal schemes $\widehat{\mathcal{Y}} \to \widehat{Y}$ compatible with the scheme morphism $\mathcal{Y} \to \widetilde{X}$, and with the isomorphism $\mathcal{Y}_1 \to Y_1 = \widetilde{Y}$.

Now consider the following diagram:

$$\begin{array}{ccccc} & & \mathrm{Pic}(\widehat{\mathcal{Y}}) & & \\ & \nearrow & \uparrow & \searrow & \\ \mathrm{Pic}(\widetilde{X}) & & & & \mathrm{Pic}(\widetilde{Y}). \\ & \searrow & \uparrow & \nearrow & \\ & & \mathrm{Pic}(\widehat{Y}) & & \end{array} \tag{1}$$

Definition 1. We say that the *condition* FNL *holds* for $(\widetilde{X}, \widetilde{Y})$ if (with the above notation)

$$\mathrm{Im}\big(\mathrm{Pic}(\widehat{Y}) \to \mathrm{Pic}(\widetilde{Y})\big) = \mathrm{Im}\big(\mathrm{Pic}(\widehat{\mathcal{Y}}) \to \mathrm{Pic}(\widetilde{Y})\big).$$

We say that the *condition FNL holds for* $(\widetilde{X}, |V|)$ if it holds for all $(\widetilde{X}, \widetilde{Y})$, where Y runs over a non-empty open subset of $|V|$ (regarded as a projective space).

The above condition says that if a line bundle on $\widetilde{Y}$ can be lifted to every infinitesimal thickening of the corresponding fiber in the universal family, it can also be lifted to the formal completion $\widehat{Y}$ of $\widetilde{X}$ along $\widetilde{Y}$, after which the conditions Lef and ALeff imply that, upto modification by an exceptional divisor for $\widetilde{Y}$, it lifts to a line bundle on $\widetilde{X}$.

By a result of Grothendieck (see [7, II, Prop. 9.6 and Ex. 9.6]), we have isomorphisms

$$\mathrm{Pic}(\widehat{Y}) \cong \varprojlim_n \mathrm{Pic}(Y_n),$$
$$\mathrm{Pic}(\widehat{\mathcal{Y}}) \cong \varprojlim_n \mathrm{Pic}(\mathcal{Y}_n).$$

We also have exact sheaf sequences

$$0 \to \mathcal{I}/\mathcal{I}^n \xrightarrow{\exp} \mathcal{O}_{Y_n}^{\times} \to \mathcal{O}_{\widetilde{Y}}^{\times} \to 0, \tag{2}$$

$$0 \to q^*\big(\mathfrak{m}/\mathfrak{m}^n\big) \xrightarrow{\exp} \mathcal{O}_{\mathcal{Y}_n}^{\times} \to \mathcal{O}_{\widetilde{Y}}^{\times} \to 0 \tag{3}$$

where we identify $\mathcal{Y}_1$ with $\widetilde{Y}$ (here *exp* denotes the exponential map, well-defined on any nilpotent ideal sheaf, since we are working in characteristic 0). Since $\mathcal{I} \cong \mathcal{O}_{\widetilde{X}}(-1)$, we see that $\mathcal{I}/\mathcal{I}^n$ is filtered by $\mathcal{I}^r/\mathcal{I}^{r+1} \cong \mathcal{O}_{\widetilde{Y}}(-r)$, $1 \leqslant r \leqslant n-1$, and these sheaves have vanishing H^1, since $\mathcal{O}_{\widetilde{Y}}(1)$ is big and base-point free on $\widetilde{Y}$ (Ramanujam vanishing theorem [11]). Hence we have an exact sequence

$$0 \to \mathrm{Pic}(Y_n) \to \mathrm{Pic}(\widetilde{Y}) \to H^2\big(\widetilde{X}, \mathcal{I}/\mathcal{I}^n\big).$$

Similarly we have an exact sequence

$$0 \to \mathrm{H}^1\big(\mathcal{Y}, q^*\big(\mathfrak{m}/\mathfrak{m}^n\big)\big) \to \mathrm{Pic}(\mathcal{Y}_n) \to \mathrm{Pic}(\widetilde{Y}) \to \mathrm{H}^2\big(\mathcal{Y}, q^*\big(\mathfrak{m}/\mathfrak{m}^n\big)\big).$$

One has the following commutative diagram, which allows us to compare the Picard groups of the two completions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^{-1}\mathcal{I}/\mathcal{I}^n & \longrightarrow & p^{-1}\mathcal{O}_{Y_n}^{\times} & \longrightarrow & p^{-1}\mathcal{O}_{\widetilde{Y}}^{\times} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & q^*(\mathfrak{m}/\mathfrak{m}^n) & \longrightarrow & \mathcal{O}_{\mathcal{Y}_n}^{\times} & \longrightarrow & \mathcal{O}_{\mathcal{Y}_1}^{\times} & \longrightarrow & 0 \end{array} \tag{4}$$

which yields (when taken along with the sequences (2), (3)) the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Pic}(Y_n) & \longrightarrow & \operatorname{Pic}(\widetilde{Y}) & \longrightarrow & \mathrm{H}^2(\widetilde{X}, \mathcal{I}/\mathcal{I}^n) \\ & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & \frac{\operatorname{Pic}(\mathcal{Y}_n)}{\mathrm{H}^1(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}/\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}})} & \longrightarrow & \operatorname{Pic}(\widetilde{Y}) & \longrightarrow & \mathrm{H}^2(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}/\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}). \end{array} \tag{5}$$

Definition 2. We say that the nth Infinitesimal Noether–Lefschetz (INL_n) condition for $(\widetilde{X}, \widetilde{Y})$ is satisfied if

$$\operatorname{Pic}(Y_n) \cong \frac{\operatorname{Pic}(\mathcal{Y}_n)}{\mathrm{H}^1(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}/\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}})}.$$

Proposition 1. *If $\widetilde{Y}$ is any smooth divisor in $|V|$, then* INL_n *holds for $(\widetilde{X}, \widetilde{Y})$ for all large enough n. In particular*: (i) FNL *holds for $(\widetilde{X}, \widetilde{Y})$,* (ii) FNL *holds for $(\widetilde{X}, |V|)$.*

The rest of this section is devoted to a proof of the above proposition. First, for a given pair $(\widetilde{X}, \widetilde{Y})$, the validity of the condition INL_n for all large n implies the validity of FNL for $(\widetilde{X}, \widetilde{Y})$. Hence, if we prove INL_n for $(\widetilde{X}, \widetilde{Y})$ for any smooth divisor $\widetilde{Y} \in |V|$, we get that FNL holds for $(\widetilde{X}, |V|)$.

Now notice that, from the diagram (5), if the map

$$\mathrm{H}^2\big(\widetilde{X}, \mathcal{I}/\mathcal{I}^n\big) \to \mathrm{H}^2\big(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}/\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}\big) \tag{6}$$

is injective, then clearly the condition INL_n is satisfied for $(\widetilde{X}, \widetilde{Y})$.

The rest of this section will be devoted to proving that the above map (6) is injective, for all large n.

Lemma 1. *Let $p : \mathcal{Y} \to \widetilde{X}$ be as above. Then*

(1) $\mathrm{R}^i p_*(\mathcal{O}_{\mathcal{Y}}) = 0\ \forall i > 0$.
(2) $p_*(\mathcal{O}_{\mathcal{Y}}) \cong \mathcal{O}_{\widetilde{X}}$.
(3) $p_*(\mathcal{O}_{\mathcal{Y}_1}) \cong \mathcal{O}_{\widetilde{Y}}$.
(4) $\mathrm{H}^i(\mathcal{Y}, p^*\mathcal{F}) \cong \mathrm{H}^i(\widetilde{X}, \mathcal{F})$ *for all i and any coherent sheaf $\mathcal{F}$ on $\widetilde{X}$.*
(5) $p_*\mathfrak{m}\mathcal{O}_{\mathcal{Y}} \cong \mathcal{I}$.

(6) $\mathrm{R}^i\, p_*\mathfrak{m}\mathcal{O}_{\mathcal{Y}} = 0\ \forall i > 0$.
(7) *The map* $\mathcal{O}_{Y_n} \to p_*(\mathcal{O}_{\mathcal{Y}_n})$ *is an injection.*
(8) $p_*(\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}) = \mathcal{I}^n$.

Proof. Since $\mathcal{Y} \to X$ is a projective bundle, we have $Rp_*p^*\mathcal{F} \cong \mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on $\widetilde{Y}$. This yields (1)–(4). For the statements (5) and (6), consider the sequence

$$0 \to \mathfrak{m}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}_1} (\cong \mathcal{O}_{\widetilde{Y}}) \to 0.$$

Applying Rp_*, since $Rp_*\mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_{\widetilde{X}}$, and p induces an isomorphism of $\mathcal{Y}_1$ with $\widetilde{Y}$, we get that $Rp_*\mathfrak{m}\mathcal{O}_{\mathcal{Y}} \cong \mathcal{I}$.

Now we prove the remaining statements. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_*(\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & p_*(\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & p_*(\mathcal{O}_{\mathcal{Y}_n}) & \longrightarrow & \mathrm{R}^1\, p_*(\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}) \longrightarrow 0 \\ & & \uparrow & & \cong\uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{I}^n & \longrightarrow & \mathcal{O}_{\widetilde{X}} & \longrightarrow & \mathcal{O}_{Y_n} & \longrightarrow & 0. \end{array} \tag{7}$$

It suffices to show $\mathcal{O}_{Y_n} \to p_*(\mathcal{O}_{\mathcal{Y}_n})$ is injective, which we may do by induction on n; this is clear for $n = 1$, since the map is an isomorphism.

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widetilde{Y}}(-n+1) & \longrightarrow & \mathcal{O}_{Y_n} & \longrightarrow & \mathcal{O}_{Y_{n-1}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p_*(\mathfrak{m}^{n-1}\mathcal{O}_{\mathcal{Y}}/\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & p_*(\mathcal{O}_{\mathcal{Y}_n}) & \longrightarrow & p_*(\mathcal{O}_{\mathcal{Y}_{n-1}}) & \longrightarrow & 0. \end{array}$$

The first vertical map is obtained by applying p_* to the analogous map

$$\left(p^*\mathcal{I}^{n-1}/\mathcal{I}^n\right) \otimes \mathcal{O}_{Y_1} \to \mathfrak{m}^{n-1}\mathcal{O}_{\mathcal{Y}}/\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}.$$

This map is a locally split (hence injective) map of locally free sheaves on Y_1, with locally free cokernel, since $\mathfrak{m}\mathcal{O}_{\mathcal{Y}}$ is the ideal sheaf of Y_1 in $\mathcal{Y}$, $p^*\mathcal{I}$ the ideal sheaf of $p^{-1}\widetilde{Y}$ in $\mathcal{Y}$, and $Y_1 \subset p^{-1}\widetilde{Y} \subset \mathcal{Y}$ are inclusions of smooth subvarieties; hence the stalks of their ideal sheaves at points of Y_1 are generated by appropriate subsets of a suitable regular system of parameters in a regular local ring.

Now by induction, we see that $\mathcal{O}_{Y_n} \to p_*\mathcal{O}_{\mathcal{Y}_n}$ is injective. □

Remark 1. For later use, we note that the sheaf

$$\operatorname{coker}(\mathcal{O}_{Y_n} \to p_*\mathcal{O}_{\mathcal{Y}_n})$$

has homological dimension 1 on $\widetilde{X}$, for each $n \geqslant 1$, since it is set-theoretically supported on $\widetilde{Y}$, and has a filtration with subquotients which are locally free sheaves on $\widetilde{Y}$.

Corollary 1. *With notation as above,*

$$\mathrm{H}^i(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}) \cong \mathrm{H}^i(\widetilde{X}, \mathcal{I}), \quad \forall i.$$

Proof. Since $Rp_*\mathfrak{m}\mathcal{O}_{\mathcal{Y}} = \mathcal{I}$, this follows from the Leray spectral sequence. □

Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{I}^n & \longrightarrow & p^*\mathcal{I} & \longrightarrow & p^*\mathcal{I}/\mathcal{I}^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{m}^n\mathcal{O}_{\mathcal{Y}} & \longrightarrow & \mathfrak{m}\mathcal{O}_{\mathcal{Y}} & \longrightarrow & \frac{\mathfrak{m}\mathcal{O}_{\mathcal{Y}}}{\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}} & \longrightarrow & 0. \end{array} \tag{8}$$

Taking cohomology as above, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^2(\widetilde{X}, \mathcal{I}/\mathcal{I}^n) & \longrightarrow & \mathrm{H}^3(\widetilde{X}, \mathcal{I}^n) & \longrightarrow & \mathrm{H}^3(\widetilde{X}, \mathcal{I}) & \longrightarrow & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(\mathcal{Y}, \frac{\mathfrak{m}\mathcal{O}_{\mathcal{Y}}}{\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}}) & \longrightarrow & \mathrm{H}^3(\mathcal{Y}, \mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & \mathrm{H}^3(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & \cdot \end{array} \tag{9}$$

From Corollary 1, we have $\mathrm{H}^i(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}) \cong \mathrm{H}^i(\widetilde{X}, \mathcal{I}) \cong \mathrm{H}^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-1))$ for all i. The latter vanishes for $i < 3$ by the Grauert–Riemenschneider vanishing theorem. This proves the exactness on the left, for the rows. We also have that $\mathrm{H}^3(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}) \cong \mathrm{H}^3(\widetilde{X}, \mathcal{I})$ is an isomorphism.

Thus, by a diagram chase, INL_n follows if we prove the injectivity of the map

$$\mathrm{H}^3(\widetilde{X}, \mathcal{I}^n) \to \mathrm{H}^3(\mathcal{Y}, \mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}). \tag{10}$$

2.1. Vanishing of a differential of a Leray spectral sequence

The Leray spectral sequence for $p : \mathcal{Y} \to \widetilde{X}$ associated to the cohomology $\mathrm{H}^*(\mathcal{Y}, \mathfrak{m}^n\mathcal{O}_{\mathcal{Y}})$ has $E_2^{p,q} \cong \mathrm{H}^p(\widetilde{X}, \mathrm{R}^q\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}})$. The map in (10) is the edge homomorphism

$$E_2^{3,0} \to \mathrm{H}^3(\mathcal{Y}, \mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}).$$

Hence the injectivity in (10) follows if we show that the differential

$$\mathrm{H}^1(\widetilde{X}, \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}) = E_2^{1,1} \to E_2^{3,0} = \mathrm{H}^3(\widetilde{X}, \mathcal{I}^n)$$

vanishes. To do so, we shall first give a more explicit description of this differential.

From Lemma 1, we get a four term sequence

$$0 \to \mathcal{I}^n \to \mathcal{O}_{\widetilde{X}} \to p_*(\mathcal{O}_{\mathcal{Y}_n}) \to \mathrm{R}^1\, p_*(\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}) \to 0$$

which we break up into two short exact sequences

$$
\begin{array}{c}
0 \longrightarrow \mathcal{I}^n \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_{Y_n} \longrightarrow 0, \\
\\
0 \longrightarrow \mathcal{O}_{Y_n} \longrightarrow p_*(\mathcal{O}_{\mathcal{Y}_n}) \longrightarrow \mathrm{R}^1\, p_*(\mathfrak{m}^n \mathcal{O}_{\mathcal{Y}}) \longrightarrow 0.
\end{array} \tag{11}
$$

One checks now that the map

$$\mathrm{H}^1\big(\widetilde{X}, \mathrm{R}^1\, p_* \mathfrak{m}^n \mathcal{O}_{\mathcal{Y}}\big) \to \mathrm{H}^2(Y_n, \mathcal{O}_{Y_n}) \to \mathrm{H}^3\big(\widetilde{X}, \mathcal{I}^n\big)$$

obtained by composing the (co)boundary maps in the cohomology sequences associated to the short exact sequences above is the differential $d_{1,1}^{3,0} : E_2^{1,1} \to E_2^{3,0}$.

However, to compute the differential, we use a different factorization which is obtained as follows.

Consider the following *nine diagram*:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathfrak{m}^n \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathfrak{m}^n \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathfrak{m}^n \mathcal{O}_{\mathcal{Y}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_{\mathcal{Y}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathcal{O}_{\mathcal{X}_n} & \longrightarrow & \mathcal{O}_{\mathcal{X}_n} & \longrightarrow & \mathcal{O}_{\mathcal{Y}_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Applying the higher derived functor $\mathrm{R}\, p_*$ to the nine diagram we get the following:

- It is easy to check that the leftmost vertical row gives

$$\mathrm{R}^1\, p_*\big(\mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathfrak{m}^n \mathcal{O}_{\mathcal{X}}\big) \cong p_*\big(\mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathcal{O}_{\mathcal{X}_n}\big)$$

and by the Kunneth formula, there is a natural isomorphism of vector bundles

$$p_*\big(\mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathcal{O}_{\mathcal{X}_n}\big) \cong \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes_k \mathcal{O}_{\widetilde{X}}(-1).$$

Thus

$$\mathrm{R}^1\, p_*\big(\mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathfrak{m}^n \mathcal{O}_{\mathcal{X}}\big) \cong \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes_k \mathcal{O}_{\widetilde{X}}(-1). \tag{12}$$

- The middle vertical sequence gives rise to a short exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}} \to \mathrm{H}^0\big(S, \mathcal{O}_S/\mathfrak{m}^n\big) \otimes_k \mathcal{O}_{\widetilde{X}} \to \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{X}} \to 0. \tag{13}$$

  This sequence is split for any $n \geqslant 1$, through the natural map

$$\mathrm{H}^0\big(S, \mathcal{O}_S/\mathfrak{m}^n\big) \to \mathrm{H}^0(S, \mathcal{O}_S/\mathfrak{m}) = k.$$

  Hence $\mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{X}}$ is a locally free sheaf on $\widetilde{X}$ (in fact it is a free $\mathcal{O}_{\widetilde{X}}$-module).
- The top horizontal sequence gives rise to a four term sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-n) \to \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes_k \mathcal{O}_{\widetilde{X}}(-1) \to \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{X}} \to \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}} \to 0.$$

  Here, by the Kunneth formula, $p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{X}} = 0$ for $n > 0$, while $p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}} = \mathcal{I}^n \cong \mathcal{O}_{\widetilde{X}}(-n)$, as seen above; the second term is given by (12).
- The lower horizontal sequence gives rise to a three term sequence

$$0 \to \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes \mathcal{O}_{\widetilde{X}}(-1) \to \mathrm{H}^0\big(S, \mathcal{O}_S/\mathfrak{m}^n\big) \otimes \mathcal{O}_{\widetilde{X}} \to p_*\mathcal{O}_{\mathcal{Y}_n} \to 0.$$

- As we have seen before, the right vertical sequence gives rise to the four term sequence

$$0 \to \mathcal{I}^n \to \mathcal{O}_{\widetilde{X}} \to p_*(\mathcal{O}_{\mathcal{Y}_n}) \to \mathrm{R}^1\, p_*\big(\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}\big) \to 0.$$

The four term sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-n) \to \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes \mathcal{O}_{\widetilde{X}}(-1) \to \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{X}} \to \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}} \to 0$$

may now be broken down into two short exact sequences

$$0 \to \mathcal{O}_{\widetilde{X}}(-n) \to \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes \mathcal{O}_{\widetilde{X}}(-1) \to \mathcal{F}_n \to 0 \tag{14}$$

and

$$0 \to \mathcal{F}_n \to \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{X}} \to \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}} \to 0. \tag{15}$$

Thus, from the exactness of the rows and columns of the nine diagram, we see that the map

$$\mathrm{H}^1\big(\widetilde{X}, \mathrm{R}^1\, p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}\big) \to \mathrm{H}^3\big(\widetilde{X}, \mathcal{I}^n\big)$$

factors via $\mathrm{H}^2(\widetilde{X}, \mathcal{F}_n)$, and so for this map to be zero (equivalently for the injectivity of the map in (10)), it suffices to show that the map on cohomology

$$\mathrm{H}^3\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-n)\big) \to \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes \mathrm{H}^3\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-1)\big), \tag{16}$$

obtained from the sequence (14), is injective.

By Serre duality on the smooth projective 3-fold $\widetilde{X}$, injectivity in Eq. (16) is equivalent to the surjectivity of

$$\phi_n^\vee : \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big)^* \otimes \mathrm{H}^0\big(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(1)\big) \to \mathrm{H}^0\big(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(n)\big). \tag{17}$$

To understand this, we need to identify the sheaf map

$$\mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big)^* \otimes \mathcal{O}_{\widetilde{X}}(1) \to \mathcal{O}_{\widetilde{X}}(n) \tag{18}$$

dual to the inclusion in the sequence (14). This is done by considering it as the restriction of an analogous sheaf map on the ambient projective space $\mathbb{P}(V)$, which in turn is identified using Lemma 2 below.

Remark 2. Note that from the sequence

$$0 \to \mathcal{O}_{Y_n} \to p_*\mathcal{O}_{\mathcal{Y}_n} \to \mathrm{R}^1 p_*\big(\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}\big) \to 0$$

the sheaf $\mathrm{R}^1 p_*(\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}})$ has homological dimension 1 on $\widetilde{X}$ (see Remark 1). We had also noted that $R^1 p_*\mathfrak{m}^n\mathcal{O}_{\mathcal{X}}$ is locally free, since (13) is split. From the sequence (15), we deduce that $\mathcal{F}_n$ is locally free on $\widetilde{X}$; thus (14) is an exact sequence of locally free sheaves on $\widetilde{X}$.

2.2. An invariance argument

In this section we shall exploit the geometry of our situation to better describe the sheaf map in (14) which leads to the map (16) on cohomology.

Let $\mathcal{P}$ be the subgroup (a maximal parabolic) of $\mathrm{PGL}(V^*) = \mathrm{Aut}(\mathbb{P}(V^*))$ which is the isotropy group of the distinguished point $s \in S = \mathbb{P}(V^*)$.

The $\mathrm{PGL}(V^*)$ action on $\mathbb{P}(V^*)$ lifts to a $\mathrm{GL}(V^*)$ action on $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ in a unique way so that the induced action on global sections is the standard representation. If $\widetilde{\mathcal{P}} \subset \mathrm{GL}(V^*)$ is the inverse image of $\mathcal{P}$ under the natural homomorphism $\mathrm{GL}(V^*) \to \mathrm{PGL}(V^*)$, then $\widetilde{\mathcal{P}}$ acts on $\mathcal{O}_{\mathbb{P}(V^*)}(1)$, and so also on the fiber over the fixed point s for the $\mathcal{P}$-action; the action on this fiber defines a character of $\widetilde{\mathcal{P}}$, which gives a splitting of the natural exact sequence

$$0 \to k^* \to \widetilde{\mathcal{P}} \to \mathcal{P} \to 0.$$

Using this splitting, we have an induced action of $\mathcal{P}$ on $\mathcal{O}_{\mathbb{P}(V^*)}(1)$, making it an equivariant invertible sheaf on $\mathbb{P}(V^*)$.

More concretely, this means that if we consider the exact sequence of vector spaces

$$0 \to W^* \to V^* \to k \to 0 \tag{19}$$

corresponding to the point $s \in \mathbb{P}(V^*)$, then the group $\mathcal{P}$ is identified with the matrix group

$$\big\{\varphi \in \mathrm{GL}\big(V^*\big) \,\big|\, \varphi\big(W^*\big) \subset W^* \text{ and } (1_{V^*} - \varphi)\big(V^*\big) \subset W^*\big\},$$

which defines actions on the sheaves $\mathcal{O}_{\mathbb{P}(V^*)}(m)$ for all m; further, through the contragredient representation of $\mathrm{GL}(V^*)$ on V, it also acts naturally on the sheaves $\mathcal{O}_{\mathbb{P}(V)}(m)$ making these $\mathcal{P}$-equivariant sheaves on $\mathbb{P}(V)$.

We shall now use this extra structure of $\mathcal{P}$-action on $S = \mathbb{P}(V^*)$ and $\mathbb{P}(V)$ to understand the sheaf map (the inclusion in (14))

$$\mathcal{O}_{\widetilde{X}}(-n) \to \mathrm{R}^1 p_*\big(\mathcal{O}_{\mathcal{X}}(-\mathcal{Y}) \otimes \mathfrak{m}^n \mathcal{O}_{\mathcal{X}}\big) \cong \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes \mathcal{O}_{\widetilde{X}}(-1)$$

(and thus also its dual sheaf map).

First, from the construction of this sheaf map using the nine-diagram in the preceding section, it is obtained by restriction from an analogous map

$$\mathcal{O}_{\mathbb{P}(V)}(-n) \xrightarrow{\phi_n} \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1). \tag{20}$$

The target vector bundle for the map ϕ_n above should be interpreted as

$$\mathrm{R}^1 p_*\big(\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V^*)}(-\mathcal{W}) \otimes q^*\mathfrak{m}^n\big)$$

where $\mathcal{W}$ is the incidence locus in the product variety $\mathbb{P}(V) \times \mathbb{P}(V^*)$ such that $\mathcal{W} \cap \mathcal{X} = \mathcal{Y}$, and

$$p : \mathbb{P}(V) \times \mathbb{P}\big(V^*\big) \to \mathbb{P}(V), \qquad q : \mathbb{P}(V) \times \mathbb{P}\big(V^*\big) \to \mathbb{P}\big(V^*\big) = S$$

are the projections. The sheaves $\mathcal{O}_{\mathbb{P}(V)}(-n)$, $\mathcal{O}_{\mathbb{P}(V)}(-1)$ are thought of as $\mathcal{I}_H^n$ and $\mathcal{I}_H$, respectively, where $H = \mathbb{P}(W) \subset \mathbb{P}(V)$ is the hyperplane corresponding to $s \in S = \mathbb{P}(V^*)$, and $\mathcal{I}_H$ is its ideal sheaf.

Now this "universal" map of sheaves ϕ_n is clearly $\mathcal{P}$-equivariant, since the corresponding nine-diagram of sheaves on $\mathbb{P}(V) \times \mathbb{P}(V^*)$ is a diagram of $\mathcal{P}$-sheaves and $\mathcal{P}$-equivariant morphisms. In fact, the "universal" version of (14) is an exact sequence of locally free $\mathcal{P}$-equivariant sheaves on $\mathbb{P}(V)$.

The map ϕ_n may be equivalently viewed as a sheaf map

$$\mathcal{O}_{\mathbb{P}(V)}(-n+1) \to \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big) \otimes \mathcal{O}_{\mathbb{P}(V)},$$

and so its $\mathcal{O}_{\mathbb{P}(V)}$-dual $\phi_n^\vee$ is a sheaf map

$$\phi_n^\vee : \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big)^* \otimes_k \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(n-1).$$

Since $\mathcal{F}_n = \operatorname{coker} \phi_n$ is locally free, we see that $\phi_n^\vee$ is surjective.

Lemma 2. *There is a $\mathcal{P}$-equivariant isomorphism*

$$\mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big)^* \cong S^{n-1}(V)$$

under which $\phi_n^\vee$ is transported to the evaluation map

$$S^{n-1}(V) \otimes_k \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(n-1).$$

Proof. The lemma is obvious when $n = 1$. When $n = 2$, the facts that $H^0(S, \mathcal{O}_S(-1)/\mathfrak{m}^2)$ has the same dimension as $V = \mathrm{H}^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$, and that no proper subspace of V generates $\mathcal{O}_{\mathbb{P}(V)}(1)$ on projective space, imply that there is an isomorphism

$$\mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^2\big)^* \cong V = \mathrm{H}^0\big(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)\big).$$

Since the $\mathcal{P}$-action on V is the one induced from the $\mathcal{P}$-structure of $\mathcal{O}_{\mathbb{P}(V)}(1)$, this vector space isomorphism must be $\mathcal{P}$-equivariant.

The exact sequences (14) for n and $n-1$ fit into a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{H}^0(S, \frac{\mathfrak{m}^{n-1}\mathcal{O}_S(-1)}{\mathfrak{m}^n}) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}(V)}(-n+1) & \xrightarrow{\phi_n} & \mathrm{H}^0(S, \frac{\mathcal{O}_S(-1)}{\mathfrak{m}^n}) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}(V)}(-n+2) & \xrightarrow{\phi_{n-1}} & \mathrm{H}^0(S, \frac{\mathcal{O}_S(-1)}{\mathfrak{m}^{n-1}}) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) & \longrightarrow & \mathcal{F}_{n-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where the rows, as well as the column, are exact sequences of locally free sheaves on $\mathbb{P}(V)$.

Taking duals, and twisting by $\mathcal{O}_{\mathbb{P}(V)}(-1)$, we see that the twisted duals of the two rows are exact, as is the twisted dual of the middle column. The twisted dual of the left vertical inclusion $\mathcal{O}_{\mathbb{P}(V)}(-n+1) \to \mathcal{O}_{\mathbb{P}(V)}(-n+2)$ of sheaves is the ($\mathcal{P}$-equivariant) inclusion $\mathcal{O}_{\mathbb{P}(V)}(n-2) \to \mathcal{O}_{\mathbb{P}(V)}(n-1)$ whose cokernel is $\mathcal{O}_H(n-1)$, where $H = \mathbb{P}(W) \subset \mathbb{P}(V)$ is the chosen hyperplane (recall that in the definition of ϕ_n, $\mathcal{O}_{\mathbb{P}(V)}(-n)$ is considered as the nth power of the ideal sheaf $\mathcal{I}_H$ of H).

Thus, if $\mathcal{H} = \ker \mathcal{F}_n \to \mathcal{F}_{n-1}$, we obtain a diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_{n-1}^{\vee}(-1) & \longrightarrow & \mathrm{H}^0(S, \frac{\mathcal{O}_S(-1)}{\mathfrak{m}^{n-1}})^* \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & \mathcal{O}_{\mathbb{P}(V)}(n-2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_{n}^{\vee}(-1) & \longrightarrow & \mathrm{H}^0(S, \frac{\mathcal{O}_S(-1)}{\mathfrak{m}^{n}})^* \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & \mathcal{O}_{\mathbb{P}(V)}(n-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}^{\vee}(-1) & \longrightarrow & \mathrm{H}^0(S, \frac{\mathfrak{m}^{n-1}\mathcal{O}_S(-1)}{\mathfrak{m}^{n}})^* \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & \mathcal{O}_{H}(n-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Taking global sections, we see that by induction on n, the lemma is reduced to the statement that the surjective map

$$\mathrm{H}^0\left(S, \frac{\mathfrak{m}^{n-1}\mathcal{O}_S(-1)}{\mathfrak{m}^n}\right)^* \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_H(n-1)$$

induces an isomorphism on global sections (it is already $\mathcal{P}$-equivariant by construction). But in fact this map on global sections is a $\mathcal{P}$-equivariant map

$$\left(S^{n-1}W^*\right)^* \to S^{n-1}W.$$

The map is not zero, since the image generates $\mathcal{O}_H(n-1)$ as a sheaf. But $S^{n-1}(W)$ is an irreducible $\mathcal{P}$-representation (the representation factors through the quotient $\mathcal{P} \to \mathrm{GL}(W)$, and $S^r(W)$ is an irreducible $\mathrm{GL}(W)$-module for any $r > 0$, since we are working in characteristic 0—see for example [6], Theorem 6.3(4) combined with formula (6.1)). So this $\mathcal{P}$-equivariant linear map must be surjective, and hence an isomorphism. □

Remark 3. We note that the induced perfect $\mathrm{GL}(W^*)$-equivariant pairing $S^{n-1}W^* \otimes S^{n-1}W \to k$ is perhaps a non-zero constant multiple of the "standard" one; for example, for $n = 2$, one computes that it is the negative of the standard pairing $W^* \otimes W \to k$. The "standard" pairing is the one obtained from the natural identification

$$S^{n-1}V^* = H^0\big(S, \mathcal{O}_S(n-1)\big) \cong H^0\big(S, \mathcal{O}_S(n-1)/\mathfrak{m}^n\big);$$

we are however pairing with $H^0(S, \mathcal{O}_S(-1)/\mathfrak{m}^n)$, which is a "different" $\mathcal{P}$-module (though, after the fact, isomorphic to $H^0(S, \mathcal{O}_S(n-1)/\mathfrak{m}^n)$).

By Serre duality, injectivity in Eq. (16) is equivalent to the surjectivity of

$$\phi_n^\vee : \mathrm{H}^0\big(S, \mathcal{O}_S(-1)/\mathfrak{m}^n\big)^* \otimes \mathrm{H}^0\big(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(1)\big) \to \mathrm{H}^0\big(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(n)\big).$$

In fact, if one uses the isomorphism obtained from Lemma 2, we see that this dual map factors as follows.

$$\begin{array}{ccc} \mathrm{H}^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(n-1)) \otimes \mathrm{H}^0(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(1)) & \xrightarrow{\phi_n^\vee} & \mathrm{H}^0(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(n)) \\ \Big\downarrow \text{restriction} & \nearrow \text{multiplication} & \\ \mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(n-1)) \otimes \mathrm{H}^0(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(1)). & & \end{array} \tag{21}$$

2.3. Global generation of the twisted canonical bundle

Lemma 3. *The map in* (16) *is injective for* $n \gg 0$, *if the sheaf* $f_* \mathrm{K}_{\widetilde{X}} \otimes \mathcal{O}_{\mathbb{P}^N}(1)$ *is generated by global sections.*

Proof. Let $\mathcal{E}$ be defined by the exact sequence

$$0 \to \mathcal{E} \to \mathrm{H}^0\big(\mathbb{P}^N, f_* \mathrm{K}_{\widetilde{X}} \otimes \mathcal{O}_{\mathbb{P}^N}(1)\big) \otimes \mathcal{O}_{\mathbb{P}^N} \to f_* \mathrm{K}_{\widetilde{X}} \otimes \mathcal{O}_{\mathbb{P}^N}(1) \to 0.$$

The surjectivity on the right is because of our global generation hypothesis.

Now tensor the above sequence with $\mathcal{O}_{\mathbb{P}^N}(n-1)$ for $n \gg 0$. By the Serre vanishing theorem, $\mathrm{H}^1(\mathbb{P}^N, \mathcal{E}(n-1))$ vanishes for $n \gg 0$ and hence we get

$$\mathrm{H}^0\big(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n-1)\big) \otimes \mathrm{H}^0\big(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(1)\big) \to \mathrm{H}^0\big(\widetilde{X}, \mathrm{K}_{\widetilde{X}}(n)\big)$$

is surjective. Thus we are done. □

Remark 4. Note that the condition above is verified if $X = \widetilde{X}$, $\mathcal{O}_X(1)$ is very ample, $|V|$ is the corresponding complete linear system, and $\mathrm{K}_X(1)$ is globally generated. This is true for instance if $X \cong \mathbb{P}^3$, and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^3}(d)$ with $d \geqslant 4$. In this case, we recover the classical Noether–Lefschetz theorem i.e. $\mathrm{Pic}(Y) \cong \mathbb{Z}$ for a very general hypersurface of degree at least 4.

Remark 5. It is possible that the hypothesis that $f_*(\mathrm{K}_{\widetilde{X}})(1)$ is generated by global sections is not quite necessary. This is due to the following: what one needs to actually prove for the condition FNL to hold is that the map

$$\mathrm{H}^2\big(\widetilde{X}, \mathcal{I}/\mathcal{I}^n\big) \to \mathrm{H}^2\big(\mathcal{Y}, \mathfrak{m}\mathcal{O}_{\mathcal{Y}}/\mathfrak{m}^n\mathcal{O}_{\mathcal{Y}}\big) \oplus \mathrm{H}^2\big(Y_n, \mathcal{O}^*_{Y_n}\big)$$

is injective. This map is certainly injective if the map into the first factor is injective. But a priori, the latter is a stronger statement. Furthermore, one requires the injectivity of the latter map only when restricted to image $\mathrm{Pic}(\widetilde{Y})$ i.e. only at the level of the Neron–Severi group.

3. The global Noether–Lefschetz theorem

If X is a smooth proper 3-fold over k (with k algebraically closed of characteristic 0), and $|V|$ a base-point free linear system on X, given by a big line bundle, then we know that for an open set of divisors Y of the linear system, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ induces an isomorphism on Picard varieties, and thus has finitely generated kernel and cokernel, which are invariant under base change to a larger algebraically closed field.

The pair (X, Y) will be said to *satisfy the Noether–Lefschetz condition* if the kernel and cokernel are spanned by classes of exceptional divisors for the morphisms induced by $|V|$. This is equivalent to saying that if $(\overline{X}, \overline{Y})$ are the corresponding normal projective varieties obtained by Stein factorization of the morphism given by $|V|$, then the map on divisor class groups $\mathrm{Cl}(\overline{X}) \to \mathrm{Cl}(\overline{Y})$ is an isomorphism.

As mentioned in the introduction, we say that Y is a "very general" member of the linear system if either (i) Y corresponds to the geometric generic member of the linear system, over the algebraic closure of the function field of the parametrizing projective space, or (ii) k is uncountable, and Y is a divisor in the linear system lying outside a countable union of subvarieties of the parameter space, thought of as a projective space over k. We now explain how these conditions are related (this is a standard argument, which we reproduce here).

In fact, if k is uncountable, we can find a countable algebraically closed subfield k_0, a projective k_0-variety X_0, and a linear system $|V_0|$ on X_0, so that the original given data are obtained by base change to k from k_0. Now consider the parameter projective space $|V| = |V_0| \times_{k_0} k$, which contains a countable family of divisors: those obtained by base-change from divisors in $|V_0|$. Any (closed) point $t \in |V|$ which lies outside this countable union must map, under the projection $|V| \to |V_0|$, to the generic point of $|V_0|$. If $K_0 = k_0(|V_0|)$ is the function field of the parameter variety $|V_0|$, and $Y_{K_0} \subset X_{K_0} = X_0 \times_{k_0} K_0$ the generic member of the linear system, then we have an inclusion $K_0 \hookrightarrow k(t)$, and an identification

$$Y_t = Y_{K_0} \times_{K_0} k(t) \subset X_{K_0} \times_{K_0} k(t) = X_0 \times_{k_0} k = X$$

(where we may identify $k(t)$ with k). On the other hand, we have also $K = k(|V|)$, the function field of V over k, and the corresponding pair (X_K, Y_K). Again this may be viewed as obtained from (X_{K_0}, Y_{K_0}) by a base change (with respect to the natural inclusion $K_0 \to K$).

Thus, it is equivalent to say that any of the 3 pairs (X, Y_t), $(X_{\overline{K}_0}, Y_{\overline{K}_0})$, $(X_{\overline{K}}, Y_{\overline{K}})$ satisfy the Noether–Lefschetz condition (where the overbar denotes algebraic closure).

In other words, if we prove a Noether–Lefschetz theorem for the geometric generic member of our linear system (where we make no further hypothesis on k), then in the case when k is uncountable, it follows also for a "very general member" Y_t of the linear system, in the other sense.

Theorem 4. *Let X be a smooth projective 3-fold over k and $V \subset H^0(X, \mathcal{O}_X(1))$ be a big, base-point free linear system on X. Further assume that the condition* FNL *holds for the pair $(X, |V|)$. Then $(X_{\overline{K}}, Y_{\overline{K}})$ satisfy the Noether–Lefschetz condition, where $Y_{\overline{K}}$ is the geometric generic member of the linear system, defined over $\overline{K}$, the algebraic closure of the function field K of $|V|$.*

Proof. We argue as in [10]. It suffices to prove that $\operatorname{coker} \operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(Y_{\overline{K}})$ is generated by exceptional divisors for the morphism $Y_{\overline{K}} \to \mathbb{P}^N_{\overline{K}}$ induced by our linear system, base changed to $\overline{K}$ (in the case when the line bundle $\mathcal{O}_X(1)$ is ample, this amounts to saying that $\operatorname{Pic}(X_{\overline{K}}) \to \operatorname{Pic}(Y_{\overline{K}})$ is surjective, which is what is considered in [10]).

Let $\alpha \in \operatorname{Pic}(Y_{\overline{K}})$ be any line bundle. Then we can find a finite subextension L of K in its algebraic closure $\overline{K}$, and a line bundle $\alpha_L \in \operatorname{Pic}(Y_L)$ so that α is the base change under $L \hookrightarrow \overline{K}$ of α_L. By increasing L if necessary, we may assume all the irreducible exceptional divisors for $Y_{\overline{K}} \to \mathbb{P}^N_{\overline{K}}$ are base changes of geometrically irreducible curves on Y_L; let $(E_1)_L, \ldots, (E_r)_L$ denote these curves.

Let U be a smooth k-variety with function field L, with a morphism $f : U \to |V|$ inducing $\operatorname{Spec} L \to \operatorname{Spec} K$ on generic points. We have a pull-back family of divisors $\mathcal{Y}_U \to U$ obtained from the original family $\mathcal{Y} \to |V|$ (we regard $\mathcal{Y}_U$ as a divisor on $X \times_k U$, so that for any point $t \in U$, we get an induced divisor $(\mathcal{Y}_U)_t \subset X \times_k k(t)$). Replacing U by a non-empty open subscheme, we may assume without loss of generality that $\mathcal{Y}_U$ is non-singular, f is étale, and there is a line bundle $\alpha_U \in \operatorname{Pic}(\mathcal{Y}_U)$ which restricts to α_L on the generic fiber Y_L of $\mathcal{Y}_U \to U$. Let $(E_1)_U, \ldots, (E_r)_U$ denote the Zariski closures in $\mathcal{Y}_U$ of the curves $(E_i)_L$ on the generic fiber $Y_L \to \operatorname{Spec} L$ of $\mathcal{Y}_U \to U$. Then, further shrinking U if necessary, we may assume the $(E_i)_U$ are irreducible divisors on $\mathcal{Y}_U$, each smooth over U with geometrically irreducible fibers.

If $t \in U$ is a closed point, $s = f(t) \in |V|$, then the smooth surfaces $(\mathcal{Y}_U)_t \subset X$ and $Y_s = \mathcal{Y}_s \subset X$ coincide. Let $(E_1)_t, \ldots, (E_r)_t$ denote the divisors on $(\mathcal{Y}_U)_t \cong Y_s$ obtained by restriction of the divisors $(E_i)_U$. Shrinking U if necessary, we see may assume that the irreducible exceptional divisors for the morphism $Y_s \to \mathbb{P}^N_k$ (obtained by restriction from $g : X \to \mathbb{P}^N_k$) are the curves $(E_i)_t$ (the labeling depends on the choice of t lying over the point s, though the collection of all the curves $(E_i)_t$ depends only on Y_s).

Since f is étale, the formal completions $\widehat{(\mathcal{Y}_U)}_t$ of $\mathcal{Y}_U$ along $(\mathcal{Y}_U)_t \cong Y_s$, and $\widehat{\mathcal{Y}}_s$ of $\mathcal{Y}$ along Y_s, are also naturally identified. Hence, by the FNL property, the restriction to Y_s of the formal line bundle $\widehat{\alpha_U}$ must be isomorphic to a line bundle in the group generated by the image of $\mathrm{Pic}(X)$ and the exceptional divisors.

Thus, after changing α_U by tensoring it with a line bundle pulled back from X, and then by another given by a linear combination of the divisors $(E_i)_U$, we can arrange that α_U restricts to the trivial bundle on the closed fiber $(\mathcal{Y}_U)_t$ of the smooth projective family of surfaces $\mathcal{Y}_U \to U$. This modified α_U then must have numerically trivial restriction to any geometric fiber of $\mathcal{Y}_U \to U$: this follows from the general property of preservation of intersection numbers under specialization, but can be seen here easily from the Hodge index theorem, for example.

In particular, our original line bundle α, upto tensoring by a line bundle pulled back from X, and one given by an exceptional divisor, is a numerically trivial line bundle; hence some non-zero integer multiple of α is algebraically equivalent to 0, and thus a divisible element in the group $\mathrm{Pic}(Y_{\overline{K}})$. However, from Theorem 3, the cokernel of the map

$$\mathrm{Pic}(X_{\overline{K}}) \to \mathrm{Pic}(Y_{\overline{K}})$$

is a finitely generated, torsion free abelian group. Hence α must have trivial image in the cokernel, which is what we wanted to prove. $\square$

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