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Arithmetically Cohen–Macaulay bundles on complete intersection varieties of sufficiently high multidegree

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Abstract Recently it has been proved that any arithmetically Cohen–Macaulay (ACM) bundle of rank two on a general, smooth hypersurface of degree at least three and dimension at least four is a sum of line bundles. When the dimension of the hypersurface is three, a similar result is true provided the degree of the hypersurface is at least six. We extend these results to complete intersection subvarieties by proving that any ACM bundle of rank two on a general, smooth complete intersection subvariety of sufficiently high multi-degree and dimension at least four splits. We also obtain partial results in the case of threefolds.

1 Introduction

We work over the field of complex numbers which shall be denoted by $\mathbb{C}$.

The motivation for the results of this article lie in the study of certain conjectures of Griffiths and Harris on the structure of curves in hypersurfaces in $\mathbb{P}^4$. These conjectures can be viewed as a generalisation of the Noether–Lefschetz theorem which we recall now.

Theorem 1 (Noether–Lefschetz theorem) *Let $X \subset \mathbb{P}^3$ be a smooth, very general hypersurface of degree $d \geq 4$. Then any curve $C \subset X$ is a complete intersection, i.e., $C = X \cap S$ where $S \subset \mathbb{P}^3$ is a hypersurface.*

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Inspired by the above theorem, Griffiths and Harris (see [14]) made a series of conjectures in decreasing strength about the structure of 1-cycles in X, the strongest one of which is the following.

Conjecture 1 *Let $X \subset \mathbb{P}^4$ be a general, smooth hypersurface of degree $d \geq 6$, and $C \subset X$ be any curve. Then $C = X \cap S$ where $S \subset \mathbb{P}^4$ is a surface.*

This conjecture was proved to be false by Voisin (see [27]). In fact, she showed that pursuing a certain line of thought, which we describe below, weaker versions of this conjecture are also false.

Notice that in the Noether–Lefschetz situation, for a smooth curve $C \subset X$, the normal bundle sequence

$$0 \to N_{C/X} \to N_{C/\mathbb{P}^3} \to \mathcal{O}_C(d) \to 0 \tag{1}$$

splits. Griffiths and Harris investigated the splitting of this normal bundle sequence and proved (see [15]) the following characterisation.

Theorem 2 *Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree d and let $C \subset X$ be a smooth curve. Then the normal bundle sequence (1) splits if and only if $C \subset X$ is a complete intersection.*

Unfortunately, the situation is not as simple in higher dimensions. Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d and $C \subset X$ be any smooth curve. It is not hard to see that if $C = X \cap S$ where $S \subset \mathbb{P}^4$ is a surface, then the normal bundle sequence for the inclusion $C \subset X \subset \mathbb{P}^4$ splits.

Furthermore if X_2 denotes the first order thickening of X in $\mathbb{P}^4$, then the splitting of the above normal bundle sequence implies the splitting of the sequence

$$0 \to N_{C/X} \to N_{C/X_2} \to N_{X/X_2}|_C \to 0.$$

By Lemma 1 in [24], the splitting of this normal bundle sequence implies that there exists $D \subset X_2$, a one dimensional subscheme "extending" C i.e., $C = D \cap X$.

It is this weaker splitting that Voisin investigates. In [27], she proves the following

Proposition 1 *Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree $d > 1$. There exist smooth curves $C \subset X$ such that the normal bundle sequence for the inclusions $C \subset X \subset X_2$ (and hence for the inclusions $C \subset X \subset \mathbb{P}^4$) does not split. In fact, C does not extend to X_2. Consequently, it is not an intersection of the form $C = X \cap S$ for any surface $S \subset \mathbb{P}^4$.*

At this point, what would seem to be missing in a more complete understanding of the conjecture of Griffiths and Harris is firstly, whether the existence of the "special" curves of Voisin which disprove it are indeed as special as they seem. Secondly, one would like to know that in spite of this conjecture being false, whether there is a "weaker" *generalised Noether–Lefschetz theorem*.

As explained below, *arithmetically Cohen–Macaulay* (ACM) vector bundles and subvarieties on hypersurfaces provide answers to both these questions. Let $(X, \mathcal{O}_X(1))$ be a smooth polarised variety and $\mathcal{F}$ be any coherent sheaf. Let $\mathrm{H}^i_*(X, \mathcal{F}) := \oplus_{\nu \in \mathbb{Z}} \mathrm{H}^i(X, \mathcal{F}(\nu))$. Recall that a vector bundle E on X is said to be ACM if $\mathrm{H}^i_*(X, E) = 0$ for $0 < i < \dim X$. A subvariety $Z \subset X$ is said to be ACM if $\mathrm{H}^i_*(X, I_{Z/X}) = 0$ for $0 < i \leq \dim Z$. Furthermore, a codimension two subvariety $Z \subset X$ is said to be *arithmetically Gorenstein*, if it is the zero locus of a section of a rank two ACM bundle E on X.

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface. Given a codimension two ACM subvariety $Z \subset X$ with dualising sheaf ω_Z, one can associate an ACM vector bundle E of rank $r+1$ where r is the minimal number of generators of the $\mathrm{H}^0_*(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$-module $\mathrm{H}^0_*(Z, \omega_Z)$. The isomorphism $\omega_Z \cong \mathcal{E}xt^1_{\mathcal{O}_X}(I_{W/X}, K_X)$, where K_X is the canonical bundle of X, gives rise to an isomorphism

$$\mathrm{H}^0(X, \oplus_{i=1}^r \omega_Z(a_i)) \cong \mathrm{H}^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(I_{W/X}, \oplus_{i=1}^r K_X(a_i))) \cong Ext^1(I_{W/X}, \oplus_{i=1}^r K_X(a_i)).$$

This isomorphism takes a minimal set of generators to a rank $r+1$ ACM bundle (the fact that it is a bundle follows from the Auslander-Buchsbaum formula since Y is locally Cohen–Macaulay) and hence in the case when Z is arithmetically Gorenstein, this is just Serre's construction. Conversely (see [18]), given any ACM bundle E of rank $r+1$ on X and r general sections in sufficiently high degree, one can obtain an ACM subvariety $Z \subset X$ such that E is the ACM bundle associated to it. In [24], using the above correspondence, the following was proved:

Proposition 2 *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 1$. If a codimension two ACM subvariety $Z \subset X$ extends to X_2, then the associated ACM vector bundle E splits into a sum of line bundles.*

Examples of non-split ACM bundles on smooth hypersurfaces of degree $d > 1$ can be found in [2] (see [24] for another construction). The existence of such bundles, together with the above proposition, immediately implies that there exist plenty of curves in X, which disprove the conjecture of Griffiths and Harris. It can be easily checked that Voisin's curves are in fact ACM, thus providing a conceptual explanation why the conjecture is false.

The following theorem which can be viewed as a weak generalisation of Theorem 1 was proved in [23] and [26].

Theorem 3 *Let $X \subset \mathbb{P}^4$ be a smooth, general hypersurface of degree $d \geq 6$ with defining polynomial $f \in \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d))$. The following equivalent statements hold true.*

(1) *Any rank two ACM bundle on X is a sum of line bundles.*
(2) *f cannot be expressed as the Pfaffian of a minimal skew-symmetric matrix of size $2k \times 2k$, $2 \leq k \leq d$, whose entries are homogeneous polynomials in five variables.*
(3) *A curve $C \subset X$ is a complete intersection if and only if C is arithmetically Gorenstein i.e., C is the zero locus of a non-zero section of a rank two ACM bundle on X.*

ACM bundles on hypersurfaces have been studied earlier. To add some history, Kleppe showed in [19] that any rank two ACM bundle E on a smooth hypersurface $X \subset \mathbb{P}^n$, $n \geq 6$, splits as a sum of line bundles. When $n = 5$, $3 \leq d \leq 6$ or $n = 4$ and $d = 6$, and X is a general smooth hypersurface, the above splitting result was first obtained by Chiantini and Madonna (see [6,5]). The first general results on ACM bundles, which subsumed these results were first proved in [22]. These in turn led to the proof of Theorem 3, as given in [23].

An important ingredient of the proof of Theorem 3 used in [26], is the following theorem of Green (see [9]) and Voisin (unpublished).

Theorem 4 *Let $X \subset \mathbb{P}^4$ be a smooth, general hypersurface of degree $d \geq 6$. Then the image of the Abel-Jacobi map*

$$\mathrm{CH}^2(X)_{\mathbb{Q}} \to J^2(X)_{\mathbb{Q}}$$

from the (rational) Chow group of codimension two cycles on X to the intermediate Jacobian modulo torsion, is zero.

Notice that Noether–Lefschetz type questions can be asked more generally for complete intersection subvarieties in projective space. Theorem 1 for instance, is well understood in a more general situation (see [7]).

Theorem 5 *Let Y be a smooth projective threefold and L a sufficiently ample, base point free line bundle on Y. Let $X \in |L|$ be a smooth, very general member of the linear system $|L|$. Then the restriction map $\iota^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is an isomorphism.*

In the above theorem, we need L sufficiently ample to imply that the map $\mathrm{H}^2(\mathcal{O}_Y) \to \mathrm{H}^2(\mathcal{O}_X)$ is not surjective. In case $Y = \mathbb{P}^3$, this translates to the condition $L = \mathcal{O}_{\mathbb{P}^3}(d)$ with $d \geq 4$. In particular, this gives an extension of the Noether–Lefschetz theorem to complete intersection surfaces of multi-degree $(d_1, \ldots, d_{n-2})$ in $\mathbb{P}^n$ (here the corresponding condition on the multi-degree of X is $\sum_{i=1}^{n-2} d_i \geq n+1$). Using (the infinitesimal version of) this theorem and some explicit analysis, Harris and Hulek (see [16]) extended Theorem 2 to smooth complete intersection surfaces.

Finally, Green and Müller-Stach (see [10]) have proved a generalisation of Theorem 4 to complete intersection subvarieties of sufficiently high multidegree.

Theorem 6 *Let X be a general complete intersection subvariety in $\mathbb{P}^n$ of sufficiently high multi-degree and dimension at least three. The image of the cycle class map $\mathrm{CH}^2(X)_{\mathbb{Q}} \to \mathrm{H}^4_{\mathcal{D}}(X, \mathbb{Q})$ into Deligne cohomology is just the image of the hyperplane class in $\mathbb{P}^n$. In particular, the image of the Abel-Jacobi map of X is contained in the torsion points of $\mathrm{J}^2(X)$.*

In view of these above two theorems, it is natural to seek extensions of Theorem 3 to complete intersection subvarieties of projective space.

2 Main results

The main results of this note are the following.

Theorem A *Let $X \subset \mathbb{P}^n$ be a general smooth complete intersection subvariety of dimension at least four and sufficiently high multidegree. Then any ACM vector bundle of rank two on X is a direct sum of line bundles.*

This can be viewed as a generalisation to complete intersections of the main result of [22].

Let $d_1 \leq d_2 \leq \cdots \leq d_{n-3}$ be a sequence of positive integers, with $d_i \gg 0$ for $1 \leq i \leq n-3$ and $d_{n-3} > \max\{d_i + d_j\}$ for $1 \leq i,\ j \leq n-4$. Recall that a bundle E is said to be *normalised* if $h^0(E(-1)) = 0$ but $h^0(E) \neq 0$. With this notation, we have

Theorem B *Let $X \subset \mathbb{P}^n$ be a general smooth complete intersection threefold of sufficiently high multidegree $(d_1, \ldots, d_{n-3})$ as above. Then any arithmetically Cohen–Macaulay, normalised rank two vector bundle E on X is a direct sum of line bundles provided $c_1(E) < d_{n-3} - 1$.*

Theorem A is obtained as a consequence of Theorem B. A word about the inequality satisfied by the first Chern class and about the condition $d_i \gg 0$ in the above theorems: since any bundle E splits iff $E(m) := E \otimes \mathcal{O}_X(m)$ splits for some m, we may assume that E is normalised. Madonna showed in [21] that on any smooth three dimensional complete intersection $X \subset \mathbb{P}^n$ a normalised rank two ACM bundle splits as a direct sum of two line bundles unless

$$-\sum_{i=1}^{n-3} d_i + n - 1 \leq c_1(E) \leq \sum_{i=1}^{n-3} d_i + 3 - n.$$

We notice that when $n = 4$ and X is a general hypersurface of degree $d \gg 0$, combining our result and Madonna's bound, it follows that the only possible first Chern class of a normalised and indecomposable rank two ACM bundle on X is $d-1$. This case is classical. Indeed, the existence of an indecomposable ACM bundle of rank two on a general smooth hypersurface in $\mathbb{P}^4$ is equivalent to the fact that a general homogeneous polynomial in five variables can be obtained as the Pfaffian of a (minimal) skew-symmetric matrix of linear forms. It is well known (see [1]), by a simple dimension count, that when $d \geq 6$, this is not possible. Thus Theorem B can be seen as a generalisation of Theorem 3 above. Finally, to prove Theorem B, we argue as in [26] for the cases of general three dimensional hypersurfaces of degree at least six. Indeed we use here the result of Green and Müller-Stach (Theorem 6) which generalised the corresponding result of Green and Voisin (Theorem 4). In doing so, we need to restrict to the cases of complete intersections of sufficiently high multi-degree and dimension at least 3.

We have then the following interesting:

Corollary C *Let $X \subset \mathbb{P}^n$ be a general smooth complete intersection subvariety of sufficiently high multidegree.*

(1) *Any arithmetically Gorenstein subvariety $T \subset X$ of codimension two is a complete intersection in X provided* $\dim X \geq 4$.
(2) *Suppose X is a threefold, and $C \subset X$ is any arithmetically Gorenstein curve. Then C is the intersection of X with a codimension two subscheme $S \subset \mathbb{P}^n$ if and only if C is a complete intersection in X. In addition, if the rank two bundle E associated to C via Serre's correspondence is normalised, and $c_1(E) < d_{n-3} - 1$, then C is a complete intersection in X.*

Finally, we should mention that another motivation for the questions on ACM vector bundles comes from the conjectures of Buchweitz-Gruel-Schreyer (see [2], Conjecture B) on the triviality of low rank ACM bundles on hypersurfaces (see [26] for more details).

The present paper builds on results proved and techniques developed in [22,23,26,29,30], some of which have been included here for the sake of completeness. The non-degeneracy of the infinitesimal invariant in particular, is shown by refining Xian Wu's proof in [29].

3 The infinitesimal invariant associated to a normal function

Let Y be a smooth projective variety of dimension $2m$, $m \geq 1$. Let $\mathcal{X} \to S$ be the universal family of smooth, degree d hypersurfaces of in Y. Let $\mathcal{C} \subset \mathcal{X}$ be a family of codimension m subvarieties over S. If $l = l(s)$ is the degree of C_s and D_s is a codimension m linear section (i.e. D_s is an intersection of m hyperplanes in X_s), then the family of cycles $\mathcal{Z}$ with fibre $\mathcal{Z}_s := d\,\mathcal{C}_s - l D_s$ for $s \in S$ defines a fibre-wise null-homologous cycle, i.e. an element in $\mathrm{CH}^m(\mathcal{X}/S)_{\mathrm{hom}}$. Let $\mathcal{J} := \{J(X_s)\}_{s\in S}$ be the family of intermediate Jacobians. In such a situation, Griffiths (see [11]) has defined a holomorphic function $\nu_{\mathcal{Z}} : S \to \mathcal{J}$, called the *normal function*, which is given by $\nu_{\mathcal{Z}}(s) = \mu_s(Z_s)$ where $\mu_s : \mathrm{CH}^m(X_s)_{\mathrm{hom}} \to J(X_s)$ is the *Abel-Jacobi* map from the group of null-homologous cycles to the intermediate Jacobian.

This normal function satisfies a "quasi-horizontal" condition (see [28], Definition 7.4). Associated to the normal function $\nu_{\mathcal{Z}}$ above, Griffiths (see [12] or [28] Definition 7.8) has defined the infinitesimal invariant $\delta\nu_{\mathcal{Z}}$. Later Green [9] generalised this definition and showed that Griffiths' original infinitesimal invariant is just one of the many infinitesimal invariants that one can associate to a normal function. For a point $s_0 \in S$, let $X = X_{s_0}$, $C := \mathcal{C}_{s_0} \subset X$

and $D := D_{s_0}$. Green showed that in particular $\delta\nu_{\mathcal{Z}}(s_0)$ is an element of the dual of the middle cohomology of the following (Koszul) complex

$$\overset{2}{\wedge}\mathrm{H}^1(X, T_X)\otimes\mathrm{H}^{m+1,m-2}(X)\to\mathrm{H}^1(X,T_X)\otimes\mathrm{H}^{m,m-1}(X)\to\mathrm{H}^{m-1,m}(X). \qquad (2)$$

We now specialise to the case $m = 2$ where $X \subset Y$ is a smooth hypersurface of dimension 3 and $C \subset X$ is a curve of degree l. Then $Z := dC - lD$ is a nullhomologous 1-cycle with support $W := C \bigcup D$. At a point $s \in S$, this infinitesimal invariant is therefore a functional

$$\delta\nu_{\mathcal{Z}}(s) : \ker\left(\mathrm{H}^1(X, T_X)\otimes\mathrm{H}^1(X,\Omega_X^2)\to\mathrm{H}^2(X,\Omega_X^1)\right)\to\mathbb{C}.$$

Consider the composite map

$$\gamma : \mathrm{H}^1(X, T_X)\otimes\mathrm{H}^1(X,\mathcal{I}_{W/X}\otimes\Omega_X^2)\to\mathrm{H}^1(X,T_X)\otimes\mathrm{H}^1(X,\Omega_X^2)\to\mathrm{H}^2(X,\Omega_X^1).$$

By abusing notation, we will let

$$\delta\nu_{\mathcal{Z}}(s) : \ker\gamma\to\mathbb{C}$$

denote the composite map. On the other hand, starting with the short exact sequence

$$0\to I_{W/X}\to\mathcal{O}_X\to\mathcal{O}_W\to 0,$$

and tensoring with Ω_X^1, yields a long exact sequence of cohomology

$$\cdots\to\mathrm{H}^1(\Omega_X^1)\to\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_W)\to\mathrm{H}^2(I_{W/X}\otimes\Omega_X^1)\to\mathrm{H}^2(\Omega_X^1)\to 0.$$

Combining this sequence with the Koszul complex (2), we get a commutative diagram:

$$\begin{array}{ccccccc} & & & \mathrm{H}^1(T_X)\otimes\mathrm{H}^1(I_{W/X}\otimes\Omega_X^2) & & & \\ & & & \downarrow\beta & \searrow\gamma & & \\ 0\to & \frac{\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_W)}{\mathrm{H}^1(\Omega_X^1)} & \xrightarrow{\lambda} & \mathrm{H}^2(I_{W/X}\otimes\Omega_X^1) & \to & \mathrm{H}^2(\Omega_X^1) & \to 0 \\ & \downarrow\chi & & & & & \\ & \mathbb{C} & & & & & \end{array} \qquad (3)$$

where χ is given by integration over the cycle Z and β is a Koszul map. As a result, one has an induced map

$$\ker\gamma\to\frac{\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_W)}{\mathrm{H}^1(\Omega_X^1)}.$$

The following is the main result that we shall use in this paper.

Theorem 7 (Griffiths [12,13]) *Let $\nu_{\mathcal{Z}}$ be the normal function as described above. Then $\delta\nu_{\mathcal{Z}}(s_0)$, the infinitesimal invariant evaluated at the point $s_0\in S$, is the composite*

$$\ker\gamma\to\frac{\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_W)}{\mathrm{H}^1(\Omega_X^1)}\xrightarrow{\chi}\mathbb{C}.$$

Remark 1 The map χ can be understood as follows: Since D is a general plane section of X, by Bertini $C\cap D=\emptyset$. Thus $\mathcal{O}_W\cong\mathcal{O}_C\oplus\mathcal{O}_D$ and so

$$\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_W)\cong\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_C)\oplus\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_D).$$

For any irreducible curve $T\subset X$, let $r_T : \mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_T)\to\mathrm{H}^1(\Omega_T^1)\cong\mathbb{C}$ be the natural restriction map. For any element $(a,b)\in\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_W)$, $\chi(a,b) := dr_C(a) - lr_D(b)\in\mathbb{C}$. Clearly, this map factors via the quotient $\mathrm{H}^1(\Omega_X^1\otimes\mathcal{O}_W)/\,\mathrm{H}^1(\Omega_X^1)$.

The main result in this situation is Theorem 6 which implies in particular that the normal function is zero on an open subset of the parameter space. By Theorem 1.1 in [9] (or Proposition 1.2.3 of [29]), we then have the following.

Theorem 8 *Let X be a general complete intersection subvariety in $\mathbb{P}^n$ of sufficiently high multi-degree and dimension at least three. If $\mathcal{Z} \to S$ is a family of codimension two, degree zero cycles contained in the universal complete intersection $\mathcal{X} \subset \mathbb{P}^n \times S$, then the infinitesimal invariant $\delta\nu_{\mathcal{Z}}$ associated to $\mathcal{Z}$ vanishes at a general point $s \in S$.*

4 ACM bundles on a smooth subvariety $X \subset Y$

Let $X = \bigcap_{i=1}^{n-3} Y_i$ be a general complete intersection of smooth hypersurfaces $Y_i \subset \mathbb{P}^n$ of degree d_i. Let $E \to X$ be an indecomposable, normalised ACM bundle of rank two. In this section, we shall establish several lemmas which will enable us to prove the non-degeneracy of the infinitesimal invariant coming from a family of arithmetically Gorenstein curves. The criterion is a refinement of Wu's criterion (see [29]).

Lemma 1 *Let E be as above with first Chern class α. Then the zero locus of every non-zero section of E has codimension 2 in X. If $C \subset X$ is the zero locus of a section of E, then we have the exact sequence*

$$0 \to \mathcal{O}_X(-\alpha) \to E^\vee \to I_{C/X} \to 0. \tag{4}$$

Furthermore, E is $((\sum_{i=1}^{n-3} d_i) - n + 3 - \alpha)-$ regular, $n - 1 - \sum_{i=1}^{n-3} d_i \le \alpha \le (\sum_{i=1}^{n-3} d_i) - n + 3$, and $K_C = \mathcal{O}_C(\sum_{i=1}^{n-3} d_i - n - 1 + \alpha)$.

Proof See [26] for the proof of the first part of the Lemma. The regularity ρ of E can be computed easily (see *op. cit.*). For the inequality satisfied by α, see [21]. □

Notation: Let $d_1 \le d_2 \le \cdots \le d_{n-3}$ be a sequence of sufficiently large positive integers (i.e. $d_i \gg 0$), and $d_{n-3} > \max\{d_i + d_j\}$ for $1 \le i, \ j \le n-4$. For the rest of the paper, we will denote by

$$Y := \bigcap_{i=1}^{n-4} Y_i, \quad d = d_{n-3} \quad \text{and} \quad X \in |\mathcal{O}_Y(d)|$$

a general member. Also, we will let E denote a normalised, indecomposable ACM bundle of rank two on X and $C \subset X$ to denote the zero locus of a non-zero section of E (which is a curve by Lemma 1).

4.1 Towards the surjectivity of the map χ

The main result of this subsection is the surjectivity of the map χ. This is achieved by identifying a subspace of $\mathrm{H}^1(W, \Omega^1_{X|W})$, restricted to which χ is surjective (see Corollary 2). The proof crucially depends on the following:

Lemma 2 *Let $C \subset X \subset Y \subset \mathbb{P}^n$ be as above. Then the natural map*

$$\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^n}(d)_{|X}) \to \mathrm{H}^1(C, \Omega^2_{\mathbb{P}^n}(d)_{|C})$$

is zero.

Before we prove this lemma, we shall need several results which we shall prove now. Let

$$0 \to G_Y \to F_{0,Y} \to E \to 0, \tag{5}$$

be a minimal resolution of E by vector bundles on Y. So $F_{0,Y} := \bigoplus \mathcal{O}_Y(-a_i)$, where $a_i \geq 0$ and the kernel G_Y is ACM. The fact that G_Y is a bundle follows from the Auslander-Buchsbaum formula (see [8, Chap. 19]).

Applying $\mathcal{H}om_{\mathcal{O}_Y}(\cdot\,, \mathcal{O}_Y)$ to sequence (5), we get

$$0 \to F_{0,Y}^\vee \to G_Y^\vee \to E^\vee(d) \to 0. \tag{6}$$

(see [22] where it is proved when $X \subset \mathbb{P}^n$ is a hypersurface: the same argument works on replacing $\mathbb{P}^n$ by Y).

Lemma 3 (see also [26]) *There exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 \to & F_{0,Y}^\vee & \to & G_Y^\vee & \to & E^\vee(d) & \to 0 \\ & \downarrow \phi & & \downarrow & & \downarrow s^\vee & \\ 0 \to & \mathcal{O}_Y & \to & \mathcal{O}_Y(d) & \to & \mathcal{O}_X(d) & \to 0 \end{array} \tag{7}$$

where under the isomorphism

$$\mathrm{Hom}(F_{0,Y}^\vee, \mathcal{O}_Y) \cong \mathrm{H}^0(F_{0,Y}) \cong \mathrm{H}^0(E), \quad \phi \mapsto s.$$

In addition, $F_{0,Y}^\vee \xrightarrow{\phi} \mathcal{O}_Y$ *is a split surjection (i.e.* ϕ *is the projection onto one of the factors).*

Proof We consider the following push-out diagram:

$$\begin{array}{ccccccccc} 0 \to & F_{0,Y}^\vee & \to & G_Y^\vee & \to & E^\vee(d) & \to 0 \\ & \downarrow \phi & & \downarrow & & || & \\ 0 \to & \mathcal{O}_Y & \to & \mathcal{K} & \to & E^\vee(d) & \to 0 \end{array} \tag{8}$$

Since $F_{0,Y}^\vee = \oplus \mathcal{O}_Y(a_i), a_i \geq 0$, any such diagram corresponds to a section $\phi \in \mathrm{H}^0(F_{0,Y})$. Next consider the pull-back diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathcal{O}_Y & \to & \mathcal{K} & \to & E^\vee(d) & \to 0 \\ & || & & \downarrow & & \downarrow s^\vee & \\ 0 \to & \mathcal{O}_Y & \to & \mathcal{O}_Y(d) & \to & \mathcal{O}_X(d) & \to 0 \end{array} \tag{9}$$

Any such diagram corresponds to a section $s \in \mathrm{H}^0(E)$. Since $\mathrm{H}^0(F_{0,Y}) \cong \mathrm{H}^0(E)$, there is a bijective correspondence between the diagrams above. Combining them, we get the desired commutative diagram. The morphism ϕ is a split surjection since $a_i \geq 0$, $\forall\, i$. □

Tensoring the exact sequence (6) with $\Omega^2_{\mathbb{P}^n}$ and taking cohomology, we get

$$\to \mathrm{H}^1(E^\vee(d) \otimes \Omega^2_{\mathbb{P}^n}) \to \mathrm{H}^2(F_{0,Y}^\vee \otimes \Omega^2_{\mathbb{P}^n}) \to \mathrm{H}^2(G_Y^\vee \otimes \Omega^2_{\mathbb{P}^n}) \to \tag{10}$$

Lemma 4 *The map* $\mathrm{H}^2(F_{0,Y}^\vee \otimes \Omega^2_{\mathbb{P}^n}) \to \mathrm{H}^2(G_Y^\vee \otimes \Omega^2_{\mathbb{P}^n})$ *in diagram (10) is the zero map.*

Proof Let $F_1 \to G_Y^\vee$ be a surjection from a sum of line bundles on $\mathbb{P}^n$ to $G_Y^\vee$, induced by a minimal set of generators of $G_Y^\vee$. Let F_0 be a sum of line bundles on $\mathbb{P}^n$ such that $F_0 \otimes \mathcal{O}_Y = F_{0,Y}$. The map $F_{0,Y}^\vee \to G_Y^\vee$ lifts to a map $\Phi : F_0^\vee \to F_1$, since $G_Y^\vee$ is ACM. Hence we have a commuting square

$$\begin{array}{ccc} \mathrm{H}^2(F_0^\vee \otimes \Omega^2_{\mathbb{P}^n}) & \to & \mathrm{H}^2(F_1 \otimes \Omega^2_{\mathbb{P}^n}) \\ \downarrow \cong & & \downarrow \\ \mathrm{H}^2(F_{0,Y}^\vee \otimes \Omega^2_{\mathbb{P}^n}) & \to & \mathrm{H}^2(G_Y^\vee \otimes \Omega^2_{\mathbb{P}^n}). \end{array}$$

To prove the lemma, it is enough to prove that the top horizontal map, which is given by the matrix Φ, is zero. In other words, we need to show that Φ has no non-zero scalar entries.

Suppose there was such an entry, then we would have a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_Y & = & \mathcal{O}_Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F_{0,Y}^{\vee} & \to & G_Y^{\vee} & \to & E^{\vee}(d) & \to & 0 \\
 & & \uparrow\downarrow & & \downarrow & & || & & \\
0 & \to & \bar{F} & \to & \bar{G} & \to & E^{\vee}(d) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here $\bar{F}$ (resp. $\bar{G}$) is defined as the cokernel of the inclusion $\mathcal{O}_Y \hookrightarrow F_{0,Y}^{\vee}$ (resp. $\mathcal{O}_Y \hookrightarrow G_Y^{\vee}$). Applying $\mathcal{H}om_{\mathcal{O}_Y}(.\,, \mathcal{O}_Y)$ to the diagram above, we get

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \bar{G}^{\vee} & \to & \bar{F}^{\vee} & \to & Ext^1_{\mathcal{O}_Y}(E^{\vee}(d), \mathcal{O}_Y) = E & \to & \\
 & & \downarrow & & \downarrow & & || & & \\
0 & \to & G_Y & \to & F_{0,Y} & \to & Ext^1_{\mathcal{O}_Y}(E^{\vee}(d), \mathcal{O}_Y) = E & \to & 0 \\
 & & \downarrow & & \uparrow\downarrow & & & & \\
 & & \mathcal{O}_Y & = & \mathcal{O}_Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Since $\mathcal{O}_Y$ is a summand of $F_{0,Y}$ which is in the image of the map $G_Y \to F_{0,Y}$, the composite map $\mathcal{O}_Y \to F_{0,Y} \to E$ is zero. In particular, this implies that $G_Y \to \mathcal{O}_Y$ is a surjection and so

$$0 \to \bar{G}^{\vee} \to \bar{F}^{\vee} \to E \to 0$$

is also a resolution. This contradicts the minimality of sequence (5). □

Proof of Lemma 2 We have the following resolution for $\mathcal{O}_X$ on $\mathbb{P}^n$:

$$0 \to \mathcal{O}_{\mathbb{P}^n}\left(-\sum_{i=1}^{n-3} d_i\right) \to \cdots \to \bigoplus_{i=1}^{n-3} \mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0.$$

Using this resolution and the fact that $d = d_{n-3} > \max_{1\le i,j\le n-4}\{d_i + d_j\}$, one can show that

$$\mathrm{H}^1(\mathcal{O}_X(d) \otimes \Omega^2_{\mathbb{P}^n}) \cong \bigoplus_{i=1}^{n-3} \mathrm{H}^2(\Omega^2_{\mathbb{P}^n}(d - d_i)) \cong \mathrm{H}^2(\Omega^2_{\mathbb{P}^n}).$$

Tensoring diagram (7) with $\Omega^2_{\mathbb{P}^n}$ yields a commuting square

$$\begin{array}{ccc} \mathrm{H}^1(E^\vee(d)\otimes\Omega^2_{\mathbb{P}^n}) & \twoheadrightarrow & \mathrm{H}^2(F^\vee_{0,Y}\otimes\Omega^2_{\mathbb{P}^n}) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(\mathcal{O}_X(d)\otimes\Omega^2_{\mathbb{P}^n}) & \cong & \mathrm{H}^2(\mathcal{O}_Y\otimes\Omega^2_{\mathbb{P}^n}) \end{array} \tag{11}$$

Here the top horizontal map is a surjection by Lemma 4, and the right vertical arrow is a surjection by Lemma 3. Hence the map $\mathrm{H}^1(E^\vee(d)\otimes\Omega^2_{\mathbb{P}^n}) \to \mathrm{H}^1(\mathcal{O}_X(d)\otimes\Omega^2_{\mathbb{P}^n})$ is a surjection. Since this map factors via $\mathrm{H}^1(I_{C/X}(d)\otimes\Omega^2_{\mathbb{P}^n})$, the natural map $\mathrm{H}^1(I_{C/X}(d)\otimes\Omega^2_{\mathbb{P}^n}) \to \mathrm{H}^1(\mathcal{O}_X(d)\otimes\Omega^2_{\mathbb{P}^n})$ is a surjection. Hence we are done. □

Let $h_Y \in \mathrm{H}^1(\Omega^1_Y)$ be the restriction of the generator $h \in \mathrm{H}^1(\Omega^1_{\mathbb{P}^n})$ and consider the class $h^2_Y \in \mathrm{H}^2(\Omega^2_Y)$. This is the image of h_X, the hyperplane class in X, under the Gysin map $\mathbb{C} \cong \mathrm{H}^1(\Omega^1_X) \to \mathrm{H}^2(\Omega^2_Y)$. Furthermore, since $d \gg 0$, by Serre vanishing, we have $\mathrm{H}^i(\Omega^2_Y(d)) = 0$ for $i = 1, 2$. Hence the coboundary map $\mathrm{H}^1(\Omega^2_Y(d)_{|X}) \to \mathrm{H}^2(\Omega^2_Y)$ is an isomorphism. By abuse of notation, we will denote the inverse image of h^2_Y under this isomorphism by h^2_Y.

Corollary 1 *Under the natural map*

$$\mathrm{H}^1(\Omega^2_Y(d)_{|X}) \to \mathrm{H}^1(\Omega^2_Y(d)_{|C}), \qquad h^2_Y \mapsto 0.$$

Proof One has a commutative square

$$\begin{array}{ccccc} \mathrm{H}^1(I_{C/X}(d)\otimes\Omega^2_{\mathbb{P}^n}) & \twoheadrightarrow & \mathrm{H}^1(\mathcal{O}_X(d)\otimes\Omega^2_{\mathbb{P}^n}) & \to & \mathrm{H}^1(\mathcal{O}_C(d)\otimes\Omega^2_{\mathbb{P}^n}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^1(I_{C/X}(d)\otimes\Omega^2_Y) & \to & \mathrm{H}^1(\mathcal{O}_X(d)\otimes\Omega^2_Y) & \to & \mathrm{H}^1(\mathcal{O}_C(d)\otimes\Omega^2_Y) \end{array}$$

The first horizontal arrow in the top row is surjection, and the middle vertical map $\mathrm{H}^1(\mathcal{O}_X(d)\otimes\Omega^2_{\mathbb{P}^n}) \to \mathrm{H}^1(\mathcal{O}_X(d)\otimes\Omega^2_Y)$ can be identified with the map $\mathrm{H}^2(\Omega^2_{\mathbb{P}^n}) \to \mathrm{H}^2(\Omega^2_Y)$ which takes the element $h^2 \mapsto h^2_Y$. Hence $h^2_Y \mapsto 0$ under the map $\mathrm{H}^2(\Omega^2_Y) \cong \mathrm{H}^1(\Omega^2_Y(d)_{|X}) \to \mathrm{H}^1(\Omega^2_Y(d)_{|C})$. □

Now we are in a position to prove the first step i.e., the surjectivity of the map χ. Consider the exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega^1_{Y|X} \to \Omega^1_X \to 0.$$

Taking second exteriors and tensoring the resulting sequence by $\mathcal{O}_X(d)$, we get a short exact sequence

$$0 \to \Omega^1_X \to \Omega^2_Y(d)_{|X} \to \Omega^2_X(d) \to 0. \tag{12}$$

For the inclusion $C \subset X$, the natural map $\Omega^1_{X|C} \to \Omega^1_C$ yields a push out diagram:

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1_{X|C} & \to & \Omega^2_Y(d)_{|C} & \to & \Omega^2_X(d)_{|C} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \Omega^1_C & \to & \mathcal{F} & \to & \Omega^2_X(d)_{|C} & \to & 0. \end{array}$$

where $\mathcal{F}$ is defined by the diagram.

Lemma 5 *The map $\mathrm{H}^1(C, \Omega^1_C) \to \mathrm{H}^1(C, \mathcal{F})$ in the associated cohomology sequence of the bottom row in the above diagram is zero. Thus we have a surjection*

$$\mathrm{V}_C := \ker[\mathrm{H}^1(\Omega^1_{X|C}) \to \mathrm{H}^1(\Omega^2_Y(d)_{|C})] \twoheadrightarrow \mathrm{H}^1(C, \Omega^1_C).$$

Proof We have a commutative diagram

$$\begin{array}{ccc} H^1(\Omega^1_X) & \to & H^1(\Omega^2_Y(d)_{|X}) \\ \downarrow & & \downarrow \\ H^1(\Omega^1_{X|C}) & \to & H^1(\Omega^2_Y(d)_{|C}) \\ \downarrow & & \downarrow \\ H^1(\Omega^1_C) & \to & H^1(\mathcal{F}) \end{array}$$

The composite of the vertical maps on the left is the map which takes the class $h_X \mapsto h_C$. Since these are the respective generators of these cohomology groups both of which are one dimensional, this composite is an isomorphism. On the other hand, the composite

$$H^1(\Omega^1_X) \to H^1(\Omega^2_Y(d)_{|X}) \to H^1(\Omega^2_Y(d)_{|C})$$

is zero: this is because the map $H^1(\Omega^1_X) \to H^1(\Omega^2_Y(d)_{|X})$ can be identified with the Gysin map $H^1(\Omega^1_X) \to H^2(\Omega^2_Y)$, and so by the above Corollary, the generator $h_X \mapsto 0$ under the composite. This implies that the map $H^1(C, \Omega^1_C) \to H^1(C, \mathcal{F})$ is zero and so we have a surjection $V_C \twoheadrightarrow H^1(C, \Omega^1_C)$. □

Corollary 2 (Surjectivity of χ) *The composite map*

$$V_C \hookrightarrow \ker[H^1(\Omega^1_{X|W}) \to H^1(\Omega^2_Y(d)_{|W})] \xrightarrow{\chi} \mathbb{C}$$

is a surjection. Hence χ is a surjection.

Proof This first inclusion follows from the fact that $\mathcal{O}_W \cong \mathcal{O}_C \oplus \mathcal{O}_D$. The surjectivity of the composite follows from the definition of χ and the above lemma. □

4.2 Some vanishing lemmas

In this subsection, we shall prove vanishing of certain cohomologies. The technical condition in Theorem B is required for these vanishings to hold and that is the only reason for its appearance in in the statement of the theorem. The main result here is Lemma 7 and the reader may skip the details which are pretty standard arguments if s/he so wishes.

Lemma 6 *With notation as above and $\alpha < d - 1$, we have*

$$H^j(T_Y \otimes K_Y \otimes I_{C/X}(2d - j)) = 0, \quad j = 1, 2.$$

Proof From the exact sequence

$$0 \to \mathcal{O}_X(-\alpha) \to E^\vee \to I_{C/X} \to 0,$$

it is enough to prove the following

(1) $H^{j+1}(T_Y \otimes K_Y \otimes \mathcal{O}_X(2d - j - \alpha)) = 0 \quad j = 1, 2.$
(2) $H^j(T_Y \otimes K_Y \otimes E(2d - j - \alpha)) = 0 \quad j = 1, 2.$

The vanishings in (1) above follow, on tensoring the exact sequence

$$0 \to T_Y \otimes \mathcal{O}_X \to T_{\mathbb{P}^n} \otimes \mathcal{O}_X \to \oplus_{i=1}^{n-4} \mathcal{O}_X(d_i) \to 0,$$

with $K_Y \otimes \mathcal{O}_Y(2d - \alpha - j)$ and using the vanishing of the following terms

(A) $H^j(K_Y(d_i + 2d - \alpha - j)_{|X})$ for $j = 1, 2$. Since $K_Y \cong \mathcal{O}_Y(\sum_{i=1}^{n-4} d_i - n - 1)$ and X is a complete intersection, this follows.

(B) $\mathrm{H}^{j+1}(T_{\mathbb{P}^n} \otimes K_Y(2d - \alpha - j)_{|X})$ for $j = 1, 2$.

Using the Euler sequence restricted to X:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \otimes \mathcal{O}_X \to 0,$$

the vanishing in (B) follows from the vanishing of

- $\mathrm{H}^{j+1}(K_Y(2d - \alpha - j + 1)_{|X})$ for $j = 1, 2$ and
- $\mathrm{H}^{j+2}(K_Y(2d - \alpha - j)_{|X})$ for $j = 1, 2$.

The only non-trivial cases are when $j = 2$ in the first case and $j = 1$ in the second case. These vanishings hold provided $\alpha < d - 1$.

For the vanishings in (2), we use the minimal resolution of E on Y:

$$0 \to G_Y \to F_{0,Y} \to E \to 0.$$

Then it suffices to show that

(C) $\mathrm{H}^j(T_Y \otimes K_Y \otimes F_{0,Y}(2d - \alpha - j)) = 0$ for $j = 1, 2$.
(D) $\mathrm{H}^{j+1}(T_Y \otimes K_Y \otimes G_Y(2d - \alpha - j))$ for $j = 1, 2$.

For (C): $F_{0,Y} = \bigoplus \mathcal{O}_Y(-a_i)$ where $-a_i + \mathrm{reg}\, E \geq 0$. So the above term is $\bigoplus_i \mathrm{H}^j(T_Y(b_i))$ where $b_i \geq d - j - 4$. Since $d \gg 0$, this is true by Serre vanishing.

For (D): From the tangent bundle sequence

$$0 \to T_Y \to T_{\mathbb{P}^n} \otimes \mathcal{O}_Y \to \oplus_{i=1}^{n-4} \mathcal{O}_Y(d_i) \to 0,$$

the required vanishings follow from

- $\mathrm{H}^j(\mathcal{O}_Y(d_i) \otimes K_Y \otimes G_Y(2d - \alpha - j)) = 0$ for $j = 1, 2$ since G_Y is ACM, and
- $\mathrm{H}^{j+1}(T_{\mathbb{P}^n} \otimes K_Y \otimes G_Y(2d - \alpha - j)) = 0$ for $j = 1, 2$. For this use the Euler sequence to reduce this statement to vanishing like the above and then use the fact that G_Y is ACM.

□

Recall that $W = C \cup D$.

Lemma 7 $\mathrm{H}^1(T_Y \otimes K_Y \otimes \mathcal{I}_{W/Y}(2d)) = 0$.

Proof From the exact sequence

$$0 \to \mathcal{O}_Y(-d) \to \mathcal{I}_{W/Y} \to I_{W/X} \to 0$$

we have, since $d \gg 0$, $\mathrm{H}^1(T_Y \otimes K_Y \otimes \mathcal{I}_{W/Y}(2d)) \cong \mathrm{H}^1(T_Y \otimes K_Y \otimes I_{W/X}(2d))$. We shall prove that the latter vanishes. For that we use the exact sequence

$$0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X(-1)^{\oplus 2} \to I_{D/X} \to 0.$$

Tensoring this with $I_{C/X}(2d)$ yields

$$0 \to I_{C/X}(2d - 2) \to I_{C/X}(2d - 1)^{\oplus 2} \to I_{W/X}(2d) \to 0.$$

Since $\mathrm{H}^j(T_Y \otimes K_Y \otimes I_{C/X}(2d - j)) = 0$ for $j = 1, 2$ from Lemma 6 above, we are done.

□

4.3 An auxiliary vector space

In this subsection, we shall construct an auxiliary vector space which surjects onto the domain of the map χ. This construction is a crucial refinement of condition (1) in [29] which was first proved in [26].

Lemma 8 *Let* $U := \ker[\mathrm{H}^0(T_Y \otimes K_Y(2d)) \to \mathrm{H}^0(K_Y(3d)_{|W})]$ *and* $V := \ker[\mathrm{H}^1(\Omega^1_{X|W}) \to \mathrm{H}^1(\Omega^2_Y(d)_{|W})]$. *Then the natural map* $U \to V$ *is a surjection.*

Proof Tensoring the short exact sequence

$$0 \to T_X \to T_{Y|X} \to \mathcal{O}_X(d) \to 0$$

with $K_Y(2d)_{|W}$ and taking cohomology, we get

$$0 \to \mathrm{H}^0(T_X \otimes K_Y(2d)_{|W}) \to \mathrm{H}^0(T_Y \otimes K_Y(2d)_{|W}) \to \mathrm{H}^0(K_Y(3d)_{|W})$$

Since $T_X \otimes K_Y(d) \cong \Omega^2_X$, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & \mathrm{U} & \to & \mathrm{H}^0(T_Y \otimes K_Y(2d)) & \to & \mathrm{H}^0(K_Y(3d)_{|W}) \\ & \downarrow & & \downarrow & & || \\ 0 \to & \mathrm{H}^0(\Omega^2_X(d)_{|W}) & \to & \mathrm{H}^0(T_Y \otimes K_Y(2d)_{|W}) & \to & \mathrm{H}^0(K_Y(3d)_{|W}). \end{array} \tag{13}$$

The middle vertical arrow can be seen to be a surjection by using the fact that the cokernel of this map injects into $\mathrm{H}^1(T_Y \otimes K_Y \otimes \mathcal{I}_{W/Y}(2d))$ which vanishes by Lemma 7. By the snake lemma, the first map is also a surjection. Since

$$\mathrm{Image}[\mathrm{H}^0(\Omega^2_X(d)_{|W}) \to \mathrm{H}^1(\Omega^1_{X|W})] = \ker[\mathrm{H}^1(\Omega^1_{X|W}) \to \mathrm{H}^1(\Omega^2_Y(d)_{|W})] = \mathrm{V},$$

we have a surjection $\mathrm{U} \twoheadrightarrow \mathrm{H}^0(\Omega^2_X(d)_{|W}) \twoheadrightarrow \mathrm{V}$. □

4.4 The final lifting

All that remains to be done now is to lift the elements from the auxiliary vector space U constructed above to $\ker \gamma$ for which we need the following

Lemma 9 *With notation as above, the multiplication map*

$$\mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(2d)) \otimes \mathrm{H}^0(\mathcal{O}_Y(d)) \to \mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(3d))$$

is surjective.

Proof Tensoring the exact sequence

$$0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X(-1)^{\oplus 2} \to I_{D/X} \to 0, \tag{14}$$

by E, we have

$$0 \to E(-2) \to E(-1)^{\oplus 2} \to I_{D/X}E \to 0. \tag{15}$$

Let $T_m := \mathrm{H}^0(\mathcal{O}_X(m))$. The exact sequence above gives rise to a diagram with exact rows where the vertical arrows are all multiplication maps:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{H}^0(E(k-2)) \otimes T_m & \to & \mathrm{H}^0(E(k-1))^{\oplus 2} \otimes T_m & \to & \mathrm{H}^0(I_{D/X}E(k)) \otimes T_m & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \mathrm{H}^0(E(m+k-2)) & \to & \mathrm{H}^0(E(m+k-1))^{\oplus 2} & \to & \mathrm{H}^0(I_{D/X}E(m+k)) & \to & 0. \end{array}$$

Since E is $(\sum_{i=1}^{n-3} d_i - \alpha - n + 3)$-regular by Lemma 1, the middle vertical arrow is a surjection for $k \geq (\sum_{i=1}^{n-3} d_i - \alpha - n + 4)$ and $m \geq 0$. It follows that the multiplication map

$$\mathrm{H}^0(I_{D/X}E(k)) \otimes \mathrm{H}^0(\mathcal{O}_X(m)) \to \mathrm{H}^0(I_{D/X}E(m+k))$$

is surjective for $k \geq (\sum_{i=1}^{n-3} d_i - \alpha - n + 4)$ and $m \geq 0$. Next consider the exact sequence $0 \to I_{D/X} \to I_{D/X}E \to I_{W/X}(\alpha) \to 0$ obtained by tensoring sequence (4) by $I_{D/X}(\alpha)$. Repeating the previous argument, it is easy to check that the multiplication map

$$\mathrm{H}^0(I_{W/X}(k)) \otimes \mathrm{H}^0(\mathcal{O}_X(m)) \to \mathrm{H}^0(I_{W/X}(m+k))$$

is surjective for $k \geq (\sum_{i=1}^{n-3} d_i - n + 4)$ and $m \geq 0$. Now using the exact sequence

$$0 \to \mathcal{O}_Y(-d) \to \mathcal{I}_{W/Y} \to I_{W/X} \to 0,$$

and the fact that the regularity of $\mathcal{O}_Y$ is $\sum_{i=1}^{n-4} d_i - n - 2$, we can conclude by repeating the argument above, that the multiplication map

$$\mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(2d)) \otimes \mathrm{H}^0(\mathcal{O}_Y(d)) \to \mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(3d))$$

is surjective. □

5 Proofs of the main results

Assume that a general, smooth complete intersection threefold $X \subset \mathbb{P}^n$ of sufficiently high multi-degree $(d_1, \ldots, d_{n-3})$ supports an indecomposable ACM rank two vector bundle E with first Chern class $\alpha < d_{n-3} - 1$. This implies that there exists a rank two bundle $\mathcal{E}$ on the universal hypersurface $\mathcal{X} \subset Y \times S'$ where S' is a Zariski open subset of S, the moduli space of smooth, degree d hypersurfaces of Y, such that for a general point $s \in S'$, $\mathcal{E}_{|X_s}$ is normalised, indecomposable, ACM and its first Chern class $\alpha_s = \alpha$ satisfies the inequality above. Furthermore, from the construction of this family, one sees that there exists a family of curves $\mathcal{C} \to S'$ such that $\mathcal{C}_s$ is the zero locus of a section of $\mathcal{E}_{|X_s}$. Let $\mathcal{Z}$ be a family of 1-cycles with fibre $\mathcal{Z}_s := d\mathcal{C}_s - lD_s$ where, as before, D_s is plane section of X_s and $l = l(s)$ is the degree of $\mathcal{C}_s$.

Proposition 3 *In the situation above, $\delta\nu_{\mathcal{Z}} \not\equiv 0$.*

Proof We shall show that $\delta\nu_{\mathcal{Z}}(s) \neq 0$ at any point $s \in S$ parametrising a smooth hypersurface $X \subset Y$. To do this, we shall lift elements of U to $\ker\gamma$ in Diagram 3. Since we have surjections $U \twoheadrightarrow V \twoheadrightarrow \mathbb{C}$, we will be done.

Let $\partial_f : \Omega_Y^3(2d) \to K_Y(3d)$ be the derivative map where f is the degree d polynomial defining X. Composing with the quotient $K_Y(3d)/K_Y(2d)$, we get a map $\bar{\partial}_f : \Omega_Y^3(2d) \to K_Y(3d)/K_Y(2d)$. Using the identification $\Omega_Y^3 \cong T_Y \otimes K_Y$, and taking cohomology, we get

$$\mathrm{H}^0(T_Y \otimes K_Y(2d)) \xrightarrow{\partial_f} \mathrm{H}^0(K_Y(3d)) \to \frac{\mathrm{H}^0(K_Y(3d))}{\mathrm{H}^0(K_Y(2d))}.$$

The cokernel of the composite map above can be identified with $\mathrm{H}^2(\Omega_X^1)$ (see [3, p. 174] or [20, Chap. 9] for details).

The key ingredient in this lifting is the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{O}_Y(d)) \otimes \mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(2d)) & \xrightarrow{\gamma'} & \frac{\mathrm{H}^0(K_Y(3d))}{\partial_f \mathrm{H}^0(T_Y \otimes K_Y(2d))} \\ \downarrow & & \downarrow \\ \mathrm{H}^1(T_X) \otimes \mathrm{H}^1(\mathcal{I}_{W/Y} \otimes \Omega^2_X) & \xrightarrow{\gamma} & \mathrm{H}^2(\Omega^1_X). \end{array} \tag{16}$$

Here the right vertical map is the one explained above. The horizontal maps γ and γ' are (essentially) cup product maps. The vertical map on the left is a tensor product of two maps: The first factor is the composite $\mathrm{H}^0(\mathcal{O}_Y(d)) \to \mathrm{H}^0(\mathcal{O}_X(d)) \to \mathrm{H}^1(T_X)$. The normal bundle of $X \subset Y$ is $\mathcal{O}_X(d)$ and $\mathrm{H}^0(\mathcal{O}_X(d)) \to \mathrm{H}^1(T_X)$ is the natural coboundary map in the cohomology sequence of the tangent bundle sequence for this inclusion. The second factor $\mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(2d)) \to \mathrm{H}^1(\mathcal{I}_{W/Y} \otimes \Omega^2_X)$ is also obtained as above by observing that $T_X \otimes K_Y(d) \cong \Omega^2_X$.

This diagram yields a map $\ker\gamma' \to \ker\gamma$. To complete the lifting, recall that by Lemma 9, the map $\mathrm{H}^0(\mathcal{O}_Y(d)) \otimes \mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(2d)) \twoheadrightarrow \mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(3d))$ is a surjection. Restricting this map to $\ker\gamma'$, we get a surjection

$$\ker\gamma' \twoheadrightarrow \bar{U} := \partial_f \mathrm{H}^0(T_Y \otimes K_Y(2d)) \cap \mathrm{H}^0(\mathcal{I}_{W/Y} \otimes K_Y(3d)).$$

Let $\widetilde{U}$ be the kernel of the map $\mathrm{H}^0(T_Y \otimes K_Y(2d)) \to \mathrm{H}^0(K_Y(3d)_{|X})$. Looking at the diagram analogous to (13) obtained by replacing W by X, we see that there is a map $\widetilde{U} \to \mathrm{H}^0(\Omega^2_X(d))$. The boundary map $\mathrm{H}^0(\Omega^2_X(d)) \to \mathrm{H}^1(\Omega^1_X)$ in the cohomology sequence associated to Diagram (12) is the zero map (this is because the composite map $\mathrm{H}^1(\Omega^1_X) \to \mathrm{H}^1(\Omega^2_Y(d)_{|X}) \cong \mathrm{H}^2(\Omega^2_Y)$ is the Gysin inclusion). This implies that the surjection $U \twoheadrightarrow V$ of Lemma 8 factors as $U \twoheadrightarrow \bar{U} \twoheadrightarrow U/\widetilde{U} \twoheadrightarrow V$ and thus we have surjections $\ker\gamma' \twoheadrightarrow \bar{U} \twoheadrightarrow V \xrightarrow{\chi} \mathbb{C}$. By the compatibility of these maps with the map $\ker\gamma' \to \ker\gamma$ and those in Diagram (3), we conclude (using Griffiths' formula) that $\delta\nu_{\mathcal{Z}}(s) \neq 0$. □

Proof of Theorem B Assume that a general complete intersection threefold X supports an indecomposable normalised ACM bundle E, with $\alpha < d - 1$. Let $\mathcal{Z}$ be the family of degree zero 1-cycles defined earlier. By the refined Wu's criterion $\delta\nu_{\mathcal{Z}} \not\equiv 0$: this contradicts the theorem of Green and Müller-Stach. Thus we are done. □

Proof of Theorem A Let X be a complete intersection subvariety of dimension four. Let E be an ACM bundle of rank two on X. As mentioned above, we may assume E to be normalised with first Chern class α. Now choose a general hypersurface $T \subset X$ of degree $d >> 0$, satisfying $d > \alpha + 1$. Since $E \otimes \mathcal{O}_T$ is ACM, and $\alpha < d - 1$, it follows from Theorem B that $E \otimes \mathcal{O}_T$ splits. This implies by a standard argument that E itself splits. The case for dimension greater than four now follows in a similar way. □

Proof of Corollary C The proof of the first part is trivial. The proof of the second part is as follows: Suppose $C = X \cap \tilde{S}$ where $\tilde{S} \subset \mathbb{P}^n$ is a codimension two subscheme. Then $C = X \cap S$ where $S := \tilde{S} \cap Y$ is a surface in Y. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^2_Y & \to & \Omega^2_Y(d) & \to & \Omega^2_Y(d)_{|X} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^2_{Y\,|S} & \to & \Omega^2_Y(d)_{|S} & \to & \Omega^2_Y(d)_{|C} & \to & 0 \end{array}$$

Taking cohomology, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Omega^2_Y(d)_{|X}) & \cong & \mathrm{H}^2(\Omega^2_Y) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(\Omega^2_Y(d)_{|C}) & \to & \mathrm{H}^2(\Omega^2_{Y\ |S}) \end{array}$$

The map $\mathrm{H}^2(\Omega^2_Y) \to \mathrm{H}^2(\Omega^2_{Y\ |S})$ is non-zero since the composite

$$\mathrm{H}^2(\Omega^2_Y) \to \mathrm{H}^2(\Omega^2_{Y\ |S}) \to \mathrm{H}^2(\Omega^2_S)$$

is a surjection which sends $h_Y^2 \mapsto h_S^2$ where h_Y and h_S are the classes of hyperplane sections in Y and S respectively. Thus the image of h_X under the composite map

$$\mathrm{H}^1(\Omega^1_X) \to \mathrm{H}^1(\Omega^2_Y(d)_{|X}) \to \mathrm{H}^2(\Omega^2_{Y\ |S})$$

is non-zero and hence its image under the map

$$\mathrm{H}^1(\Omega^1_X) \to \mathrm{H}^1(\Omega^2_Y(d)_{|X}) \to \mathrm{H}^1(\Omega^2_Y(d)_{|C})$$

is also non-zero. By the proof of Lemma 2, if E were indecomposable, then the above map is zero. This implies when $\alpha < d - 1$, that the associated rank two bundle splits, hence C is a complete intersection. □

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