ON CODIMENSION TWO SUBVARIETIES IN HYPERSURFACES

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Abstract. We show that for a smooth hypersurface $X \subset \mathbb{P}^n$ of degree at least 2, there exist arithmetically Cohen-Macaulay (ACM) codimension two subvarieties $Y \subset X$ which are not an intersection $X \cap S$ for a codimension two subvariety $S \subset \mathbb{P}^n$. We also show there exist $Y \subset X$ as above for which the normal bundle sequence for the inclusion $Y \subset X \subset \mathbb{P}^n$ does not split.

Dedicated to Spencer Bloch

1. Introduction

In this note, we revisit some questions of Griffiths and Harris from 1985 [GH]:

Questions (Griffiths and Harris). *Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree $d \geq 6$ and $C \subset X$ be a curve.*

(1) *Is the degree of C a multiple of d?*
(2) *Is $C = X \cap S$ for some surface $S \subset \mathbb{P}^4$?*

The motivation for these questions comes from trying to extend the Noether-Lefschetz theorem for surfaces to threefolds. Recall that the Noether-Lefschetz theorem states that if X is a very general surface of degree $d \geq 4$ in $\mathbb{P}^3$, then $\mathrm{Pic}(X) = \mathbb{Z}$, and hence every curve C on X is the complete intersection of X and another surface S.

C. Voisin very soon [Vo] proved that the second question had a negative answer by constructing counter-examples on any smooth hypersurface of degree at least 2. She also considered a third question:

Question. *With the same terminology and when C is smooth:*

(3) *Does the exact sequence of normal bundles associated to the inclusions $C \subset X \subset \mathbb{P}^4$:*

$$0 \to N_{C/X} \to N_{C/\mathbb{P}^4} \to \mathcal{O}_C(d) \to 0$$

split?

Her counter-examples provided a negative answer to this question as well. The first question, the Degree Conjecture of Griffiths-Harris, is still open. Strong evidence for this conjecture was provided by some elementary but ingenious examples of Kollár ([BCC],Trento examples). In particular he shows that if $\gcd(d, 6) = 1$ and $d \geq 4$ and X is a very general hypersurface of degree d^2 in $\mathbb{P}^4$, then every curve on X has degree a multiple of d. In the same vein, Van Geemen shows that if $d > 1$ is an odd number and X is a very general hypersurface of degree $54d$, then every curve on X has degree a multiple of $3d$.

The main result of this note is the existence of a large class of counterexamples which subsumes Voisin's counterexamples and places them in the context of arithmetically Cohen-Macaulay

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(ACM) vector bundles on X. It is well known that ACM bundles which are not sums of line bundles can be found on any hypersurface of degree at least 2 [BGS], and for such a bundle, say of rank r, on X, ACM subvarieties of codimension two can be created on X by considering the dependency locus of $r-1$ general sections. These subvarieties fail to satisfy Questions 2 and 3. We will be working on hypersurfaces in $\mathbb{P}^n$ for any $n \geq 4$ and our constructions of ACM subvarieties may not give smooth ones. Hence in Question 3, we will consider the splitting of the conormal sheaf sequence instead.

2. Main results

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 2$ and let $Y \subset X$ be a codimension 2 subscheme. Recall that Y is said to be an arithmetically Cohen-Macaulay (ACM) subscheme of X if $\mathrm{H}^i(X, I_{Y/X}(\nu)) = 0$ for $0 < i \leq \dim Y$ and for any $\nu \in \mathbb{Z}$. Similarly, a vector bundle E on X is said to be ACM if $\mathrm{H}^i(X, E(\nu)) = 0$ for $i \neq 0, \dim X$ and for any $\nu \in \mathbb{Z}$.

Given a coherent sheaf $\mathcal{F}$ on X, let $s_i \in \mathrm{H}^0(\mathcal{F}(m_i))$ for $1 \leq i \leq k$ be generators for the $\oplus_{\nu\in\mathbb{Z}} \mathrm{H}^0(\mathcal{O}_X(\nu))$-graded module $\oplus_{\nu\in\mathbb{Z}} \mathrm{H}^0(\mathcal{F}(\nu))$. These sections give a surjection of sheaves $\oplus_{i=1}^k \mathcal{O}_X(-m_i) \twoheadrightarrow \mathcal{F}$ which induces a surjection of global section $\oplus_{i=1}^k \mathrm{H}^0(\mathcal{O}_X(\nu - m_i)) \twoheadrightarrow \mathrm{H}^0(\mathcal{F}(\nu))$ for any $\nu \in \mathbb{Z}$.

Applying this to the ideal sheaf $I_{Y/X}$ of an ACM subscheme of codimension 2 in X, we obtain the short exact sequence

$$0 \to G \to \oplus_{i=1}^k \mathcal{O}_X(-m_i) \to I_{Y/X} \to 0,$$

where G is some ACM sheaf on X of rank $k-1$. Since Y is ACM as a subscheme of X, it is also ACM as a subscheme of $\mathbb{P}^n$. In particular, Y is locally Cohen-Macaulay. Hence G is a vector bundle by the Auslander-Buchsbaum Theorem (see [Mat] page 155). We will loosely say that G is associated to Y.

Conversely, the following Bertini type theorem which goes back to arguments of Kleiman in [Kl] (see also [Ban]) shows that given an ACM bundle G on X, we can use G to construct ACM subvarieties Y of codimension 2 in X:

Proposition 1. *(Kleiman). Given a bundle G of rank $k-1$ on X, a general map $G \to \oplus_{i=1}^k \mathcal{O}_X(m_i)$ for sufficiently large m_i will determine the ideal sheaf (up to twist) of a subvariety Y of codimension 2 in X with a resolution of sheaves:*

$$0 \to G \to \oplus_{i=1}^k \mathcal{O}_X(m_i) \to I_{Y/X}(m) \to 0.$$

Since the conclusion of Question 2 implies that of Question 3, we will look at just Question 3, in the conormal sheaf version.

Let X be a hypersurface of degree d in $\mathbb{P}^n$ defined by the equation $f = 0$. Let X_2 be the thickening of X defined by $f^2 = 0$ in $\mathbb{P}^n$. Given a subvariety Y of codimension 2 in X, let $I_{Y/\mathbb{P}}$ (resp. $I_{Y/X}$) denote the ideal sheaf of $Y \subset \mathbb{P}^n$ (resp. $Y \subset X$). The conormal sheaf sequence is

$$0 \to \mathcal{O}_Y(-d) \to I_{Y/\mathbb{P}}/I_{Y/\mathbb{P}}^2 \to I_{Y/X}/I_{Y/X}^2 \to 0. \tag{1}$$

Lemma 1. *For the inclusion $Y \subset X \subset \mathbb{P}^n$, if the sequence of conormal sheaves (1) splits, then there exists a subscheme $Y_2 \subset X_2$ containing Y such that*

$$I_{Y_2/X_2}(-d) \xrightarrow{f} I_{Y_2/X_2} \to I_{Y/X} \to 0$$

is exact. Furthermore, $f I_{Y_2/X_2}(-d) = I_{Y/X}(-d)$.

Proof. Suppose sequence (1) splits: then we have a surjection

$$I_{Y/\mathbb{P}} \twoheadrightarrow I_{Y/\mathbb{P}}/I^2_{Y/\mathbb{P}} \twoheadrightarrow \mathcal{O}_Y(-d)$$

where the first map is the natural quotient map and the second is the splitting map for the sequence. The kernel of this composition defines a scheme Y_2 in $\mathbb{P}^n$. Since this kernel $I_{Y_2/\mathbb{P}}$ contains $I^2_{Y/\mathbb{P}}$ and hence f^2, it is clear that $Y \subset Y_2 \subset X_2$.

The splitting of (1) also means that $f \in I_{Y/\mathbb{P}}(d)$ maps to $1 \in \mathcal{O}_Y$. We get the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \to & I_{Y_2/\mathbb{P}} & \to & I_{Y/\mathbb{P}} & \to & \mathcal{O}_Y(-d) & \to & 0 \\
 & & \uparrow f^2 & & \uparrow f & & \uparrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}}(-2d) & \stackrel{f}{\to} & \mathcal{O}_{\mathbb{P}}(-d) & \to & \mathcal{O}_X(-d) & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

This induces

$$0 \to I_{Y/X}(-d) \to I_{Y_2/X_2} \to I_{Y/X} \to 0.$$

In particular, note that $I_{Y/X}(-d)$ is the image of the multiplication map $f : I_{Y_2/X_2}(-d) \to I_{Y_2/X_2}$. □

Now assume that Y is an ACM subvariety on X of codimension 2. The ideal sheaf of Y in X has a resolution

$$0 \to G \to \oplus_{i=1}^k \mathcal{O}_X(-m_i) \to I_{Y/X} \to 0,$$

for some ACM bundle G on X associated to Y.

Lemma 2. *Suppose the conditions of the previous lemma hold, and in addition Y is an ACM subvariety. Then there is an extension of the ACM bundle G (associated to Y) on X to a bundle $\mathcal{G}$ on X_2. ie. there is a vector bundle $\mathcal{G}$ on X_2 such that the multiplication map $f : \mathcal{G}(-d) \to \mathcal{G}$ induces the exact sequence $0 \to G(-d) \to \mathcal{G} \to G \to 0$.*

Proof. Since Y is ACM, $H^1(I_{Y/X}(-d+\nu)) = 0, \forall \nu$, hence in the sequence stated in the previous lemma, the right hand map is surjective on the level of sections. Therefore, the map $\oplus_{i=1}^k \mathcal{O}_X(-m_i) \to I_{Y/X}$ can be lifted to a map $\oplus_{i=1}^k \mathcal{O}_{X_2}(-m_i) \to I_{Y_2/X_2}$. Since a global section of $I_{Y_2/X_2}(\nu)$ maps to zero in $I_{Y/X}$ only if it is a multiple of f, by Nakayama's lemma, this lift is surjective at the level of global sections in different twists, and hence on the level of sheaves. Hence there is a commuting diagram of exact sequences:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
I_{Y_2/X_2}(-d) & \to & I_{Y_2/X_2} & \to & I_{Y/X} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\oplus_{i=1}^k \mathcal{O}_{X_2}(-m_i-d) & \to & \oplus_{i=1}^k \mathcal{O}_{X_2}(-m_i) & \to & \oplus_{i=1}^k \mathcal{O}_X(-m_i) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{G}(-d) & \to & \mathcal{G} & \to & G & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

where the sheaf $\mathcal{G}$ is defined as the kernel of the lift, and the map from the left column to the middle column is multiplication by f. It is easy to verify that the lowest row induces an exact sequence

$$0 \to G(-d) \to \mathcal{G} \to G \to 0.$$

By Nakayama's lemma, $\mathcal{G}$ is a vector bundle on X_2.

□

Proposition 2. *Let E be an ACM bundle on X. If E extends to a bundle $\mathcal{E}$ on X_2, then E is a sum of line bundles.*

Proof. There is an exact sequence $0 \to E(-d) \to \mathcal{E} \to E \to 0$, where the left hand map is induced by multiplication by f on $\mathcal{E}$. Let $F_0 = \oplus \mathcal{O}_{\mathbb{P}^n}(a_i) \twoheadrightarrow E$ be a surjection induced by the minimal generators of E. Since E is ACM, this lifts to a map $F_0 \twoheadrightarrow \mathcal{E}$. This lift is surjective on global sections by Nakayama's lemma (since the sections of $\mathcal{E}$ which are sent to 0 in E are multiples of f). Thus we have a diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & E(-d) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & F_1 & \to & F_0 & \to & \mathcal{E} & \to & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \to & G_1 & \to & F_0 & \to & E & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & E(-d) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

G_1 and F_1 are sums of line bundles on $\mathbb{P}^n$ by Horrocks' Theorem. Furthermore, $G_1 \cong F_0(-d)$. Thus $0 \to F_0(-d) \xrightarrow{\Phi} F_0 \to E \to 0$ is a minimal resolution for E on $\mathbb{P}^n$. As a consquence of this, one checks that $\det \Phi = f^{\operatorname{rank} E}$. On the other hand, the degree of $\det \Phi = d \operatorname{rank} F_0$ and so we have $\operatorname{rank} F_0 = \operatorname{rank} E$. Restricting, this resolution to X, we get a surjection $F_0 \otimes \mathcal{O}_X \twoheadrightarrow E$. The ranks of both vector bundles being the same, this implies that this is an isomorphism. □

Corollary 1. *Let $Y \subset X$ be a codimension 2 ACM subvariety. If the conormal sheaf sequence (1) splits, then*

- *the ACM bundle G associated to Y is a sum of line bundles,*
- *there is a codimension 2 subvariety S in $\mathbb{P}^n$ such that $Y = X \cap S$.*

Proof. The first statement follows from Lemma 2 and Proposition 2. For the second statement, since the bundle G associated to Y is a sum of line bundles $\oplus_{i=1}^{k-1} \mathcal{O}_X(-l_i)$ on X, the map $G \to \oplus_{i=1}^{k} \mathcal{O}_X(-m_i)$ can be lifted to a map $\oplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}}(-l_i) \to \oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}}(-m_i)$. The determinantal variety S of codimension 2 in $\mathbb{P}^n$ determined by this map has the property that $Y = X \cap S$. □

In conclusion, we obtain the following collection of counterexamples:

Corollary 2. *If G is an ACM bundle on X which is not a sum of line bundles, and if Y is a subvariety of codimension 2 in X constructed from G as in Proposition 1, then Y does not satisfy the conclusion of either Question 2 or Question 3.*

Buchweitz-Greuel-Schreyer have shown [BGS] that any hypersurface of degree at least 2 supports (usually many) non-split ACM bundles. We will give another construction in the next section.

3. Remarks

3.1. The infinitesimal Question 3 was treated by studying the extension of the bundle to the thickened hypersurface X_2. This method goes back to Ellingsrud, Gruson, Peskine and Strømme [EGPS]. If we are not interested in the infinitesimal Question 3, but just in the more geometric Question 2, a geometric argument gives an even easier proof of the existence of codimension 2 ACM subvarieties $Y \subset X$ which are not of the form $Y = X \cap Z$ for some codimension 2 subvariety $Z \subset \mathbb{P}^n$.

Proposition 3. *Let E be an ACM bundle on a hypersurface X in $\mathbb{P}^n$ which extends to a sheaf $\mathcal{E}$ on $\mathbb{P}^n$; i.e. there is an exact sequence*

$$0 \to \mathcal{E}(-d) \xrightarrow{f} \mathcal{E} \to E \to 0 \tag{2}$$

Then E is a sum of line bundles.

Proof. At each point p on X, over the local ring $\mathcal{O}_{\mathbb{P},p}$ the sheaf $\mathcal{E}$ is free, of the same rank as E. Hence $\mathcal{E}$ is locally free except at finitely many points. Let $\mathbb{H}$ be a general hyperplane not passing through these points. Let $X' = X \cap \mathbb{H}$, and $\mathcal{E}', E'$ be the restrictions of $\mathcal{E}, E$ to $\mathbb{H}, X'$.

It is enough to show that E' is a sum of line bundles on X'. This is because any isomorphism $\oplus\mathcal{O}_{X'}(a_i) \to E'$ can be lifted to an isomorphism $\oplus\mathcal{O}_X(a_i) \to E$, as $H^1(E(\nu)) = 0, \forall\, \nu \in \mathbb{Z}$. The bundle E' on X' is ACM and from the sequence

$$0 \to \mathcal{E}'(-d) \to \mathcal{E}' \to E' \to 0,$$

it is easy to check that $H^i(\mathcal{E}'(\nu)) = 0, \forall\, \nu \in \mathbb{Z}$, for $2 \le i \le n-2$. Since $\mathcal{E}'$ is a vector bundle on $\mathbb{H}$, we can dualize the sequence to get

$$0 \to \mathcal{E}'^{\vee}(-d) \to \mathcal{E}'^{\vee} \to E'^{\vee} \to 0.$$

$E'^{\vee}$ is still an ACM bundle, hence $H^i(\mathcal{E}'^{\vee}(\nu)) = 0, \forall\, \nu \in \mathbb{Z}$, and $2 \le i \le n-2$.

By Serre duality, we conclude that $\mathcal{E}'$ is an ACM bundle on $\mathbb{H}$, and by Horrocks' theorem, $\mathcal{E}'$ is a sum of line bundles. Hence, its restriction E' is also a sum of line bundles on X'. □

Proposition 4. *Let Y be an ACM subvariety of codimension 2 in the hypersurface X such that the associated ACM bundle G is not a sum of line bundles. Then there is no pure subvariety Z of codimension 2 in $\mathbb{P}^n$ such that $Z \cap X = Y$.*

Proof. Suppose there is such a Z. Then there is an exact sequence $0 \to I_{Z/\mathbb{P}}(-d) \to I_{Z/\mathbb{P}} \to I_{Y/X} \to 0$, where the inclusion is multiplication by f, the polynomial defining X. Since Z has no embedded points, $H^1(I_{Z/\mathbb{P}}(\nu)) = 0$ for $\nu << 0$. Combining this with $H^1(I_{Y/X}(\nu)) = 0, \forall\, \nu \in \mathbb{Z}$, and using the long exact sequence of cohomology, we get $H^1(I_{Z/\mathbb{P}}(\nu)) = 0, \forall\, \nu \in \mathbb{Z}$.

Now suppose Y has the resolution $0 \to G \to \oplus\mathcal{O}_X(-m_i) \to I_{Y/X} \to 0$. From the vanishing just proved, the right hand map can be lifted to a map $\oplus\mathcal{O}_{\mathbb{P}}(-m_i) \to I_{Z/\mathbb{P}}$, which is easily checked to be surjective (at the level of global sections). It follows that if $\mathcal{G}$ is the kernel of this lift, $\mathcal{G}$ is an extension of G to $\mathbb{P}^n$. By the previous proposition, G is a sum of line bundles. This is a contradiction. □

3.2. Voisin's original example was as follows. Let P_1 and P_2 be two planes meeting at a point p in $\mathbb{P}^4$. The union Σ is a surface which is not locally Cohen-Macaulay at p. Let X be a smooth hypersurface of degree $d > 1$ which passes through p. $X \cap \Sigma$ is a curve Z in X with an embedded point at p. The reduced subscheme Y has the form $Y = C_1 \cup C_2$, where C_1 and C_2 are plane curves. Voisin argues that Y itself does not have the form $X \cap S$ for any surface S in $\mathbb{P}^4$.

We can treat this example from the point of view of ACM bundles. $I_{Z/X}$ has a resolution on X which is just the restriction of the resolution of the ideal of the union $P_1 \cup P_2$ in $\mathbb{P}^4$, *viz.*

$$0 \to \mathcal{O}_X(-4) \to 4\mathcal{O}_X(-3) \to 4\mathcal{O}_X(-2) \to I_{Z/X} \to 0.$$

From the sequence $0 \to I_{Z/X} \to I_{Y/X} \to k_p \to 0$, it is easy to see that Y is ACM, with a resolution

$$0 \to G \to 4\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-d) \to I_{Y/X} \to 0.$$

G is an ACM bundle. If it were a sum of line bundles, comparing the two resolutions, we find that $h^0(G(2)) = 0$ and $h^0(G(3)) = 4$, hence $G = 4\mathcal{O}_X(-3)$. But then $G \to 4\mathcal{O}_X(-2) \oplus \mathcal{O}_X(-d)$ cannot be an inclusion. Thus G is an ACM bundle which is not a sum of line bundles.

Voisin's subsequent smooth examples were obtained by placing Y on a smooth surface T contained in X and choosing divisors Y' in the linear series $|Y + mH|$ on T. When m is large, Y' can be chosen smooth. In fact, such curves Y' are doubly linked to the original curve Y in X, hence they have a similar resolution $G' \to L \to I_{D'/X} \to 0$, where L is a sum of line bundles and where G' equals G up to a twist and a sum of line bundles.

The fact that G above is not a sum of line bundles is related (via the mapping cone of the map of resolutions) to the fact that k_p itself cannot have a finite resolution by sums of line bundles on X. This follows from the following proposition which provides another argument for the existence of ACM bundles on arbitrary smooth hypersurfaces of degree ≥ 2.

Proposition 5. *Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree ≥ 2 with homogeneous coordinated ring S_X. Let L be a linear space (possibly a point or even empty) inside X of codimension r, with homogeneous ideal $I(L)$ in S_X. A free presentation of $I(L)$ of length $r-2$ will have a kernel whose sheafification is an ACM bundle on X which is not a sum of line bundles.*

Proof. It should first be understood that the homogeneous ideal $I(L)$ of the empty linear space will be taken as the irrelevant ideal $(X_0, X_1, \ldots, X_n)$. Let the free presentation of $I(L)$ together with the kernel be

$$0 \to M \to F_{r-2} \to \cdots \to F_0 \to I(L) \to 0,$$

where F_i are free graded S_X modules. Its sheafification looks like

$$0 \to \tilde{M} \to \tilde{F}_{r-2} \to \cdots \to \tilde{F}_0 \to I_{L/X} \to 0.$$

Since L is locally Cohen-Macaulay, $\tilde{M}$ is a vector bundle on X, and since L is ACM, so is $\tilde{M}$. M equals $\oplus_{\nu \in \mathbb{Z}} H^0(\tilde{M}(\nu))$. Hence, $\tilde{M}$ is a sum of line bundles only if M is a free S_X module.

If $\mathbb{H}$ is a general hyperplane in $\mathbb{P}^n$ which meets X and L transversally along $X_{\mathbb{H}}$ and $L_{\mathbb{H}}$ respectively, the above sequences of modules and sheaves can be restricted to give similar sequences in $\mathbb{H}$. The restriction $\tilde{M}_{\mathbb{H}}$ is an ACM bundle on $X_{\mathbb{H}}$.

Repeat this successively to find a maximal and general linear space $\mathbb{P}$ in $\mathbb{P}^n$ which does not meet L. If $X' = X \cap \mathbb{P}$, the restriction of the sequence of S_X modules to X' gives a resolution

$$0 \to M' \to F'_{r-2} \to \cdots \to F'_0 \to S_{X'} \to k \to 0.$$

Localize this sequence of graded $S_{X'}$ modules at the irrelevant ideal $I(L) \cdot S_{X'}$, to look at its behaviour at the vertex of the affine cone over X'. k is the residue field of this local ring. Since X and hence X' has degree ≥ 2, the cone is not smooth at the vertex. By Serre's theorem ([Se], IV-C-3-Cor 2), k cannot have finite projective dimension over this local ring. Hence M' is not a free module. Therefore neither is M. $\square$

3.3. We make a few concluding remarks about Question 1, the Degree Conjecture of Griffiths and Harris. A vector bundle G on a smooth hypersurface X in $\mathbb{P}^4$ has a second Chern class $c_2(G) \in A^2(X)$, the Chow group of codimension 2 cycles. If $h \in A^1(X)$ is the class of the hyperplane section of X, the degree of any element $c \in A^2(X)$ will be defined to be the degree of the zero cycle $c \cdot h \in A^3(X)$. (Note that by the Lefschetz theorem, all classes in $A^1(X)$ are multiples of h.)

With this notation, if E is any bundle on X and Y is a curve obtained from E with the sequence (*vide* Proposition 1)

$$0 \to E \to \oplus_{i=1}^{k} \mathcal{O}_X(m_i) \to I_{Y/X}(m) \to 0,$$

a calculation tells us that the degree d of X divides the degree of Y if and only if d divides the degree of $c_2(E)$.

More generally: let Y be any curve in X and resolve $I_{Y/X}$ to get

$$0 \to E \to \oplus_{i=1}^{l} \mathcal{O}_X(b_i) \to \oplus_{i=1}^{k} \mathcal{O}_X(a_i) \to I_{Y/X} \to 0,$$

where E is an ACM bundle on X. Then a similar calculation tells us that the degree d of X divides the degree of Y if and only if d divides the degree of $c_2(E)$.

Hence we may ask the following question which is equivalent to the Degree Conjecture:

ACM Degree Conjecture. *If X is a general hypersurface in $\mathbb{P}^4$ of degree $d \geq 6$, then for any indecomposable ACM vector bundle E on X, d divides the degree of $c_2(E)$.*

The examples created above in Proposition 5 satisfy this, when L has codimension > 2 in X. In [MRR], this conjecture is settled for ACM bundles of rank 2 on X.

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