LEFSCHETZ THEOREMS FOR TORSION ALGEBRAIC CYCLES IN CODIMENSION 2

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ABSTRACT. Let Y be a smooth projective variety over $\mathbb{C}$, and X be a smooth hypersurface in Y. We prove that the natural restriction map on Chow groups of codimension two cycles is an isomorphism when restricted to the torsion subgroups provided $\dim Y \geqslant 5$. We prove an analogous statement for a very general hypersurface $X \subset \mathbb{P}^4$ of degree $\geqslant 5$. In the more general setting of a very general hypersurface X of sufficiently high degree in a fixed smooth projective four-fold Y, under some additional hypothesis, we prove that the restriction map is an isomorphism on ℓ-primary torsion for almost all primes ℓ. As a consequence, we obtain a weak Lefschetz theorem for torsion in the Griffiths groups of codimension 2 cycles, and prove the injectivity of the Abel-Jacobi map when restricted to torsion in this Griffiths group, thereby providing a partial answer to a question of Nori.

CONTENTS

1. Introduction

1.1. **Motivation.** We begin by recalling the Lefschetz conjectures for Chow groups. Below, we work over the field of complex numbers.

Conjecture 1.1 (Weak Lefschetz conjecture)**.** Let Y be a smooth, projective variety of dimension $m+1$ and X be a smooth, ample divisor in Y. The restriction map of rational Chow groups

$$\mathrm{CH}^i(Y)_{\mathbb{Q}} \to \mathrm{CH}^i(X)_{\mathbb{Q}}$$

is an isomorphism for $i < m/2$, and a monomorphism for $i = m/2$.

Conjecture 1.2 (Noether-Lefschetz conjecture)**.** Let Y be a smooth, projective variety of dimension $m+1$, and X be a very general, sufficiently ample divisor in Y. Then

$$\mathrm{CH}^i(Y)_{\mathbb{Q}} \to \mathrm{CH}^i(X)_{\mathbb{Q}}$$

is an isomorphism for $i < m$, and a monomorphism for $i = m$.

Conjecture 1.1 is a consequence of the Bloch-Beilinson conjectures (see [15] for e.g.), and Conjecture 1.2, which is due to Nori (see [14], Conjecture 7.2.5), similarly follows from the Bloch-Beilinson conjectures and Nori's connectivity theorem (see *op. cit.*).

On the other hand, Totaro (see [20]) has conjectured that the isomorphisms in Conjecture 1.2 should hold integrally. In particular, Totaro's conjecture implies that the natural restriction map on torsion in Chow groups

$$\mathrm{CH}^i(Y)_{\mathrm{tors}} \to \mathrm{CH}^i(X)_{\mathrm{tors}}$$

is an isomorphism for $i < m$.

1.2. **Statement of results.** The results in this note mark some progress for codimension 2 cycles. More precisely, we prove the following *Lefschetz theorems for torsion cycles in codimension two.*

Theorem 1.3. *Let Y be a smooth projective variety of dimension at least 4 and $X \subset Y$ be a smooth, ample divisor. Then the natural restriction map*

$$\mathrm{CH}^2(Y)_{\mathrm{tors}} \to \mathrm{CH}^2(X)_{\mathrm{tors}}$$

is an isomorphism for $\dim Y \geqslant 5$, and an injection for $\dim Y = 4$.

Remark 1. Assume now that in the theorem above, Y is defined over a separably closed field of characteristic $p > 0$. Then the proof of the above theorem yields that the restriction map above is an isomorphism on prime - to - p torsion.

For hypersurfaces in $\mathbb{P}^4$, we prove an analogous statement:

Theorem 1.4. *For a very general hypersurface $X \subset \mathbb{P}^4$ of degree $\geqslant 5$, one has*

$$\mathrm{CH}^2(X)_{\mathrm{tors}} = 0.$$

This immediately implies the following obvious

Corollary 1.5. *For a very general hypersurface $X \subset \mathbb{P}^4$ of degree $\geqslant 5$, one has*

$$\mathrm{Grif}^2(X)_{\mathrm{tors}} = 0.$$

Here $\mathrm{Grif}^2(X)$ *is the Griffiths group of codimension* 2 *cycles.*

The above Corollary answers in the affirmative the case $m = 2$ of the following question due to Chad Schoen ([17], 3.4.4):

For a sufficiently general hypersurface $X \subset \mathbb{P}^{2m}$, is $\mathrm{Grif}^m(X)_{\mathrm{tors}} = 0$?

More generally, using the methods of the proof of Theorem 1.4, we prove a Noether-Lefschetz theorem for torsion cycles in codimension 2.

Theorem 1.6. *Let Y be a smooth projective variety of dimension at least 4 and $\mathcal{O}_Y(1)$ be a sufficiently ample line bundle on Y. Assume that the universal family $\mathcal{X} \subset Y \times S$, where S is the parameter space of smooth members of the linear system $|\mathcal{O}_Y(1)|$, satisfies hypothesis* (**H**) *(see §4.4 for details). Then, for a very general member X in the linear system $|\mathcal{O}_Y(1)|$, the restriction map $\mathrm{CH}^2(Y) \to \mathrm{CH}^2(X)$ is an isomorphism on p-torsion for almost all primes p.*

Theorem 1.7. *Let Y be a smooth projective variety of dimension at least 5, and X be a general ample hyperplane section in Y. Then the restriction map*

$$\mathrm{Grif}^2(Y)_{\mathrm{tors}} \to \mathrm{Grif}^2(X)_{\mathrm{tors}}$$

is an isomorphism.

Remark 2. We note that hypothesis (**H**) is not so easy to verify. However, results of Illusie ([8]) give several examples (see §4 for more details). In particular, global complete intersections in $\mathbb{P}^N$ satisfy these hypotheses (4.13).

1.3. **Method of proof.** The basic ingredient in the proofs of Theorems 1.3 and 1.6 is the identification of ℓ-primary torsion in $\mathrm{CH}^2(X)$ with level 1 coniveau of $\mathrm{H}^3_{et}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ following the work of Merkurjev-Suslin [12], see Theorem 2.5. The theorems then are consequences of the strictness of the coniveau filtration. However, the techniques for proving these strictness statements (Theorems 1.3 and 1.6) are very different. The former is an easy consequence of an open version of the usual weak Lefschetz theorem combined with a Bertini argument. However, for the latter we use reduction mod p and a theorem of Bloch-Esnault (see [3]) which states that under some hypothesis, coniveau level 1 is a strict subset of the whole cohomology group. A monodromy argument then allows one to conclude that this must be zero.

We briefly describe the contents of the following sections. In §2, we recall some background material on the coniveau filtration and its relation to torsion in codimension two Chow groups. In §3, we give a proof of Theorem 1.3, which follows from a simple application of a Bertini theorem and an open version of the weak Lefschetz theorem. In §4, we give a proof of Theorems 1.4 and 1.6 which, as mentioned above, follow from a reduction mod p argument and an application of a theorem of Bloch-Esnault. Finally, in §5 we give a proof of Theorem 1.7.

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2. Preliminaries

2.1. **Coniveau Filtration.** Let X denote a scheme of finite type over a fixed field $k \subset \mathbb{C}$. Then the coniveau filtration on the singular cohomology $\mathrm{H}^i(X, A)$ of X (i.e. singular cohomology of the base change $X_{\mathbb{C}}$) with coefficients in an abelian group A is defined as follows:

$$\begin{aligned} \mathrm{N}^j\mathrm{H}^i(X,A) \quad &:= \quad \textstyle\sum_{\operatorname{codim} Z \geqslant j} \ker[\mathrm{H}^i(X,A) \to \mathrm{H}^i(X \backslash Z, A)] \\ &\cong \quad \textstyle\sum_{\operatorname{codim} Z \geqslant j} \mathrm{Image}[\mathrm{H}^{i-2j}(\widetilde{Z}, A) \to \mathrm{H}^i(X,A)]. \end{aligned}$$

Here $\tilde{Z} \to Z$ is a desingularization. This gives rise to the descending *coniveau filtration*

$$\mathrm{N}^0\mathrm{H}^i(X,A) = \mathrm{H}^i(X,A) \supset \mathrm{N}^1\mathrm{H}^i(X,A) \supset \mathrm{N}^2\mathrm{H}^i(X,A) \supset \mathrm{N}^3\mathrm{H}^i(X,A) \supset \cdots .$$

Remark 3. One has an analogously defined coniveau filtration for the étale cohomology of X with coefficients in any torsion abelian group A with torsion prime to the characteristic of k.

We shall think of $\mathrm{H}^i(X, A)$ as a filtered A-module with the filtration given by coniveau. One has the following basic functoriality result:

Any morphism $f : X \to Y$ of smooth projective varieties induces a filtered morphism on cohomology. In particular, one has

$$f^*(\mathrm{N}^j\mathrm{H}^i(Y,A)) \subset \mathrm{N}^j\mathrm{H}^i(X,A).$$

One has a similar statement for étale cohomlogy. In the setting of singular cohomology, this is proved in [2]. The same proof works also for étale cohomology. On the other hand, recall that the coniveau filtration can also be defined as the filtration induced by the coniveau spectral sequence. Since the coniveau spectral sequence is contravariantly functorial in f, so is the resulting filtration on its abutment. In particular, the result holds for any Bloch-Ogus cohomology theory. We also have the following standard compatibility of coniveau with respect to the comparison isomorphism.

Lemma 2.1. *Suppose $k \subset \mathbb{C}$ is algebraically closed. For $A = \mathbb{Z}/\ell^r\mathbb{Z}$, the comparison isomorphism between étale cohomology and singular cohomology*

$$c : \mathrm{H}^i_{et}(X, A) \to \mathrm{H}^i(X, A),$$

is a morphism of filtered modules.

Proof. First note that étale cohomology and the coniveau filtratrion are invariant under extensions of algebraically closed fields ([1], Expose XX, 2.2). Hence, we may assume that $k = \mathbb{C}$. The result now follows from the fact that the comparison isomorphism is functorial. In particular, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}^i_{et}(X,A) & \longrightarrow & \mathrm{H}^i_{et}(X\backslash Z,A) \\ \downarrow c & & \downarrow c \\ \mathrm{H}^i(X,A) & \longrightarrow & \mathrm{H}^i(X\backslash Z,A). \end{array}$$

□

Finally, we recall that the integral ℓ-adic cohomology (with Tate twists) of X is defined as the inverse limit

$$\mathrm{H}^i_{et}(X,\mathbb{Z}_\ell(n)) = \varprojlim_m \mathrm{H}^i_{et}(X,\mathbb{Z}/\ell^m\mathbb{Z}(n))$$

and the rational ℓ-adic cohomology of X is

$$\mathrm{H}^i_{et}(X,\mathbb{Q}_\ell(n)) := \mathrm{H}^i_{et}(X,\mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In this setting, we define the coniveau filtration as the inverse limit of the previously defined coniveau filtration:

$$\mathrm{N}^j\mathrm{H}^i_{et}(X,\mathbb{Z}_\ell(n)) := \varprojlim_m \mathrm{N}^j\mathrm{H}^i_{et}(X,\mathbb{Z}/\ell^m\mathbb{Z}(n)) \quad \text{and} \quad \mathrm{N}^j\mathrm{H}^i_{et}(X,\mathbb{Q}_\ell(n)) := \mathrm{N}^j\mathrm{H}^i_{et}(X,\mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Note that one could alternatively define the coniveau filtration on ℓ-adic cohomology by repeating the definition of the coniveau filtration for $\mathrm{H}^i_{et}(X,\mathbb{Z}/\ell^m\mathbb{Z})$. In particular, this is the filtration one would get from the Bloch-Ogus machinery. However, this is not the same as the one defined above for ℓ-adic cohomology (see [18]). On the other hand, it is the inverse limit filtration which is naturally related to the torsion in codimension two Chow groups.

2.2. **Strictness of coniveau and GHC.** In this paragraph, we briefly recall the generalized Hodge conjecture and its relation to coniveau.

Let X denote a smooth projective variety over $\mathbb{C}$. Recall that the cohomology groups of X carry a pure Hodge structure given by the Hodge decomposition

$$\mathrm{H}^i(X,\mathbb{C}) \cong \oplus_{p+q=i}\mathrm{H}^q(X,\Omega^p_X),$$

and one has the usual Hodge filtration defined by $\mathrm{F}^p\mathrm{H}^i(X,\mathbb{C}) := \oplus_{p'\geqslant p}\mathrm{H}^{i-p'}(X,\Omega^{p'}_X)$. Moreover, it is easy to see that $\mathrm{N}^j\mathrm{H}^i(X,\mathbb{Q}) \subset \mathrm{F}^j\mathrm{H}^i(X,\mathbb{C}) \cap \mathrm{H}^i(X,\mathbb{Q})$. However, as pointed out by Grothendieck (see [7]), while the LHS is always a pure Hodge structure, the RHS is in general not a Hodge structure, and hence the inclusion need not be an equality in general. This led to his formulation of the generalized Hodge conjecture:

Conjecture 2.2 (Generalized Hodge conjecture)**.** With notation as above,

$\mathrm{GHC}(j,i,X):$ $\mathrm{N}^j\mathrm{H}^i(X,\mathbb{Q})$ *is the largest sub Hodge structure of* $\mathrm{F}^j\mathrm{H}^i(X,\mathbb{C}) \cap \mathrm{H}^i(X,\mathbb{Q})$.

We refer to [11] for a survey of results related to the GHC.

Recall, a morphism of filtered vector spaces $f : (\mathrm{V}, \mathrm{N}^{\cdot}) \to (\mathrm{W}, \mathrm{N}^{\cdot})$ is **strict** if

$$f(\mathrm{N}^j\mathrm{V}) = \mathrm{N}^j W \cap \mathrm{Im}(f).$$

As an immediate corollary of Conjecture 2.2 one obtains the following:

Conjecture 2.3. Let $f : X \to Y$ be a morphism of smooth projective complex algebraic varieties. Then the induced morphism of coniveau filtered $\mathbb{Q}$-vector spaces

$$f : \mathrm{H}^i(Y, \mathbb{Q}) \to \mathrm{H}^i(X, \mathbb{Q})$$

is a strict morphism.

Suppose now that $X \hookrightarrow Y$ is a smooth ample divisor. Then the weak Lefschetz Theorem combined with Conjecture 2.3 implies the following weak Lefschetz conjecture for coniveau:

Conjecture 2.4. Let $X \hookrightarrow Y$ be as above. Then the induced map

$$\mathrm{N}^j\mathrm{H}^i(Y, \mathbb{Q}) \to \mathrm{N}^j\mathrm{H}^i(X, \mathbb{Q})$$

is an isomorphism for all j and $i < \dim(X)$.

A proof of this conjecture is given in the next section.

2.3. **Bloch-Quillen theory and the Merkurjev-Suslin theorem.** We recall a result relating coniveau level one with torsion in CH^2 (see [4], Prop 3.1). Let X denote a smooth proper variety over a separably closed field k. For ℓ prime to the characteristic of k, we set

$$\mathrm{H}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \varinjlim_m \mathrm{H}^i(X, \mathbb{Z}/\ell^m(n)), \text{and}$$

$$\mathrm{N}^j\mathrm{H}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) := \varinjlim_m \mathrm{N}^j\mathrm{H}^i(X, \mathbb{Z}/\ell^m(n)).$$

Theorem 2.5. *For X/k a smooth, proper, connected variety over a separably closed field, and ℓ prime, $\ell \neq \mathrm{char}.k$, there is a natural isomorphism*

$$\mathrm{N}^1\mathrm{H}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \cong \mathrm{CH}^2(X)\{\ell\}.$$

Here $\mathrm{CH}^2(X)\{\ell\} := \bigcup_{\ell^m} \mathrm{CH}^2(X)_{\ell^m-tors}$ is the ℓ-primary subgroup.

2.4. **A Bertini Theorem.** The following Bertini theorem will be useful in the following.

Theorem 2.6 ([10], Theorem 1)**.** *Let $X \hookrightarrow \mathbb{P}^n$ denote a smooth projective variety over an infinite field k. Let $Z \hookrightarrow X$ denote a closed subvariety and $\mathcal{I}_{Z,\mathbb{P}^n}$ denote the ideal sheaf of Z in $\mathbb{P}^n$. Let a be an integer such that $\mathcal{I}_{Z,\mathbb{P}^n}(a)$ is generated by global sections. Then a general hyperplane section of X of degree $a+1$ is smooth away from Z. In particular, the intersection of a general element of $|\mathcal{I}_{Z,\mathbb{P}^n}(a+1)|$ with X is smooth away from Z and does not contain X.*

We record the following easy consequence for future reference:

Corollary 2.7. *Let $Y \subset \mathbb{P}^N$ be a smooth projective variety, and let $X \subset Y$ denote a smooth member of $|\mathcal{O}_Y(1)|$. Let $Z \subset X$ be a closed subvariety and a be the smallest integer such that $\mathcal{I}_{Z,\mathbb{P}^n}(a)$ is generated by global sections. Then a general hypersurface V in the linear system $|\mathcal{I}_{Z,\mathbb{P}^n}(a+1)|$ intersects both X and Y smoothly away from Z.*

Remark 4. Let X, Y, and Z be as above such that codimension of Z in X is p. A repeated application of the Bertini theorem above shows that there exist hypersurfaces $V_1, \ldots, V_p$ such that each V_i contains Z, and the intersection of the V_i has codimension p in X, and $p+1$ in Y.

3. Lefschetz type results for coniveau

Theorem 3.1. *Let $(Y, \mathcal{O}_Y(1))$ be a smooth complex projective variety and X a smooth member of the linear system $|\mathcal{O}_Y(1)|$. Then the natural restriction map*

$$\mathrm{N}^1\mathrm{H}^i(Y,\mathbb{Z}) \to \mathrm{N}^1\mathrm{H}^i(X,\mathbb{Z})$$

is an isomorphism for all $i < \dim(X)$ and an injection for $i = \dim(X)$.

Proof. First, note that functoriality of coniveau and the usual weak Lefschetz implies that the map is injective in the desired range. Suppose now that $i < \dim(X)$ and let $\alpha \in \mathrm{N}^1\mathrm{H}^i(X,\mathbb{Z})$. Then, again by the usual weak Lefschetz, there exists a unique lift $\tilde{\alpha} \in \mathrm{H}^i(Y,\mathbb{Z})$ of α. Therefore, we must show that $\tilde{\alpha} \in \mathrm{N}^1\mathrm{H}^i(Y,\mathbb{Z})$. By definition of N^1, there is a closed codimension 1 subvariety $Z \subset X$ such that α maps to zero in $\mathrm{H}^i(X\backslash Z,\mathbb{Z})$. By Corollary 2.7, for $a \gg 0$, we may choose V in $|\mathrm{H}^0(Y,\mathcal{I}_Z(a))|$ such that $W := V \cap X$ is proper and contains Z. Since $W \supset Z$, one has a factorization:

$$\mathrm{H}^i(X,\mathbb{Z}) \to \mathrm{H}^i(X\backslash Z,\mathbb{Z}) \to \mathrm{H}^i(X\backslash W,\mathbb{Z}).$$

It follows that α maps to zero in $\mathrm{H}^i(X\backslash W,\mathbb{Z})$. On the other hand, Lemma 3.2 below implies that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i(Y,\mathbb{Z}) & \longrightarrow & \mathrm{H}^i(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(Y\backslash V,\mathbb{Z}) & \longrightarrow & \mathrm{H}^i(X\backslash W,\mathbb{Z}), \end{array}$$

where the horizontal maps are isomorphisms. It follows that $\tilde{\alpha}$ is in the kernel of the left vertical, and hence lies in coniveau level one. □

The following lemma is probably well known to the experts and follows easily from Artin vanishing. We provide a proof due to the lack of an appropriate reference.

Lemma 3.2. *Let Y and X be as before, and $V \in |\mathrm{H}^0(Y,\mathcal{I}_Z(a))|$ as above so that in particular, V intersects X properly. Then the natural restriction map*

$$\mathrm{H}^i(Y\backslash V,\mathbb{Z}) \to \mathrm{H}^i(X\backslash V,\mathbb{Z})$$

is an isomorphism for all $i < \dim(X)$ and injective when $i = \dim(X)$.

Proof. If $V = \varnothing$, this is the usual weak Lefschetz theorem. Moreover, the same proof goes through verbatim in the setting of the proposition. Namely, consider the long exact sequence in cohomology:

$$\cdots \to \mathrm{H}^i_c((Y\backslash X)\backslash V,\mathbb{Z}) \to \mathrm{H}^i(Y\backslash V,\mathbb{Z}) \to \mathrm{H}^i(X\backslash V,\mathbb{Z}) \to \cdots.$$

Since $(Y\backslash X)\backslash V$ is a smooth affine variety, the left-most group in the above exact sequence vanishes for all $i < \dim(Y)$. This follows from usual Artin vanishing, and duality. □

Remark 5. By Remark 4, the same proof goes through to show more generally that an element in coniveau level j lifts to one in coniveau level j. In particular, Theorem 3.1 holds if we take coniveau level j instead of level 1. For example, an element $\alpha \in \mathrm{N}^j\mathrm{H}^i(X,\mathbb{Z})$ iff there is a codimiension j subvariety $Z \subset X$ such that $\tilde{\alpha}$ maps to zero in $\mathrm{H}^i(X\backslash Z,\mathbb{Z})$. Then Remark 4 allows us to find a codimension j subvariety $W \subset Y$ such that $W = V_1 \cap \cdots \cap V_j$ is the intersection of j hyperplanes (of large enough degree) all containing Z, and such that V intersects X properly. Moreover, Lemma 3.2 continues to hold for $V = \cup V_i$ since (by separatedness) $(Y\backslash X)\backslash V$ is still affine. One can then deduce Lemma 3.2 for $W = \cap V_i$ via an application of Mayer-Vietoris. This then allows one to argue as in the proof of Theorem 3.1. However, it is only the level 1 coniveau that plays a role in the applications to torsion codimension two cycles discussed below.

Corollary 3.3. *Let Y be a smooth projective variety over a separably closed field k, $X \in |\mathcal{O}_Y(1)|$ as before, and ℓ prime to char(k). Then the natural restriction map*

$$\mathrm{N}^j\mathrm{H}^i(Y,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \to \mathrm{N}^j\mathrm{H}^i(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$

is an isomorphism for all $i < \dim(X)$ and injective when $i = \dim(X)$.

Proof. We first note that Theorem 3.1 and Lemma 3.2 (with same proof) hold for Y over a separably closed field k if one considers instead étale cohomology with coefficients in a torsion ring A such that $char(k)$ is invertible in A. The result now follows by taking direct limits and Remark 5. □

Corollary 3.4. *Let Y be a smooth projective variety over a separably closed field k of characteristic zero, and $X \in |\mathcal{O}_Y(1)|$ as before. Suppose furthermore that $\dim(Y) \geqslant 4$. Then the natural restriction map*

$$\mathrm{CH}^2(Y)_{\mathrm{tors}} \to \mathrm{CH}^2(X)_{\mathrm{tors}}$$

is an isomorphism if $\dim(Y) \geqslant 5$ and an injection if $\dim(Y) = 4$. If $char(k) = \ell > 0$, then one has an analogous statement on prime to ℓ torsion.

Proof. This is a consequence of the previous Corollary and Theorem 2.5. □

4. Noether-Lefschetz type results for strictness of Coniveau

In this section, we investigate Noether-Lefschetz type theorems for strictness of coniveau and its application to torsion codimension 2 cycles. Much of this section is based on [19]. We begin by recalling some background which will be used in the following subsection.

4.1. **Ordinary Reduction.** Let k be a perfect field of characteristic p. Then a smooth proper variety X over k is *ordinary* if $\mathrm{H}^q(X, B\Omega^r_{X/k}) = 0$ for all q and r. Here $B\Omega^r_{X/k} \subset \Omega^r_{X/k}$ denotes the exact r-forms. More generally, let $\pi : X \to S$ denote a smooth proper morphism of schemes over k. Then X is ordinary over S if $R^q\pi_*B\Omega^r_{X/S} = 0$ for all q and r. By Illusie ([9], 1.2), being ordinary is stable under base change. Moreover, the set of points where the fibers X_s are ordinary form an open subset U of S. Finally, the condition can be checked stalk-wise.

It may happen that the open set U above is the empty set. However, the main results of ([9]) show that there exist hypersurfaces in $\mathbb{P}^n$ which are ordinary. In particular, the generic hypersuface in projective space is ordinary. More generally, Illusie proves the following result.

Theorem 4.1. *Let Y be a smooth projective variety over k such that every complete intersection of Y of multidegree $(1,\ldots,1)$ is ordinary. Then the general hypersurface of multi-degree $(a_1,\ldots,a_d)$ is ordinary for all a_i.*

The hypothesis in the Theorem is not so easy to check in practice. However, they are trivially verified for $Y = \mathbb{P}^n$, as well as for quadrics.

4.2. **Coniveau level one theorems.** Let R denote a complete discrete valuation ring with a perfect residue field k of characteristic p, and fraction field K of characteristic 0. The following theorem of Bloch-Esnault will be used in the following to show the triviality of various coniveau level one pieces.

Theorem 4.2 ([3],Theorem 1.2)**.** *Let X be a smooth proper scheme over $S = \mathrm{Spec}(R)$. Suppose that X_k (the special fiber) is ordinary, the Hodge groups $\mathrm{H}^s(X,\Omega^t_{X/S})$ are torsion free for all t and s, and that $\Gamma(X_k,\Omega^m_{X_k}) \neq 0$. Then*

$$\mathrm{N}^1\mathrm{H}^m(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z}) \neq \mathrm{H}^m(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z}).$$

Remark 6. In *loc. cit.,* the theorem is stated with the hypothesis that the crystalline cohomology (rather than the Hodge) groups are torsion free. However, note that the current hypotheses imply that the crystalline cohomology is torsion free.

We now prove a slightly general version of the previous theorem in dimension 4. More precisely, suppose $Y \to \mathrm{Spec}(R)$ is smooth projective with (relatively) ample bundle $\mathcal{O}_Y(1)$ and that X is a smooth element of the associated (relative) linear system. In particular, X_K is a smooth hyperplane section of Y_K. Finally, we shall assume $\dim(Y_K) = 4$.

Recall that, in this setting, one has the vanishing cycles $\delta \in \mathrm{H}^3(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z})$, and the vanishing cycles cohomology $\mathrm{H}^3_{ev}(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z})$, which is by definition the $\mathbb{Z}/p\mathbb{Z}$-span of the vanishing cycles δ in $\mathrm{H}^3(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z})$. We refer to ([1], Exposes XIII, XVIII) for details on vanishing cycles. Below we set

$$\mathrm{N}^1\mathrm{H}^3_{ev}(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z}) := \mathrm{H}^3_{ev}(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z}) \cap \mathrm{N}^1\mathrm{H}^3(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z}).$$

Then we have the following analog of the theorem of Bloch-Esnault.

Theorem 4.3. *With notation as above, suppose furthermore:*

(i) X has ordinary good reduction.
(ii) Either $3 < (p-1)/\gcd(e,p-1)$ (where e is the absolute ramification degree of R) or that the Hodge groups of X relative to S have no torsion.
(iii) Suppose that $\mathrm{H}^0(Y_k,\Omega^3_{Y_k}) \to \mathrm{H}^0(X_k,\Omega^3_{X_k})$ is not surjective.
(iv) The hard Lefschetz map $\mathrm{H}^3(Y_{\bar{K}},\mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^5(Y_{\bar{K}},\mathbb{Z}/p\mathbb{Z})$ is an isomorphism.

Then

$$\mathrm{N}^1\mathrm{H}^3_{ev}(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z}) \neq \mathrm{H}^3_{ev}(X_{\bar{K}},\mathbb{Z}/p\mathbb{Z}).$$

Proof. Since the proof follows closely the proof of Bloch-Esnault ([3], Theorem 1.2), we give an outline and refer to loc. cit. for some of the details. Let $i_X : X_k \hookrightarrow X$ (resp. $i_Y : Y_k \hookrightarrow Y$) denote the natural closed embedding and similarly $j_X : X_K \hookrightarrow X$ (resp. $j_Y : Y_K \hookrightarrow Y$) denote the corresponding open immersion. In the following, we write bars over various objects to denote their base extension to the algebraic closure. Then one has the following Bloch-Kato spectral sequence:

$$\mathrm{E}_2^{s,t} := \mathrm{H}^s(Y_{\bar{k}}, (\bar{i_Y}^*\mathbb{R}^t\bar{j}_{Y*}(\mathbb{Z}/p\mathbb{Z}(t)))(-t)) \Longrightarrow \mathrm{H}^{s+t}(Y_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$$

One has a similar spectral sequence for $X_{\bar{k}}$ and $X_{\bar{K}}$. Now under the assumption of ordinarity and (ii), one has a diagram:

$$\mathrm{H}^3(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) \twoheadrightarrow \mathrm{H}^0(X_{\bar{k}}, (\bar{i_X}^*\mathbb{R}^3\bar{j}_{X*}(\mathbb{Z}/p\mathbb{Z}(3)))(-3)) \hookrightarrow \mathrm{H}^0(X_{\bar{k}}, \Omega^3_{X_{\bar{k}}}(-3)).$$

We note that such a diagram always exists (in particular also for $Y_{\bar{K}}$). However, here the hypothesis (ii) guarantees that the above spectral sequence degenerates at E_2, which gives that the left map in the above diagram is surjective. The injection on the right is a consequence of ordinarity. Moreover, under this assumption, the injection on the right becomes an isomorphism if we tensor the middle term with $\bar{k}$. We denote the compositions in the above diagram by α_X (resp. α_Y). This gives rise to a commutative diagram:

(4.2.3)
$$\begin{array}{ccc}
\mathrm{H}^3(Y_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}^3(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) \\
\downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_X} \\
\mathrm{H}^0(Y_{\bar{k}}, \Omega^3_{Y_{\bar{k}}}(-3)) & \longrightarrow & \mathrm{H}^0(X_{\bar{k}}, \Omega^3_{X_{\bar{k}}}(-3)) \\
\downarrow & & \downarrow \\
\Omega^3_{\mathrm{L}_Y}(-3) & \longrightarrow & \Omega^3_{\mathrm{L}_X}(-3)
\end{array}$$

Here L_X is the separable closure of the function field of $X_{\bar{k}}$ and similarly for L_Y. The horizontal maps are just the restriction maps. It follows from loc. cit. 1.2.7, that one has a commutative diagram such that the right vertical is injective:

$$\begin{array}{ccc}
\mathrm{H}^3(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\alpha_X} & \mathrm{H}^0(X_{\bar{k}}, \Omega^3_{X_{\bar{k}}}(-3)) \\
\downarrow{\scriptstyle \beta_X} & & \downarrow{\scriptstyle \gamma} \\
\mathrm{H}^3(\mathrm{K}(X_{\bar{K}}), \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \Omega^3_{\mathrm{L}_X}(-3)
\end{array}$$

It follows that if $\beta_X(\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})) = 0$ then $\alpha_X(\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})) = 0$. Moreover, the former is equivalent to $\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) = \mathrm{N}^1\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$. Therefore, we are reduced to showing that $\alpha_X(\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})) \neq 0$. By assumption (iv), one has $\mathrm{H}^3(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) = \mathrm{H}^3(Y_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) + \mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$. Therefore, if $\alpha_X(\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})) = 0$ then the middle horizontal in (4.2.3) must be surjective (since α_X is surjective after tensoring with $\bar{k}$). On the other hand, the middle arrow is not surjective by hypothesis (iii). □

Remark 7. Note that, in the proof, we do not require any information on α_Y except its existence. And, in particular, one only needs the assumptions (i) and (ii) of the Theorem for X.

Remark 8. In practice, hypothesis (iv) of the theorem is rare. However, it is true for principally polarized abelian varieties. On the other hand, we shall consider varieties over $\mathbb{C}$, and then spread them out over number fields. In that case, the hypothesis will hold for the reduction modulo p for almost all primes p.

4.3. **The case $Y = \mathbb{P}^4$.** In this subsection, we prove a Noether-Lefschetz type theorem for torsion codimension two cycles on a very general hypersurface $X \subset \mathbb{P}^4_{\mathbb{C}}$ of high enough degree.

Theorem 4.4. *Let X be a very general degree d hypersurface in $\mathbb{P}^4_{\mathbb{C}}$. Then for $d \geqslant 5$, one has $\mathrm{CH}^2(X)_{\mathrm{tors}} = 0$.*

The previous theorem will follow from the following theorem. Before stating it, we set-up some notation. Let $X \to S$ denote the universal family of (smooth) hypersurfaces of degree d in $\mathbb{P}^4_{\mathbb{C}}$, where S is the parameter space of such hypersurfaces. In the following, let η denote the generic point of S and $\bar{\eta}$ the geometric generic point. Note that S is the moduli of smooth hypersurfaces of degree d, and is defined over $\mathbb{Z}$. Moreover, the universal family $X \to S$ is also defined over $\mathbb{Z}$, and we let $\mathcal{X} \to \mathcal{S}$ denote the universal family over $\mathbb{Z}$. Its base change to $\mathbb{C}$ gives the universal family over $\mathbb{C}$, and similarly its reduction modulo a prime p gives the universal family of hypersurfaces in $\mathbb{P}^4_{\mathbb{F}_p}$. Finally, note that $\mathcal{S}$ is smooth over $\mathbb{Z}$ ([6], 1.9).

Theorem 4.5. *With notation as above and $d \geqslant 5$, one has $\mathrm{CH}^2(X_{\bar{\eta}})_{\mathrm{tors}} = 0$.*

We first recall how Theorem 4.5 implies 4.4. This is a standard argument which we reproduce here for the reader's convenience (see also Lemma 2.1, [21] or §3, [16]). Suppose we have contravariant functors F on the category of smooth projective varieties over a field k (for any k) with values in abelian groups equipped with base change maps (along field extensions $k \subset k'$). Suppose that F satisfies rigidity. Namely, for any extension of algebraically closed fields $K \subset L$, the natural map $F(X) \to F(X_L)$ is an isomorphism for any X a smooth projective variety over K. In this case, one has the following standard result.

Lemma 4.6. *Let $k = \bar{k}$ be an uncountable field. Suppose we are given a pair (Y, V) where Y is a smooth projective variety over k, and V is a linear system given by a very ample line bundle on Y. Let $\mathcal{S}$ denote the open set in V parametrizing smooth sections and $\mathcal{X} \subset Y \times \mathcal{S}$ the corresponding universal family. Let K denote the the function field of $\mathcal{S}$, $\overline{K}$ its algebraic closure, and $\eta := \mathrm{Spec}(K)$, $\bar{\eta} := \mathrm{Spec}(\overline{K})$ be the generic and geometric generic points of $\mathcal{S}$ respectively. Then the following are equivalent:*
(1) The restriction map $F(Y_{\bar{\eta}}) \to F(\mathcal{X}_{\bar{\eta}})$ is an isomorphism.
(2) For a very general element X of the linear system V, the restriction map $F(Y) \to F(X)$ is an isomorphism.

Before we prove the Lemma, we note that an application of the lemma with $F(-) = \mathrm{CH}^2(-)_{\mathrm{tors}}$ together with Theorem 4.5 gives a proof of Theorem 4.4. The rigidity for F is a direct consequence of the following result due to Lecomte-Suslin:

Proposition 4.7. *(Lecomte-Suslin) Let X denote a smooth proper variety over an algebraically closed field L, and suppose K is an algebraically closed field extension of L. Then the natural extension of scalars map*

$$\mathrm{CH}^p(X)_{\mathrm{tors}} \to \mathrm{CH}^p(X_K)_{\mathrm{tors}}$$

is an isomorphism.

Proof of Lemma 4.6. There exists a countable algebraically closed field $k_0 \subset k$ such that the whole data (Y, V) can be descended to k_0. In particular, there exists a pair (Y_0, V_0) (defined over k_0) whose base change to k is (Y, V). In the following, $|V|$ and $|V_0|$ will denote the corresponding projective spaces. Let $t \in |V|$ be a closed point such that t lies in the complement of the (countable) union of divisors in $|V|$ coming from base change of divisors in $|V_0|$. Any such t maps to the generic point of $|V_0|$ under the projection map $|V| \to |V_0|$. Let K_0 (resp. K) denote the function field of $|V_0|$ (resp. $|V|$). (Note that $\eta = \mathrm{Spec}(K)$). It follows that $K_0 \hookrightarrow k(t) \cong k$. Let $\mathcal{X}_0$ denote the universal family over $|V_0|$, $X_{K_0} := (\mathcal{X}_0)_{K_0}$ and $Y_{K_0} := (Y_0)_{K_0}$. Then (Y, X_t) is the base change of (Y_{K_0}, X_{K_0}). Moreover, $(Y_K, \mathcal{X}_K)$ is obtained via base change from (Y_{K_0}, X_{K_0}). The results now follows by passing to algebraic closures. □

Proof of Theorem 4.5. In the following, we fix a prime p.

Step 1: Recall, by the remarks above, there is a smooth projective family $\mathcal{X} \to \mathcal{S}$ smooth over $\mathbb{Z}$ whose base change to $\mathbb{C}$ gives the universal family. Let $\eta_{\mathcal{S}}$ denote the generic point of $\mathcal{S}$ and $\bar{\eta}_{\mathcal{S}}$ its geometric generic point. Note that $\overline{k(\eta_{\mathcal{S}})} \hookrightarrow \overline{k(\eta)}$ is an extension of algebraically closed fields. Since $(\mathcal{X}_{\bar{\eta}_{\mathcal{S}}})_{\bar{\eta}} \cong X_{\bar{\eta}}$, by Lecomte-Suslin rigidity (4.7) it is enough to show that $\mathrm{CH}^2(\mathcal{X}_{\bar{\eta}_{\mathcal{S}}})_{\mathrm{tors}} = 0$.

Step 2: By the theorem of Colliot-Thélène – Raskind (2.5) it is enough to show that, for all p, $\mathrm{N}^1\mathrm{H}^3_{et}(\mathcal{X}_{\bar{\eta}_{\mathcal{S}}}, \mathbb{Z}/p\mathbb{Z}) = 0$. Since we work over separably closed fields, we drop the Tate twists from our notation.

Step 3: Again by rigidity, we may localize the universal family to $\mathbb{Z}_{(p)}$. Specifically, after localizing and base change, we may assume $\mathcal{S} = \mathrm{Spec}(R)$ where R is a discrete valuation ring of mixed charactersitic $(0, p)$. Since ordinarity is an open condition, we can also assume (by further localizing if necessary) that our universal family over $\mathcal{S}$ has good ordinary reduction. Applying rigidity again, we may assume that our DVR R is a complete DVR with perfect residue field k.

Step 4: Suppose now that we are in the setting of Step 3. Note that $\mathcal{X}$ is just a family of hypersurfaces (with ordinary reduction) in $\mathbb{P}^4$ over $\mathcal{S}$ and similarly for its reduction over k. We are reduced to showing that

$$\mathrm{N}^1\mathrm{H}^3_{et}(\mathcal{X}_{\bar{\eta}_{\mathcal{S}}}, \mathbb{Z}/p\mathbb{Z}) = 0$$

where $\bar{\eta}_{\mathcal{S}}$ is the geometric generic point of $\mathcal{S}$ and $\mathcal{S}$ satisfies the hypothesis of the Theorem 4.2. Moreover, $\mathcal{X}_k$ is ordinary by construction, and (up to localizing S further) the Hodge groups are all torsion free. Finally, if the degree $d \geqslant 5$, then $\mathrm{H}^0(\mathcal{X}_k, \Omega^3_{\mathcal{X}_k}) \neq 0$. Therefore,

$$\mathrm{N}^1\mathrm{H}^3_{et}(\mathcal{X}_{\bar{\eta_S}}, \mathbb{Z}/p\mathbb{Z}) \neq \mathrm{H}^3_{et}(\mathcal{X}_{\bar{\eta_S}}, \mathbb{Z}/p\mathbb{Z}).$$

It is now enough to note that the monodromy action on $\mathrm{H}^3_{et}(X_{\bar{\eta_S}}, \mathbb{Z}/p\mathbb{Z})$ is absolutely irreducible. This is recalled in the following Lemma. □

The following Lemma is well known. We include it here since we could not find a reference in the literature.

Lemma 4.8. *Let $X \to S$ denote the universal hypersurface of degree d in $\mathbb{P}^4_{\mathbb{C}}$ and η denote the generic point of S. Then the monodromy action (i.e. the action of $Gal(\bar{\eta}/\eta)$) on $\mathrm{H}^3_{et}(X_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ is absolutely irreducible for all p.*

Proof. This is a direct consequence of ([6], 1.11). By loc. cit., for $s_0 \in S$, the action of the fundamental group $\pi_1(S, s_0)$ on the singular cohomology $\mathrm{H}^3(X_{s_0}, \mathbb{Z}/p\mathbb{Z})$ is irreducible. Since the action of $\pi_1(S, s_0)$ factors through its profinite completion, we see that the action of the etale fundamental group $\pi_1^{et}(S, s_0)$ on $\mathrm{H}^3_{et}(X_{s_0}, \mathbb{Z}/p\mathbb{Z})$ is irreducible. Since S is smooth, we have that the natural map $Gal(\bar{\eta}/\eta) \to \pi_1^{et}(S, s_0)$ is surjective. It follows that the induced action of $Gal(\bar{\eta}/\eta)$ on $\mathrm{H}^3(X_{s_0}, \mathbb{Z}/p\mathbb{Z})$ is irreducible. On the other hand, by the smooth and proper base change theorem one has a $Gal(\bar{\eta}/\eta)$-equivariant isomorphism $\mathrm{H}^i_{et}(X_{s_0}, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{H}^i_{et}(X_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$. The result now follows. □

Remark 9. The previous lemma also holds in the case of multidegree global complete intersections. This follows since Deligne's result ([6], 1.11) also holds in this generality.

4.4. **Some remarks in the general case.** In this sub-section, we discuss a partial generalization of the results of the last sub-section to the case of hyperplane sections of an arbitrary fourfold Y. In particular, let $Y \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective 4-fold. Let S denote the parameter space for smooth elements of $|\mathcal{O}_Y(a)|$, and let $\pi : X \to S$ denote the universal family. By the usual spreading technique, we can find a smooth projective model $\mathcal{Y} \to \mathcal{T}$, of relative dimension 4, of Y over $\mathcal{T} = \mathrm{Spec}(A)$ where $A \subset \mathbb{C}$ is a finitely generated $\mathbb{Z}$-algebra. We have a relatively ample bundle $\mathcal{O}_{\mathcal{Y}/\mathcal{T}}(1)$ on $\mathcal{Y}$ whose base change to $\mathbb{C}$ gives $\mathcal{O}_Y(1)$. The relative proj of $\mathcal{O}_{\mathcal{Y}/\mathcal{T}}(a)$ gives the parameter space for (relative) degree d hyperplane sections. Let $\mathcal{S}$ denote the locus corresponding to smooth hyperplane sections. Then $\mathcal{S}$ is a model of S over $\mathcal{T}$, and the incidence scheme $\mathcal{X} \hookrightarrow \mathcal{Y} \times_{\mathcal{T}} \mathcal{S}$ is a model for X. In particular, we have a diagram:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \times_{\mathcal{T}} \mathcal{S} \\ & \searrow^{\pi_{\mathcal{T}}} & \downarrow \\ & & \mathcal{S} \end{array}$$

which gives

$$\begin{array}{ccc} X & \longrightarrow & Y \times_{\mathbb{C}} S \\ & \searrow^{\pi} & \downarrow \\ & & S \end{array}$$

when we base change to $\mathbb{C}$. Moreover, we can assume that all the morphisms in the diagram are smooth. Finally, we note that up to further shrinking $\mathcal{T}$, we may assume that the Hodge cohomology sheaves $R^q\pi_*\Omega^p_{\mathcal{Y}/\mathcal{T}}$ as well as $R^q\pi_*\Omega^p_{\mathcal{X}/\mathcal{T}}$ are locally free. Given a closed point $s \in \mathrm{Spec}(A)$, we may pull back our diagram to the residue field $k(s)$. Note that this is a finite field. Finally, below we shall assume the following:

(**H**) The generic member of the family $\mathcal{X} \to \mathcal{S}$ has smooth ordinary specialization at all closed points in an open sub-set of $\mathrm{Spec}(A)$.

As before, we let η denote the generic point of S and $\bar{\eta}$ the geometric generic point.

Theorem 4.9. *Suppose Y is as above and that Y has a model satisfying the hypothesis* (**H**). *The the natural restriction map*

$$\mathrm{N}^1\mathrm{H}^3(Y_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{N}^1\mathrm{H}^3(X_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for almost all p if the degree a is large.

Before proving the Theorem, we state the following two lemmas which will be used in proving the Theorem.

Lemma 4.10. *Let Y be a smooth complex projective variety of dimension n. Then the Hard Lefschetz map*

$$\mathrm{H}^{n-1}(Y, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^{n+1}(Y, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for almost all p.

Proof. Note that the integral Hard Lefschetz map on homology

$$A := \mathrm{H}_{n+1}(Y, \mathbb{Z}) \to \mathrm{H}_{n-1}(Y, \mathbb{Z}) =: B$$

has finite torsion kernel and cokernel. Suppose p doesn't divide the product of the orders of the kernel and cokernel. Then the induced map

$$\mathrm{Hom}(B, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism. Moreover, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_n(Y, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-2}(Y, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) = 0$ for almost all primes p. It follows by the universal coefficient sequence that $\mathrm{H}^{n-1}(Y, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^{n+1}(Y, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism for almost all primes p.

□

Lemma 4.11. *Let X denote a smooth hyperplane section of a smooth projective variety $Y \subset \mathbb{P}^N_{\mathbb{C}}$ of dimension n. Then*

$$\mathrm{N}^1\mathrm{H}^{n-1}(X, \mathbb{Z}/p\mathbb{Z}) = \mathrm{N}^1\mathrm{H}^{n-1}(Y, \mathbb{Z}/p\mathbb{Z}) + \mathrm{N}^1\mathrm{H}^{n-1}_{ev}(X, \mathbb{Z}/p\mathbb{Z})$$

for almost all p.

Proof. This follows from the previous lemma since one has a commutative diagram with exact row:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^{n-1}_{ev}(X, \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}^{n-1}(X, \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{H}^{n+1}(Y, \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & 0 \\
 & & & & \uparrow & \nearrow & & & \\
 & & & & \mathrm{H}^{n-1}(Y, \mathbb{Z}/p\mathbb{Z}) & & & &
\end{array}$$

□

Proof of Theorem 4.9. We may proceed as in the proof of Theorem 4.5. In particular, it is enough to show that for almost all primes p (and $a >> 0$):

$$\mathrm{N}^1\mathrm{H}^3(\mathcal{Y}_{\bar{\eta}_{\mathcal{S}}}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{N}^1\mathrm{H}^3(\mathcal{X}_{\bar{\eta}_{\mathcal{S}}}, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism. Let $U \subset \mathrm{Spec}(A)$ denote the open subset as in the property (**H**) above. First note that, for almost all primes p, there is a closed point $s \in U$ with residue field of char. p. By arguing as before, we can find a complete DVR R with perfect residue field k and quotient field K with $\mathrm{Spec}(K) \to \eta_{\mathcal{S}}$ (i.e. $k(\eta_{\mathcal{S}}) \subset K$) such that the following holds: We have $X_R \subset Y_R$ smooth proper over R such that the base change of the pair $(\mathcal{X}_{\bar{\eta}_{\mathcal{S}}}, \mathcal{Y}_{\bar{\eta}_{\mathcal{S}}})$ to $\bar{K}$ is the base change of the pair (X_R, Y_R) to $\bar{K}$. As before, we are reduced to showing that the natural restriction map $\mathrm{N}^1\mathrm{H}^3(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{N}^1\mathrm{H}^3(Y_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism. We shall show that this is the case for almost all p under our assumptions.

Step 1: First, note that $\mathrm{N}^1\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$ is a strict subset of $\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$ for almost all p. To see this, we apply Theorem 4.3 to the present situation. The hypothesis (**H**) above guarantees that assumption (i) of that theorem is satisfied for almost all primes. Similarly, we may assume (ii) by taking primes large enough (since $\dim(X) = 3$, we may take $p \geqslant 5$).

Furthermore, by taking the degree of X to be large, we may also assume the assumption (iii) of that theorem holds. Finally, assumption (iv) of the theorem follows for almost all primes from Lemma 4.10. We can simply choose an algebraically closed field which contains both $\bar{K}$ and $\mathbb{C}$, and then apply the Lemma while noting that étale cohomology is invariant under algebraically closed field extensions.
Step 2: Next note that Lemma 4.11 (and arguing as above) implies that

$$\mathrm{N}^1\mathrm{H}^3(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) = \mathrm{N}^1\mathrm{H}^3(Y_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) + \mathrm{N}^1\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$$

for almost all p.
Step 3: Finally, the irreducibility of the monodromy action on vanishing cycles (for almost all primes p and X with large degree) implies that $\mathrm{N}^1\mathrm{H}^3_{ev}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all almost all primes p, and so the result follows from Step 2. □

Combining everything gives the following result:

Corollary 4.12. *Let Y be as above* $(\dim(Y) = 4)$*. Furthermore, suppose that the hypothesis* (**H**) *is satisfied. Then the natural restriction map*

$$\mathrm{CH}^2(Y)\{p\} \to \mathrm{CH}^2(X)\{p\}$$

is an isomorphism for very general X of large enough degree and almost all primes p.

As remarked earlier, Illusie's Theorem 4.1 allows one, in principal, to construct some examples where (**H**) is satisfied. This is the case if Y is a quadric or if it is $\mathbb{P}^4$ (which was already considered before). More generally, Illusie's theorem implies that for any smooth complete intersection $Y \subset \mathbb{P}^N$ its generic hyperplane section is ordinary. In particular, one has the following corollary:

Corollary 4.13. *Let $Y \subset \mathbb{P}^N_{\mathbb{C}}$ denote a smooth complete intersection of multidegree $(d_1, \ldots, d_k)$ of* $\dim(Y) = 4$*, and $X \subset Y$ a very general hyperplane section of high degree. If Y is general, then $\mathrm{CH}^2(X)\{p\} = 0$ for almost all primes p.*

Proof. By the previous remarks and Corollary 4.12, the restriction map

$$\mathrm{CH}^2(Y)\{p\} \to \mathrm{CH}^2(X)\{p\}$$

is an isomorphism for almost all primes p and X very general of high enough degree. On the other hand, by Corollary 3.4 applied to $Y = \mathbb{P}^N$ and $X = Y$, we have that for a general Y as above, $\mathrm{CH}^2(Y)\{p\} = 0$. □

Remark 10. Suppose that we start with a complete intersection Y as in Corollary 4.13 which is defined over $\mathbb{Q}$. Then the set of primes which might be 'problematic' (i.e. where $\mathrm{CH}^2(X)\{p\} \neq 0$ for X very general of high degree in $Y_{\mathbb{C}}$) are precisely the primes where Y has bad reduction. While the result should still be true in those cases, the strategy of proof employed above does not allows us to conclude anything for such primes.

It would also be interesting to find examples of fourfolds Y which satisfy hypothesis (**H**) and have non-vanishing $\mathrm{H}^3(Y)$.

5. Torsion in the Griffiths group

For a smooth, projective variety V over $\mathbb{C}$, and for any integer $0 \leqslant i \leqslant \dim V$, recall that one has cycle class maps into Betti cohomology

$$\mathrm{cl}^i : \mathrm{CH}^i(V) \to \mathrm{H}^{2i}(V, \mathbb{Z}).$$

Let $\mathrm{CH}^i_{\mathrm{hom}}(V)$ denote the kernel of the above map; this is the subgroup of cycle classes modulo *homological equivalence*. Besides this and the notion of *rational equivalence*, a third important equivalence relation for cycles is the notion of *algebraic equivalence*. We say two codimension i cycles Z_1 and Z_2 are *algebraically equivalent* to each other, written $Z_1 \sim_{alg} Z_2$, if there exists a curve C, and a cycle $\mathcal{Z}$ on $V \times C$, flat over C, such that for two points $c_1, c_2 \in C$, we have $[Z_1] - [Z_2] = [\mathcal{Z} \cap (X \times \{c_1\})] - [\mathcal{Z} \cap (X \times \{c_2\})]$. Let $\mathrm{A}^i(V)$ denote the group of cycle classes modulo algebraic equivalence. It is a standard fact that $\mathrm{A}^i(V) \subset \mathrm{CH}^i_{\mathrm{hom}}(V)$, and the quotient is called the *Griffiths group*:

$$0 \to \mathrm{A}^i(Z) \to \mathrm{CH}^i_{\mathrm{hom}}(Z) \to \mathrm{Grif}^{\,i}(Z) \to 0.$$

5.1. **The weak Lefschetz theorem for torsion in the Griffiths group.** We set up some notation and recall some results. For a smooth projective variety Z:

- $\mathrm{A}^2(Z)$ will denote the subgroup of cycles algebraically equivalent to zero in $\mathrm{CH}^2(Z)$.
- $\mathrm{J}^2(Z) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2), \mathrm{H}^3(Z))$ will denote the Griffiths intermediate Jacobian.
- $\mathrm{J}^2_0(Z)$ will denote the intermediate Jacobian of the largest integral Hodge structure in $\mathrm{H}^{1,2}(Z) \oplus \mathrm{H}^{2,1}(Z)$. Equivalently, $\mathrm{J}^2_0(Z)$ is the *largest abelian variety contained in* $\mathrm{J}^2(Z)$.

Proposition 5.1. *Let $Y \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth, projective variety and X be a general ample hyperplane section in Y. If $\dim Y \geqslant 5$, then the restriction map*

$$\mathrm{A}^2(Y)\{p\} \to \mathrm{A}^2(X)\{p\}$$

is an isomorphism for all primes p. If $\dim(Y) = 4$, then the restriction map above is an injection for all primes p.

Proof. We first claim that the hypothesis will ensure that the restriction map

$$\mathrm{CH}^2_{\mathrm{hom}}(Y)\{p\} \to \mathrm{CH}^2_{\mathrm{hom}}(X)\{p\} \tag{5.1.1}$$

is an isomorphism if $\dim(Y) \geqslant 5$ and an injection if $\dim(Y) = 4$. The injectivity follows since the image of the composition

$$\mathrm{CH}^2_{\mathrm{hom}}(Y)\{p\} \hookrightarrow \mathrm{CH}^2(Y)\{p\} \xrightarrow{\cong} \mathrm{CH}^2(X)\{p\}$$

lands in $\mathrm{CH}^2_{\mathrm{hom}}(X)\{p\}$. For surjectivity, suppose $\dim(Y) \geqslant 5$, and let $\xi \in \mathrm{CH}^2_{\mathrm{hom}}(X)\{p\} \subset \mathrm{CH}^2(X)\{p\}$. Then ξ lifts to a cycle $\xi' \in \mathrm{CH}^2(Y)\{p\}$ by Corollary 3.4. On the other hand, the usual weak Lefschetz now shows that ξ' must be homologically equivalent to zero. Now, the isomorphism in 5.1.1 implies that $\mathrm{A}^2(Y)\{p\}$ injects into $\mathrm{A}^2(X)\{p\}$. Hence we need to prove that $\mathrm{A}^2(Y)\{p\} \to \mathrm{A}^2(X)\{p\}$ is a surjection. By Theorem 10.3 [13], this in turn is equivalent to proving that the injection $\mathrm{J}^2_0(Y)\{p\} \to \mathrm{J}^2_0(X)\{p\}$ is also a surjection. On the other hand, $\mathrm{J}^2(Y) \to \mathrm{J}^2(X)$ is an isomorphism by the weak Lefschetz theorem.

∎

Proof of Theorem 1.7. For a smooth projective variety Z, we first claim that the exact sequence

$$0 \to \mathrm{A}^2(Z) \to \mathrm{CH}^2_{\mathrm{hom}}(Z) \to \mathrm{Grif}^2(Z) \to 0$$

restricts to an exact sequence

$$0 \to \mathrm{A}^2(Z)_{\mathrm{tors}} \to \mathrm{CH}^2_{\mathrm{hom}}(Z)_{\mathrm{tors}} \to \mathrm{Grif}^2(Z)_{\mathrm{tors}} \to 0.$$

Observe that it is enough to prove the sequence is right exact. So let $z \in \mathrm{CH}^2_{\mathrm{hom}}(Z)$ be such that $\bar{z} := z + A^2(Z)$ is a non-trivial torsion element in $\mathrm{Grif}^2(Z)$. Then there exists an $N > 0$ such that $Nz \in \mathrm{A}^2(Z)$. If $Nz \neq 0$, then since $\mathrm{A}^2(Z)$ is divisible, this means that $z \in \mathrm{A}^2(Z)$ which in turn will imply $\bar{z} = 0$, a contradiction. Thus any lift of a cycle in $\mathrm{Grif}^2(Z)_{\mathrm{tors}}$ lies in $\mathrm{CH}^2_{\mathrm{hom}}(Z)_{\mathrm{tors}}$.

One concludes the proof by applying the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{A}^2(Y)_{\mathrm{tors}} & \to & \mathrm{CH}^2_{\mathrm{hom}}(Y)_{\mathrm{tors}} & \to & \mathrm{Grif}^2(Y)_{\mathrm{tors}} & \to & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ 0 & \to & \mathrm{A}^2(X)_{\mathrm{tors}} & \to & \mathrm{CH}^2_{\mathrm{hom}}(X)_{\mathrm{tors}} & \to & \mathrm{Grif}^2(X)_{\mathrm{tors}} & \to & 0, \end{array}$$

where the two left most vertical maps are isomorphisms if $\dim(Y) \geqslant 5$ by Proposition 5.1 and its proof. □

Remark 11. Let $\dim Y = 4$, and assume that $\mathrm{CH}^2(Y)_{\mathrm{tors}} \subset \mathrm{CH}^2(Y)_{\mathrm{hom}}$ – for eg. assume $\mathrm{H}^4(Y, \mathbb{Z})_{\mathrm{tors}} = 0$. Then with hypotheses as in Theorem 1.6, one also obtains the following Noether-Lefschetz theorem: for X a very general, sufficiently ample hypersurface in Y, the restriction map $\mathrm{Grif}^2(Y)\{p\} \to \mathrm{Grif}^2(X)\{p\}$ is an isomorphism for almost all primes p. The proof is similar to the proof of Theorem 1.7.

5.2. **A question of Nori.** Griffiths' Abel-Jacobi map $\theta : \mathrm{CH}^2_{\mathrm{hom}}(Z) \to \mathrm{J}^2(Z)$ descends to a map $\overline{\theta} : \mathrm{Grif}^2(Z) \to \mathrm{J}^2(Z)/\mathrm{J}^2_0(Z)$. Nori (see [14]) asks if $\overline{\theta}$ is an injection. The techniques used in the above subsection yields a partial answer to this question.

Proposition 5.2. *Let Z be any smooth, projective variety over $\mathbb{C}$. The map $\overline{\theta}$ when restricted to the torsion subgroup $\mathrm{Grif}^2(Z)_{\mathrm{tors}}$ is an injection.*

Proof. Consider the diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{A}^2(Z)_{\mathrm{tors}} & \to & \mathrm{CH}^2_{\mathrm{hom}}(Z)_{\mathrm{tors}} & \to & \mathrm{Grif}^2(Z)_{\mathrm{tors}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{J}^2_0(Z)_{\mathrm{tors}} & \to & \mathrm{J}^2(Z)_{\mathrm{tors}} & \to & (\mathrm{J}^2(S)/\mathrm{J}^2_0(Z))_{\mathrm{tors}} & \to & 0. \end{array}$$

Here the vertical maps are the Abel-Jacobi maps. The left most vertical map is bijective by [13], Theorem 10.3, and the middle vertical is injective by [5], Corollaire 5. The injectivity of $\overline{\theta}$ on $\mathrm{Grif}^2(Z)_{\mathrm{tors}}$ follows. □

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