RANK 3 ACM BUNDLES ON GENERAL HYPERSURFACES IN $\mathbb{P}^5$

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ABSTRACT. We prove that a general hypersurface in $\mathbb{P}^5$ of degree $d \geqslant 3$ does not support an indecomposable rank 3 arithmetically Cohen-Macaulay (ACM) bundle. This settles the base case of a generic version of a conjecture of Buchweitz, Greuel and Schreyer.

1. INTRODUCTION

We work over an algebraically closed field of characteristic zero.

Let Y be a smooth, projective variety and $\mathcal{O}_Y(1)$ denote an ample line bundle. The Lefschetz theorems state that for any smooth hyperplane section $X \subset Y$, we have an isomorphism of Picard groups $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X)$ provided that $\dim X \geqslant 3$. When $\dim X = 2$, the above isomorphism holds if, in addition, $\mathcal{O}_Y(1)$ is *sufficiently ample* and X is *very general* (i.e., X belongs to the complement of a countable union of closed subvarieties of the parameter space $|\mathcal{O}_Y(1)|$).

One may view these theorems as providing conditions under which line bundles on X extend to line bundles on Y. Our motivation behind the results of this article is to formulate a generalisation of the Lefschetz theorems to higher rank bundles when $Y = \mathbb{P}^{n+1}$. To do so, we restrict ourselves to the class of *arithmetically Cohen Macaulay* (ACM) bundles on X. Recall that a bundle E on X is said to be ACM if

$$H^i_*(X, E) := \bigoplus_{a \in \mathbb{Z}} H^i(X, E(a)) = 0, \ \ 0 < i < \dim X.$$

ACM bundles are ubiquitous – for any smooth hypersurface $X \subset \mathbb{P}^{n+1}$, and a bundle E, let $\overline{F}_0 \to E$ be a map from a sum of line bundles to E such that the map of global sections $H^0_*(X, \overline{F}_0) \to H^0_*(X, E)$ is surjective. Now consider a minimal resolution of the form

$$0 \to E_{n-1} \to \overline{F}_{n-2} \to \overline{F}_{n-3} \to \cdots \to \overline{F}_0 \to E \to 0, \tag{1}$$

where $\overline{F}_i := \oplus_j \mathcal{O}_X(a_{ij})$ for $0 \leqslant i < n - 1$, and such that if $E_{i+1} := \mathrm{Image}(\overline{F}_{i+1} \to \overline{F}_i)$ is the $(i+1)$-st syzygy bundle, then $H^0_*(X, \overline{F}_{i+1}) \to H^0_*(X, E_{i+1})$ is surjective. It follows then that E_{n-1} is ACM.

In the context of ACM bundles, the following conjecture due to Buchweitz, Greuel and Schreyer [4] has been a guiding light:

Conjecture 1 (BGS conjecture). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Any ACM bundle E on X of rank $r < 2^e$ for $e := \left\lfloor \frac{n-1}{2} \right\rfloor$, is a sum of line bundles.*

The first non-trivial instances of the BGS conjecture for higher degree hypersurfaces are when $n = 3$ and $n = 4$; in these cases, the conjecture predicts that any ACM line bundle is the restriction of a line bundle on $\mathbb{P}^{n+1}$ and hence follows from the *Grothendieck-Lefschetz theorem*. When $n \geqslant 5$, the conjecture states that any ACM bundle of rank $r < 4$ is a sum of line

bundles. The case of rank 2 bundles was first proved by Kleppe [12] and independently as part of a more general theorem in [13]. More recently, [23] (and independently, [16]) establishes this result for rank 3 ACM bundles thereby settling another higher rank case of this conjecture. The conjecture was established for quadric hypersurfaces in [11].

The BGS conjecture may be viewed as an extension theorem in the spirit of the Lefschetz theorems for Picard groups. To see this, one notes that if an ACM bundle E extends to a bundle $\widetilde{E}$ on $\mathbb{P}^{n+1}$, then $\widetilde{E}$ is also ACM, and hence by Horrocks' result [9], is a sum of line bundles. Consequently, we regard this conjecture as providing us a higher rank analogue of the Grothendieck-Lefschetz theorem. This point of view immediately suggests that one ought to have an analogue of the Noether-Lefschetz theorem for ACM bundles on projective hypersurfaces of sufficiently high degree as well. Furthermore, the results in [13, 14, 15, 17, 3], and especially the techniques used in [17] and the results proved in [15] draw a direct connection with traditional Noether-Lefschetz theory and its generalisations. The BGS conjecture and various results referred to above also suggest that it seems likely that if there are no non-split ACM bundles of rank $r \leqslant 2^s$ for some s, then there are none of rank $r < 2^{s+1}$. Putting all this together, and as a first step, we propose the following

Conjecture 2 (Generic BGS conjecture). *Let $X \subset \mathbb{P}^{n+1}$ be a general[1] hypersurface of sufficiently high degree and E be an ACM bundle of rank r on X. If $r < 2^s$, where $s := \left\lfloor \frac{n+1}{2} \right\rfloor$, then E is a sum of line bundles.*

The above was posed as a question in [21] where the case of rank 3 ACM bundles on hypersurfaces in $\mathbb{P}^5$ was proved under the additional hypothesis that the bundles have fewer than 8 generators. In this article, we prove that result in complete generality. Consequently, the main result here, and the corresponding result for rank 2 ACM bundles proved in [13], and by different methods in [17] (this uses the results in [5]), prove the base case of the above conjecture for hypersurfaces in $\mathbb{P}^5$. A more optimistic conjecture can be found in [6].

Here is a precise statement of our main results:

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface.*

  (i) *If $n \geqslant 5$, then any ACM rank 3 bundle on X is a sum of line bundles.*
  (ii) *If $n = 4$, and X is general of degree $d \geqslant 3$, then any ACM rank 3 bundle on X is a sum of line bundles.*
  (iii) *If $n = 3$, and X is general of degree $d \geqslant 5$, then any ACM rank 3 bundle E on X is rigid, i.e., $H^1(X, \mathcal{E}nd E) = 0 = H^2(X, \mathcal{E}nd E)$.*

The above result extends, word for word, the results in [13] where these statements were established for rank 2 ACM bundles. Indeed, the ideas introduced in *op. cit.* serve as the basis for the proof here as well. However, the proof here is far more technical, and depends on some key insights into the structure of vector bundles on hypersurfaces, their resolutions and filtrations involving their exterior powers. In fact, the results in §3 and in §4 hold for any vector bundle on a hypersurface. Part (i) of the theorem was earlier proved by the second author (see [23]) and relied on a criterion due to Huneke and Wiegand (see [10]). The proof here is self-contained and is a consequence of our method which proves results for rank 3 ACM bundles on hypersurfaces in $\mathbb{P}^5$ and $\mathbb{P}^4$ as well.

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[1]A Noether-Lefschetz type statement is usually stated for a *very general* hypersurface; i.e., a hypersurface which is parametrised by a point outside a countable union of closed, proper subvarieties in the parameter space. However, it follows from Lemma 3.3 in [13] that there are only finitely many closed subvarieties which parametrise indecomposable ACM bundles of a fixed rank. Thus we may replace *very general* by general in the statement of Conjecture 2.

For related work and a brief history of prior results, we refer the reader to [13], [17], [23] and the references therein.

Outline of the proof. Let X be a smooth hypersurface in $\mathbb{P}^{n+1}$, $n \geqslant 3$, and E be a rank 3 ACM bundle on X. By Ex. II.5.16 of [8], we note that for a short exact sequence of vector bundles $0 \to E' \to F \to E'' \to 0$, there is a decreasing filtration on $\wedge^r F$,

$$\wedge^r F = E_{r,r} \supseteq E_{r,r-1} \supseteq \cdots \supseteq E_{r,0} = \wedge^r E' \supseteq E_{r,-1} = 0$$

with associated graded pieces $Q_{r,i,i-1} \cong \wedge^i E' \otimes \wedge^{r-i} E''$.

If we let $0 \to G \to \overline{F}_0 \to E \to 0$ be a minimal resolution of E (see §2 for details), then we have the following commutative diagram for the third exterior powers:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & E_{3,1} & = & E_{3,1} & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & E_{3,2} & \to & \wedge^3 \overline{F}_0 & \to & \wedge^3 E & \to & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \to & \wedge^2 E \otimes G & \to & Q_{3,3,1} & \to & \wedge^3 E & \to & 0. \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
\tag{2}
$$

Since $\operatorname{rank} E = 3$, $\wedge^3 E \cong \mathcal{O}_X(c)$, where $c = c_1(E) \in \operatorname{Pic}(X) \cong \mathbb{Z}$. Taking cohomology, we get a commutative square from the middle and bottom rows above:

$$
\begin{array}{ccc}
H^0_*(X, \wedge^3 E) & \twoheadrightarrow & H^1_*(X, E_{3,2}) \\
\| & & \downarrow \\
H^0_*(X, \wedge^3 E) & \to & H^1_*(X, \wedge^2 E \otimes G).
\end{array}
$$

Here the horizontal arrows are the coboundary maps for the two rows in the commutative diagram above.

The right vertical arrow is a surjection provided that

$$H^2_*(X, E_{3,1}) = 0. \tag{3}$$

Assuming this, we see that the bottom horizontal arrow is then a surjection. This implies that $H^1_*(X, \wedge^2 E \otimes G)$ is generated as a graded module by the image of $1 \in H^0_*(X, \wedge^3 E)$ under this map. We denote this as $\zeta \in H^1(X, \wedge^2 E \otimes G(-c))$ where $c = c_1(E)$. It turns out that the element ζ is the Yoneda class of the sequence $0 \to G \to \overline{F}_0 \to E \to 0$ under the identification

$$H^1(X, \wedge^2 E(-c) \otimes G) \cong H^1(X, E^\vee \otimes G) \cong \operatorname{Ext}^1(E, G).$$

By Remark 1 in §2.2, $H^1_*(X, \wedge^2 E \otimes G) \cong H^2_*(X, \mathcal{E}nd E(c - d))$ and so the latter is also an 1-generated module. Thus by Corollary 3.8 of [13] or Theorem 2 of [19], the fact that E is supported on a general hypersurface of degree $d \geqslant 3$ means that $H^2(X, \mathcal{E}nd E(a)) = 0$ for $a \geqslant 0$. However, when $\dim X = 4$, we have, $H^2(X, \mathcal{E}nd E(-d)) \cong H^2(X, \mathcal{E}nd E(2d - 6))$ by Serre duality. Thus, when $2d - 6 \geqslant 0$, or equivalently, $d \geqslant 3$, we see that this group vanishes and so $\zeta = 0$.

Results for $\dim X = 3$ and $\dim X \geqslant 5$ require a complete analysis of cohomologies of various graded pieces. This is done in Theorem 5.7 where using the vanishing in (3), we examine the filtered piece $E_{3,0} \cong \wedge^3 G$ and the diagram obtained from the inclusion $E_{3,0} \to E_{3,2}$; see diagram (68). When $\dim X \geqslant 5$, we show that the group $H^2(X, \mathcal{E}nd E(-d))$ vanishes

for any smooth X in Theorem 6.1. The vanishing of this cohomology group implies that the sequence $0 \to G \to \overline{F}_0 \to E \to 0$ splits, and hence E is a sum of line bundles.

The technical heart of this paper lies in sections 3, 4, and 5 which contain various results which are used in proving Theorem 5.7.

2. PRELIMINARIES

We recall some standard results here. More details can be found in §3 of [19] and §2 of [13].

2.1. Generalities about vector bundles on hypersurfaces.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d with ideal sheaf $\mathcal{I} := \mathcal{O}_{\mathbb{P}}(-d)$. We let X_r be the r-fold thickening of X in $\mathbb{P}^{n+1}$ with sheaf of ideals $\mathcal{I}^r$. Let E be a bundle on X of rank u. Since $\dim X \geqslant 3$, we have $\mathrm{Pic}(X) \cong \mathbb{Z}$ by the Grothendieck-Lefschetz theorem. Using this isomorphism, we let $c := c_1(E) \in \mathbb{Z}$, so that $\wedge^u E \cong \mathcal{O}_X(c)$. The bundle E has a minimal resolution over $\mathbb{P}^{n+1}$ of the form

$$0 \to \widetilde{F}_1 \xrightarrow{\Phi} \widetilde{F}_0 \to E \to 0, \tag{4}$$

where $\widetilde{F}_0$ is sum of line bundles and $\widetilde{F}_1$ is a bundle on $\mathbb{P}^{n+1}$. When E is ACM, an application of Horrocks' splitting criterion shows that $\widetilde{F}_1$ is also a sum of line bundles.

Restricting the above resolution to X, we get a 4-term exact sequence

$$0 \to E(-d) \to \overline{F}_1 \xrightarrow{\overline{\Phi}} \overline{F}_0 \to E \to 0.$$

Let $G := \mathrm{Image}(\overline{\Phi})$. Breaking this up into short exact sequences, we get

$$0 \to G \to \overline{F}_0 \to E \to 0 \text{ and,} \tag{5}$$

$$0 \to E(-d) \to \overline{F}_1 \to G \to 0. \tag{6}$$

When E is an ACM bundle, G is also an ACM bundle. The surjection $\overline{F}_1 \twoheadrightarrow G$ lifts to a surjective map $\widetilde{F}_1 \twoheadrightarrow G$ and gives rise to the following minimal resolution of G:

$$0 \to \widetilde{F}_0(-d) \xrightarrow{\Psi} \widetilde{F}_1 \to G \to 0. \tag{7}$$

Dualizing (4), we get

$$0 \to \widetilde{F}_0^{\vee} \xrightarrow{\Phi^t} \widetilde{F}_1^{\vee} \to E^{\vee}(d) \to 0, \tag{8}$$

and on restricting to X, we get the exact sequences

$$0 \to G^{\vee} \to \overline{F}_1^{\vee} \to E^{\vee}(d) \to 0 \text{ and,} \tag{9}$$

$$0 \to E^{\vee} \to \overline{F}_0^{\vee} \to G^{\vee} \to 0. \tag{10}$$

2.2. The extension class $\zeta \in \mathrm{Ext}_X^1(E, G) \cong H^1(X, E^{\vee} \otimes G)$.

The exact sequence

$$0 \to G \to \overline{F}_0 \to E \to 0$$

defines an element $\zeta \in \mathrm{Ext}_X^1(E, G) \cong H^1(X, E^{\vee} \otimes G)$. It is clear that $\zeta = 0$ if and only if this sequence splits. This, by the Krull-Schmidt theorem [1], (see also Remarks 1 and 2 below), is equivalent to the splitting of E (and G). Tensoring the above sequence with $E^{\vee}$, and taking cohomology, we obtain the following long exact sequence of cohomology:

$$0 \to H^0(X, G \otimes E^{\vee}) \to H^0(X, \overline{F}_0 \otimes E^{\vee}) \to H^0(X, E \otimes E^{\vee}) \to H^1(X, G \otimes E^{\vee}) \to \cdots.$$

It is standard that, under the coboundary map,

$$H^0(X, E \otimes E^{\vee}) \to H^1(X, G \otimes E^{\vee}),$$

the identity 1 is mapped to the element ζ.

Similarly, tensoring with $E^\vee$, the sequence

$$0 \to E(-d) \to \overline{F}_1 \to G \to 0,$$

and taking cohomology, we get a boundary map $H^1(X, E^\vee \otimes G) \to H^2(X, \mathcal{E}nd E(-d))$. Under this map, ζ is mapped to the element η, where η is the *obstruction class* of E (see Remark 2 below).

Remark 1. When $\dim X \geqslant 3$, the boundary map $H^1_*(X, E^\vee \otimes G) \to H^2_*(X, \mathcal{E}nd E(-d))$ is an isomorphism.

Remark 2. For an arbitrary bundle E on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, the vanishing of the class $\eta \in H^2(X, \mathcal{E}nd E(-d))$ is necessary and sufficient for E to extend to a bundle E_2 on X_2, the second order thickening of X (see, for instance, [19] for details). In the case when E is an ACM bundle (of arbitrary rank), it is shown in [15], by elementary arguments, that E splits if and only if E extends to a bundle E_2 on X_2. This fact was used to generalize Voisin's counter example (see [24]) to a generalised Noether-Lefschetz conjecture of Griffiths and Harris (see [7]).

2.3. **Two cokernel sheaves.** Notice that, affine locally on $\mathbb{P}^{n+1}$, the map Φ in sequence (4) is the diagonal matrix $(f, \cdots, f, 1, \cdots, 1)$ where the first u entries are all f's, and the remaining $m - u$ entries are 1's, so that (4) is of the form

$$0 \to R^m \xrightarrow{\text{diag}(f, \cdots, f, 1, \cdots, 1)} R^m \to (R/(f))^u \to 0.$$

Here $\text{Spec}(R)$ is an affine open set in $\mathbb{P}^{n+1}$ and $m = \text{rank}(F_0) = \text{rank}(F_1)$.

The r-th exterior power of Φ has the following local description:

$$\wedge^r \Phi := \text{diag}(f^r, f^{r-1}, \cdots, f^{r-1}, f^{r-2}, \cdots, f^{r-2}, f^{r-3}, \cdots, 1),$$

where the number of f^{r-i}'s along the diagonal is $\binom{u}{r-i}\binom{m-u}{i}$. This is a consequence of the fact that an r-fold product of the diagonal entries consists of choosing $(r - i)$ f's and the remaining i 1's from among u f's and $(m - u)$ 1's.

We also have a similar description for the map Ψ in (7) and its exterior powers.

Letting $\mathcal{E}_r$ and $\mathcal{G}_r$ denote the cokernels of $\wedge^r \Phi$ and $\wedge^r \Psi$ respectively, we get the exact sequences

$$0 \to \wedge^r \widetilde{F}_1 \to \wedge^r \widetilde{F}_0 \to \mathcal{E}_r \to 0, \quad \text{and} \tag{11}$$

$$0 \to \wedge^r \widetilde{F}_0(-rd) \to \wedge^r \widetilde{F}_1 \to \mathcal{G}_r \to 0. \tag{12}$$

Definition 1. We will say that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ has no intermediate cohomology if

$$H^i_*(\mathbb{P}^n, \mathcal{F}) = 0, \quad 0 < i < \dim \text{Supp}(\mathcal{F}).$$

Remark 3. When E is an ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ then the sheaves $\mathcal{E}_r$ and $\mathcal{G}_r$ have no intermediate cohomology.

We also recall the following result:

Lemma 2.1. *Let* $\text{rank} \, F_0 = m$. *The sheaves* $\mathcal{E}_r$ *and* $\mathcal{G}_r$ *are supported on* X_r, *the r-fold thickening of* X. *Moreover, affine locally on* $\mathbb{P}^{n+1}$, $\mathcal{E}_r$ *is of the form*

$$\mathcal{O}_{X_r}^{\oplus \binom{u}{r}} \oplus \mathcal{O}_{X_{r-1}}^{\oplus \binom{u}{r-1} \cdot \binom{m-u}{1}} \oplus \ldots \oplus \mathcal{O}_{X_{r-i}}^{\oplus \binom{u}{r-i} \cdot \binom{m-u}{i}} \ldots \oplus \mathcal{O}_X^{\oplus \binom{u}{1} \cdot \binom{m-u}{r-1}}.$$

Proof. The proof follows using the local description of the map $\wedge^r \Phi$ above. For details, see Lemma 3.1 of [23]. $\qquad\square$

2.3.1. *A convention.* A tilde e.g. $\widetilde{\mathsf{F}}$ will be used to denote a vector bundle or a sheaf on projective space $\mathbb{P}^{n+1}$. The restriction to the hypersurface will be denoted as $\mathsf{F}|_X$ or $\overline{\mathsf{F}}$.

3. TWO FILTRATIONS

Let Y be a smooth projective variety and let $X \subset Y$ be a smooth hypersurface cut out by a section $f \in \mathcal{O}_Y(d)$ where $\mathcal{O}_Y(1)$ is an ample line bundle. Let E be a vector bundle on X. Then E has a minimal resolution of the form $0 \to \widetilde{\mathsf{F}}_1 \to \widetilde{\mathsf{F}}_0 \to E \to 0$, where as before $\widetilde{\mathsf{F}}_0 \cong \oplus \mathcal{O}_Y(a_i)$, and $\widetilde{\mathsf{F}}_1$ is an ACM vector bundle on $(Y, \mathcal{O}_Y(1))$. Restricting this sequence to X gives us, as before, exact sequences (5) and (6). Associated to these exact sequences are two filtrations on the r-th exterior power of the bundles $\overline{\mathsf{F}}_0$ and $\overline{\mathsf{F}}_1$ which we recall now.

3.1. **The first filtration.** On $\wedge^r \overline{\mathsf{F}}_0$, we have the following filtration via sequence (5):

$$\wedge^r G = E_{r,0} \subset E_{r,1} \subset \ldots \subset E_{r,r-1} \subset E_{r,r} = \wedge^r \overline{\mathsf{F}}_0,$$

such that for $i > j$, if we set

$$\mathcal{Q}_{r,i,j} := \operatorname{coker}(E_{r,j} \to E_{r,i}),$$

then in particular, we have (see Ex. II.5.16 of [8])

$$\mathcal{Q}_{r,i,i-1} = \operatorname{coker}(E_{r,i-1} \to E_{r,i}) = \wedge^i E \otimes \wedge^{r-i} G. \tag{13}$$

Thus we have diagrams (for $r \geqslant i > j > k$):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
E_{r,k} & === & E_{r,k} \\
\downarrow & & \downarrow \\
0 \longrightarrow E_{r,j} \longrightarrow E_{r,i} \longrightarrow \mathcal{Q}_{r,i,j} \longrightarrow 0 \\
\downarrow & & \downarrow & \| \\
0 \longrightarrow \mathcal{Q}_{r,j,k} \longrightarrow \mathcal{Q}_{r,i,k} \longrightarrow \mathcal{Q}_{r,i,j} \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array} \tag{14}$$

3.2. **The second filtration.** On $\wedge^r \overline{\mathsf{F}}_1$, we have a similar filtration via sequence (6):

$$\wedge^r E(-rd) = G_{r,0} \subset G_{r,1} \subset \ldots \subset G_{r,r-1} \subset G_{r,r} = \wedge^r \overline{\mathsf{F}}_1,$$

where, for $i > j$, we set $\mathcal{P}_{r,i,j} := \operatorname{coker}(G_{r,j} \to G_{r,i})$. Here again, we have

$$\mathcal{P}_{r,i,i-1} = \wedge^i G \otimes \wedge^{r-i} E(-(r-i)d).$$

Remark 4. For filtrations obtained from the dual sequences (9) and (10), we will denote by $\mathcal{P}'_{r,i,j}$ and $\mathcal{Q}'_{r,i,j}$, the respective associated graded pieces.

Lemma 3.1. *Let* $\operatorname{rank} \mathsf{F}_0 = m$. *Then*
(1) $\operatorname{rank} E_{r,i} = \binom{u}{i}\binom{m-u}{r-i} + \binom{u}{i-1}\binom{m-u}{r-i+1} + \cdots + \binom{u}{1}\binom{m-u}{r-1} + \binom{m-u}{r}$,
(2) $\operatorname{rank} G_{r,i} = \binom{m-u}{i}\binom{u}{r-i} + \binom{m-u}{i-1}\binom{u}{r-i+1} + \cdots + \binom{m-u}{1}\binom{u}{r-1} + \binom{u}{r}$,
(3) $\operatorname{rank} \mathcal{Q}_{r,i,j} = \binom{u}{i}\binom{m-u}{r-i} + \binom{u}{i-1}\binom{m-u}{r-i+1} + \cdots + \binom{u}{j+1}\binom{m-u}{r-j-1}$, *and*

(4) $\operatorname{rank} \mathcal{P}_{r,i,j} = \binom{m-u}{i}\binom{u}{r-i} + \binom{m-u}{i-1}\binom{u}{r-i+1} + \cdots + \binom{m-u}{j+1}\binom{u}{r-j-1}$.

Proof. We will prove the first part. The rest follow in a similar fashion. The first and second filtrations locally yield a direct sum decomposition of the exterior powers of the bundles $\overline{F}_0$ and $\overline{F}_1$. More precisely, there is a covering of X by affine open subsets such that for each open set $U \subset X$ in this covering, one has $\overline{F}_0 \otimes \mathcal{O}_U \cong E_U \oplus G_U$, where E_U (respectively G_U) is the restriction of E (respectively G) to U. In this case, we have

$$E_{r,i} \otimes \mathcal{O}_U \cong \bigoplus_{p=0}^{i} \wedge^p E_U \otimes \wedge^{r-p} G_U.$$

The rank computation now follows. $\qquad\square$

3.3. **Some multilinear algebra.**
Recall that, for any vector bundle V on X, we have a natural map

$$\omega_{k,k-1} : \wedge^k V \to \wedge^{k-1} V \otimes V.$$

Since a similar map exists for the dual bundle $V^\vee$, it follows that the above map is a split injection. Thus we also have a split surjection

$$\omega'_{k,k-1} : \wedge^{k-1} V \otimes V \to \wedge^k V.$$

We have similar maps involving the dual $V^\vee$ as well.

The surjection $\overline{F}_0 \twoheadrightarrow E$ gives rise to the commutative square

$$
\begin{array}{ccc}
\wedge^r \overline{F}_0 & \twoheadrightarrow & \wedge^r E \\
\downarrow & & \downarrow \\
\wedge^{r-1} \overline{F}_0 \otimes \overline{F}_0 & \twoheadrightarrow & \wedge^{r-1} E \otimes E.
\end{array}
$$

The bottom horizontal map factors via $\wedge^{r-1} E \otimes \overline{F}_0$, where the map $\wedge^{r-1} E \otimes \overline{F}_0 \to \wedge^{r-1} E \otimes E$ is induced by the surjection $\overline{F}_0 \twoheadrightarrow E$. Hence we have a diagram

$$
\begin{array}{ccccccccc}
0 & \to & E_{r,r-1} & \to & E_{r,r} & \to & \wedge^r E & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^{r-1} E \otimes G & \to & \wedge^{r-1} E \otimes \overline{F}_0 & \to & \wedge^{r-1} E \otimes E & \to & 0.
\end{array}
\tag{15}
$$

The kernel of the left vertical map is $E_{r,r-2}$ and so it yields the push-forward diagram

$$
\begin{array}{ccccccccc}
0 & \to & E_{r,r-1} & \to & E_{r,r} & \to & \wedge^r E & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \wedge^{r-1} E \otimes G & \to & \mathcal{Q}_{r,r,r-2} & \to & \wedge^r E & \to & 0.
\end{array}
\tag{16}
$$

By the universal property of push-forward diagrams, we see that the vertical maps in (15) factor via the bottom row of (16). Thus we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \wedge^{r-1} E \otimes G & \to & \mathcal{Q}_{r,r,r-2} & \to & \wedge^r E & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^{r-1} E \otimes G & \to & \wedge^{r-1} E \otimes \overline{F}_0 & \to & \wedge^{r-1} E \otimes E & \to & 0.
\end{array}
\tag{17}
$$

In particular, if we let $r = \operatorname{rank}(E)$, and tensor the above diagram with $(\wedge^r E)^{-1}$, then the coboundary maps in the cohomology diagram associated to (17) yields the following commutative square:

$$
\begin{array}{ccc}
H^0(X, \mathcal{O}_X) & \to & H^1(X, E^\vee \otimes G) \\
\downarrow & & \| \\
H^0(X, \mathcal{E}nd(E)) & \to & H^1(X, E^\vee \otimes G).
\end{array}
\tag{18}
$$

Thus, it follows that under the top horizontal map the identity element 1 is mapped to the element ζ as well. In particular, this means that the top horizontal row in diagram (17) also represents the class ζ.

A similar analysis starting with the bundle G yields a diagram analogous to (17):

$$
\begin{array}{ccccccccc}
0 & \to & \wedge^{r-1}G \otimes E(-d) & \to & \mathcal{P}_{r,r,r-2} & \to & \wedge^r G & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^{r-1}G \otimes E(-d) & \to & \wedge^{r-1}G \otimes \bar{F}_1 & \to & \wedge^{r-1}G \otimes G & \to & 0.
\end{array}
\tag{19}
$$

3.4. **Maps between filtered pieces.**

Lemma 3.2. *Let notation be as above.*

(1) For $r \geqslant k > 1$, *there exist commutative diagrams:*

$$
\begin{array}{ccccccccc}
0 & \to & E_{r,k-1} & \to & E_{r,k} & \to & \wedge^k E \otimes \wedge^{r-k}G & \to & 0 \\
& & \downarrow \alpha_{r,k-1} & & \downarrow \alpha_{r,k} & & \downarrow & & \\
0 & \to & E_{r-1,k-2} \otimes E & \to & E_{r-1,k-1} \otimes E & \to & \wedge^{k-1}E \otimes \wedge^{r-k}G \otimes E & \to & 0.
\end{array}
\tag{20}
$$

Here the right vertical map is the composition

$$
\wedge^k E \otimes \wedge^{r-k}G \xrightarrow{\omega_{k,k-1}\otimes \mathrm{id}} \wedge^{k-1}E \otimes E \otimes \wedge^{r-k}G \cong \wedge^{k-1}E \otimes \wedge^{r-k}G \otimes E.
$$

(2) For $r > j \geqslant 1$, *there exists a diagram:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_{r-1,j-1} \otimes G & \longrightarrow & E_{r-1,j} \otimes G & \longrightarrow & \wedge^j E \otimes \wedge^{r-1-j}G \otimes G & \longrightarrow & 0 \\
& & \downarrow \gamma_{r,j-1} & & \downarrow \gamma_{r,j} & & \downarrow & & \\
0 & \longrightarrow & E_{r,j-1} & \longrightarrow & E_{r,j} & \longrightarrow & \wedge^j E \otimes \wedge^{r-j}G & \longrightarrow & 0.
\end{array}
\tag{21}
$$

Here the right vertical map is the composition

$$
\wedge^j E \otimes \wedge^{r-j-1}G \otimes G \xrightarrow{\mathrm{id}\otimes \omega'_{r-j,r-j-1}} \wedge^j E \otimes \wedge^{r-j}G.
$$

Proof. We prove the assertion (1) by decreasing induction on k. For the base case $k = r$, we first recall that $E_{r,r} \cong \wedge^r \bar{F}_0$, and $E_{r-1,r-1} \cong \wedge^{r-1}\bar{F}_0$, and so the map

$$
\alpha_{r,r} : E_{r,r} \to E_{r-1,r-1} \otimes E
$$

is the composite

$$
\wedge^r \bar{F}_0 \to \wedge^{r-1}\bar{F}_0 \otimes \bar{F}_0 \to \wedge^{r-1}\bar{F}_0 \otimes E.
$$

Thus we have a diagram

$$
\begin{array}{ccccccccc}
0 & \to & E_{r,r-1} & \to & E_{r,r} & \to & \wedge^r E & \to & 0 \\
& & & & \downarrow \alpha_{r,r} & & \downarrow & & \\
0 & \to & E_{r-1,r-2} \otimes E & \to & E_{r-1,r-1} \otimes E & \to & \wedge^{r-1}E \otimes E & \to & 0.
\end{array}
\tag{22}
$$

Since the square on the right obviously commutes, we have an induced map $\alpha_{r,r-1}$ on the kernels.

Now assume, we have defined $\alpha_{r,k}$ for some $k < r$. Then we have a diagram

$$
\begin{array}{ccccccccc}
0 & \to & E_{r,k-1} & \to & E_{r,k} & \to & \wedge^k E \otimes \wedge^{r-k}G & \to & 0 \\
& & & & \downarrow \alpha_{r,k} & & \downarrow & & \\
0 & \to & E_{r-1,k-2} \otimes E & \to & E_{r-1,k-1} \otimes E & \to & \wedge^{k-1}E \otimes \wedge^{r-k}G \otimes E & \to & 0.
\end{array}
\tag{23}
$$

The square on the right commutes, as can be readily verified locally, and so induces a map between the kernels. This proves the inductive step. The proof of (2) is similar. For the base case of $j = r - 1$, recalling that $E_{r-1,r-1} \cong \wedge^{r-1}\overline{F}_0$, we have the following diagram

$$
\begin{array}{ccccccccc}
0 & \to & \wedge^{r-1}\overline{F}_0 \otimes G & \to & \wedge^{r-1}\overline{F}_0 \otimes \overline{F}_0 & \to & \wedge^{r-1}\overline{F}_0 \otimes E & \to & 0 \\
& & & & \downarrow^{\gamma_{r,r}} & & \downarrow & & \\
0 & \to & E_{r,r-1} & \to & E_{r,r} = \wedge^r\overline{F}_0 & \to & \wedge^r E & \to & 0.
\end{array}
\tag{24}
$$

Here the top row is obtained by tensoring (5) with $\wedge^{r-1}\overline{F}_0$, and the vertical arrow on the right is the composite $\wedge^{r-1}\overline{F}_0 \otimes E \to \wedge^{r-1}E \otimes E \to \wedge^r E$. Commutativity of the right square induces a map between the kernels, namely

$$
\gamma_{r,r-1} : E_{r-1,r-1} \otimes G \to E_{r,r-1}.
$$

Suppose now that we have the desired diagram involving the maps $\gamma_{r,j}$ and $\gamma_{r,j+1}$. Then $\gamma_{r,j-1}$ is defined by the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & E_{r-1,j-1} \otimes G & \to & E_{r-1,j} \otimes G & \to & \wedge^j E \otimes \wedge^{r-1-j}G \otimes G & \to & 0 \\
& & \downarrow^{\gamma_{r,j-1}} & & \downarrow^{\gamma_{r,j}} & & \downarrow & & \\
0 & \to & E_{r,j-1} & \to & E_{r,j} & \to & \wedge^j E \otimes \wedge^{r-j}G & \to & 0.
\end{array}
\tag{25}
$$

$\square$

4. SOME HOMOLOGICAL ALGEBRA

Let Y be a smooth projective variety and let $X \subset Y$ be a smooth hypersurface as before. Starting with the maps Φ and Ψ which define the bundles E and G, we have seen in the previous sections that we obtain, on one hand, sheaves $\mathcal{E}_r$ and $\mathcal{G}_r$ (which are supported on the r-th thickening X_r) and their restrictions to various thickenings X_j, $0 < j < r$, and on the other hand, the filtered pieces $E_{r,i}$ and $G_{r,i}$. In this section, we will establish results which will relate these two sets of objects.

4.1. The cokernel sheaves $\mathcal{E}_r$, $\mathcal{G}_r$ and their restrictions. The main result of this section is Proposition 4.5 which generalizes Proposition 3.5 of [23].

For any $r > j \geqslant 1$, we have short exact sequences

$$
0 \to \mathcal{O}_{X_{r-j}}(-jd) \to \mathcal{O}_{X_r} \to \mathcal{O}_{X_j} \to 0,
\tag{26}
$$

$$
0 \to \mathcal{O}_Y(-jd) \to \mathcal{O}_Y \to \mathcal{O}_{X_j} \to 0.
\tag{27}
$$

These sequences fit in the diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}_Y(-rd) & = & \mathcal{O}_Y(-rd) \\
\downarrow & & \downarrow \\
0 \longrightarrow \mathcal{O}_Y(-jd) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{X_j} \longrightarrow 0 \\
\downarrow \quad\quad\quad \downarrow \quad\quad\quad \| \\
0 \longrightarrow \mathcal{O}_{X_{r-j}}(-jd) \longrightarrow \mathcal{O}_{X_r} \longrightarrow \mathcal{O}_{X_j} \longrightarrow 0 \\
\downarrow \quad\quad\quad \downarrow \\
0 \quad\quad\quad 0.
\end{array} \tag{28}
$$

In what follows, we will often write $\operatorname{Tor}^i_Y(\mathcal{F}, \mathcal{G})$ when we mean $\operatorname{Tor}^i_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})$ where $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_Y$-modules. The following fact is used throughout our computations.

Lemma 4.1. *Let $\mathcal{F}$ be any coherent $\mathcal{O}_{X_j}$-module. Then*

$$\operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_j}) \cong \mathcal{F}(-jd) \ \ and \ \ \operatorname{Tor}^i_Y(\mathcal{F}, \mathcal{O}_{X_j}) = 0 \ for \ i > 1.$$

Proof. Tensoring the short exact sequence of sheaves of $\mathcal{O}_Y$-modules (27) with $\mathcal{F}$ gives

$$\cdots \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_Y) \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_j}) \to \mathcal{F}(-jd) \to \mathcal{F} \to \mathcal{F} \to 0.$$

The last surjection is an isomorphism and $\operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_Y) = 0$; this proves the first claim. For the second claim, we observe from (27) that $\mathcal{O}_{X_j}$ has homological dimension 1 as an $\mathcal{O}_Y$-module and so the higher Tor terms vanish. $\square$

Lemma 4.2. *Let $\mathcal{F}$ be an $\mathcal{O}_{X_r}$-module and let $1 \leqslant j < r$. Then*

$$\operatorname{Ker}(\operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_j}) \to \operatorname{Tor}^1_{X_r}(\mathcal{F}, \mathcal{O}_{X_j})) \cong \operatorname{Ker}(\mathcal{F}(-jd) \to \mathcal{F}|_{X_{r-j}}(-jd)).$$

Proof. Upon tensoring sequence (26) with $\mathcal{F}$ over Y, we get the long exact sequence

$$0 \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_{r-j}})(-jd) \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_r}) \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_j}) \to \ker[\mathcal{F}(-jd)|_{X_{r-j}} \to \mathcal{F}|_{X_r}] \to 0.$$

Next, we tensor (26) with $\mathcal{F}$ over X_r to get the identification,

$$\operatorname{Tor}^1_{X_r}(\mathcal{F}, \mathcal{O}_{X_j}) \cong \ker[\mathcal{F}(-jd)|_{X_{r-j}} \to \mathcal{F}|_{X_r}].$$

Thus we have the 4-term sequence

$$0 \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_{r-j}})(-jd) \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_r}) \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_j}) \to \operatorname{Tor}^1_{X_r}(\mathcal{F}, \mathcal{O}_{X_j}) \to 0. \tag{29}$$

Tensoring the leftmost column in diagram (28) with $\mathcal{F}$ over Y gives

$$0 \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_{r-j}}(-jd)) \to \mathcal{F}(-rd) \to \mathcal{F}(-jd) \to \mathcal{F}|_{X_{r-j}}(-jd) \to 0. \tag{30}$$

Since $\operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_r}) \cong \mathcal{F}(-rd)$, the lemma follows from (29) and (30). $\square$

Lemma 4.3. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a sequence of $\mathcal{O}_{X_r}$-modules. For $1 \leqslant j < r$, assume that $\operatorname{Tor}^1_{X_r}(\mathcal{F}, \mathcal{O}_{X_j}) = 0$. Then there exists an exact sequence*

$$0 \to \operatorname{Tor}^1_Y(\mathcal{F}', \mathcal{O}_{X_j}) \to \operatorname{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_j}) \to \operatorname{Tor}^1_Y(\mathcal{F}'', \mathcal{O}_{X_j}) \to \operatorname{Tor}^1_{X_r}(\mathcal{F}'', \mathcal{O}_{X_j}) \to 0.$$

Proof. Tensoring $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ respectively by $\mathcal{O}_{X_j}$ over Y and X_r gives the two sequences below:

$$0 \to \mathrm{Tor}^1_Y(\mathcal{F}', \mathcal{O}_{X_j}) \to \mathrm{Tor}^1_Y(\mathcal{F}, \mathcal{O}_{X_j}) \to \mathrm{Tor}^1_Y(\mathcal{F}'', \mathcal{O}_{X_j}) \to \mathcal{F}'|_{X_j} \to \mathcal{F}|_{X_j} \to \mathcal{F}''|_{X_j} \to 0,$$

$$0 \to \mathrm{Tor}^1_{X_r}(\mathcal{F}'', \mathcal{O}_{X_j}) \to \mathcal{F}'|_{X_j} \to \mathcal{F}|_{X_j} \to \mathcal{F}''|_{X_j} \to 0. \tag{31}$$

The desired sequence now follows. $\qquad\square$

For $j < r$, we also define

$$\mathcal{G}_{r,j} := \mathrm{Ker}(\wedge^r F_0|_{X_j} \twoheadrightarrow \mathcal{E}_r|_{X_j}) \text{ and } \mathcal{E}_{r,j} := \mathrm{Ker}(\wedge^r F_1|_{X_j} \twoheadrightarrow \mathcal{G}_r|_{X_j}). \tag{32}$$

Restricting sequence (11) to X_r and by a repeated use of Lemma 4.1, we get

$$\cdots \to \wedge^r F_1(-rd) \to \wedge^r F_0(-rd) \to \wedge^r F_1 \to \wedge^r F_0 \to \mathcal{E}_r \to 0, \tag{33}$$

which breaks up into two short exact sequences

$$0 \to \mathcal{G}_r \to \wedge^r F_0 \to \mathcal{E}_r \to 0, \text{ and}$$
$$0 \to \mathcal{E}_r(-rd) \to \wedge^r F_1 \to \mathcal{G}_r \to 0. \tag{34}$$

Since $\mathrm{Im}(\wedge^r \overline{F}_1 \to \wedge^r \overline{F}_0) = \wedge^r G$, upon restricting the first sequence to X, we get the sequence

$$0 \to \wedge^r G \to \wedge^r \overline{F}_0 \to \overline{\mathcal{E}}_r := \mathcal{E}_r \otimes \mathcal{O}_X \to 0. \tag{35}$$

In particular, we have

Lemma 4.4. *With notation as above,*

(i) $\mathcal{Q}_{r,r,0} := \mathrm{Coker}(E_{r,0} \to E_{r,r}) = \mathrm{Coker}(\wedge^r G \to \wedge^r \overline{F}_0) \cong \overline{\mathcal{E}}_r$, *and*
(ii) $\mathcal{G}_{r,1} := \mathrm{Ker}(\wedge^r \overline{F}_0 \twoheadrightarrow \overline{\mathcal{E}}_r) \cong \wedge^r G = E_{r,0} = \mathcal{P}_{r,r,r-1}$.

Proposition 4.5. *Let* $1 \leqslant j < r$. *There exist short exact sequences*

(1) $0 \to \mathcal{E}_{r,j} \to \mathcal{E}_r(-jd) \to \mathcal{E}_r|_{X_{r-j}}(-jd) \to 0$.
(2) $0 \to \mathcal{G}_{r,j}(-(r-j)d) \to \mathcal{G}_r \to \mathcal{G}_r|_{X_{r-j}} \to 0$.

Proof. On restricting the $\mathcal{O}_{X_r}$-sequence

$$0 \to \mathcal{G}_r \to \wedge^r F_0 \to \mathcal{E}_r \to 0$$

to X_j, we get the long exact sequence

$$0 \to \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_j}) \to \mathcal{G}_r|_{X_j} \to \wedge^r F_0|_{X_j} \to \mathcal{E}_r|_{X_j} \to 0.$$

Breaking this up into short exact sequences yields

$$0 \to \mathcal{G}_{r,j} \to \wedge^r F_0|_{X_j} \to \mathcal{E}_r|_{X_j} \to 0, \text{ and}$$

$$0 \to \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_j}) \to \mathcal{G}_r|_{X_j} \to \mathcal{G}_{r,j} \to 0.$$

We also have a 4-term sequence

$$0 \to \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_{X_j}) \to \wedge^r F_1|_{X_j} \to \wedge^r F_0|_{X_j} \to \mathcal{E}_r|_{X_j} \to 0$$

obtained by restricting (11) to X_j. The injection above yields the exact sequence

$$0 \to \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_{X_j}) \to \wedge^r F_1|_{X_j} \to \mathcal{G}_{r,j} \to 0.$$

These last two short exact sequences, and snake lemma yield a diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{E}_{r,j} & & = & & \mathcal{E}_{r,j} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_{X_j}) & \to & \wedge^r F_1|_{X_j} & \to & \mathcal{G}_{r,j} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_j}) & \to & \mathcal{G}_r|_{X_j} & \to & \mathcal{G}_{r,j} & \to & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{36}
$$

Thus we get an exact sequence

$$
0 \to \mathcal{E}_{r,j} \to \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_{X_j}) \to \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_j}) \to 0.
\tag{37}
$$

Letting $\mathcal{F} = \mathcal{E}_r$ in Lemma 4.2 completes the proof of part (1). The proof of part (2) is similar.
$\square$

4.2. Cokernel sheaves and filtrations.

In this section, we relate the cokernel sheaves $\mathcal{E}_r$, $\mathcal{G}_r$ and their restrictions to the thickenings X_j, $1 \leqslant j \leqslant r$, with the filtered pieces $E_{r,i}$ and $G_{r,i}$. The former are more amenable to cohomology computations, while the latter are what we need to understand to carry out the proof of the main theorem.

The main results of this section are Lemma 4.7 and Proposition 4.13.

Lemma 4.6. *We have the following isomorphisms:*

(1) $\mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) \cong G_{r,r-1}$ *and* $\mathrm{Tor}^1_Y(\mathcal{G}_r, \mathcal{O}_X) \cong E_{r,r-1}(-rd)$.
(2) $\mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_X) \cong \mathcal{P}_{r,r-1,0}$ *and* $\mathrm{Tor}^1_{X_r}(\mathcal{G}_r, \mathcal{O}_X) \cong \mathcal{Q}_{r,r-1,0}(-rd)$.
(3) $\mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_{r-1}}) \cong \mathcal{Q}_{r,r-1,0}(-(r-1)d)$
(4) $\mathrm{Tor}^1_{X_r}(\mathcal{G}_r, \mathcal{O}_{X_{r-1}}) \cong \mathcal{P}_{r,r-1,0}(-(r-1)d)$

Proof. (1) Restricting the sequence (11)

$$
0 \to \wedge^r \widetilde{F}_1 \to \wedge^r \widetilde{F}_0 \to \mathcal{E}_r \to 0
$$

to X gives

$$
0 \to \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) \to \wedge^r \overline{F}_1 \to \wedge^r \overline{F}_0 \to \overline{\mathcal{E}}_r \to 0.
$$

The image of the map $\wedge^r \overline{F}_1 \to \wedge^r \overline{F}_0$ is $\wedge^r G$, and hence by definition $\mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) \cong G_{r,r-1}$. The proof of the other isomorphism is similar.

(2) We have just seen that we have a sequence

$$
0 \to \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) \to \wedge^r \overline{F}_1 \to \wedge^r G \to 0.
\tag{38}
$$

The sequence $0 \to \mathcal{G}_r \to \wedge^r F_0 \to \mathcal{E}_r \to 0$ on restricting to X gives a short exact sequence

$$
0 \to \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_X) \to \overline{\mathcal{G}}_r \to \wedge^r G \to 0.
\tag{39}
$$

Together these short exact sequences yield the diagram

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \wedge^r E(-rd) & = & \wedge^r E(-rd) & & \\
& & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) & \to & \wedge^r \overline{F}_1 & \to \wedge^r G \to & 0 \\
& & \downarrow & & \downarrow & & \| \\
0 & \to & \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_X) & \to & \overline{\mathcal{G}}_r & \to \wedge^r G \to & 0. \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}
\tag{40}
$$

In the left column, we have by definition, $G_{r,0} := \wedge^r E(-rd)$, and from (1) above $\mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) \cong G_{r,r-1}$. Thus the injection in the left column is the map $G_{r,0} \hookrightarrow G_{r,r-1}$ and so we have $\mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_X) \cong \mathcal{P}_{r,r-1,0}$.

The isomorphism $\mathrm{Tor}^1_{X_r}(\mathcal{G}_r, \mathcal{O}_X) \cong \mathcal{Q}_{r,r-1,0}(-rd)$ follows in a similar fashion.

(3) Applying $\mathcal{E}_r \otimes_{\mathcal{O}_{X_r}}$ to the sequence

$$
0 \to \mathcal{O}_X(-(r-1)d) \to \mathcal{O}_{X_r} \to \mathcal{O}_{X_{r-1}} \to 0,
$$

we get

$$
0 \to \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_{r-1}}) \to \overline{\mathcal{E}}_r(-(r-1)d) \to \mathcal{E}_r \to \mathcal{E}_r|_{X_{r-1}} \to 0.
$$

Using Proposition 4.5 (1), we get

$$
0 \to \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_{r-1}}) \to \overline{\mathcal{E}}_r(-(r-1)d) \to \mathcal{E}_{r,1}(d) \to 0.
$$

Recall that

    (i) $\mathcal{Q}_{r,r,0} := \mathrm{Coker}(E_{r,0} \to E_{r,r}) = \mathrm{Coker}(\wedge^r G \to \wedge^r \overline{F}_0) = \overline{\mathcal{E}}_r$, and that

    (ii) $\mathcal{Q}_{r,r,r-1} := \mathrm{Coker}(E_{r,r-1} \to E_{r,r}) = \wedge^r E.$

Thus we have

$$
\mathcal{E}_{r,1} = \mathrm{Ker}(\wedge^r \overline{F}_1 \to \overline{\mathcal{G}}_r) = \wedge^r E(-rd) = \mathcal{Q}_{r,r,r-1}(-rd).
\tag{41}
$$

The proof now follows by observing that

$$
\mathrm{Ker}\left(\overline{\mathcal{E}}_r(-(r-1)d) \to \mathcal{E}_{r,1}(d)\right) = \mathrm{Ker}\left(\mathcal{Q}_{r,r,0}(-(r-1)d) \to \mathcal{Q}_{r,r,r-1}(-(r-1)d)\right)
$$

$$
=: \mathcal{Q}_{r,r-1,0}(-(r-1)d).
$$

Finally, (4) follows by a similar argument. $\square$

Lemma 4.7. *Let $1 < j < r$. There exist short exact sequences*

    (1) $0 \to E_{r,r-1}(-(r-1)d) \to \mathcal{G}_r \to \mathcal{G}_{r,r-1} \to 0.$

    (2) $0 \to E_{r,j-1}(-(j-1)d) \to \mathcal{G}_{r,j} \to \mathcal{G}_{r,j-1} \to 0.$

    (3) $0 \to \mathcal{Q}_{r,r-1,0}(-(r-1)d) \to \mathcal{G}_r|_{X_{r-1}} \to \mathcal{G}_{r,r-1} \to 0.$

Proof. We note that part (1) is a special case of (2) (for $j = r$). However, since the proof is direct, we present it here. We have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{G}_r & \to & \wedge^r F_0 & \to & \mathcal{E}_r & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{G}_{r,r-1} & \to & \wedge^r F_0|_{X_{r-1}} & \to & \mathcal{E}_r|_{X_{r-1}} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0. & &
\end{array}
\tag{42}
$$

Here the middle and right vertical maps are restriction maps, and the left vertical map is the induced map. By the snake lemma, we have an exact sequence of the kernels of the vertical maps:
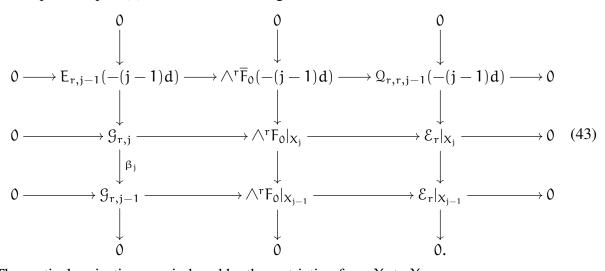
$$0 \to \ker(\mathcal{G}_r \twoheadrightarrow \mathcal{G}_{r,r-1}) \to \wedge^r \overline{F}_0(-(r-1)d) \to \ker(\mathcal{E}_r \twoheadrightarrow \mathcal{E}_r|_{X_{r-1}}) \to 0.$$

The last term is $\mathcal{E}_{r,1}(d)$, which is isomorphic to $\wedge^r E(-(r-1)d)$ by (41) in Proposition 4.5. Thus we have

$$\ker(\mathcal{G}_r \twoheadrightarrow \mathcal{G}_{r,r-1}) \cong E_{r,r-1}(-(r-1)d).$$

This proves (1).

The proof of part (2) follows from the diagram below:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{r,j-1}(-(j-1)d) & \longrightarrow & \wedge^r \overline{F}_0(-(j-1)d) & \longrightarrow & \mathcal{Q}_{r,r,j-1}(-(j-1)d) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_{r,j} & \longrightarrow & \wedge^r F_0|_{X_j} & \longrightarrow & \mathcal{E}_r|_{X_j} & \longrightarrow & 0 \quad (43) \\
& & \downarrow{\scriptstyle \beta_j} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_{r,j-1} & \longrightarrow & \wedge^r F_0|_{X_{j-1}} & \longrightarrow & \mathcal{E}_r|_{X_{j-1}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0. & &
\end{array}
$$

The vertical surjections are induced by the restriction from X_j to X_{j-1}.

All that needs to be proved is the exactness of the first and third columns. By the local description of $\mathcal{E}_r$ (see Lemma 2.1), it follows that locally $\mathcal{G}_{r,j}$ is of the form

$$\mathcal{O}_{X_j}^{\binom{m-u}{r}} \oplus \mathcal{O}_{X_{j-1}}^{\binom{m-u}{r-1}\binom{u}{1}} \oplus \mathcal{O}_{X_{j-2}}^{\binom{m-u}{r-2}\binom{u}{2}} \oplus \cdots \oplus \mathcal{O}_{X_2}^{\binom{m-u}{r-j+2}\binom{u}{j-2}} \oplus \mathcal{O}_X^{\binom{m-u}{r-j+1}\binom{u}{j-1}}. \qquad (44)$$

Exactness can now be checked locally by using the local description of the maps between sheaves $\mathcal{G}_{r,j} \twoheadrightarrow \mathcal{G}_{r,j-1}$ and $\mathcal{E}_r|_{X_j} \twoheadrightarrow \mathcal{E}_r|_{X_{j-1}}$.

For part (3), we restrict the sequence

$$0 \to \mathcal{G}_r \to \wedge^r F_0 \to \mathcal{E}_r \to 0$$

to X_{r-1} to get

$$0 \to \mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_{X_{r-1}}) \to \mathcal{G}_r|_{X_{r-1}} \to \wedge^r F_0|_{X_{r-1}} \to \mathcal{E}_r|_{X_{r-1}} \to 0. \qquad (45)$$

The injection above, using Lemma 4.6 (3), yields the exact sequence

$$0 \to \mathcal{Q}_{r,r-1,0}(-(r-1)d) \to \mathcal{G}_r|_{X_{r-1}} \to \mathcal{G}_{r,r-1} \to 0.$$

This is the desired statement. $\qquad \square$

Proposition 4.8. $\mathcal{G}_{r,j}|_X \cong \mathcal{P}_{r,r,r-j}$ and $\mathcal{E}_{r,j}|_X \cong \mathcal{Q}_{r,r,r-j}(-rd)$

Proof. We will only prove the first isomorphism (which is what we really need). The proof of the second assertion follows by arguing in a similar fashion. On restricting the left vertical sequence in the diagram (43), we get a sequence

$$E_{r,j-1}(-(j-1)d) \to \mathcal{G}_{r,j}|_X \xrightarrow{\overline{\beta}_j} \mathcal{G}_{r,j-1}|_X \to 0.$$

Once again, using the local description of the sequence above, we see that the left arrow factors via the quotient $E_{r,j-1}(-(j-1)d) \twoheadrightarrow \mathfrak{Q}_{r,j,j-1}(-(j-1)d)$. Since $\mathfrak{Q}_{r,j,j-1}(-(j-1)d) \cong \mathcal{P}_{r,r-j+1,r-j}$, we get an exact sequence

$$0 \to \mathcal{P}_{r,r-j+1,r-j} \to \mathcal{G}_{r,j}|_X \xrightarrow{\overline{\beta}_j} \mathcal{G}_{r,j-1}|_X \to 0.$$

The proof now follows by decreasing induction on j; for the base case $j = r$, we have $\mathcal{G}_{r,r} = \mathcal{G}_r$ and $\mathcal{G}_r|_X = \mathcal{P}_{r,r,0}$. $\qquad\square$

Remark 5. The filtration $G_{r,0} \subset G_{r,1} \subset \cdots \subset G_{r,r}$ can be recovered from the maps $G_{r,r} \twoheadrightarrow \mathcal{P}_{r,r,j}$. Proposition 4.8 is crucial as it shows that these maps of bundles on X have a natural lift to X_{r-j} in the form of maps of sheaves $\wedge^r F_1|_{X_{r-j}} \twoheadrightarrow \mathcal{G}_{r,r-j}$.

Lemma 4.9. $\mathrm{Tor}^1_{X_j}(\mathcal{E}_r|_{X_j}, \mathcal{O}_X) \cong \mathcal{P}_{r,r-1,r-j}$ *and* $\mathrm{Tor}^1_{X_j}(\mathcal{G}_r|_{X_j}, \mathcal{O}_X) \cong \mathfrak{Q}_{r,r-1,r-j}(-rd)$.

Proof. We have a sequence

$$0 \to \mathrm{Tor}^1_{X_j}(\mathcal{E}_r|_{X_j}, \mathcal{O}_X) \xrightarrow{\beta} \overline{\mathcal{G}}_{r,j} \xrightarrow{\alpha} \wedge^r \overline{F}_0 \to \overline{\mathcal{E}}_r \to 0,$$

obtained by restricting to X the exact sequence

$$0 \to \mathcal{G}_{r,j} \to \wedge^r F_0|_{X_j} \to \mathcal{E}_r|_{X_j} \to 0.$$

Recall that $\mathcal{G}_{r,j} = \mathrm{Image}(\wedge^r \Phi|_{X_j})$, and hence the image of the map $\overline{\mathcal{G}}_{r,j} \xrightarrow{\alpha} \wedge^r \overline{F}_0$ is equal to $\mathrm{Image}(\wedge^r \Phi|_X) = \wedge^r G$. By Proposition 4.8, we have $\overline{\mathcal{G}}_{r,j} \cong \mathcal{P}_{r,r,r-j}$. Since taking exterior powers commutes with restriction maps, we see that the map $\overline{\mathcal{G}}_{r,j} \to \wedge^r G$ is the same as the natural map $\mathcal{P}_{r,r,r-j} \to \mathcal{P}_{r,r,r-1}$ whose kernel is $\mathcal{P}_{r,r-1,r-j}$.

The proof of the other statement is similar, and so we omit it. $\qquad\square$

Lemma 4.10. *We have the following identifications:*
  (i) $\mathrm{Tor}^1_Y(\mathcal{E}_{r,r-1}, \mathcal{O}_X) \cong G_{r,r-2}(-(r-1)d)$.
  (ii) $\mathrm{Tor}^1_Y(\mathcal{G}_{r,r-1}, \mathcal{O}_X) \cong E_{r,r-2}(-(r-1)d)$.

Proof. We prove part (i), and omit the proof for part (ii) as it is identical. From Proposition 4.5, by setting $j = r - 1$, we have an exact sequence

$$0 \to \mathcal{E}_{r,r-1} \to \mathcal{E}_r(-(r-1)d) \to \mathcal{E}_r|_X(-(r-1)d) \to 0.$$

Restricting this sequence to X, we get

$$0 \to \mathrm{Tor}^1_Y(\mathcal{E}_{r,r-1}, \mathcal{O}_X) \to \mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X)(-(r-1)d) \xrightarrow{\alpha} \mathrm{Tor}^1_Y(\overline{\mathcal{E}}_r, \mathcal{O}_X)(-(r-1)d) \to \overline{\mathcal{E}}_{r,r-1} \to 0.$$

We note that the map α factors as below:

$$
\begin{array}{ccc}
\mathrm{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) & \xrightarrow{\makebox[4cm]{α}} & \mathrm{Tor}^1_Y(\overline{\mathcal{E}}_r, \mathcal{O}_X) \\
\downarrow & & \| \\
\mathrm{Tor}^1_{X_r}(\mathcal{E}_r, \mathcal{O}_X) \longrightarrow \mathrm{Tor}^1_{X_2}(\mathcal{E}_r|_{X_2}, \mathcal{O}_X) \longrightarrow & & \mathrm{Tor}^1_{X_2}(\overline{\mathcal{E}}_r, \mathcal{O}_X)
\end{array}
\tag{46}
$$

By Lemma 4.9, $\mathrm{Tor}^1_{X_2}(\mathcal{E}_r|_{X_2}, \mathcal{O}_X) \cong \mathcal{P}_{r,r-1,r-2} \cong \mathfrak{Q}_{r,1,0}(-d)$, and so the rightmost map on the bottom sequence $\mathrm{Tor}^1_{X_2}(\mathcal{E}_r|_{X_2}, \mathcal{O}_X) \to \mathrm{Tor}^1_{X_2}(\overline{\mathcal{E}}_r, \mathcal{O}_X)$ is the composite injective map

$$\mathcal{P}_{r,r-1,r-2} \cong \mathfrak{Q}_{r,1,0}(-d) \hookrightarrow \mathfrak{Q}_{r,r,0}(-d).$$

This, together with the isomorphism $\text{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) \cong G_{r,r-1}$ from Lemma 4.6, shows that the map $\text{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X) \xrightarrow{\alpha} \text{Tor}^1_Y(\mathcal{E}_r|_X, \mathcal{O}_X)$ is the same as the composite map

$$G_{r,r-1} \twoheadrightarrow \mathcal{P}_{r,r-1,r-2} \cong \mathcal{Q}_{r,1,0}(-d) \hookrightarrow \mathcal{Q}_{r,r,0}(-d).$$

Hence we have

$$\begin{aligned}
\text{Tor}^1_Y(\mathcal{E}_{r,r-1}, \mathcal{O}_X) &\cong \ker(\text{Tor}^1_Y(\mathcal{E}_r, \mathcal{O}_X)(-(r-1)d) \to \text{Tor}^1_Y(\mathcal{E}_r|_{X_2}, \mathcal{O}_X)(-(r-1)d)) \\
&\cong \ker(G_{r,r-1}(-(r-1)d) \to \mathcal{P}_{r,r-1,r-2}(-(r-1)d)) \\
&= G_{r,r-2}(-(r-1)d),
\end{aligned}$$

where the last equality follows from the definition of $\mathcal{P}_{r,r-1,r-2}$.

$\square$

We state the following standard result from homological algebra whose proof we omit.

Lemma 4.11. *Let $0 \to M' \to M \to M'' \to 0$ and $0 \to N' \to N \to N'' \to 0$ be two exact sequences of R-modules. Then we have a commutative diagram*

$$
\begin{array}{ccc}
\text{Tor}^{i+1}(M'', N'') & \longrightarrow & \text{Tor}^i(M'', N') \\
\downarrow & & \downarrow \\
\text{Tor}^i(M', N'') & \longrightarrow & \text{Tor}^{i-1}(M', N')
\end{array}
\tag{47}
$$

where all the maps are boundary maps. The vertical maps come from tensoring the first exact sequence with N'' and N' respectively, while the horizontal maps come from tensoring the second exact sequence with M'' and M' respectively.

Proposition 4.12. *We have a short exact sequence*

$$0 \to \wedge^r G(-(j-1)d) \to \mathcal{G}_{r,j} \to \mathcal{G}_{r,j}|_{X_{j-1}} \to 0.$$

Proof. Applying Lemma 4.11 to following sequences on X_j

$$0 \to \mathcal{O}_X(-(j-1)d) \to \mathcal{O}_{X_j} \to \mathcal{O}_{X_{j-1}} \to 0, \tag{48}$$

and

$$0 \to \mathcal{G}_{r,j} \to \wedge^r F_0|_{X_j} \to \mathcal{E}_r|_{X_j} \to 0 \tag{49}$$

gives the following diagram (with $i = 1$)

$$
\begin{array}{ccc}
\text{Tor}^2_{X_j}(\mathcal{E}_r|_{X_j}, \mathcal{O}_{X_{j-1}}) & \xrightarrow{\cong} & \text{Tor}^1_{X_j}(\mathcal{E}_r|_{X_j}, \mathcal{O}_X(-(j-1)d)) \\
\downarrow{\cong} & & \downarrow{\beta} \\
\text{Tor}^1_{X_j}(\mathcal{G}_{r,j}, \mathcal{O}_{X_{j-1}}) & \xrightarrow{\gamma} & \overline{\mathcal{G}}_{r,j}(-(j-1)d).
\end{array}
\tag{50}
$$

The right vertical map β is obtained by restricting (49) to X, and hence is the same as the map in Lemma 4.9. The bottom horizontal map can be seen to be an injection. Since $\text{Coker}(\beta) \cong \wedge^r G(-(j-1)d)$, we get $\text{Coker}(\gamma) \cong \wedge^r G(-(j-1)d)$. $\square$

Proposition 4.13. *We have a short exact sequence*

$$0 \to E_{3,1}(-d) \to \mathcal{G}_{3,2} \to \wedge^3 G \to 0. \tag{51}$$

Proof. Tensoring the sequence

$$0 \to \mathcal{O}_Y(-(j-1)d) \to \mathcal{O}_Y \to \mathcal{O}_{X_{j-1}} \to 0$$

with $\mathcal{G}_{r,j}$ over Y, we get

$$0 \to \mathrm{Tor}_Y^1(\mathcal{G}_{r,j}, \mathcal{O}_{X_{j-1}}) \to \mathcal{G}_{r,j}(-(j-1)d) \to \mathcal{G}_{r,j} \to \mathcal{G}_{r,j}|_{X_{j-1}} \to 0.$$

For $r = 3, j = 2$, Lemma 4.10 yields the isomorphism $\mathrm{Tor}_Y^1(\mathcal{G}_{3,2}, \mathcal{O}_X) \cong E_{3,1}(-2d)$, while from Proposition 4.12, we see that the kernel of the surjection in the sequence above is $\wedge^3 G(-d)$. Breaking up the 4-term sequence above into short exact sequences yields the desired sequence. $\qquad\square$

5. COHOMOLOGY COMPUTATIONS FOR ACM BUNDLES

Henceforth, E will be an ACM bundle on a smooth, degree d hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geqslant 3$.

The main result of this section is Theorem 5.7. As mentioned in the introduction, the key idea in this paper is to work our way up from the bottom most term $E_{3,0} \cong \wedge^3 G$ of the filtration $\{E_{3,i}\}$ on $\wedge^3 \overline{F}_0$. It turns out that to understand this filtration, we will need to use the filtration $\{E_{2,j}\}$ on $\wedge^2 \overline{F}_0$, and the maps between the filtered pieces in the two filtrations (see §3.4) in an essential way (Lemma 5.5).

The inclusion $E_{3,0} \hookrightarrow E_{3,2}$ gives us diagram (68), and using Lemma 5.5, we first prove that $\wedge^3 G$ is ACM which implies that the associated graded piece $\mathcal{Q}_{3,2,0}$ is 1-generated. Using Lemma 5.5 once again, we prove that the surjection $\mathcal{Q}_{3,2,0} \to \mathcal{Q}_{3,2,1} := \wedge^2 E \otimes G$ induces a surjection $H_*^1(X, \mathcal{Q}_{3,2,0}) \to H_*^1(X, \wedge^2 E \otimes G)$, and thus the latter is also 1-generated.

5.1. **The bundle** $E_{r,0} \cong \wedge^r G$. In this section, we will prove the necessary results required to prove Lemma 5.5 by analyzing the bottom most filtered piece $E_{r,0} \subset \wedge^r \overline{F}_0$.

Lemma 5.1. *Assume that E is a rank r ACM vector bundle on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geqslant 3$. Then the sheaf $\mathcal{G}_{r,r-1}$ has no intermediate cohomology (see definition 1).*

Proof. From Lemma 4.7, we have an exact sequence

$$0 \to E_{r,r-1}(-(r-1)d) \to \mathcal{G}_r \to \mathcal{G}_{r,r-1} \to 0.$$

We first note that $\mathcal{G}_r$ is ACM by looking at its defining sequence

$$0 \to \wedge^r \widetilde{\overline{F}}_0(-rd) \to \wedge^r \widetilde{\overline{F}}_1 \to \mathcal{G}_r \to 0.$$

Next, by definition, we have

$$0 \to E_{r,r-1} \to \wedge^r \overline{F}_0 \to \wedge^r E \to 0.$$

Since $\mathrm{rank} E = r$, $\wedge^r E \cong \mathcal{O}_X(a)$ for some $a \in \mathbb{Z}$, and hence ACM. Thus

$$H_*^i(X, E_{r,r-1}) = 0 \text{ for } 2 \leqslant i \leqslant n-1.$$

As a consequence, we get

$$H_*^i(X, \mathcal{G}_{r,r-1}) = 0, \ 1 \leqslant i \leqslant n-2.$$

All that remains to be shown is that $H_*^{n-1}(X, \mathcal{G}_{r,r-1}) = 0$. For this, we first note that from the sequence (cf. Propositions 4.5 (i))

$$0 \to \mathcal{E}_{r,1}(d) \to \mathcal{E}_r \to \mathcal{E}_r|_{X_{r-1}} \to 0,$$

we have $H_*^{n-2}(X, \mathcal{E}_r|_{X_{r-1}}) = 0$, using the fact that (i) $\mathcal{E}_{r,1}(d) \cong \wedge^r E(-(r-1)d)$ (cf. (41)), and (ii) $\mathcal{E}_r$ has no intermediate cohomology by Remark 3. The desired vanishing then follows from the sequence

$$0 \to \mathcal{G}_{r,r-1} \to \wedge^r F_0|_{X_{r-1}} \twoheadrightarrow \mathcal{E}_r|_{X_{r-1}} \to 0.$$

$\square$

The following result was proved in [23]. As the proof is very short, we include it here to help keep this article essentially self-contained.

Proposition 5.2. *Let E be an ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ with $n \geqslant 3$. Then $\wedge^2 E$ is ACM if and only if $\wedge^2 G$ is ACM. In particular, when $\mathrm{rank}(E) = 3$, we have $\wedge^2 G$ is ACM.*

Proof. By Proposition 4.5 (for $r = 2, j = 1$) and equation (35), we have short exact sequences

$$0 \to \wedge^2 E(d) \to \mathcal{E}_2 \to \overline{\mathcal{E}}_2 \to 0,$$
$$0 \to \wedge^2 G \to \wedge^2 \overline{F}_0 \to \overline{\mathcal{E}}_2 \to 0.$$

Since $\mathcal{E}_2$ is ACM and $\wedge^2 \overline{F}_0$ is a direct sum of line bundles, it follows that

$$H_*^i(X, \wedge^2 E(d)) \cong H_*^i(X, \wedge^2 G), \ 2 \leqslant i \leqslant n-1.$$

For $i = 1$, we apply this result again to $E^\vee$ instead of E and use Serre duality. In this case G will be replaced by $G^\vee(-d)$. Finally, when $\mathrm{rank} E = 3$, we have $\wedge^2 E \cong E^\vee(c)$, and so is ACM. $\square$

Lemma 5.3. *Suppose that E and $\wedge^{r-1} G$ are ACM. We have short exact sequences*

$$0 \to H_*^i(X, \wedge^r G) \to H_*^{i+1}(X, \wedge^{r-1} G \otimes E(-d)) \to H_*^{i+1}(X, \mathcal{P}_{r,r,r-2}) \to 0, \ 1 \leqslant i \leqslant n-2.$$

We also have a surjection

$$H_*^1(X, \wedge^{r-1} G \otimes E(-d)) \twoheadrightarrow H_*^1(X, \mathcal{P}_{r,r,r-2}).$$

Proof. Recall, from §3.3, that associated to the bundle $\mathcal{P}_{r,r,r-2}$, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \wedge^{r-1} G \otimes E(-d) & \to & \mathcal{P}_{r,r,r-2} & \to & \wedge^r G & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^{r-1} G \otimes E(-d) & \to & \wedge^{r-1} G \otimes \overline{F}_1 & \to & \wedge^{r-1} G \otimes G & \to & 0.
\end{array}
\tag{52}
$$

Taking cohomology, we get a commutative square

$$
\begin{array}{ccc}
H^i(X, \mathcal{P}_{r,r,r-2}) & \to & H^i(X, \wedge^r G) \\
\downarrow & & \downarrow \\
H^i(X, \wedge^{r-1} G \otimes \overline{F}_1) & \to & H^i(X, \wedge^{r-1} G \otimes G).
\end{array}
\tag{53}
$$

Since $\wedge^{r-1} G$ is ACM and $\overline{F}_1$ is a direct sum of line bundles, the term on the left in the bottom row is zero. The fact that $\wedge^r G \to \wedge^{r-1} G \otimes G$ is a split inclusion implies now that the map

$$H^i(X, \mathcal{P}_{r,r,r-2}) \to H_*^i(X, \wedge^r G)$$

is zero for $1 \leqslant i \leqslant n-1$. The desired statements now follow. $\square$

Corollary 5.4. *Suppose that E and $\wedge^{r-1} G$ are ACM. We have short exact sequences*

$$0 \to H_*^i(X, E_{r,1}) \to H_*^i(X, \wedge^{r-1} G \otimes E) \to H_*^{i+1}(X, \wedge^r G) \to 0, \ 1 \leqslant i \leqslant n-2.$$

We also have an injection

$$H_*^{n-1}(X, E_{r,1}) \to H_*^{n-1}(X, \wedge^{r-1} G \otimes E).$$

Proof. Applying Lemma 5.3 to the dual exact sequence $0 \to E^\vee \to \overline{F}_0^\vee \to G^\vee \to 0$ gives

$$0 \to H_*^i(X, \wedge^r G^\vee) \to H_*^{i+1}(X, \wedge^{r-1} G^\vee \otimes E^\vee) \to H_*^{i+1}(X, \mathcal{P}'_{r,r,r-2}) \to 0, \quad 1 \leqslant i \leqslant n-2,$$

and a surjection

$$H_*^1(X, \wedge^{r-1} G^\vee \otimes E^\vee) \twoheadrightarrow H_*^1(X, \mathcal{P}'_{r,r,r-2}). \tag{54}$$

By definition (see Remark 4), the bundle $\mathcal{P}'_{r,r,r-2}$ sits in an exact sequence

$$0 \to \mathcal{P}'_{r,r-1,r-2} \to \mathcal{P}'_{r,r,r-2} \to \mathcal{P}'_{r,r,r-1} \to 0.$$

Since the extreme terms are the associated graded pieces in the filtration, this may be rewritten as

$$0 \to \wedge^{r-1} G^\vee \otimes E^\vee \to \mathcal{P}'_{r,r,r-2} \to \wedge^r G^\vee \to 0. \tag{55}$$

On the other hand the filtration $\{E_{r,i}\}$ on $\wedge^r \overline{F}_0$ yields the exact sequence

$$0 \to E_{r,0} \to E_{r,1} \to Q_{r,1,0} \to 0.$$

This again may be rewritten as

$$0 \to \wedge^r G \to E_{r,1} \to \wedge^{r-1} G \otimes E \to 0. \tag{56}$$

It follows then that (55) and (56) are dual to each other, and hence we get $\mathcal{P}'^\vee_{r,r,r-2} \cong E_{r,1}$.

Dualizing the sequence (54) and using the above isomorphism gives the desired result. $\square$

5.2. **Rank 3 case.**

Henceforth, we assume that E is a rank 3 ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geqslant 3$. From Lemma 3.2 (2), we have the following diagram for $r = 3$, $j = 1$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \wedge^2 G \otimes G & \longrightarrow & E_{2,1} \otimes G & \overset{\alpha}{\longrightarrow} & G \otimes E \otimes G & \longrightarrow & 0 \\
& & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\pi} & & \\
0 & \longrightarrow & \wedge^3 G & \longrightarrow & E_{3,1} & \overset{\beta}{\longrightarrow} & \wedge^2 G \otimes E & \longrightarrow & 0.
\end{array} \tag{57}$$

Lemma 5.5. *In the diagram above, the composition*[2]

$$H^i(X, E_{2,1} \otimes G) \overset{\alpha}{\to} H^i(X, G \otimes G \otimes E) \overset{\pi}{\to} H^i(X, \wedge^2 G \otimes E)$$

is the zero map for $1 \leqslant i \leqslant n-1$.

Proof. We first recall the two ways of defining the bundle $E_{2,1}$. In the first definition, as a filtered piece of $\wedge^2 \overline{F}_0$ (see §3), it occurs as part of the exact sequences

$$0 \to E_{2,1} \to \wedge^2 \overline{F}_0 \to \wedge^2 E \to 0, \tag{58}$$

$$0 \to \wedge^2 G \to E_{2,1} \to G \otimes E \to 0. \tag{59}$$

In the second definition, the 4-term sequence

$$0 \to S^2 G \to G \otimes F_0 \to \wedge^2 \overline{F}_0 \to \wedge^2 E \to 0,$$

breaks up into short exact sequences

$$0 \to S^2 G \to G \otimes F_0 \to E_{2,1} \to 0, \tag{60}$$

$$0 \to E_{2,1} \to \wedge^2 \overline{F}_0 \to \wedge^2 E \to 0. \tag{61}$$

---

[2]We will abuse notation, and use the same symbols for a map between vector bundles as well as the induced map between their cohomologies.

Together with the split exact sequence

$$0 \to S^2 G \xrightarrow{i} G \otimes G \xrightarrow{p} \wedge^2 G \to 0, \tag{62}$$

we get a commutative diagram as follows:

$$\tag{63}$$

Here the top row is (60) tensored with G, the middle row is (5) tensored with $G \otimes G$ and the last row is (5) tensored with $\wedge^2 G$. The top vertical arrows are induced by the injection $S^2 G \hookrightarrow G \otimes G$ for the left map, associativity of tensor products for the equality in the middle, and the surjection $E_{2,1} \twoheadrightarrow G \otimes E$ for the right map. The bottom vertical arrows are all induced by the surjection $G \otimes G \twoheadrightarrow \wedge^2 G$ and uses the associativity of tensor products.

The cohomology sequence associated to the above diagram yields the following diagram where the horizontal arrows are the boundary maps:

$$\tag{64}$$

$$
\begin{array}{ccc}
H^i(X, E_{2,1} \otimes G) & \xrightarrow{d_1} & H^{i+1}(X, S^2 G \otimes G) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \tilde{\imath}} \\
H^i(X, G \otimes E \otimes G) & \xrightarrow{d_2} & H^{i+1}(X, G \otimes G \otimes G) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{p}} \\
H^i(X, \wedge^2 G \otimes E) & \xrightarrow{d} & H^{i+1}(X, \wedge^2 G \otimes G).
\end{array}
$$

To prove the Lemma, we first note that the boundary map d is an injection for $1 \leqslant i \leqslant n-1$, since $\wedge^2 G \otimes F_0$ is ACM. Hence, to prove $\pi \circ \alpha = 0$, it is enough to prove that $d \circ \pi \circ \alpha = 0$. By the commutativity of the above diagram, it is enough to prove that $\tilde{p} \circ \tilde{\imath} \circ d_1 = 0$ in that range. This is obvious since $\tilde{p} \circ \tilde{\imath} = 0$ as the right column is exact. $\qquad \square$

Proposition 5.6. *With notation as above, we have*

    (i) $H^i_*(X, E_{3,2}) = 0$, $2 \leqslant i \leqslant n-1$.
    (ii) $H^i_*(X, E_{3,1}) = 0$, $2 \leqslant i \leqslant n-1$.

Proof. The bundle $E_{3,2}$ is defined by the short exact sequence

$$0 \to E_{3,2} \to \wedge^2 \overline{F}_0 \to \wedge^2 E \to 0.$$

Taking cohomology yields the proof for (i).

For (ii), we first note that (57) yields a commutative square

$$\tag{65}$$

$$
\begin{array}{ccc}
H^i_*(X, E_{2,1} \otimes G) & \xrightarrow{\alpha} & H^i(X, G \otimes E \otimes G) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \pi} \\
H^i_*(X, E_{3,1}) & \xrightarrow{\beta} & H^i_*(X, \wedge^2 G \otimes E).
\end{array}
$$

By Lemma 5.5 and the commutativity of the above square, we have $\beta \circ \delta = \pi \circ \alpha = 0$. Further, since $\operatorname{rank} E = 3$, we have $\wedge^2 G$ is ACM by Proposition 5.2, and so by Corollary 5.4 the map β is injective for $1 \leqslant i \leqslant n - 1$. Thus we see that the map

$$\delta : H^i_*(X, E_{2,1} \otimes G) \to H^i_*(X, E_{3,1}) \tag{66}$$

is the zero map for $1 \leqslant i \leqslant n - 1$.

Next, by Lemma 3.2 (2) for $r = 3, j = 2$ we have the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_{2,1} \otimes G & \longrightarrow & \wedge^2 F_0 \otimes G & \longrightarrow & \wedge^2 E \otimes G & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \delta} & & \downarrow & & \| & & \\
0 & \longrightarrow & E_{3,1} & \longrightarrow & E_{3,2} & \longrightarrow & \wedge^2 E \otimes G & \longrightarrow & 0.
\end{array}
\tag{67}
$$

Thus we see that the boundary map

$$\partial : H^i_*(X, \wedge^2 E \otimes G) \to H^{i+1}_*(X, E_{3,1})$$

factors via the map $\delta : H^{i+1}_*(X, E_{2,1} \otimes G) \to H^{i+1}_*(X, E_{3,1})$, and hence is zero for $0 \leqslant i \leqslant n - 2$. In particular, the map

$$H^i_*(X, E_{3,1}) \to H^i_*(X, E_{3,2})$$

is injective for $1 \leqslant i \leqslant n - 1$. The result now follows from (i). $\qquad \square$

Theorem 5.7. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface with $n \geqslant 3$. Let E be a rank 3 ACM bundle. Then*

(a) *$\wedge^3 G$ is ACM.*
(b) *The graded modules $H^1_*(X, \mathcal{Q}_{3,2,0})$, $H^1_*(X, \wedge^2 E \otimes G)$ and $H^2_*(X, \wedge^2 E \otimes E(-d))$ are generated by a single element.*

Proof. The sequence

$$0 \to E_{3,1}(-d) \to \mathcal{G}_{3,2} \to \wedge^3 G \to 0,$$

in Proposition 4.13 yields, on taking cohomology, the long exact sequence

$$\cdots \to H^i_*(X, \mathcal{G}_{3,2}) \to H^i_*(X, \wedge^3 G) \to H^{i+1}_*(X, E_{3,1}(-d)) \to \cdots.$$

The vanishings from Lemma 5.1 and Proposition 5.6 imply that

$$H^i_*(X, \wedge^3 G) = 0 \text{ for } 1 \leqslant i \leqslant n - 2.$$

Arguing similarly with the duals $E^\vee$ and $G^\vee$ instead, we get

$$H^i_*(X, \wedge^3 G^\vee) = 0 \text{ for } 1 \leqslant i \leqslant n - 2.$$

By Serre duality, it follows that $\wedge^3 G$ is an ACM bundle whenever $n - 2 \geqslant 1$ or equivalently, $n \geqslant 3$ which proves (a).

For (b), we first observe that the exterior powers of the maps in the exact sequence $0 \to G \to \overline{F}_0 \to E \to 0$ yield a diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & (68)\\
& & \downarrow & & \downarrow & & & &\\
0 & \longrightarrow & E_{3,0} & \longrightarrow & E_{3,2} & \longrightarrow & \mathcal{Q}_{3,2,0} & \longrightarrow & 0\\
& & \| & & \downarrow & & \downarrow & &\\
0 & \longrightarrow & \wedge^3 G & \longrightarrow & \wedge^3 \overline{F}_0 & \longrightarrow & \overline{\mathcal{E}}_3 & \longrightarrow & 0\\
& & & & \downarrow & & \downarrow & &\\
& & & & \wedge^3 E & = & \wedge^3 E & &\\
& & & & \downarrow & & \downarrow & &\\
& & & & 0 & & 0. & &
\end{array}
$$

From the middle row of the above diagram, we have $H^i_*(X, \overline{\mathcal{E}}_3) = 0$ when $1 \leqslant i \leqslant n-2$. Thus the right column yields the surjection

$$H^0_*(X, \wedge^3 E) \twoheadrightarrow H^1_*(X, \mathcal{Q}_{3,2,0}).$$

In particular, $H^1_*(X, \mathcal{Q}_{3,2,0})$ is 1-generated.

Next, we recall the following special case of diagram (14) (with $r = 3, i = 2, j = 1, k = 0$)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_{3,1} & \longrightarrow & E_{3,2} & \longrightarrow & \wedge^2 E \otimes G & \longrightarrow & 0\\
& & \downarrow & & \downarrow & & \| & &\\
0 & \longrightarrow & \wedge^2 G \otimes E & \longrightarrow & \mathcal{Q}_{3,2,0} & \longrightarrow & \wedge^2 E \otimes G & \longrightarrow & 0.
\end{array} \qquad (69)
$$

In the bottom horizontal sequence, we have used the fact that $\mathcal{Q}_{r,i,i-1} \cong \wedge^i E \otimes \wedge^{r-i} G$, for $r = 3$ and $i = 1, 2$ (see equation (13)).

We claim that the map

$$H^1_*(X, \mathcal{Q}_{3,2,0}) \to H^1_*(X, \wedge^2 E \otimes G)$$

obtained by taking cohomology in the above diagram is a surjection. To see this, we note that from diagrams (67) and (69), the boundary map

$$H^i_*(X, \wedge^2 E \otimes G) \to H^{i+1}_*(X, \wedge^2 G \otimes E) \qquad (70)$$

factors as

$$H^i_*(X, \wedge^2 E \otimes G) \to H^{i+1}_*(X, E_{2,1} \otimes G) \xrightarrow{\delta} H^{i+1}_*(X, E_{3,1}) \to H^{i+1}_*(X, \wedge^2 G \otimes E).$$

Since $\delta = 0$ when $0 \leqslant i \leqslant n-2$ (see (66)), the boundary map in (70) vanishes for $0 \leqslant i \leqslant n-2$. This proves our claim.

To prove that $H^2_*(X, \wedge^2 E \otimes E(-d))$ is 1-generated, we will show that there is an isomorphism

$$H^1_*(X, \wedge^2 E \otimes G) \cong H^2_*(X, \wedge^2 E \otimes E(-d)).$$

To do so, we first note that since E is a rank 3 ACM bundle, $\wedge^2 E$ is also ACM. We have the exact sequence (6): $0 \to E(-d) \to \overline{F}_1 \to G \to 0$; tensoring this with $\wedge^2 E$ and taking cohomology, we get a long exact sequence

$$\cdots H^1_*(X, \wedge^2 E \otimes \overline{F}_1) \to H^1_*(X, \wedge^2 E \otimes G) \to H^2_*(X, \wedge^2 E \otimes E(-d)) \to H^2_*(X, \wedge^2 E \otimes \overline{F}_1) \cdots.$$

The extreme terms vanish, and so we have our desired isomorphism. This completes the proof of (b).

$\square$

6. PROOF OF THE MAIN RESULTS

Theorem 6.1 (= Theorem 1.1 (i))**.** *Let* $X \subset \mathbb{P}^6$ *be a smooth hypersurface. Then any rank 3 ACM bundle on* X *is split.*

Proof. From Proposition 5.6, Theorem 5.7 and the sequence (top row of (68))

$$0 \to \wedge^3 G \to E_{3,2} \to \mathcal{Q}_{3,2,0} \to 0,$$

it follows that $H^i_*(X, \mathcal{Q}_{3,2,0}) = 0$ when $i = 2, 3$. Similarly the sequence

$$0 \to \wedge^3 G \to E_{3,1} \to \wedge^2 G \otimes E \to 0$$

gives $H^i_*(X, \wedge^2 G \otimes E) = 0$ when $i = 2, 3$. The last two vanishings together with the sequence

$$0 \to \wedge^2 G \otimes E \to \mathcal{Q}_{3,2,0} \to \wedge^2 E \otimes G \to 0$$

imply $H^2_*(X, \wedge^2 E \otimes G) = 0$. Using Serre duality, and a similar analysis with $E^\vee$ and $G^\vee$ we also get that $H^3_*(X, \wedge^2 E \otimes G) = 0$. Using this in the sequence

$$0 \to \wedge^2 E \otimes G \to \wedge^2 E \otimes F_0 \to \wedge^2 E \otimes E \to 0,$$

we then get that $H^2_*(X, \wedge^2 E \otimes E) = 0$. In particular, $H^2(X, \mathcal{E}nd(E)(-d)) = 0$.

Recall that under the boundary isomorphism $H^1(X, E^\vee \otimes G) \to H^2(X, \mathcal{E}nd E(-d))$, the extension class ζ maps to the class η (§2.2). Consequently both η and ζ vanish. This means that E must be a split bundle (Remark 2). $\square$

Theorem 6.2 (= Theorem 1.1 (ii))**.** *Let* $X \subset \mathbb{P}^5$ *be a general hypersurface of degree* $d \geqslant 3$. *Then any rank 3 ACM bundle* E *splits.*

Proof. The proof is exactly as in [13]. Since E is supported on a general hypersurface X, this means that the multiplication map

$$g : H^2(X, \mathcal{E}nd E(-d)) \to H^2(X, \mathcal{E}nd E)$$

is zero for any $g \in H^0(X, \mathcal{O}_X(d))$ (see Theorem 3.7 in *op. cit.*). However, from Theorem 5.7 and its proof, we have

$$H^0_*(X, \wedge^3 E) \twoheadrightarrow H^1_*(X, \wedge^2 E \otimes G) \cong H^2_*(X, \wedge^2 E \otimes E(-d)) \cong H^2_*(X, \mathcal{E}nd E(c - d)).$$

Hence the last group is 1-generated, with its generator in degree $-d$. Together, this means that $H^2(X, \mathcal{E}nd E(a)) = 0$ for $a \geqslant 0$. On the other hand, by Serre duality we have

$$H^2(X, \mathcal{E}nd E(-d)) \cong H^2(X, \mathcal{E}nd E(2d - 6)) = 0 \text{ whenever } d \geqslant 3.$$

This means that when $d \geqslant 3$, $H^2(X, \mathcal{E}nd E(-d)) = 0$, and so in particular, $\eta = 0$. The vanishing of η is equivalent to the splitting of E as noted before. This finishes the proof. $\square$

Remark 6. We note that the main result of [21] shows that the above theorem follows from Theorem 5.7 (a). However, part (b) of the same theorem yields the far simpler proof above.

Theorem 6.3 (= Theorem 1.1 (iii))**.** *Let* $X \subset \mathbb{P}^4$ *be a general hypersurface of degree* $d \geqslant 5$. *Then any rank 3 ACM bundle is rigid.*

Proof. We need to show that $H^1(X, \mathcal{E}nd E) = 0 = H^2(X, \mathcal{E}nd E) = 0$. Arguing as above, we see that $H^2(X, \mathcal{E}nd E(a)) = 0$ for $a \geqslant 0$. Furthermore, by Serre duality, $H^1(X, \mathcal{E}nd E) \cong H^2(X, \mathcal{E}nd E(d - 5))$, and hence $H^1(X, \mathcal{E}nd E) = 0$ for $d \geqslant 5$. $\square$

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